South American Journal of Logic
Vol. 3, n. 2, pp. 225–272, 2017
ISSN: 2446-6719

Duality Patterns in 2-PCD Fragments

Hans Smessaert and Lorenz Demey

Abstract

The two main types of logical relations and diagrams studied in logical geometry are the Aristotelian and the duality diagrams. The central aim of this paper is to establish a typology of duality patterns exhibited by fragments of four formulas which are closed under negation, i.e. which consist of two pairs of contradictory formulas (PCDs). These duality patterns are computed in two steps. First of all, each of the PCDs in the four-formula fragment is shown to generate its own — possibly collapsed — duality square. Secondly, these two 'intermediate' squares are superimposed onto one another, yielding seven major types of duality patterns for the four-formula fragment as a whole. Furthermore, these seven duality patterns are related to the complexity of the semantic representations assigned to the four formulas in the fragment, as expressed in terms of their bitstring encodings.

Keywords: duality diagram, self-duality, degenerate duality square, collapsed duality square, Boolean algebras, logical geometry.

1 Introduction

Duality phenomena occur in nearly all mathematically formalized disciplines, such as algebra, geometry, logic and natural language semantics [17, 23]. However, many of these disciplines use the term 'duality' in vastly different — sometimes even entirely unrelated — senses. The present paper exclusively focuses on duality patterns involving the interaction between an 'external' and an 'internal' negation of some kind, which primarily arise in logic and linguistics. A well-known example from logic is the duality between conjunction and disjunction in classical propositional logic. A well-known example from linguistics concerns the duality between the aspectual particles *already* and *still* in natural language [28, 30, 46, 48].[1]

[1] In particular, *already outside* means the same as *not still inside*, and hence, *not already outside* means the same as *still inside* (where *inside* is taken to be synonymous to *not outside*).

Since duality phenomena are ubiquitous both in formal logical languages and in natural languages, it has been suggested that duality is a semantic universal, which can be of great heuristic value in comparative linguistic research [47]. Furthermore, duality also plays a central role in *artificial* languages, which can be viewed as occupying an intermediate position between formal and natural languages. For example, Lincos, which was developed by Freudenthal [14] for the purpose of cosmic communication, contains duality principles for conjunction/disjunction (1.36.8), universal/existential quantification (1.36.9), necessity/possibility (3.25.1) and obligation/permission (3.32.3). It is important to stress that many authors employ the notion of duality as a means to describe the specific details of a particular formal or natural language, without going into any systematic theorizing about the notion itself. The present paper, however, does offer a more abstract, theoretical perspective on the concept of duality.

The first main aim of this paper is to establish a typology of duality patterns exhibited by fragments of four formulas which are closed under negation, i.e. which consist of two pairs of contradictory formulas (PCDs). These duality patterns are computed in two steps. First of all, each of the PCDs in the four-formula fragment is shown to generate its own duality square. Secondly, these two 'intermediate' squares are superimposed onto one another, yielding seven major types of duality patterns for the four-formula fragment as a whole. Underlying this typology is the threefold distinction between (1) classical, (2) collapsed and (3) degenerate constellations, which correlates with two formulas respectively standing (1) in exactly one duality relation, (2) in two duality relations simultaneously, or (3) in no duality relation whatsoever. The second aim of the paper is then to investigate the relationship between the seven duality patterns and the complexity of the semantic representations assigned to the four formulas in the fragment. In the framework of logical geometry, this complexity gets expressed in terms of (the length of the) bitstring encodings.

Given the fact that we — once again [10, 41] — want to emphasize the conceptual independence of the duality relations and the Aristotelian relations, it may come as a surprise to see the two (sets of) concepts are so closely united in the present paper. By focusing exclusively on 2-PCD fragments, we essentially take the Aristotelian notion of contradiction as our starting point for the study of duality patterns. Furthermore the 2-PCD fragments that we will study in Section 4 can all be characterised as subdiagrams of a given Boolean closed Aristotelian diagram. In other words, the fragments of four *formulas* that we consider — together with their bitstring representations — are characterised in terms of (Boolean-)Aristotelian considerations,[2] whereas the logical *relations*

[2]See [12] for more mathematical details on the close relationship between Boolean and Aristotelian structure.

holding between the formulas are analysed from a duality perspective. However, this approach need not be problematic, as can be gathered from the fact that exactly the opposite approach shows up time and again in the history of logic, when duality considerations are used to delineate the set of *formulas*, but the logical *relations* between them are studied from an Aristotelian perspective. For example, already with the classical square of oppositions, the focus is on the Aristotelian relations of contradiction and (sub)contrariety, while the formulas are defined in duality terms, i.e. in terms of different syntactic positions for the negation operator, such as pre- and postnegation.[3] Furthermore, also later and more complex octagonal diagrams primarily capture constellations of Aristotelian relations, whereas the formulas decorating the eight vertices are defined in duality terms.[4]

The paper is organised as follows. As for the broader background of the typology of duality patterns, Section 2 stresses the need to draw a clear conceptual distinction between duality relations — such as internal and external negation — and Aristotelian relations — such as contradiction and (sub)contrariety. The core results of the paper are presented in Section 3, which establishes the typology of seven main duality patterns. Three subfamilies are distinguished depending on whether the two superimposed intermediate squares are (1) both classical (2) one classical and one collapsed or (3) both collapsed. Section 4 then investigates the relation between these seven duality patterns and the differences in Boolean complexity of the semantic representations for the four formulas (measured in bitstring length). Finally, Section 5 sums things up and briefly introduces two topics for further research.

2 The Aristotelian versus the Duality Relations

In this section we start off by introducing the Aristotelian relations (Subsection 2.1) and the duality relations (Subsection 2.2). Among the similarities between the Aristotelian and the duality relations, we focus on the fact that both

[3] In medieval Latin rhymes, such as *pre contradic, post contra, pre postque subalter*, external negation ('pre') is associated with contradiction, internal negation ('post') with contrariety, and dual negation ('pre postque') with subalternation. These rhymes can be found in the logical works of authors such as Peter of Spain (see [6, pp. 10–11] and [4, pp. 116–117]), William of Sherwood [27, p. 38] and John Wyclif [13, p. 22].

[4] In the case of the octagons studied by John Buridan [25, 36], the formulas contain two operators which give rise to three syntactic negation positions, namely external, intermediate and internal negation. By contrast, in the case of the octagons studied by Keynes [24], Johnson [22], Reichenbach [37] and Hacker [18], the formulas contain only one operator but its two arguments — i.c. the subject and the predicate — can be negated independently, yielding one external but two internal negation positions. For a detailed analysis of these two types of octagons and the duality cubes they give rise to, see [10].

sets of relations yield a distinction between classical and degenerate squares (Subsection 2.3). As for the — more fundamental — differences between the Aristotelian and the duality relations, a key role is played by the absence of irreflexivity, which may trigger the collapse of a square into a pair in the case of the duality relations but not in the case of the Aristotelian relations (Subsection 2.4).

2.1 The Aristotelian Relations

Logical relations such as contradiction and contrariety have become known as the *Aristotelian relations*, since they were originally defined in the logical works of Aristotle [1]. In contemporary terms, these relations are characterised relative to some background logical system S, which is assumed to have connectives expressing Boolean negation ($\neg$), conjunction ($\wedge$) and implication ($\rightarrow$), and a model-theoretic semantics ($\models$).[5] Formally, the four Aristotelian relations are defined as follows: the formulas φ and ψ are said to be

S-*contradictory* ($\textsc{cd}_\mathsf{S}$)	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
S-*contrary* ($\textsc{c}_\mathsf{S}$)	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$,
S-*subcontrary* ($\textsc{sc}_\mathsf{S}$)	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
in S-*subalternation* ($\textsc{sa}_\mathsf{S}$)	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$.

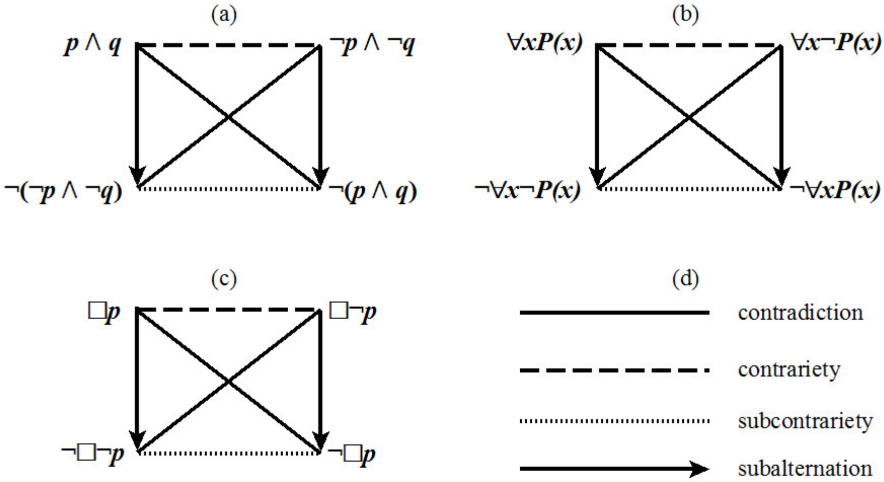
More informally, two formulas are contradictory iff they cannot be true together and cannot be false together; they are contrary iff they cannot be true together but may be false together; they are subcontrary iff they cannot be false together but may be true together; they are in subalternation iff the first one entails the second one but not vice versa.

The Aristotelian relations holding between a given set of formulas are often visualised by means of *Aristotelian diagrams* (based on graphical conventions such as the one shown in Figure 1(d)). The most widely known of these diagrams is the so-called 'square of oppositions', which comprises 4 formulas and the 6 Aristotelian relations holding between them.[6] For example, Figure 1 shows Aristotelian squares involving (a) the propositional connectives of conjunction and disjunction, (b) the universal and existential quantifiers, and (c) the modal operators of necessity and possibility.

[5]When the system S is clear from the context, it is often left implicit.

[6]For a more exhaustive historical overview, see [33] and [39, chapter 5].

Figure 1: Aristotelian squares: (a) conjunction-disjunction, (b) universal-existential, (c) necessity-possibility; (d) graphical representations of the Aristotelian relations.



2.2 The Duality Relations

In order to present a precise formal definition of duality relations such as internal and external negation, we consider Boolean algebras $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, \neg_{\mathbb{A}}, \top_{\mathbb{A}}, \bot_{\mathbb{A}}\rangle$ and $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \bot_{\mathbb{B}}\rangle$ [15],[7] and n-ary operators $O_1, O_2 \colon \mathbb{A}^n \to \mathbb{B}$. The duality relations are defined as follows: O_1 and O_2 are

- *identical* — abbreviated as $\textsc{id}(O_1, O_2)$ — iff $\forall a_1, \ldots, a_n \in A \colon O_1(a_1, \ldots, a_n) = O_2(a_1, \ldots, a_n)$,
- each other's *external negation* — abbreviated as $\textsc{eneg}(O_1, O_2)$ — iff $\forall a_1, \ldots, a_n \in A \colon O_1(a_1, \ldots, a_n) = \neg_{\mathbb{B}} O_2(a_1, \ldots, a_n)$,
- each other's *internal negation* — abbreviated as $\textsc{ineg}(O_1, O_2)$ — iff $\forall a_1, \ldots, a_n \in A \colon O_1(a_1, \ldots, a_n) = O_2(\neg_{\mathbb{A}} a_1, \ldots, \neg_{\mathbb{A}} a_n)$,
- each other's *dual* — abbreviated as $\textsc{dual}(O_1, O_2)$ — iff $\forall a_1, \ldots, a_n \in A \colon O_1(a_1, \ldots, a_n) = \neg_{\mathbb{B}} O_2(\neg_{\mathbb{A}} a_1, \ldots, \neg_{\mathbb{A}} a_n)$.

For any relation $R \in \{\textsc{id}, \textsc{ineg}, \textsc{eneg}, \textsc{dual}\}$, it holds that R is *functional*: for each O_1, there is exactly one O_2 such that $R(O_1, O_2)$. Hence we can simply switch from relational to functional notation, and write $O_2 = R(O_1)$. For example, since $\textsc{dual}(\wedge, \vee)$, we can write $\vee = \textsc{dual}(\wedge)$, and say that $\vee$ is the

[7]Note that one can also give a definition of the Aristotelian relations relative to arbitrary Boolean algebras, which has the definition in Subsection 2.1 as a special instance [11].

(unique) dual of $\wedge$. Furthermore, any relation $R \in \{\text{ID}, \text{INEG}, \text{ENEG}, \text{DUAL}\}$ is *symmetric*; this can functionally be expressed as follows: $O_2 = R(O_1)$ iff $O_1 = R(O_2)$, which is itself equivalent to the property that $R(R(O)) = O$ for all operators $O\colon \mathbb{A}^n \to \mathbb{B}$. Furthermore, the definitions of the duality relations/functions can harmlessly be transposed from *operators* $O\colon \mathbb{A}^n \to \mathbb{B}$ to the *outputs* of those operators. For example, if the operator $O_2\colon \mathbb{A}^n \to \mathbb{B}$ is the dual of the operator $O_1\colon \mathbb{A}^n \to \mathbb{B}$, then for all $a_1, \ldots, a_n \in \mathbb{A}$, it holds that $O_2(a_1, \ldots, a_n) \in \mathbb{B}$ is the dual of $O_1(a_1, \ldots, a_n) \in \mathbb{B}$. Hence, in the case of classical propositional logic (CPL), for example, we can say not only that the operator $\vee$ is the dual of the operator $\wedge$ — i.e. $\vee = \text{DUAL}(\wedge)$ —, but also that the formula $\varphi \vee \psi$ is 'the' dual (up to logical equivalence) of the formula $\varphi \wedge \psi$ — i.e. $\varphi \vee \psi = \text{DUAL}(\varphi \wedge \psi)$ — for all $\varphi, \psi \in \mathcal{L}_{\mathsf{CPL}}$.

When ID, ENEG, INEG and DUAL are viewed as functions, they map each operator $O\colon \mathbb{A}^n \to \mathbb{B}$ onto the operators $\text{ID}(O), \text{ENEG}(O), \text{INEG}(O), \text{DUAL}(O)\colon \mathbb{A}^n \to \mathbb{B}$, respectively. Since the input and output of these functions are of the same type (namely: operators $\mathbb{A}^n \to \mathbb{B}$), they can be applied repeatedly. For example, starting with an operator $O\colon \mathbb{A}^n \to \mathbb{B}$, we can apply INEG to it to obtain the operator $\text{INEG}(O)\colon \mathbb{A}^n \to \mathbb{B}$; by applying ENEG to the latter we obtain the operator $\text{ENEG}(\text{INEG}(O))\colon \mathbb{A}^n \to \mathbb{B}$. It follows immediately from the definitions of the duality relations/functions that $\text{ENEG}(\text{INEG}(O)) = \text{DUAL}(O)$. Since this holds independently of the concrete operator O, we can write $\text{ENEG} \circ \text{INEG} = \text{DUAL}$. In this way, we obtain a large number of functional identities that descibe the behavior of the duality and internal/external negation functions. These identities can be summarized by stating that the functions ID, ENEG, INEG and DUAL jointly form a group that is isomorphic to the *Klein four group* $\mathbf{V}_4$ (German: *Kleinsche Vierergruppe*).[8] Its Cayley table looks as follows:

$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

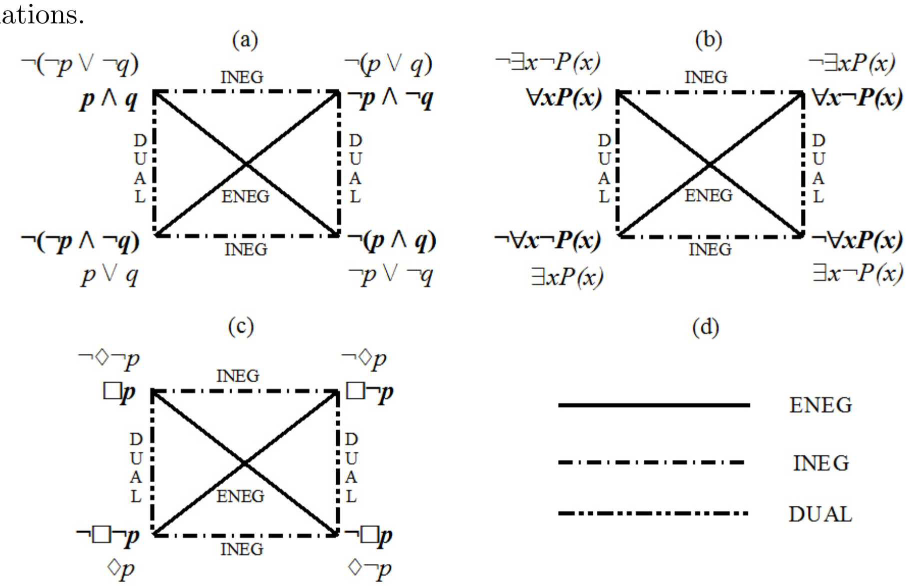
For every operator $O\colon \mathbb{A}^n \to \mathbb{B}$, one can define the set of four operators $\delta(O) := \{\text{ID}(O), \text{ENEG}(O), \text{INEG}(O), \text{DUAL}(O)\}$.[9] It is natural to view the set

[8]The fact that duality behavior can be described by means of $\mathbf{V}_4$ was already noted by authors such as Piaget [35], Gottschalk [16], Löbner [29], van Benthem [47] and Peters and Westerståhl [34], although many of them used slightly differing labels for the group elements. Furthermore, although the fact that the Klein four group $\mathbf{V}_4$ is isomorphic to the direct product of $\mathbb{Z}_2$ with itself, i.e. $\mathbf{V}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^2$, is well-known in group theory, its logico-linguistic significance has only recently begun to be explored (see [7] and [10]).

[9]In light of the correspondence between operators and formulas, one can also write $\delta(\varphi) := \{\text{ID}(\varphi), \text{ENEG}(\varphi), \text{INEG}(\varphi), \text{DUAL}(\varphi)\}$, where $\varphi = O(a_1, \ldots, a_n)$.

$\delta(O)$ as 'generated' by the operator O; however, it should be emphasized that $\delta(O)$ can be seen as generated by *any* of its elements, i.e. for any $O' \in \delta(O)$, it holds that $\delta(O') = \delta(O)$ (see [34, p. 134] and [49, p. 205]).[10] The set $\delta(O)$ is said to be 'closed under duality', in the sense that applying any of the ID-, ENEG-, INEG- or DUAL-functions to its elements only yields operators that already belong to $\delta(O)$. The operators (or formulas) in $\delta(O)$ thus constitute natural families (see [34, p. 26] and [47, p. 31]), which are often visualised by means of *duality squares*. Using the graphical conventions in Figure 2(d)), duality squares involving (a) the propositional connectives of conjunction and disjunction, (b) the universal and existential quantifiers, and (c) the modal operators of necessity and possibility are shown in Figure 2(a-c) respectively. It is important, in this respect, to emphasize that the ID-relations are not visualised explicitly in any of these duality squares, since they would simply constitute loops on all vertices of the squares.

Figure 2: Duality squares: (a) conjunction-disjunction, (b) universal-existential, (c) necessity-possibility; (d) graphical representations of the duality relations.



2.3 Similarities between Aristotelian and Duality Relations

In terms of overall layout, the duality squares in Figure 2(a-c) closely resemble the Aristotelian squares in Figure 1(a-c). In particular: (i) on the diagonals,

[10]For DUAL(O), for instance, we find that $\delta(\text{DUAL}(O)) = \{\text{ID}(\text{DUAL}(O)), \text{ENEG}(\text{DUAL}(O)), \text{INEG}(\text{DUAL}(O)), \text{DUAL}(\text{DUAL}(O))\} = \{\text{DUAL}(O), \text{INEG}(O), \text{ENEG}(O), \text{ID}(O)\} = \delta(O)$.

the duality relation ENEG corresponds to the Aristotelian relation of contradiction, (ii) on the vertical edges, the duality relation DUAL corresponds to the Aristotelian relation of subalternation, and (iii) on the horizontal edges, the duality relation INEG corresponds to the Aristotelian relations of contrariety and subcontariety (see [47, p. 31], [21, p. 148], [34, p. 25] and [49, p. 202]).[11] These strong diagrammatic similarities might explain why authors such as D'Alfonso [5], Mélès [32] and Schumann [38] have come close to straightforwardly identifying the two types of squares — for example, by using Aristotelian terminology to describe the duality square (or vice versa), or by viewing one as a generalisation of the other. In spite of the important differences between the two — which we discuss in full detail in Subsection 2.4 below — the fact remains that in many concrete cases, a single fragment of four formulas simultaneously constitutes an Aristotelian and a duality square (for a recent example involving formulas with definite descriptions, see [9]).[12]

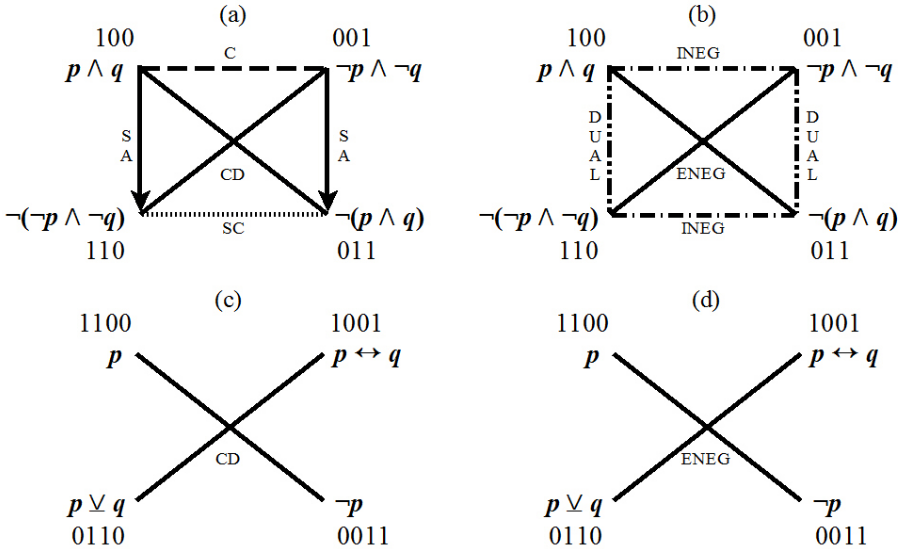
A second respect in which the sets of Aristotelian and duality relations resemble one another concerns the distinction between *classical* and *degenerate* constellations. If we focus — as will be the case in Sections 3 and 4 — on fragments of 4 formulas (from a logical language $\mathcal{L}$ for a logical system S) which are closed under negation, i.e. which consist of two pairs of contradictories, this analogy between the two sets can be summarised in the following table:

	classical square			**degenerate square**
Aristotelian	2 × CD	2 × SA	1 × C; 1 × SC	2 × CD
duality	2 × ENEG	2 × DUAL	2 × INEG	2 × ENEG

[11] Note that the visual representation of duality squares exhibits a much greater degree of variation than that of the Aristotelian squares. In [29, p. 69ff.] and [26, p. 201], for instance, the DUAL-relations occupy the diagonals, thereby graphically reflecting the fact that DUAL is the combination of ENEG on the vertical edges and INEG on the horizontal edges or vice versa. Alternatively, on the basis of his phase quantification approach to duality, Löbner has argued that INEG should be seen as the combination of ENEG and DUAL, and should thus occupy the diagonals of the square (see [30, p. 57] and [31, p. 488]).

[12] From a more linguistic point of view, both types of diagrams have been used to account for asymmetries in certain lexicalization patterns in natural languages. For example, Horn [19], Jaspers [21] and Seuren and Jaspers [40] make use of the Aristotelian relations to explain why natural languages lack primitive lexical items for particular vertices in the diagrams, whereas the phase quantification approach of Löbner [29, 31] explains the same linguistic phenomena in terms of the duality relations.

Figure 3: (a) classical Aristotelian square, (b) classical duality square, (c) degenerate Aristotelian square, and (d) degenerate duality square (see Section 4 for more details on the concrete bitstring assignments used here).



With either set of relations, a *classical* constellation consists of *six* relations, which means that any of the six pairs of distinct formulas that can be chosen from the four-formula fragment actually does stand in a relation of that set. By contrast, for each set of relations, a *degenerate* constellation consists of only *two* relations: only two out of the six pairs of distinct formulas — in particular those two pairs that we started out with to define the fragment in the first place — stand in any relation of that set. Visually speaking, the *classical* Aristotelian square in Figure 3(a) and the classical duality square in Figure 3(b) — which were already introduced in Figure 1(a-c) and Figure 2(a-c) respectively — represent two relations on the diagonals, two relations on the horizontal edges and two relations on the vertical edges. The *degenerate* Aristotelian and duality squares, by contrast, only represent the two relations on the diagonals in Figure 3(c) and Figure 3(d) respectively. This yields X-shaped patterns, with the four pairs of formulas along the edges of the squares not standing in any Aristotelian/duality relation whatsoever.[13] As for the Aristotelian relations, the difference between the classical and the degenerate square is well-understood: in order for two formulas φ and ψ to stand in no Aristotelian relation whatsoever — as in the degenerate case of in Figure 3(c) — four conditions must be met: φ and ψ (1) can be true together and (2)

[13]For the Aristotelian relations, Béziau and Payette [3, pp. 11–12] refer to this impoverished structure as "an X of opposition".

can be false together, (3) φ does not entail ψ, and (4) ψ does not entail φ either.[14] As for the duality relations, by contrast, the difference between the classical square in Figure 3(b) and the degenerate square in Figure 3(d) has hardly ever — if at all — been discussed up till now. After all, the literature exclusively focuses on fragments of the form $\delta(O)$ (see Subsection 2.2), which will, by definition, always yield *classical* duality squares. It is precisely the aim of Section 3 and 4 to offer a first broad typology of duality patterns, which will also have to take into account the fact that certain duality squares 'collapse'. This property does not have any straightforward counterpart among the Aristotelian relations, as will be argued in Subsection 2.4.

2.4 Differences between Aristotelian and Duality relations

The conceptual independence of Aristotelian and duality relations has been argued for by many authors over the past two decades (see [29], [31], [34], [49], [7], [41], [10], among others). Globally speaking, the set of Aristotelian relations is fundamentally *heterogeneous* whereas the set of duality relations is fundamentally *homogeneous*. First of all, all four duality relations ID, INEG, ENEG, and DUAL are *symmetric* (see Subsection 2.2), but not all Aristotelian relations are: contradiction (CD), contrariety (C) and subcontrariety (SC) are symmetric (due to being defined in terms of being true/false together), whereas subalternation (SA) is not (since it is defined in terms of truth propagation; see [43]). Secondly, all four duality relations ID, INEG, ENEG, and DUAL are *functional* (again see Subsection 2.2), but not all Aristotelian relations are: as a matter of fact, only CD is functional, whereas C/SC/SA are not.[15]

Furthermore, there is no overall one-to-one correspondence whatsoever between the two sets of four relations: ID does not correspond to any Aristotelian relation at all, whereas INEG corresponds to both C and SC. Although DUAL may seem to correspond to SA, the former is both functional and symmetric, whereas the latter is neither. The only true correspondence between the duality relations and the Aristotelian relations therefore holds between ENEG and CD, which are both functional and symmetric. The connection between these relations at the diagonals of the two types of square diagrams will play a crucial role — by defining the 'pairs of contradictories' (PCDs) — in the typology of

[14] In terms of the bitstring format for assigning semantic representations to formulas (see also Section 4 for more details), this relation of logical independence or 'unconnectedness' in the degenerate Aristotelian square can be shown to require bitstrings of length at least 4 in Figure 3(c), whereas the formulas in the classical Aristotelian square in Figure 3(a) can always be encoded by means of bitstrings of length 3 [43], [45], [12]).

[15] In other words, whereas any formula has exactly one contradictory, it may be (sub)contrary to or in subalternation with more than one formula.

duality patterns to be established in Sections 3 and 4 below.[16]

The heterogeneous nature of the set of Aristotelian relations — as opposed to the uniformity of the set of duality relations — has been captured in terms of the properties of symmetry and functionality. A second cluster of differences between the two sets of relations can be accounted for in terms of the notions of uniqueness and (ir)reflexivity. As for *uniqueness*, we first of all observe that two contingent formulas can only stand in at most one Aristotelian relation: although the Aristotelian relations are not jointly exhaustive, they are mutually exclusive (see [7, p. 321] and [43, Section 3.1]).[17] With the duality relations, by contrast, the situation is more complex, since two formulas may stand (1) in no duality relation with each other, (2) in exactly one such relation, or even (2) in two duality relations at the same time, as will be illustrated shortly. In other words, the duality relations are neither mutually exclusive nor jointly exhaustive.

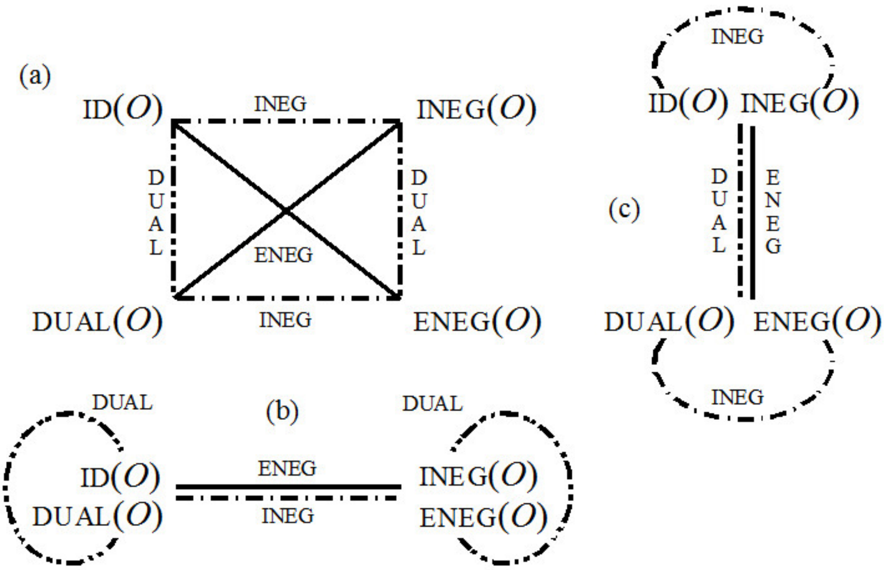
Closely related to the notion of uniqueness is that of *(ir)reflexivity.* Here as well, the picture is straightforward with the Aristotelian relations CD, C, SC and SA, since all four of them are irreflexive: no contingent formula can stand in any Aristotelian relation with itself.[18] With the duality relations, by contrast, the situation is again more complex. First of all, the duality relation ID is reflexive by definition (any operator/formula being identical to itself), whereas the duality relation ENEG is irreflexive on pain of inconsistency: if an operator $O\colon \mathbb{A}^n \to \mathbb{B}$ is its own external negation, then $\mathbb{B}$ is the trivial Boolean algebra (in which $\top_{\mathbb{B}} = \bot_{\mathbb{B}}$). The two remaining duality relations of INEG and DUAL, however, are neither reflexive nor irreflexive: some but not all contingent formulas may stand in the relation of INEG or DUAL with themselves. Let us now consider these two cases — which crucially combine the properties of non-uniqueness and non-irreflexivity — in some more technical detail.

[16] On a more abstract level, the Aristotelian relations are highly logic-sensitive, whereas the duality relations are insensitive to the underlying logic: two formulas may very well stand in different Aristotelian relations in different logical systems, but will always stand in the same duality relation, regardless of the logical system (see e.g. [8], [11] and [12]).

[17] In [43] the Aristotelian relations are argued to be hybrid between a set of four *opposition* relations and a set of four *implication* relations, both of which are mutually exclusive as well as jointly exhaustive.

[18] Note that with the two new sets of four logical relations proposed in [43], each set *does* contain a reflexive relation: as for the opposition relations, every contingent formula stands in the relationship of *non-contradiction* with itself (can be true together and can be false together), whereas among the implication relations, the relation of *bi-implication* by definition holds between any contingent formula and itself.

Figure 4: (a) Ordinary duality square, (b) collapsed duality pattern for an operator that is its own dual, (c) collapsed duality pattern for an operator that is its own internal negation.



For some operators $O\colon \mathbb{A}^n \to \mathbb{B}$, it might happen that $\textsc{dual}(O) = O = \textsc{id}(O)$, i.e. O is *self-dual*. In this case, one can also show that $\textsc{ineg}(O) = \textsc{eneg}(O)$, i.e. O's internal and external negation coincide with each other. For example, consider the identity operator $I_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$ (for any Boolean algebra $\mathbb{A}$), which is defined by $I_{\mathbb{A}}(a) := a$. For any element $a \in A$, it holds that $\textsc{dual}(I_{\mathbb{A}})(a) = \neg_{\mathbb{A}} I_{\mathbb{A}}(\neg_{\mathbb{A}} a) = \neg_{\mathbb{A}}\neg_{\mathbb{A}} a = a = I_{\mathbb{A}}(a)$, and thus $\textsc{dual}(I_{\mathbb{A}}) = I_{\mathbb{A}}$, i.e. $I_{\mathbb{A}}$ is self-dual. Similarly, for any element $a \in A$ it holds that $\textsc{ineg}(I_{\mathbb{A}})(a) = I_{\mathbb{A}}(\neg_{\mathbb{A}} a) = \neg_{\mathbb{A}} a = \neg_{\mathbb{A}} I_{\mathbb{A}}(a) = \textsc{eneg}(I_{\mathbb{A}})(a)$, and thus $\textsc{ineg}(I_{\mathbb{A}}) = \textsc{eneg}(I_{\mathbb{A}})$.

Completely analogously, for some operators $O\colon \mathbb{A}^n \to \mathbb{B}$, it can happen that $\textsc{ineg}(O) = O = \textsc{id}(O)$, i.e. O is its own internal negation. In this case, one can also show that $\textsc{dual}(O) = \textsc{eneg}(O)$, i.e. O's external negation and dual coincide with each other. Consider, for example, the contingency operator $C\colon \mathcal{L}_{\mathsf{D}} \to \mathcal{L}_{\mathsf{D}}$, which is defined by $C(\varphi) := \Diamond\varphi \wedge \Diamond\neg\varphi$. For any $\varphi \in \mathcal{L}_{\mathsf{D}}$, it holds that $\textsc{ineg}(C)(\varphi) = C(\neg\varphi) = \Diamond\neg\varphi \wedge \Diamond\neg\neg\varphi = \Diamond\varphi \wedge \Diamond\neg\varphi = C(\varphi)$, and thus $\textsc{ineg}(C) = C$. Similarly, it holds that $\textsc{dual}(C)(\varphi) = \neg C(\neg\varphi) = \neg(\Diamond\neg\varphi \wedge \Diamond\neg\neg\varphi) = \neg(\Diamond\varphi \wedge \Diamond\neg\varphi) = \textsc{eneg}(C)(\varphi)$, and thus $\textsc{dual}(C) = \textsc{eneg}(C)$.

Whenever an operator O is its own dual or internal negation, the set $\delta(O)$ — defined in Subsection 2.2 — does not contain *four*, but only *two* distinct operators (see [34, p. 134] and [49, p. 205]), and thus cannot be visualised using a classical duality square. If $O = \textsc{dual}(O)$, then $\delta(O) = \{\textsc{id}(O), \textsc{ineg}(O)\}$, and thus, the duality square in Figure 4(a) 'collapses' into the binary horizontal duality diagram in Figure 4(b). Analogously, if $O = \text{INEG}(O)$, then $\delta(O) = \{\text{ID}(O), \text{DUAL}(O)\}$, and thus, the duality square in Figure 4(a) collapses into the binary vertical duality diagram in Figure 4(c). Observe that both these *collapsed* duality squares straightforwardly visualise the systematic co-occurrence of the properties of non-uniqueness and non-irreflexivity in the case of the duality relations. With the classical duality square in Figure 4(a), the loops on the four vertices for the ID relation are not visualised explicitly. With the two collapsed duality squares in Figures 4(b-c), by contrast, the loops nicely capture the idea that certain operators or formulas can be their own DUAL or INEG respectively. At the same time, in the two collapsed duality squares, the ENEG relation (on the original diagonals of the square) turns out to coincide with the INEG or DUAL on the respective horizontal and vertical edges of the original square, again demonstrating the non-uniqueness of the duality relations.

Each vertex of the two collapsed duality squares in Figure 4(b-c) can be seen to contain two fomulas which may be (syntactically) distinct but are nevertheless logically equivalent. A collapsed duality square can thus be seen as consisting of two such *pairs of equivalent formulas*. The collapsed square in Figure 4(b) contains two *self-dual pairs* of equivalent formulas (henceforth abbreviated as SDPs), whereas the one in Figure 4(c) contains two *self-internal pairs* of equivalent formulas (henceforth abbreviated as SIPs). More technically, $\{\alpha, \text{DUAL}(\alpha)\}$ is an SDP iff $\alpha \equiv \text{DUAL}(\alpha)$ and $\{\alpha, \text{INEG}(\alpha)\}$ is an SIP iff $\alpha \equiv \text{INEG}(\alpha)$. Furthermore, if a formula is self-dual, its negation is self-dual as well, and the same holds for self-internal formulas. Hence, if $\{\alpha, \beta\}$ is an SDP, then $\{\neg\alpha, \neg\beta\}$ is an SDP as well, and similarly, if $\{\alpha, \beta\}$ is an SIP, then $\{\neg\alpha, \neg\beta\}$ is an SIP as well. In other words, a collapsed duality square by definition consists of *two* SDPs or *two* SIPs. Visually speaking, the two SDPs constitute the loops in Figure 4(b), whereas the two SIPs constitute the loops in Figure 4(c), and we will refer to these two types of collapsed squares informally as *self-dual squares* and a *self-internal squares* respectively.

From the group-theoretical perspective introduced in Subsection 2.2, the collapsed duality squares in Figures 4(b-c) no longer correspond to the Klein four group $\mathbf{V}_4$. As is shown in the Cayley tables below, with operators that are their own duals, $\text{DUAL} = \text{ID}$, and $\text{ENEG} = \text{INEG}$, and hence $\mathbf{V}_4$ collapses into a group that is isomorphic to $\mathbb{Z}_2$. And analogously, with operators that are their own internal negations, $\text{INEG} = \text{ID}$, and $\text{ENEG} = \text{DUAL}$, and thus $\mathbf{V}_4$ again collapses into a group that is isomorphic to $\mathbb{Z}_2$.

$\circ$	0	1
0	0	1
1	1	0

$\circ$	ID/DUAL	ENEG/INEG
ID/DUAL	ID/DUAL	ENEG/INEG
ENEG/INEG	ENEG/INEG	ID/DUAL

$\circ$	ID/INEG	ENEG/DUAL
ID/INEG	ID/INEG	ENEG/DUAL
ENEG/DUAL	ENEG/DUAL	ID/INEG

3 Duality Patterns in 2-PCD-Fragments

This section presents the typology of duality patterns exhibited by fragments consisting of 2 pairs of contradictory formulas (PCDs). Duality squares are no longer seen as generated by a single operator or formula, but on the basis of a PCD (Subsection 3.1). The first subfamily of 2-PCD-fragments yields constellations of two classical duality squares (Subsection 3.2). With the second subfamily, the duality configurations combine one classical and one collapsed square (Subsection 3.3), whereas with the third, two collapsed duality squares are integrated (Subsection 3.4). The overview of duality patterns draws a careful distinction between classical, degenerate and collapsed squares (Subsection 3.5).

3.1 Duality Squares Generated by PCDs

As was described in Subsection 2.2, the standard way to generate and study duality patterns is to start from one operator O, and then to define the set of four operators $\delta(O)$ as $\{\text{ID}(O), \text{ENEG}(O), \text{INEG}(O), \text{DUAL}(O)\}$. The methodology adopted in the present paper, however, is to start off from fragments $\mathcal{F}$ of four non-equivalent formulas, and to see whether or not the Klein group is at work in $\mathcal{F}$. The fragments to be considered all meet the requirement of being closed under negation, which means they consist of two *pairs of contradictory formulas* (henceforth abbreviated as PCDs). The α-PCD π_α and the β-PCD π_β are defined as $\pi_\alpha := \{\alpha, \neg\alpha\}$ and $\pi_\beta := \{\beta, \neg\beta\}$ respectively. A four-formula fragment $\mathcal{F}$ is then seen as the union of two PCDs, i.e. $\mathcal{F} = \pi_\alpha \cup \pi_\beta = \{\alpha, \neg\alpha, \beta, \neg\beta\}$.

The main justification for this restriction to negation-closed fragments is that the Aristotelian relation of contradiction (CD) corresponds with the duality relation of external negation (ENEG). As was argued in Subsection 2.4, CD and ENEG are the only relations that the Aristotelian and duality sets really have in common without any further complications. Hence, the combination of CD and ENEG counts as a good 'common core' to use as the basis for a typology of duality patterns which does justice to the (dis)similarities with the Aristotelian relations. Furthermore, remember from Figure 3 in Subsection 2.3 that, both with the Aristotelian square and with the duality square, it is the CD resp. ENEG relation that 'survives' when going from the classical to the degenerate cases. However, in view of the existence of the collapsed duality squares — which have no counterpart whatsoever among the Aristotelian relations —, the degenerate duality squares need to be further analysed and subclassified.

The general procedure for establishing such a typology of duality patterns in the next subsections consists of three steps. First of all, in order to emphasise the crucial role of the two individual PCDs π_α and π_β constituting the fragment $\mathcal{F}$, we define a new function δ^*. Unlike the original δ, this new δ^* takes a PCD as its input and yields the (possibly collapsed) duality square that can be generated by that PCD as its output:[19] $\delta^*(\pi_\alpha) := \{\text{ID}(\alpha), \text{ENEG}(\alpha), \text{INEG}(\alpha), \text{DUAL}(\alpha)\}$, and $\delta^*(\pi_\beta) := \{\text{ID}(\beta), \text{ENEG}(\beta), \text{INEG}(\beta), \text{DUAL}(\beta)\}$. In a second move, these two 'intermediate' duality squares are superimposed, yielding the extended fragment $\mathcal{F}^*$ which consists of eight formulas:

$$\begin{aligned}\mathcal{F}^* &:= \delta^*(\pi_\alpha) \ \cup \ \delta^*(\pi_\beta) \\ &= \{\text{ID}(\alpha), \text{ENEG}(\alpha), \text{INEG}(\alpha), \text{DUAL}(\alpha), \text{ID}(\beta), \text{ENEG}(\beta), \text{INEG}(\beta), \text{DUAL}(\beta)\}.\end{aligned}$$

As a third step, the overall duality constellation of the extended fragment $\mathcal{F}^*$ — including the possible collapses/equivalences — allows one to determine the duality pattern of the original fragment $\mathcal{F}$.

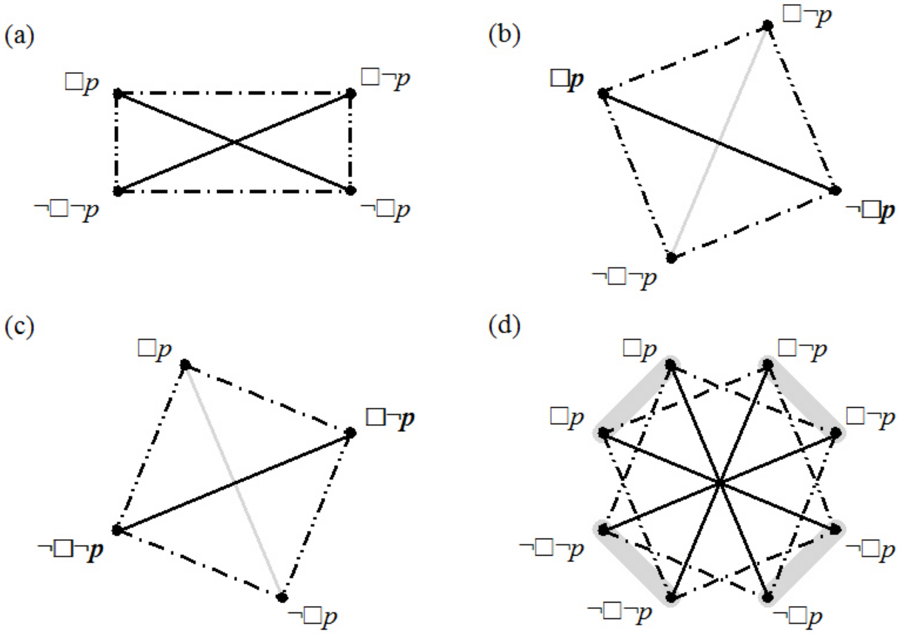
3.2 2-PCD-Fragments Generating two Classical Duality Squares

In order to apply the general three-step procedure introduced above, we consider the fragment $\mathcal{F}$ in Figure 5(a), where the formula at the top left vertex is $\alpha = \Box p$ and the formula at the bottom left vertex is $\beta = \neg\Box\neg p$. Hence, the α-PCD is the 'descending' diagonal from top left to bottom right, whereas the β-PCD is the 'ascending' diagonal from bottom left to top right, i.e. $\pi_\alpha = \{\Box p, \neg\Box p\}$ and $\pi_\beta = \{\neg\Box\neg p, \Box\neg p\}$. Next, two intermediate duality squares are generated by applying the δ^* function independently to the two PCDs of the fragment $\mathcal{F}$, yielding $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ in Figure 5(b) and (c) respectively. Note, first of all, that in both cases, an extra PCD is added as the light-grey diagonal, which is orthogonal to the original black diagonals for π_α and π_β respectively. Furthermore, the resulting duality configuration is classical in both cases: the duality relations of INEG and DUAL hold along the respective 'horizontal' and 'vertical' edges of both intermediate duality squares in Figures 5(b) and (c). In a final move, $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ are superimposed,

[19] Note that δ agrees with δ^*, in the sense that $\delta^*(\pi_\alpha) = \delta(\alpha) = \delta(\neg\alpha)$ (also recall Footnote 9).

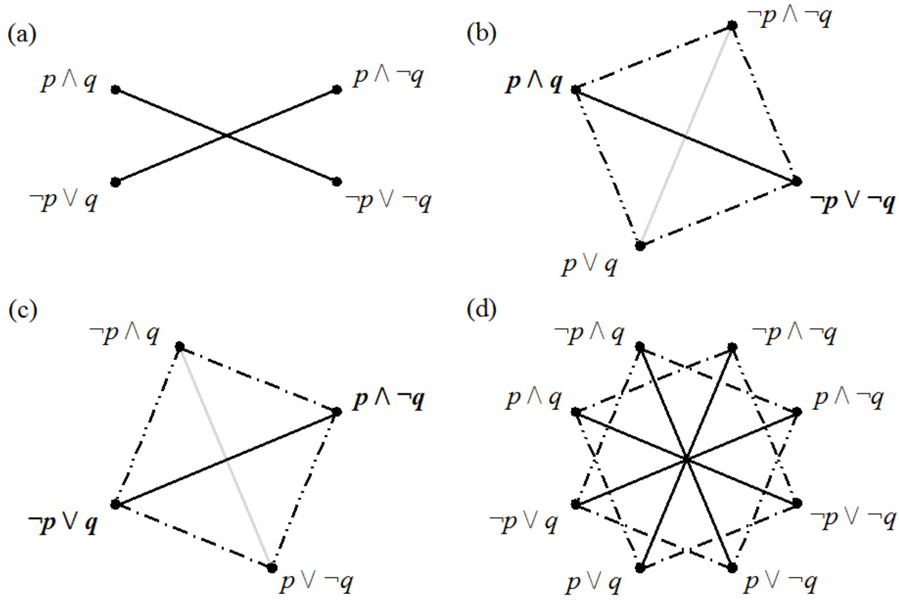
which results in the octagonal representation in Figure 5(d) for the extended fragment $\mathcal{F}^*$. This octagon reveals four pairs of equivalent formulas (indicated with the grey bars) and hence collapses into a square. In other words, the extended fragment $\mathcal{F}^*$ does not contain eight distinct formulas but instead collapses to a four-formula fragment identical to the original fragment $\mathcal{F}$. If we now focus on the two PCDs constituting the original fragment $\mathcal{F}$ inside the extended fragment $\mathcal{F}^*$, we observe that the duality diagram for $\mathcal{F}$ in Figure 5(a) contains the duality relations of INEG and DUAL along its horizontal and vertical edges, and hence is classified as a *classical* duality square. Furthermore, this classical duality pattern will be called CLCL1, since it is built from two classical duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$.

Figure 5: Example of the *classical* (CLCL1) duality pattern: (a) classical square for the fragment $\mathcal{F}$, (b) classical square for $\delta^*(\pi_\alpha)$, (c) classical square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



The second major duality pattern — illustrated in Figure 6 — closely resembles the first one, in that the two intermediate duality squares are again both classical. The fragment $\mathcal{F}$ in Figure 6(a) consists of $\pi_\alpha = \{p \wedge q, \neg p \vee \neg q\}$ and $\pi_\beta = \{\neg p \vee q, p \wedge \neg q\}$. The two duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$, which are shown in Figures 6(b-c), are both classical and hence, this duality pattern will be called CLCL2. Unlike the situation in Figures 5(b-c), however, the square for $\delta^*(\pi_\alpha)$ turns out to be different from that for $\delta^*(\pi_\beta)$. Superimposing $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ yields the octagon for the extended fragment $\mathcal{F}^*$ in Figure 6(d). As there are no logical equivalences among the eight formulas of $\mathcal{F}^*$, nothing can be collapsed. Again focusing on the two pcds constituting the original fragment $\mathcal{F}$, the four formulas turn out not to stand in any other (non-id) duality relation whatsoever — apart from the eneg diagonals. Therefore, the conclusion is that, even though, underlyingly, it gives rise to two independent classical duality squares — as revealed in the octagon for the fragment $\mathcal{F}^*$ — the original fragment $\mathcal{F}$ itself constitutes a *degenerate* duality pattern: the duality diagram for $\mathcal{F}$ in Figure 6(a) contains no duality relations along the edges of the square.

Figure 6: Example of the *simple* degenerate (clcl2) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) classical square for $\delta^*(\pi_\alpha)$, (c) classical square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.

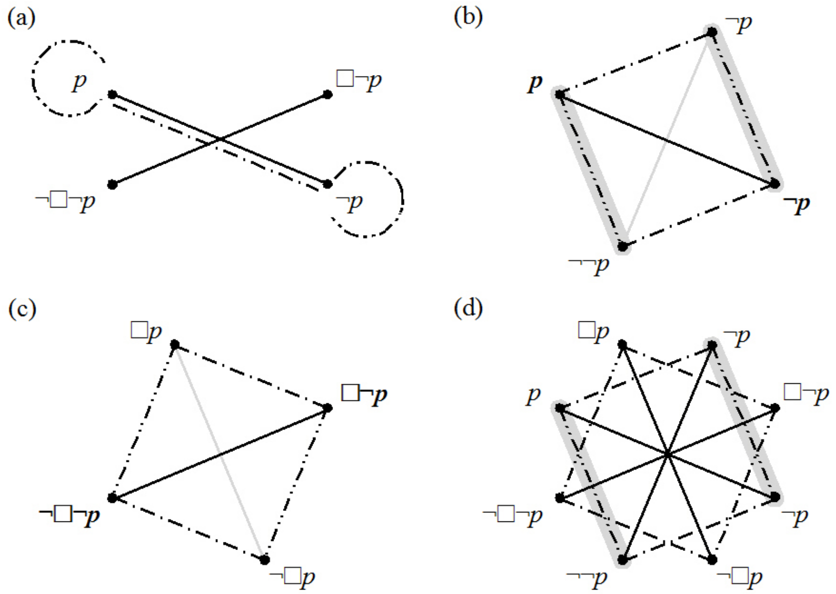


If we now compare the overall clcl1 and clcl2 duality patterns in Figures 5 and 6 respectively, we first of all observe a fundamental similarity in that both consist of two pcds that generate two *individual* classical duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ (as is reflected in their clcl naming). The major difference, however, resides in the *interaction* between these two intermediate squares within the extended fragment $\mathcal{F}^*$. In the octagon in Figure 5(d) the eight formulas are pairwise identical, whereas Figure 6(d) contains no equivalences/identities. For the original fragment $\mathcal{F}$ itself, the former constellation yields the *classical* clcl1 duality square in Figure 5(a), whereas the latter yields the *simple degenerate* CLCL2 duality square in Figure 6(a).

3.3 2-PCD-Fragments Generating one Classical and one Collapsed Duality Square

With the first subfamily of duality patterns introduced in the previous subsection, the intermediate squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ were both classical. We now turn to a second subfamily, in which one intermediate square is still a classical duality square, but the other is a collapsed duality square.[20]

Figure 7: Example of the *partially self-dual* degenerate (SDCL) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) self-dual collapsed square for $\delta^*(\pi_\alpha)$, (c) classical square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



Consider the fragment $\mathcal{F}$ in Figure 7(a), consisting of $\pi_\alpha = \{p, \neg p\}$ and $\pi_\beta = \{\neg\Box\neg p, \Box\neg p\}$. Notice that π_β in Figure 7 is identical to the second PCD in Figure 5, and hence the intermediate square in Figure 7(c) is the same classical duality square as in Figure 5(c). However, a new situation arises with the intermediate square for $\delta^*(\pi_\alpha)$ in Figure 7(b). With the so-called 'bare modalities' p and $\neg p$ in $\mathcal{L}_\mathsf{D}$, $\textsc{eneg}(p) = \textsc{ineg}(p)$ and $\textsc{eneg}(\neg p) = \textsc{ineg}(\neg p)$, and hence $\textsc{dual}(p) = (\textsc{eneg} \circ \textsc{ineg})(p) = \textsc{id}(p)$ and $\textsc{dual}(\neg p) = \textsc{id}(\neg p)$. In

[20] In the concrete examples in the present subsection, the square generated by π_α is collapsed and that generated by π_β is classical. Needless to say, the roles of π_α and π_β can be switched around without loss of generality.

other words, the bare modalities p and $\neg p$ are *self-dual*. So the grey diagonal which is added orthogonally to π_α in Figure 7(b) introduces the PCD $\{\neg\neg p, \neg p\}$, whose formulas are pairwise equivalent to those of π_α. More in particular, both $\{p, \neg\neg p\}$ and $\{\neg p, \neg p\}$ are SDPs — *self-dual pairs* of equivalent formulas (see Subsection 2.4) — and the grey bars indicate that these two SDPs will collapse. In Figure 7(d), $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ are superimposed to yield an octagon for the extended fragment $\mathcal{F}^*$. This octagon reveals two equivalences so that $\mathcal{F}^*$ no longer contains eight distinct formulas — as was the case in Figure 6(d) — but instead reduces to a six-formula fragment. As a consequence, the original fragment $\mathcal{F}$ itself constitutes a *degenerate* duality pattern: its duality diagram in Figure 7(a) contains no duality relations along the edges of the square. Nevertheless, it does contain two DUAL loops for the vertices of π_α — corresponding to the two underlying SDPs in Figure 7(b) — and an extra INEG relation for the diagonal of π_α itself. Because of this combination of a self-dual and a classical square, this duality pattern will be called SDCL, and will be referred to as a *partially self-dual degenerate* duality square.[21]

The fourth duality pattern — illustrated in Figure 8 — closely resembles the third one, in that one intermediate duality square is classical, whereas the other is collapsed. Consider the fragment $\mathcal{F}$ in Figure 8(a), consisting of $\pi_\alpha = \{p \leftrightarrow q, \neg(p \leftrightarrow q)\}$ and $\pi_\beta = \{\neg p \vee q, p \wedge \neg q\}$. Notice that the intermediate square for $\delta^*(\pi_\beta)$ in Figure 8(c) is the same classical duality square as that in Figure 6(c). Again, however, the intermediate square for $\delta^*(\pi_\alpha)$ in Figure 8(b) constitutes a collapsed duality square. The grey diagonal introduces the PCD $\{\neg p \leftrightarrow \neg q, \neg(\neg p \leftrightarrow \neg q)\}$, whose formulas are pairwise equivalent (in CPL) to those of π_α: $(p \leftrightarrow q) \equiv (\neg p \leftrightarrow \neg q)$ and $\neg(p \leftrightarrow q) \equiv \neg(\neg p \leftrightarrow \neg q)$. Hence, both $\{(p \leftrightarrow q), (\neg p \leftrightarrow \neg q)\}$ and $\{\neg(p \leftrightarrow q), \neg(\neg p \leftrightarrow \neg q)\}$ are SIPs — *self-internal pairs* of equivalent formulas (see Subsection 2.4)[22] — as indicated by the grey bars. In Figure 8(d), the octagon for the extended fragment $\mathcal{F}^*$ is generated by superimposing $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$. As was the case in Figure 7(d), the two equivalences trigger a reduction of $\mathcal{F}^*$ from an eight-formula fragment to a six-formula fragment.[23] Hence, the original fragment $\mathcal{F}$ itself constitutes a *degenerate* duality pattern: its duality diagram in Figure 8(a) contains no duality relations along the edges of the square. Nevertheless, it does contain two INEG loops for the vertices of π_α — corresponding to the two underlying SIPs in Figure 8(b) — and an extra DUAL relation for the diagonal of π_α it-

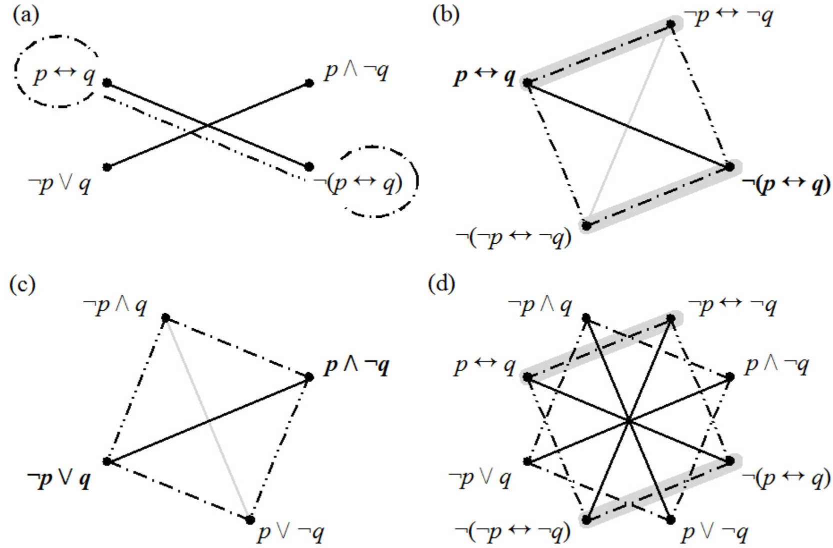
[21] Switching around π_α and π_β yields the completely analogous *partially self-dual degenerate* CLSD duality pattern.

[22] In other words, $\text{INEG}(p \leftrightarrow q) = \text{ID}(p \leftrightarrow q)$ and $\text{INEG}(\neg(p \leftrightarrow q)) = \text{ID}(\neg(p \leftrightarrow q))$ and hence $\text{ENEG}((\neg)(p \leftrightarrow q)) = \text{DUAL}((\neg)(p \leftrightarrow q))$.

[23] In the terminology of [41, p. 180], such six-formula fragments yield the so-called 'shield and spear' constellation for the duality hexagon, as opposed to the 'star' constellation for the Aristotelian hexagon.

self. Because of this combination of a self-internal and a classical square, this *partially self-internal degenerate* duality pattern will be called SICL.[24]

Figure 8: Example of the *partially self-internal* degenerate (SICL) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) self-internal collapsed square for $\delta^*(\pi_\alpha)$, (c) classical square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



When we compare the two duality patterns introduced in this subsection, the similarities are predominant. Both with the SDCL pattern in Figure 7 and with the SICL pattern in Figure 8, (1) the intermediate duality square for $\delta^*(\pi_\alpha)$ is collapsed, (2) the intermediate duality square for $\delta^*(\pi_\beta)$ is classical, (3) the extended fragment $\mathcal{F}^*$ in the octagon consists of six non-equivalent formulas, and (4) the original fragment $\mathcal{F}$ yields a degenerate duality square. The crucial difference between the two duality patterns then concerns the 'driving force' behind the collapse of the intermediate square for $\delta^*(\pi_\alpha)$. With the *partially self-dual degenerate* SDCL pattern, the square in Figure 7(b) collapses along its 'vertical' DUAL edges, yielding a π_α-PCD in Figure 7(a) with DUAL loops for the two underlying SDPs and an extra diagonal for INEG. With the *partially self-internal degenerate* SICL pattern, by contrast, the square in Figure 8(b) collapses along its 'horizontal' INEG edges, yielding a π_α-PCD in Figure 8(a) with INEG loops for the two underlying SIPs and an extra diagonal for DUAL.

When we compare the SDCL and SICL patterns with the CLCL1 and CLCL2

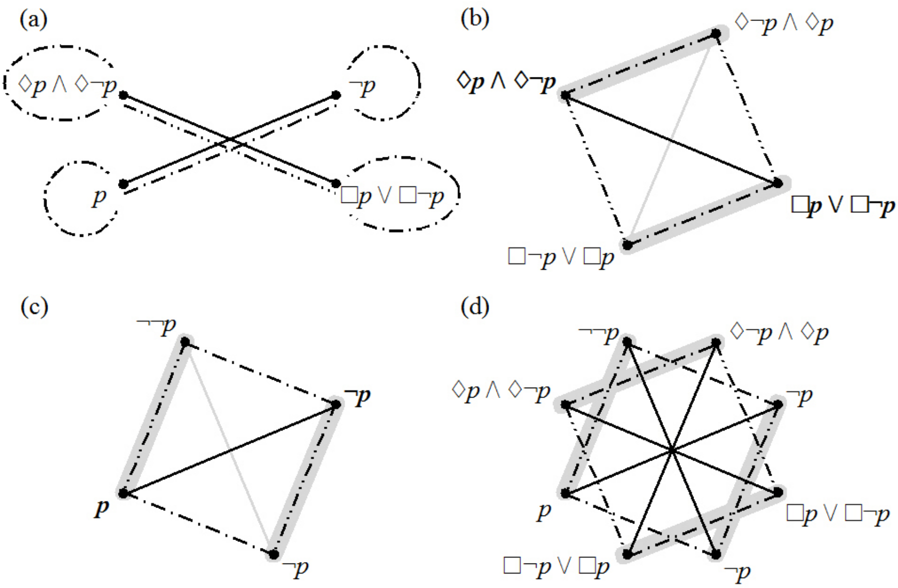
[24]Switching around π_α and π_β again yields the completely analogous *partially self-internal degenerate* CLSI duality pattern.

patterns introduced in Subsection 3.2, the major differences concern the nature of the original fragment $\mathcal{F}$ and the extended fragment $\mathcal{F}^*$. As for $\mathcal{F}^*$, the octagons in Figures 7(d) and 8(d) — with their *six* non-equivalent formulas and two equivalence pairs — can be seen to occupy an intermediate position between the CLCL1 octagon in Figure 5(d) — with its *four* non-equivalent formulas and four equivalence pairs — on the one hand, and the CLCL2 octagon in Figure 6(d) — with its *eight* non-equivalent formulas — on the other hand. As for the original fragment $\mathcal{F}$, the SDCL and SICL patterns at first sight resemble the CLCL2 pattern in that — in the absence of any duality relations along their edges — all three of them constitute a *degenerate* duality square. Nevertheless, the former two patterns manifestly reveal a much greater complexity w.r.t. the duality relations, by virtue of the non-trivial, i.e. non-ID, loops and the two coinciding duality relations along the diagonal of their π_α PCDs. In other words, unlike the Aristotelian relations — which basically have one subtype of degenerate square (see Subsection 2.3) — the duality relations have so far been shown to give rise to at least three subtypes of degenerate squares: (1) the *simple degenerate* CLCL2 square, (2) the *partially self-dual degenerate* SDCL square and (3) the *partially self-internal degenerate* SICL square.

3.4 2-PCD-Fragments Generating two Collapsed Duality Squares

Two subfamilies of duality patterns have been discussed so far: with the first subfamily — patterns CLCL1 and CLCL2 in Subsection 3.2 —, the intermediate squares generated for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ are both classical, whereas with the second subfamily — patterns SDCL and SICL in Subsection 3.3 —, one intermediate square is classical but the other is collapsed. We now turn to a third and final subfamily, consisting of three duality patterns in which *both* intermediate duality squares are collapsed.

Figure 9: Example of the *mixed self-internal/self-dual* degenerate (SISD) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) self-internal collapsed square for $\delta^*(\pi_\alpha)$, (c) self-dual collapsed square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



The duality pattern illustrated in Figure 9 contains both a self-internal collapsed square and a self-dual collapsed square. Consider the fragment $\mathcal{F}$ in Figure 9(a), consisting of $\pi_\alpha = \{\Diamond p \wedge \Diamond\neg p, \Box p \vee \Box\neg p\}$ and $\pi_\beta = \{p, \neg p\}$. First of all, the grey diagonal in the intermediate square for $\delta^*(\pi_\alpha)$ in Figure 9(b) introduces the additional PCD $\{\Diamond\neg p \wedge \Diamond p, \Box\neg p \vee \Box p\}$, whose formulas are pairwise (and trivially) equivalent to those of π_α. In particular, both $\{\Diamond p \wedge \Diamond\neg p, \Diamond\neg p \wedge \Diamond p\}$ and $\{\Box p \vee \Box\neg p, \Box\neg p \vee \Box p\}$ are SIPs, as indicated by the grey bars.[25] Notice that this *self-internal* collapsed square for $\delta^*(\pi_\alpha)$ is perfectly analogous to the one for the formula $p \leftrightarrow q$ from $\mathcal{L}_{\mathsf{CPL}}$ in Figure 8(b). Secondly, the intermediate square for $\delta^*(\pi_\beta)$ in Figure 9(c) is the same *self-dual* collapsed square (up to rotation) as that in Figure 7(b). And finally, the octagon for the extended fragment $\mathcal{F}^*$ in Figure 9(d) superimposes $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$. Combining the two pairs of equivalences — coming from each of the intermediate squares independently — triggers a reduction of $\mathcal{F}^*$ from an eight-formula fragment to a four-formula fragment. Hence, the original fragment $\mathcal{F}$ itself constitutes a *degenerate* duality pattern, as shown in Figure 9(a). Nevertheless, it yields an intricate overall duality constellation — even more so

[25]In other words, INEG$(\Diamond p \wedge \Diamond\neg p) =$ ID$(\Diamond p \wedge \Diamond\neg p)$ and INEG$(\Box p \vee \Box\neg p) =$ ID$(\Box p \vee \Box\neg p)$, and hence ENEG$(\Diamond p \wedge \Diamond\neg p) =$ DUAL$(\Diamond p \wedge \Diamond\neg p)$ and ENEG$(\Box p \vee \Box\neg p) =$ DUAL$(\Box p \vee \Box\neg p)$.

than the SDCL and SICL patterns from the previous subsection — since it not only contains two INEG loops (for the two underlying SIPs) and an extra DUAL relation for the PCD π_α, but also — conversely — two DUAL loops (for the two underlying SDPs) and an extra INEG relation for the PCD π_β. Hence, it will be called a *mixed self-internal/self-dual degenerate* SISD duality pattern.[26]

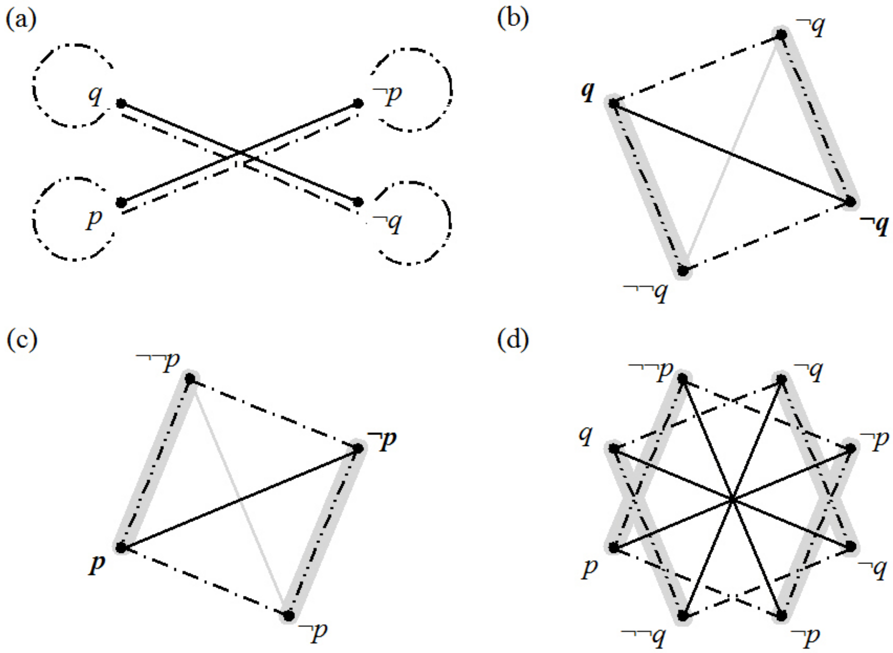
As was the case with the SISD pattern, the intermediate squares of the duality pattern illustrated in Figure 10, are both collapsed. However, since the fragment $\mathcal{F}$ consists of $\pi_\alpha = \{p, \neg p\}$ and $\pi_\beta = \{q, \neg q\}$, the collapsed duality squares for $\delta^*(\pi_\alpha)$ in Figure 10(b) and for $\delta^*(\pi_\beta)$ in Figure 10(c) are both *self-dual*, i.e. involving two SDPs each. The superimposition of $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ in the octagon for $\mathcal{F}^*$ in Figure 10(d) again involves four equivalence pairs, reducing $\mathcal{F}^*$ from an eight- to a four-formula fragment. Hence, $\mathcal{F}$ once again yields a *degenerate* duality square in Figure 10(a). The resulting overall duality constellation is equally complex as that of the SISD pattern in Figure 9(a), but more symmetrical in the sense that all four loops concern the same DUAL relation — corresponding to the four underlying SDPs — and the extra INEG relation holds for both π_α and π_β. Hence, it will be called a *fully self-dual degenerate* SDSD duality pattern.

Also with the final member of the third subfamily — illustrated in Figure 11 — the collapsed duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ are both of the same subtype. However, unlike the two self-dual squares with the SDSD pattern, the intermediate squares in Figures 11(b) and 11(c) are both *self-internal*. Consider the fragment $\mathcal{F}$ in Figure 11(a), consisting of $\pi_\alpha = \{$*more than 80% or less than 20% of the A's are B, at most 80% but at least 20% of the A's are B*$\}$ and $\pi_\beta = \{$*at least 80% or at most 20% of the A's are B, less than 80% but more than 20% of the A's are B*$\}$. These formulas are taken from the realm of *proportional quantification* in natural language.[27] Within the formal-semantic framework of Generalized Quantifier Theory [2], such formulas are considered to be of the general form Q(A,B), where Q is a (potentially complex) quantifier (such as *some*, *all*, or *more than 80% or less than 20%*), A is the subject noun (such as *children*) and B is the verbal predicate (such as *are asleep*). With formulas of the form Q(A,B), the internal negation is taken to operate on the verbal predicate argument only, i.e. INEG(Q(A,B)) := Q(A,¬B). The simple proportional quantifiers *more than 80%* and *less than 20%* can then be shown to be one another's INEG, and the same holds for *at least 80%* and *at most 20%*:

[26] Switching around π_α and π_β again yields the completely analogous *mixed self-internal/self-dual degenerate* SDSI duality pattern.

[27] In mathematical notation, *more than* corresponds to $>$, *less than* corresponds to $<$, *at least* corresponds to $\geq$ (i.e. *more than or equal*), and *at most* corresponds to $\leq$ (i.e. *less than or equal*).

Figure 10: Example of the *fully self-dual* degenerate (SDSD) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) self-dual collapsed square for $\delta^*(\pi_\alpha)$, (c) self-dual collapsed square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



$$\begin{aligned}
\text{INEG(more than 80\%(A,B))} &\equiv \text{more than 80\%(A,}\neg\text{B)}\\
&\equiv \text{less than 20\%(A,B)}\\
\text{INEG(at least 80\%(A,B))} &\equiv \text{at least 80\%(A,}\neg\text{B)}\\
&\equiv \text{at most 20\%(A,B)}
\end{aligned}$$

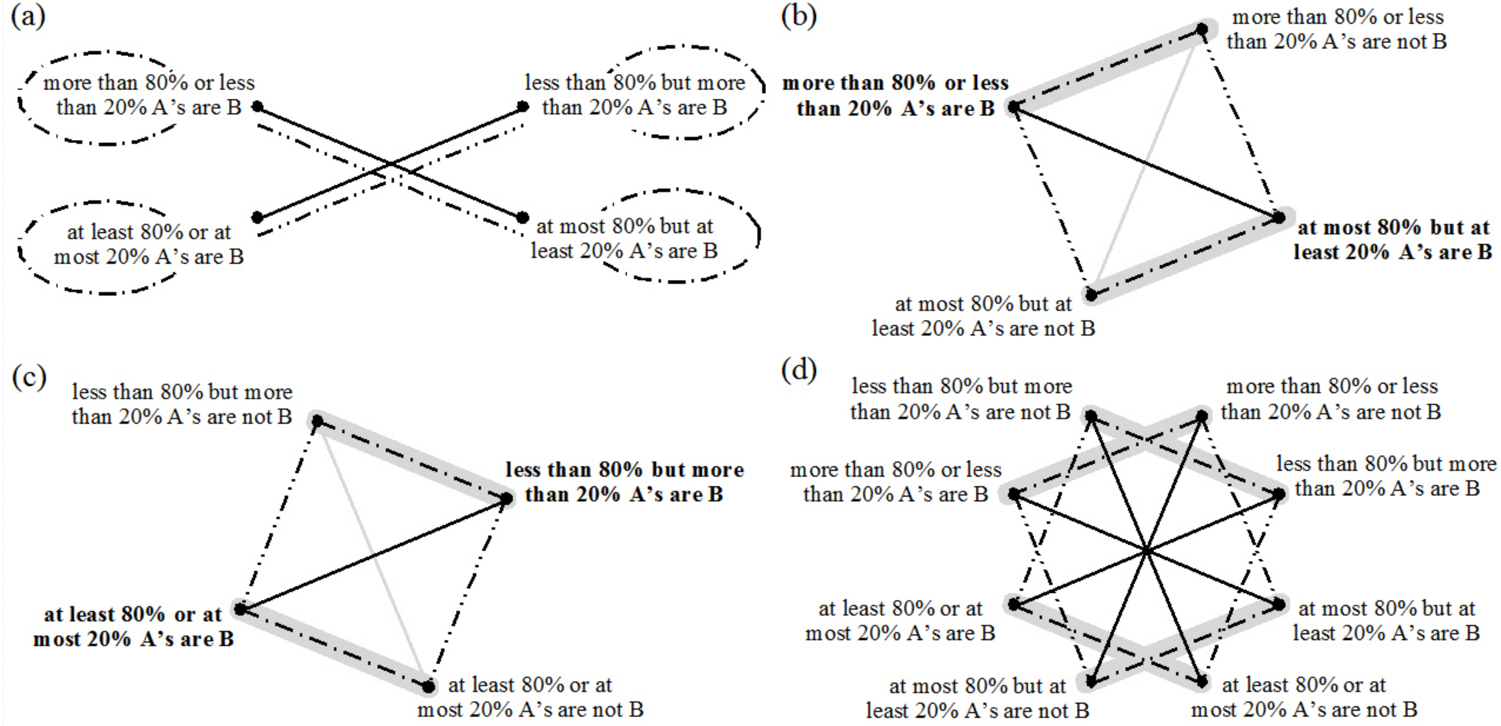
In other words, *more than 80% of the children are awake (i.e. not asleep)* is equivalent to *less than 20% of the children are asleep* and vice versa. Similarly, *at least 80% of the children are awake* is equivalent to *at most 20% of the children are asleep*. If we now consider complex proportional quantifiers — which are Boolean combinations of the simple proportional quantifiers above, such as *more than 80%* ***or*** *less than 20%* — we can observe the following equivalences:

$$\begin{aligned}
&\text{INEG[(more than 80\% } \textbf{or} \text{ less than 20\%)(A,B)]} & \\
\equiv\ & \text{(more than 80\% } \textbf{or} \text{ less than 20\%)(A,}\neg\text{B)} & (1)\\
\equiv\ & \text{more than 80\%(A,}\neg\text{B) } \textbf{or} \text{ less than 20\%(A,}\neg\text{B)} & (2)\\
\equiv\ & \text{less than 20\%(A,B) } \textbf{or} \text{ more than 80\%(A,B)} & (3)\\
\equiv\ & \text{more than 80\%(A,B) } \textbf{or} \text{ less than 20\%(A,B)} & (4)\\
\equiv\ & \text{(more than 80\% } \textbf{or} \text{ less than 20\%)(A,B)} & (5)
\end{aligned}$$

Step 1 is an application of the INEG function (as defined above), and steps 2 and 5 translate the disjunction between the level of the quantifiers and the level of the propositions, i.e. (Q1 $\vee$ Q2)(A,B) $\equiv$ Q1(A,B) $\vee$ Q2(A,B). Step 3 then reflects the key property, described above, that the two simple proportional quantifiers involved are one another's INEG. This sequence of equivalences demonstrates that *more than 80%* ***or*** *less than 20% of the A's are B* — the *disjunctive* α-formula from π_α in Figure 11(a) — is its own internal negation. A completely analogous sequence can be given to show that the *conjunctive* $\neg\alpha$-formula from π_α, namely *at most 80%* ***but*** *at least 20% of the A's are B*, is its own INEG as well. The general principle underlying these equivalences can be stated as follows: "A disjunction/conjunction as a whole is its own internal negation iff the two disjuncts/conjuncts are one another's internal negation".[28] Furthermore, and again completely analogously, both formulas constituting π_β in Figure 11(a) — namely *at least 80% or at most 20% of the A's are B* and *less than 80% but more than 20% of the A's are B* — turn out to be their own INEG.

[28]Note that this general formulation also accounts for the self-internal nature of the conjunctive formula $\Diamond p \wedge \Diamond\neg p$ and the disjunctive formula $\Box p \vee \Box\neg p$ in Figure 9.

Figure 11: Example of the *fully self-internal* degenerate (SISI) duality pattern: (a) degenerate square for the fragment $\mathcal{F}$, (b) self-internal collapsed square for $\delta^*(\pi_\alpha)$, (c) self-internal collapsed square for $\delta^*(\pi_\beta)$, (d) octagon for the fragment $\mathcal{F}^*$.



Having thus established the properties of the four proportional formulas in π_α and π_β in full detail, we can now (re)turn to the intermediate duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ in Figure 11(b) and in Figure 11(c) respectively, and observe that both squares are indeed *self-internal* collapsed duality squares, i.e. they each involve two SIPs. As a consequence, the superimposition of $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ in the octagon for $\mathcal{F}^*$ in Figure 11(d) again involves four equivalence pairs, reducing $\mathcal{F}^*$ from an eight- to a four-formula fragment. Hence, $\mathcal{F}$ once again yields a *degenerate* duality square in Figure 11(a). The resulting overall constellation is equally symmetrical as that of the SDSD pattern in Figure 10(a), the only differences being that all four loops now concern the INEG relation — corresponding to the four underlying SIPs — and that the extra relation holding for both π_α and π_β is that of DUAL. Hence, it will be called a *fully self-internal degenerate* SISI duality pattern.

If we now compare the SISD, SDSD and SISI patterns constituting the third subfamily, the similarities again prevail. With all three of them, (1) the intermediate $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ both yield collapsed duality squares, (2) the extended fragment $\mathcal{F}^*$ has four equivalence pairs and thus collapses from an eight-formula to a four-formula fragment, and (3) the original fragment $\mathcal{F}$ yields a degenerate duality square with four non-trivial (non-ID) loops on the vertices and two diagonals with double duality relations. The crucial difference between the three duality patterns then once again concerns the nature of the collapse of the intermediate squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$. Pattern SISD in Figure 9 combines a 'horizontal' INEG collapse for two SIPs with a 'vertical' DUAL collapse for two SDPs; pattern SDSD in Figure 10 consists of two vertical DUAL collapses with four SDPs; and pattern SISI in Figure 11 is the result of two horizontal INEG collapses with four SIPs. The overall conclusion is that — in addition to the three subtypes of degenerate duality squares characterised at the end of the previous subsection (i.e. *simple*, *partially self-dual* and *partially self-internal*) — the present subsection has introduced three more subtypes, namely (1) the *mixed self-internal/self-dual* SISD pattern, (2) the *fully self-dual* SDSD pattern, and (3) the *fully self-internal* SISI pattern.

3.5 Overview of Duality Patterns for 2-PCD-Fragments

The initial trigger for the proposed analysis of duality patterns was the observation in Subsection 2.3 that — for four-formula fragments consisting of two PCDs — both the Aristotelian relations and the duality relations yield *classical* and *degenerate* square constellations. In spite of this fundamental similarity, however, the situation turns out to be much more complex in the case of the duality relations, since these may also give rise to *collapsed* constellations (see Subsection 2.4), which have no counterpart among the Aristotelian relations.

Table 1: Typology of Duality Patterns for the 2-PCD-fragment $\mathcal{F}$ in terms of the numbers of (i) self-dual pairs (SDP), (ii) self-internal pairs (SIP), and (iii) non-equivalent formulas in fragment $\mathcal{F}^*$

pattern	SDP	SIP	$\mathcal{F}^*$	fragment $\mathcal{F}$
CLCL1	0	0	4	classical
CLCL2	0	0	8	*simple* degenerate
SDCL	2	0	6	*partially self-dual* degenerate
SICL	0	2	6	*partially self-internal* degenerate
SISD	2	2	4	*mixed self-internal/self-dual* degenerate
SDSD	4	0	4	*fully self-dual* degenerate
SISI	0	4	4	*fully self-internal* degenerate

As was described in full detail above, the general procedure for establishing the typology of duality patterns consists of three steps: (i) the two individual PCDs π_α and π_β, constituting the fragment $\mathcal{F}$, yield the respective duality squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$; (ii) these two intermediate duality squares are superimposed, yielding the extended fragment $\mathcal{F}^*$, consisting of eight formulas; and (iii) the overall duality constellation of $\mathcal{F}^*$ — including the possible collapses/equivalences — allows one to determine the duality pattern of the original fragment $\mathcal{F}$. The resulting typology consists of seven major types, namely one classical and six degenerate duality patterns.

At this point it is important to stress the precise relation between these seven patterns on the one hand, and the underlying tripartition into classical, collapsed and degenerate constellations on the other hand. Remember that a constellation is (1) *classical* if all pairs of formulas stand in exactly one duality relation, that it is (2) *collapsed* if all pairs of formulas stand in two duality relations simultaneously, and that it is (3) *degenerate* if at least one pair of formulas stands in no duality relation whatsoever. By definition of δ^*, the intermediate squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ can only be classical or collapsed (and never degenerate). By contrast, when assigning the duality pattern to $\mathcal{F}$ on the basis of $\mathcal{F}^*$, the result is either classical or degenerate: although it may contain one or even two collapsed squares, it can never be collapsed as a whole itself (because $\mathcal{F}$ is assumed to contain four non-equivalent formulas). In Table 1, the general procedure for distinguishing duality patterns is summarised, with the second and third columns characterising step one — the intermediate squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ —, the fourth column corresponding to step two — the superposition in fragment $\mathcal{F}^*$ — and the fifth column yielding the resulting subtype for the fragment $\mathcal{F}$ in step three.

Starting off with the characterisation of the intermediate squares in the second and third columns of Table 1, we first of all observe that the central subdivision into three subfamilies of duality patterns crucially relies on the number of collapsed squares involved. Remember that duality squares collapse when a formula and its INEG — or a formula and its DUAL — turn out to be logically equivalent to one another, thus rendering both formulas self-internal or self-dual respectively. Hence, if an intermediate square is collapsed, it consists of *two* pairs of equivalent formulas: either two SDPs (*self-dual pairs*), or else two SIPs (*self-internal pairs*). With the CLCL1 and CLCL2 patterns in the first subfamily of Table 1, neither of the two intermediate squares contains any SDPs or SIPs, i.e. they are both classical. With the SDCL and SICL patterns in the second subfamily, the intermediate square for $\delta^*(\pi_\alpha)$ is collapsed — containing two SDPs or two SIPs respectively — but the one for $\delta^*(\pi_\beta)$ is classical.[29] With the three patterns in the third subfamily, the intermediate squares are both collapsed: the SISD pattern has two pairs of each subtype of SDPs and SIPs, whereas the SDSD and SISI patterns have four pairs of the same subtype.

If we now turn to the fourth column of Table 1, the number of non-equivalent formulas in the extended fragment $\mathcal{F}^*$ exhibits a nice correlation with the number of SDPs and SIPs from the previous two columns, at least for the six degenerate patterns. Being defined as the union of $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$, fragment $\mathcal{F}^*$ contains eight formulas. With the CLCL2 pattern, neither $\delta^*(\pi_\alpha)$ nor $\delta^*(\pi_\beta)$ contains any SDPs or SIPs, and therefore *all eight* of the formulas in fragment $\mathcal{F}^*$ are non-equivalent. With the SDCL and SICL patterns, by contrast, the two equivalence pairs in $\delta^*(\pi_\alpha)$ yield *six* non-equivalent formulas for $\mathcal{F}^*$. And finally, with the SISD, SDSD and SISI patterns, combining two equivalence pairs in $\delta^*(\pi_\alpha)$ with two equivalence pairs in $\delta^*(\pi_\beta)$ results in *four* non-equivalent formulas for $\mathcal{F}^*$. In other words, with the six subtypes of degenerate duality patterns, the numbers of SDPs, SIPs and non-equivalent $\mathcal{F}^*$-formulas in the columns two through four in Table 1 systematically add up to eight. However, such a correlation does not hold for the CLCL1 pattern: it has no SDPs or SIPs and nevertheless only contains *four* non-equivalent $\mathcal{F}^*$-formulas, which is due to the fact that the four equivalence pairs involved have nothing to do with formulas being self-dual or self-internal.

Thirdly and finally, the terminology adopted in the right-most column of Table 1 for the six subtypes of degenerate duality patterns, reveals an inverse correlation between the number of non-equivalent $\mathcal{F}^*$-formulas and the complexity of the duality constellation for the original fragment $\mathcal{F}$: as the number of non-equivalent formulas decreases, the complexity of the degenerate duality pattern increases. With the *simple* degenerate CLCL2 pattern, the eight non-

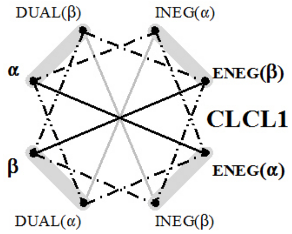
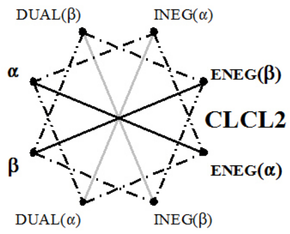
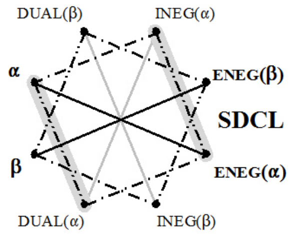
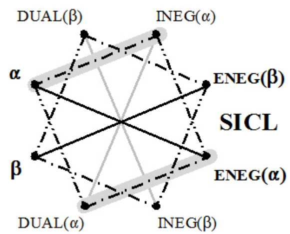
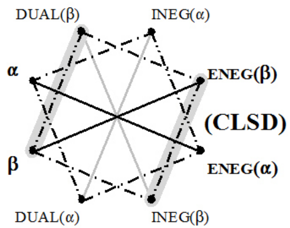
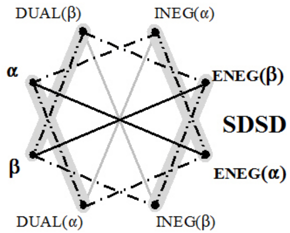
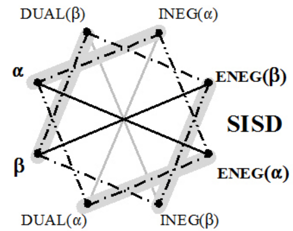
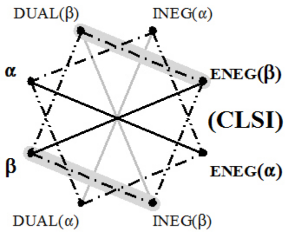
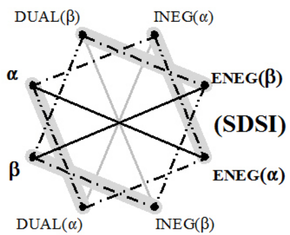
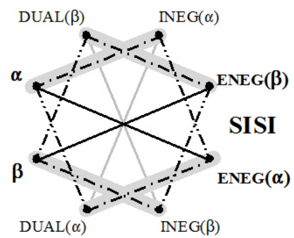
[29] Remember that the mirror-image constellation — with $\delta^*(\pi_\alpha)$ classical and $\delta^*(\pi_\beta)$ collapsed — is perfectly equivalent. We return to this issue in our discussion of Table 2.

equivalent formulas in $\mathcal{F}^*$ only yield *two* duality relations in $\mathcal{F}$, namely the two ENEG relations of the diagonals.[30] With the second subfamily, the number of non-equivalent $\mathcal{F}^*$-formulas decreases to six, but the number of duality relations in $\mathcal{F}$ increases to *five*: in addition to the two ENEG diagonals, the π_α-PCD gets two extra 'loop' relations on its vertices and one extra relation on its diagonal. The difference between the *partially self-dual* SDCL pattern and the *partially self-internal* SICL pattern is determined by which duality relation goes onto the two loops. With the third subfamily, the number of non-equivalent $\mathcal{F}^*$-formulas again decreases — from six to four —, but the number of duality relations in $\mathcal{F}$ increases from five to *eight*: in addition to the two ENEG diagonals, both the π_α- and the π_β-PCD get two extra 'loop' relations on their vertices and one extra relation on their diagonal. The difference between the *mixed self-internal/self-dual* SISD pattern, the *fully self-dual* SDSD pattern and the *fully self-internal* SISI pattern is based on whether or not the π_α- and π_β-PCDs have the same duality relations on their loops. It is interesting to observe that — from the point of view of the complexity of the overall duality constellations — the *classical* duality square in the CLCL1 pattern, with its *six* duality relations (two for the diagonals and four along the edges), occupies an intermediate position between the *five* relations of the partially self-dual/self-internal SDCL and SICL patterns on the one hand, and the *eight* relations of the mixed/fully self-dual/self-internal SISD, SDSD and SISI patterns on the other hand.

The above discussion of the seven duality patterns in Table 1 perfectly corresponds to the three steps in the general procedure going from $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ to the extended fragment $\mathcal{F}^*$ and back to the original fragment $\mathcal{F}$. One aspect of this procedure, however, has remained somewhat underrated so far, namely the individual role of $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$. Therefore, Table 2 classifies the seven duality patterns by drawing the distinction between the classical square, the self-dual collapse and the self-internal collapse independently for $\delta^*(\pi_\alpha)$ on the horizontal axis and for $\delta^*(\pi_\beta)$ on the vertical axis. The octagonal constellations for the fragment $\mathcal{F}^*$ are presented in their full generality, i.e. not with the concrete formulas decorating the vertices of Figures 5 to 11, but with a general $\pi_\alpha = \{\alpha, \text{ENEG}(\alpha)\}$ and $\pi_\beta = \{\beta, \text{ENEG}(\beta)\}$ constituting the core for the fragment $\mathcal{F}$ (represented with the black diagonals).

[30]We will ignore the 'trivial' ID-relation for the sake of convenience.

Table 2: Summary of the Duality Patterns.

	classical $\delta^*(\pi_\alpha)$		self-dual $\delta^*(\pi_\alpha)$	self-internal $\delta^*(\pi_\alpha)$
classical $\delta^*(\pi_\beta)$				
self-dual $\delta^*(\pi_\beta)$				
self-internal $\delta^*(\pi_\beta)$				

Starting with the cell at the top left of Table 2, we observe that an identical input — namely a classical square for both $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ — yields two radically different output scenarios: the *classical* square of the CLCL1 pattern collapses the two intermediate squares onto one another, whereas the *simple* degenerate square of the CLCL2 pattern contains no equivalence pairs whatsoever. Notice, however, that the four equivalence pairs of the CLCL1 pattern are fundamentally different from the four equivalence pairs in the SISD, SDSD and SISI patterns: in the latter cases, all four equivalences are situated *within* $\delta^*(\pi_\alpha)$ or $\delta^*(\pi_\beta)$ separately, i.e. collapsing the intermediate squares along their INEG or DUAL edges, whereas, in the former case, all four equivalences hold *across* $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$.

The remaining two patterns on the top row of Table 2 combine a collapsed square for $\delta^*(\pi_\alpha)$ with a classical square for $\delta^*(\pi_\beta)$: with the *partially self-dual* SDCL pattern, the two SDPs are located along the 'vertical' DUAL edges and will thus yield DUAL loops and an extra diagonal for INEG in the fragment $\mathcal{F}$, whereas with the *partially self-internal* SICL pattern, the two SIPs are situated along the 'horizontal' INEG edges, yielding INEG loops and an extra diagonal for DUAL in the fragment $\mathcal{F}$.

Turning to the third subfamily at the bottom right of Table 2, we arrive at the duality patterns with four equivalence pairs. The *mixed self-internal/self-dual* SISD pattern combines two horizontal SIPs — and two INEG loops in $\mathcal{F}$— with two vertical SDPs — and two DUAL loops in $\mathcal{F}$. With the *fully self-dual* SDSD pattern, all four equivalence pairs are located along the vertical DUAL edges — yielding four DUAL loops and two extra INEG diagonals in $\mathcal{F}$— , whereas with the *fully self-internal* SISI pattern, all four equivalence pairs concern the horizontal INEG edges — generating four INEG loops and two extra DUAL diagonals in $\mathcal{F}$.

Overlooking Table 2 as a whole, a clear axis of symmetry emerges along the diagonal from top left to bottom right. The CLCL1/2, SDSD and SISI patterns — which are all located on this diagonal — are 'symmetric' constellations in that the subtype of $\delta^*(\pi_\alpha)$ — classical, self-dual collapsed or self-internal collapsed — is identical to that of $\delta^*(\pi_\beta)$. The SDCL, SICL and SISD patterns in the top right part of the table are less symmetrical, but they all have their mirror-image counterpart in the CLSD, CLSI and SDSI patterns in the bottom left part of the table. Although we did not provide any explicit examples of the latter three patterns above, they can straighforwardly be obtained by switching around π_α and π_β.

4 Duality Patterns and Bitstring Length

This section investigates the relation between the seven duality patterns distinguished in the previous section and the type of decorations involved. In logical geometry, the formulas constituting the 2-PCD-fragments frequently get assigned a semantic representation in terms of bitstrings. We start off by briefly introducing the notion of bitstring and by presenting the general strategy for looking at duality relations in terms of operations on bitstrings (Subsection 4.1). Secondly, we compare the two well-known Boolean algebras $\mathbb{B}_3$ and $\mathbb{B}_4$, which consist of (formulas that can be represented by) bitstrings of length 3 and 4 respectively (Subsection 4.2). Then we briefly compare more complex structures with (formulas that can be represented by) bitstrings of length 5 and 6 (Subsection 4.3). The overview reveals the crucial role of the distinction between odd and even bitstring lengths (Subsection 4.4).

4.1 The Duality Relations in Finite Boolean Algebras

Bitstrings are sequences of bits (0/1) that encode the denotations of formulas or expressions from *logical systems* (e.g. classical propositional logic, first-order logic, modal logic and public announcement logic) or *lexical fields* (e.g. comparative quantification, subjective quantification, color terms and set inclusion relations), where each bit position concerns a component in a partition of logical space. For a given logical system S and a finite fragment of the language of S, i.e. $\mathcal{F} = \{\varphi_1, \ldots, \varphi_m\} \subseteq \mathcal{L}_\mathsf{S}$, the *partition of* S *induced by* $\mathcal{F}$, denoted as $\Pi_\mathsf{S}(\mathcal{F})$, is defined as the set of *anchor formulas* $\alpha \equiv \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m \in \mathcal{L}_\mathsf{S}$ (where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$), which are S-consistent. For every formula $\varphi \in \mathbb{B}(\mathcal{F})$ — the Boolean closure of the fragment $\mathcal{F}$ — the bitstring $\beta^\mathcal{F}_\mathsf{S}(\varphi) \in \{0,1\}^n$ is defined as follows:[31]

$$\text{for each bit position } 1 \leq i \leq n\colon [\beta^\mathcal{F}_\mathsf{S}(\varphi)]_i := \begin{cases} 1 & \text{if } \models_\mathsf{S} \alpha_i \to \varphi, \\ 0 & \text{if } \models_\mathsf{S} \alpha_i \to \neg\varphi. \end{cases}$$

In other words, a formula φ gets a value 1 for the i-th position of its bitstring iff it is entailed by the corresponding anchor formula α_i. In the next subsections, various examples will be presented of logical systems and their respective partitions. For the sake of convenience, we will often omit the reference to the logical system S or the fragment $\mathcal{F}$, and simply write $\beta(\varphi) = b$ if the formula φ is encoded by the bitstring b. Individual (values for) bit positions will then be

[31] A more detailed description of the mathematically precise technique for assigning bitstrings to formulas on the basis of partitions induced by arbitrary logical fragments is presented in [12]. The logical, diagrammatic and cognitive effectiveness of the earlier, informal bitstring approach is discussed in [45].

denoted using indices: b_i is the (value for) the i-th bit position in the bitstring b.

We can now define two basic operations on bitstrings of arbitrary length, namely the SWITCH-operation and the FLIP-operation, as follows:

$$\begin{array}{rcl} \text{SWITCH}(b_1, b_2, \ldots, b_n) & := & (\neg b_1, \neg b_2, , \ldots, \neg b_n) \\ \text{FLIP}(b_1, b_2, \ldots, b_n) & := & (b_n, \ldots, b_2, b_1) \end{array}$$

The SWITCH-operation switches each individual bit to its opposite value — for example, SWITCH(11010) = 00101 — whereas, with the FLIP-operation, each bit value is maintained but the left-to-right linear ordering of the bits is reversed — for example, FLIP(11010) = 01011. These two operations may interact with each other, and the order in which they are applied, is irrelevant:

$$\begin{array}{rcl} \text{SWITCH}(\text{FLIP}(b_1, b_2, \ldots, b_n)) & = & \text{SWITCH}(b_n, \ldots, b_2, b_1) \\ & = & (\neg b_n, \ldots, \neg b_2, \neg b_1) \\ & = & \text{FLIP}(\neg b_1, \neg b_2, \ldots, \neg b_n) \\ & = & \text{FLIP}(\text{SWITCH}(b_1, b_2, \ldots, b_n)) \end{array}$$

For example, SWITCH(FLIP(11010)) = SWITCH(01011) = 10100, i.e. the linear ordering of the bits has been reversed and all bit values have been switched.[32]

In all the examples that will be considered in the next subsections, the SWITCH- and FLIP-operations on bitstrings turn out to correlate straightforwardly with the duality operations of ENEG and INEG respectively:

$$\begin{array}{rcl} \beta(\text{ENEG}(\varphi)) & = & \text{SWITCH}(\beta(\varphi)) \\ \beta(\text{INEG}(\varphi)) & = & \text{FLIP}(\beta(\varphi)) \end{array}$$

Hence, computing the bitstring for an *externally* negated formula boils down to *switching* the bitstring of the original formula, and similarly, computing the bitstring for an *internally* negated formula boils down to *flipping* the bitstring of the original formula. Given the definition of the DUAL operation as the composition of ENEG and INEG, it follows that assigning a bitstring to the dual of a formula is equivalent to the combined application of SWITCH and FLIP to that formula's original bitstring: $\beta(\text{DUAL}(\varphi)) = \beta(\text{ENEG}(\text{INEG}(\varphi))) = \text{SWITCH}(\beta(\text{INEG}(\varphi))) = \text{SWITCH}(\text{FLIP}(\beta(\varphi)))$.

Remember from the previous section that formulas which are their own INEG or DUAL played an important role in establishing the typology of duality patterns. With the *self-internal* formulas, we observe that if $\varphi \equiv \text{INEG}(\varphi)$

[32]Notice, furthermore, that if SWITCH(FLIP(b)) = c then SWITCH(b) = FLIP(c) and FLIP(b) = SWITCH(c): e.g. given that SWITCH(FLIP(11010)) = 10100, we get SWITCH(11010) = FLIP(10100) = 00101 and FLIP(11010) = SWITCH(10100) = 01011.

Table 3: Types of Contingent PCDs in arbitrary finite Boolean algebras.

	SI	SD	CL	total
$\mathbb{B}_n, n$ is even: $n = 2m$	$\frac{2^m-2}{2} = 2^{m-1}-1$	$\frac{2^m}{2} = 2^{m-1}$	$(2^{n-1}-1)-(2^{m-1}-1)-2^{m-1} = 2^{n-1}-2^m$	$\frac{2^n-2}{2} = 2^{n-1}-1$
$\mathbb{B}_n, n$ is odd: $n = 2m+1$	$\frac{2^{m+1}-2}{2} = 2^m-1$	0	$(2^{n-1}-1)-(2^m-1)-0 = 2^{n-1}-2^m$	$\frac{2^n-2}{2} = 2^{n-1}-1$

then $\beta(\varphi) = \beta(\text{INEG}(\varphi)) = \text{FLIP}(\beta(\varphi))$. In other words, if a formula is self-internal, the FLIP-operation has no effect on its bitstring — $(\beta(\varphi)_1, \ldots, \beta(\varphi)_n) = (\beta(\varphi)_n, \ldots, \beta(\varphi)_1)$ — which means that this bitstring is *symmetrical*, i.e. for $1 \leq i \leq n$: $\beta(\varphi)_i = \beta(\varphi)_{n+1-i}$. Notice that the latter holds for bitstrings with an *even* length — e.g. FLIP(001100) = 001100 — as well as for those with an *odd* length — e.g. FLIP(01110) = 01110. With the *self-dual* formulas, on the other hand, we observe that if $\varphi \equiv \text{DUAL}(\varphi)$ then $\beta(\varphi) = \beta(\text{DUAL}(\varphi)) = \text{SWITCH}(\text{FLIP}(\beta(\varphi)))$. In other words, if a formula is self-dual, the combination of the SWITCH- and FLIP-operations has no effect on its bitstring. This is equivalent to saying that the two operations have the same effect on this bitstring — i.e. $\text{SWITCH}(\beta(\varphi)) = \text{FLIP}(\beta(\varphi))$. This is the case, for instance, for the bitstring 110100, since SWITCH(110100) = FLIP(110100) = 001011. Notice that, in order for a formula to be self-dual, its bitstring must have the same number of bit positions with value 1 as with value 0 — and hence have an *even* bitstring length n — and furthermore satisfy the constraint that for $1 \leq i \leq n$: $\beta(\varphi)_i = \neg\beta(\varphi)_{n+1-i}$. To prove this more formally, suppose — towards a contradiction — that a formula φ is self-dual, and the length n of its bitstring $\beta(\varphi)$ is odd. Then for its middle bit position $m := \frac{n+1}{2} \in \mathbb{N}$, it holds that $n + 1 - m = m$. Since φ is self-dual, i.e. $\varphi \equiv \text{DUAL}(\varphi)$, it follows that $\beta(\varphi) = \beta(\text{DUAL}(\varphi)) = \text{SWITCH}(\text{FLIP}(\beta(\varphi)))$, and hence $\beta(\varphi)_i = \neg\beta(\varphi)_{n+1-i}$ for $1 \leq i \leq n$. In particular, it holds that $\beta(\varphi)_m = \neg\beta(\varphi)_{n+1-m} = \neg\beta(\varphi)_m$, and hence, $0 = 1$, contradiction. Remember, finally, from Subsection 2.4, that no formula can be its own ENEG, on pain of inconsistency. Perfectly analogously, it is impossible for the SWITCH-operation to leave the original bitstring unchanged: $\beta(\varphi) \neq \beta(\text{ENEG}(\varphi)) = \text{SWITCH}(\beta(\varphi))$, since otherwise we would have $\beta(\varphi)_i = \neg\beta(\varphi)_i$, and hence $0 = 1$, for every bit position $1 \leq i \leq n$.

The distinction between bitstrings of even and odd length also plays an important role in calculating the number of contingent SI/SD/CL PCDs. First of all, if the bitstring length is *even*, say $n = 2m$, determining a bitstring that

Table 4: Types of Contingent PCDs in some small Boolean algebras.

	SI	SD	CL	total
$\mathbb{B}_3$	1	0	2	3
$\mathbb{B}_4$	1	2	4	7
$\mathbb{B}_5$	3	0	12	15
$\mathbb{B}_6$	3	4	24	31

is its own INEG boils down to determining the values of its first m bit positions. There are thus 2^m such bitstrings, of which $2^m - 2$ are contingent, yielding $\frac{2^m-2}{2}$ contingent SI PCDs. By contrast, if the bitstring length is *odd*, say $n = 2m+1$, determining a bitstring that is its own INEG boils down to determining the values of its first $m + 1$ bit positions — viz. the first m positions plus the middle position. There are thus $2^{m+1} - 2$ contingent such bitstrings, yielding $\frac{2^{m+1}-2}{2}$ contingent SI PCDs. Furthermore, if n is even, analogous considerations show that there are 2^m bitstrings that are their own DUAL — all of which are contingent — yielding $\frac{2^m}{2}$ contingent SD PCDs. By contrast, if n is odd, there are no contingent SD PCDs (cf. supra). Finally, in both cases, the number of CL PCDs is computed by subtracting the numbers of SI and SD PCDs from the total number of PCD. These calculations are summarised in Table 3, and the concrete numbers of SI, SD and CL PCDs for $n \in \{3, 4, 5, 6\}$ are shown in Table 4.

4.2 Bitstrings of Length 3 and 4

Starting off with the Boolean algebra $\mathbb{B}_3$, Table 5 illustrates how the $2^3 = 8$ bitstrings of length 3 can serve as the denotations of 8 formulas from various systems of modal logic (e.g. the system D) and from Generalised Quantifier Theory (GQT). Bitstrings belong to different *levels* according to the number of bits that have value 1. Hence, $\mathbb{B}_3$ consists of three level 1 (L1) bitstrings, three level 2 (L2) bitstrings, one L0 and one L3 bitstring, where the latter two are standardly disregarded for being non-contingent (resp. contradictory and tautological).

The set of anchor formulas constituting the partition of D in Table 5 consists of $\alpha_1 = \Box p$, $\alpha_2 = \neg\Box p \wedge \Diamond p$ and $\alpha_3 = \neg\Diamond p$. Given the definition of the β-function in the previous subsection, $\beta(\Diamond p) = 110$, for instance, since $\Diamond p$ is entailed by α_1 and α_2 but not by α_3.[33]

[33] Completely analogously, the partition for GQT in Table 5 consists of $\alpha_1 =$ *all(A,B)*, $\alpha_2 =$ *some but not all(A,B)* and $\alpha_3 =$ *no(A,B)*. Hence, $\beta($*no or all(A,B)*$) = 101$, for instance, since *no or all(A,B)* is entailed by α_1 and α_3 but not by α_2.

Table 5: Bitstrings of length 3 for the 8 formulas of the modal logic D and Generalised Quantifier Theory (GQT).

D	GQT	$\beta(\varphi)$	$\beta(\neg\varphi)$	GQT	D
$\Box p$	*all(A,B)*	100	011	*not all(A,B)*	$\neg\Box p$
$\neg\Box p \wedge \Diamond p$	*some but not all(A,B)*	010	101	*all or no(A,B)*	$\Box p \vee \neg\Diamond p$
$\neg\Diamond p$	*no(A,B)*	001	110	*some(A,B)*	$\Diamond p$
$\Box p \wedge \neg\Box p$	*all and not all(A,B)*	000	111	*some or no(A,B)*	$\Diamond p \vee \neg\Diamond p$

If we now turn to the typology of duality patterns in $\mathbb{B}_3$, we first of all generalise the notion of PCD from the level of formulas in a concrete language to the level of bitstrings. In other words, the PCD $\{\Box p, \neg\Box p\}$ is seen as one instantiation of the abstract PCD $\{100, 011\}$. The $2^3 - 2 = 6$ contingent bitstrings in $\mathbb{B}_3$ thus give rise to 3 PCDs: $\{100, 011\}$, $\{010, 101\}$ and $\{001, 110\}$. Each of these can then be characterised as *classical* (CL), *self-dual* (SD) or *self-internal* (SI);[34] cf. Table 4. The $\{010, 101\}$ PCD counts as SI, since the FLIP-operation has no effect on either of these symmetric bitstrings: FLIP(010) = 010 and FLIP(101) = 101.[35] A PCD classifies as SD iff the effect of SWITCH and FLIP is the same. However, as was demonstrated at the end of the previous subsection, for a bitstring to be SD, it must have an even bitstring length, which is obviously never the case in $\mathbb{B}_3$, i.e. in this Boolean algebra there are no SD PCDs. If a PCD is neither SD nor SI, it belongs to the default, classical type CL. To sum up, $\mathbb{B}_3$ has one PCD of type SI, viz. $\{010, 101\}$, and two PCDs of type CL, viz. $\{100, 011\}$ and $\{001, 110\}$. As illustrated in the table below, determining the duality patterns for 2-PCD-fragments is now a matter of basic combinatorics: 3 PCDs give rise to the $\frac{3\times 2}{2} = 3$ squares in the bottom left triangle, one of which is a CLCL1 pattern, and two of which are SICL:[36]

$\mathbb{B}_3$	$\{100, 011\}$ (CL)	$\{001, 110\}$ (CL)	$\{010, 101\}$ (SI)
$\{100, 011\}$ (CL)			
$\{001, 110\}$ (CL)	CLCL1		
$\{010, 101\}$ (SI)	SICL	SICL	

As for the Boolean algebra $\mathbb{B}_4$, Table 6 illustrates how the $2^4 = 16$ bitstrings

[34] Remember from Subsection 2.4 that, when a formula is self-dual or self-internal, the same holds for its negation.

[35] In terms of formulas, this corresponds to INEG$(\neg\Box p \wedge \Diamond p) = \neg\Box\neg p \wedge \Diamond\neg p \equiv \Diamond p \wedge \neg\Box p \equiv \neg\Box p \wedge \Diamond p$ and INEG$(\Box p \vee \neg\Diamond p) = \Box\neg p \vee \neg\Diamond\neg p \equiv \neg\Diamond p \vee \Box p \equiv \Box p \vee \neg\Diamond p$.

[36] The cells on the diagonal of this table remain empty, because a square consists of two *distinct* PCDs. Furthermore, we only need to consider one half of the table, because the 'order' in which two PCDs are combined with each other is irrelevant.

of length 4 can serve as the denotations of 16 formulas of the logical systems D and CPL. In both cases, the partition consists of the four L1 formulas $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, in terms of which the β-function can assign values to each of the four bits.[37]

Table 6: Bitstrings of length 4 for the 16 formulas of D and CPL.

D	CPL	$\beta(\varphi)$	$\beta(\neg\varphi)$	CPL	D
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$p \wedge \neg\Box p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\neg p \vee \Box p$
$\neg p \wedge \Diamond p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$p \vee \neg\Diamond p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\neg p \wedge \Diamond p)$	q	1010	0101	$\neg q$	$\neg\Box p \wedge (p \vee \neg\Diamond p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

In $\mathbb{B}_4$, the $2^4 - 2 = 14$ contingent bitstrings give rise to 7 PCDs, one of which consists of two symmetric bitstrings, i.e. $\{0110, 1001\}$, and is hence SI: FLIP$(0110) = 0110$ and FLIP$(1001) = 1001$.[38] Furthermore, the even bitstring length in $\mathbb{B}_4$ yields two PCDs that are SD, namely $\{1100, 0011\}$ and $\{1010, 0101\}$: SWITCH$(1100) =$ FLIP$(1100) = 0011$, and similarly, SWITCH$(1010) =$ FLIP(1010) $= 0101$.[39] The remaining four PCDs in $\mathbb{B}_4$ are of the default type CL; cf. Table 4. The table below shows the duality patterns for the 2-PCD-fragments in $\mathbb{B}_4$.

$\mathbb{B}_4$	$\{1000, 0111\}$ (CL)	$\{0100, 1011\}$ (CL)	$\{0010, 1101\}$ (CL)	$\{0001, 1110\}$ (CL)	$\{1100, 0011\}$ (SD)	$\{1010, 0101\}$ (SD)	$\{1001, 0110\}$ (SI)
$\{1000, 0111\}$ (CL)							
$\{0100, 1011\}$ (CL)	CLCL2						
$\{0010, 1101\}$ (CL)	CLCL2	CLCL1					
$\{0001, 1110\}$ (CL)	CLCL1	CLCL2	CLCL2				
$\{1100, 0011\}$ (SD)	SDCL	SDCL	SDCL	SDCL			
$\{1010, 0101\}$ (SD)	SDCL	SDCL	SDCL	SDCL	SDSD		
$\{1001, 0110\}$ (SI)	SICL	SICL	SICL	SICL	SISD	SISD	

[37]Reformulating the partition $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ for CPL as $\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ more straightforwardly reflects the fact that each of the four bit positions corresponds to one row in the classical truth table for a binary connective in CPL.

[38]In terms of CPL-formulas, this corresponds to INEG$(p \leftrightarrow q) = \neg p \leftrightarrow \neg q \equiv p \leftrightarrow q$ and INEG$(\neg(p \leftrightarrow q)) = \neg(\neg p \leftrightarrow \neg q) \equiv \neg(p \leftrightarrow q)$.

[39]Again in terms of CPL-formulas, these two SD PCDs correspond to the four SDPs $\{p, \neg\neg p\}$, $\{\neg p, \neg p\}$, $\{q, \neg\neg q\}$ and $\{\neg q, \neg q\}$ in the self-dual collapsed squares in Figure 10(b-c).

We again do some easy combinatorial calculations. The 7 contingent PCDs of $\mathbb{B}_4$ give rise to $\frac{7\times 6}{2} = 21$ squares, as shown in the bottom left triangle of the table above. Given that there are 4 PCDs of type CL, there are $\frac{4\times 3}{2} = 6$ possible ways to choose two of them to yield a CLCL square. Two of these squares are of the classical, CLCL1 type, namely the combination of $\{1000, 0111\}$ and $\{0001, 1110\}$ on the one hand, and that of $\{0100, 1011\}$ and $\{0010, 1101\}$ on the other. Two bitstring PCDs $\{b, \neg b\}$ and $\{b', \neg b'\}$ yield a CLCL1 square iff $\text{FLIP}(b) \in \{b', \neg b'\}$.[40] For the two PCDs in the first CLCL1 square, we get $\text{FLIP}(1000) = 0001$, and for those in the second square $\text{FLIP}(0100) = 0010$. The remaining four squares are then of the simple degenerate, CLCL2 type. The numbers for the next three duality patterns are obtained by simply multiplying the numbers of component PCD types: 2 SD$\times$4 CL = 8 SDCL, 1 SI$\times$4 CL = 4 SICL and 1 SI $\times$ 2 SD = 2 SISD. There being only 2 PCDs of the SD type, there is obviously only one way to build an SDSD pattern. Finally, and again obviously, the SISI pattern is excluded in principle in $\mathbb{B}_4$, since there is only one SI PCD to begin with.

4.3 Bitstrings of Length 5 and 6

In Subsection 3.4, we introduced natural language formulas expressing the notion of 'proportionality'. The Boolean algebra $\mathbb{B}_5$ provides us with bitstrings of length 5, which turns out to be the minimal length required for the denotations of such proportional formulas, since the latter concern a partition of logical space in terms of the five anchor formulas α_i below:

$\alpha_1 =$ *More than 80% of the A's are B*
$\alpha_2 =$ *Exactly 80% of the A's are B*
$\alpha_3 =$ *Less than 80% but more than 20% of the A's are B*
$\alpha_4 =$ *Exactly 20% of the A's are B*
$\alpha_5 =$ *Less than 20% of the A's are B*

We will not list all $2^5 = 32$ formulas involved, but briefly illustrate how the β-function uses $\alpha_1, \ldots, \alpha_5$ to assign values to each of the five bits in the case of the four conjunctive and disjunctive formulas from π_α and π_β in Figure 11:

β(*More than 80% or less than 20% of the A's are B*) $= 10001$
β(*At most 80% but at least 20% of the A's are B*) $= 01110$
β(*At least 80% or at most 20% of the A's are B*) $= 11011$
β(*Less than 80% but more than 20% of the A's are B*) $= 00100$

[40]Furthermore, in such a CLCL1 constellation, it can easily be shown that $\text{FLIP}(\neg b) \in \{b', \neg b'\}\backslash\{\text{FLIP}(b)\}$.

In Subsection 3.4, all four of these complex formulas were shown to be their own INEGs, and indeed all of their respective bitstrings are symmetrical as well: FLIP(10001) = 10001, FLIP(01110) = 01110, FLIP(11011) = 11011 and FLIP(00100) = 00100. In addition to these four, $\mathbb{B}_5$ has two more (contingent) symmetric bitstrings, namely 01010 and 10101: FLIP(01010) = 01010 and FLIP(10101) = 10101. In $\mathbb{B}_5$, the $2^5 - 2 = 30$ contingent bitstrings give rise to 15 PCDs. Furthermore, since the bitstring length in $\mathbb{B}_5$ is odd, PCDs of type SD are excluded in principle, and hence the 15 PCDs are either of type CL or of type SI. The six symmetric bitstrings mentioned above, yield three PCDs of type SI — viz. $\{00100, 11011\}$, $\{01010, 10101\}$ and $\{10001, 01110\}$ — and the remaining 12 PCDs are of the default type CL; cf. Table 4.

In order to establish the distribution of the duality patterns in $\mathbb{B}_5$, we again do some basic combinatorial calculations. The 15 contingent PCDs of $\mathbb{B}_4$ give rise to $\frac{15\times 14}{2} = 105$ squares. Since there are no PCDs of type SD, 3 out of the 7 duality patterns are not instantiated, namely SDCL, SISD and SDSD. Given that there are 12 PCDs of type CL, there are $\frac{12\times 11}{2} = 66$ possible ways to choose two of them to yield a CLCL square. These 66 squares then need to be further subdivided into the CLCL1 and CLCL2 types. Remember that two bitstring PCDs $\{b, \neg b\}$ and $\{b', \neg b'\}$ yield a CLCL1 square iff FLIP$(b) \in \{b', \neg b'\}$. The 12 CL PCDs turn out to combine pairwise into the 6 classical CLCL1 squares below, where FLIP(b) is systematically placed right below the bitstring b:[41]

$\{10000, 01111\}$ $\{00001, 11110\}$	$\{01000, 10111\}$ $\{00010, 11101\}$	$\{11000, 00111\}$ $\{00011, 11100\}$
$\{01100, 10011\}$ $\{00110, 11001\}$	$\{10100, 01011\}$ $\{00101, 11010\}$	$\{10010, 01101\}$ $\{01001, 10110\}$

Secondly, the remaining $66 - 6 = 60$ squares are of the simple degenerate, CLCL2 type. Thirdly, for the SICL pattern, we simply multiply the numbers of component PCD types: 3 SI $\times$ 12 CL = 36 SICL. And fourthly, with 3 PCDs of type SI, there are $\frac{3\times 2}{2} = 3$ possible ways of choosing two of them to yield an SISI pattern. So, to sum up, the 105 duality patterns for 2-PCD-fragments in $\mathbb{B}_5$ are subdivided as follows: 6 are of type CLCL1, 60 of type CLCL2, 36 of type SICL, and 3 of type SISI.

Turning to the algebra $\mathbb{B}_6$, we will refrain from providing a detailed set of formulas whose denotation can be characterised in terms of a β-function using a partition of *six* anchor formulas α. Nevertheless, Demey and Smessaert [12, Subsection 5.2] demonstrate that bitstrings of length six are required to capture the semantics of formulas involving multiple quantifiers, such as

[41] Notice that adding up the 6 symmetric bitstrings mentioned before, with the 24 bitstrings in these 6 CLCL1 squares yields the complete set of $32 - 2 = 30$ contingent bitstrings of $\mathbb{B}_5$.

$\forall x\Big(human(x) \to \forall y\big((donkey(y) \wedge own(x,y)) \to run(y)\big)\Big)$, which already show up in the works of the medieval philosopher John Buridan [20, 36] in sentences of the form "of every human, every donkey runs".[42]

In $\mathbb{B}_6$, the $2^6 - 2 = 62$ contingent bitstrings give rise to 31 PCDs. Three of them are of type SI, since they consist of symmetric bitstrings, i.e. {100001, 011110}, $\{010010, 101101\}$ and $\{001100, 110011\}$: e.g. with the first PCD, we get FLIP(100001) = 100001 and FLIP(011110) = 011110.[43] Furthermore, the even bitstring length in $\mathbb{B}_6$ yields four PCDs that are SD, viz. $\{111000, 000111\}$, $\{110100, 001011\}$, $\{101010, 010101\}$ and $\{011001, 100110\}$: taking the first two of these PCDs by way of example, we get SWITCH(111000) = FLIP(111000) = 000111, and SWITCH(110100) = FLIP(110100) = 001011. Subtracting the 3 SI PCDs and the 4 SD PCDs from the total set of 31 leaves us with 24 PCDs of the default CL type; cf. Table 4. Based on the by now familiar combinatorial and logical considerations, the table below shows how the total number of $\frac{31\times 30}{2} = 465$ squares are distributed across the seven duality patterns in $\mathbb{B}_6$:

CLCL	276	$= \frac{24 \text{ CL} \times 23 \text{ CL}}{2}$	SICL	72	$= 3 \text{ SI} \times 24 \text{ CL}$
CLCL1	12	$= \frac{24 \text{ CL}}{2}$	SISD	12	$= 3 \text{ SI} \times 4 \text{ SD}$
CLCL2	264	$= 276 \text{ CLCL} - 12 \text{ CLCL1}$	SDSD	6	$= \frac{4 \text{ SD} \times 3 \text{ SD}}{2}$
SDCL	96	$= 4 \text{ SD} \times 24 \text{ CL}$	SISI	3	$= \frac{3 \text{ SI} \times 2 \text{ SI}}{2}$

4.4 Overview of Duality Patterns and Bitstring Length

In the previous two subsections, the distribution of the seven duality patterns from Section 3 was considered in function of bitstring lengths of three through six, i.e. in the Boolean algebras $\mathbb{B}_3$ through $\mathbb{B}_6$. The numerical results we have obtained, are summarised in Table 7 below. Although a clear increase emerges from the two patterns in $\mathbb{B}_3$ to the full range of seven patterns in $\mathbb{B}_6$, the increase is not strictly monotone, in that $\mathbb{B}_4$ has *six* patterns, whereas with $\mathbb{B}_5$, the number of patterns drops back to *four*. As was demonstrated above, the crucial distinction involved is that between bitstrings with an *even* length and those with an *odd* length. The latter cannot yield any self-dual PCDs in principle, and hence the three duality patterns that make use of such SD PCDs — viz. SDCL, SISD and SDSD — are excluded in principle as well. This accounts

[42]Such formulas typically occur in sets of eight, thus yielding octagonal Aristotelian diagrams [20, 36]. The so-called 'Buridan octagon' and its internal structure have been analysed in great detail in logical geometry [12, 42, 44].

[43]Notice that these 6 symmetric bitstrings of $\mathbb{B}_6$ correspond one-to-one to those listed above for $\mathbb{B}_5$, in the sense that the *single* central bit position b_3 of $\mathbb{B}_5$ is 'split up' into the *two* central bit positions b_3b_4 — with identical bit values — of $\mathbb{B}_6$.

for all the absent types in the case of $\mathbb{B}_5$ and for three out of the five zeroes on the first row for $\mathbb{B}_3$ in Table 7.

Table 7: The Correlation between Duality patterns and Bitstring Length.

	CLCL1	CLCL2	SDCL	SICL	SISD	SDSD	SISI	
$\mathbb{B}_3$	1	0	**0**	2	**0**	**0**	0	3
$\mathbb{B}_4$	2	4	8	4	2	1	0	21
$\mathbb{B}_5$	6	60	**0**	36	**0**	**0**	3	105
$\mathbb{B}_6$	12	264	96	72	12	6	3	465

The two zeroes for $\mathbb{B}_3$ and $\mathbb{B}_4$ in the column of the SISI pattern are straightforwardly explained by the fact that both algebras only contain *one single* PCD of type SI, which is obviously insufficient to yield an SISI pattern. And finally, the absence of any CLCL2 patterns with $\mathbb{B}_3$ can be accounted for in two ways. On the one hand, there are only two PCDs of type CL available, and these necessarily 'click together' into a CLCL1 pattern. On the other hand, in order to yield a CLCL2 pattern — with two non-collapsing intermediate squares for $\delta^*(\pi_\alpha)$ and $\delta^*(\pi_\beta)$ — one needs *eight* non-equivalent, contingent formulas, whereas $\mathbb{B}_3$ only contains $2^3 - 2 = 6$ such formulas. Summing up the situation for $\mathbb{B}_3$, the five absent types need to be accounted for in terms of three different properties. First, the absence of the three patterns involving self-duality — SDCL, SISD and SDSD — is due to its odd bitstring length, a property which it has in common with $\mathbb{B}_5$. Second, the absence of the SISI pattern relates to there being only one SI PCD, a property which it has in common with $\mathbb{B}_4$. And finally, the absence of the CLCL2 pattern — a property which $\mathbb{B}_3$ does *not* share with any of the other algebras — is caused by its own small size, i.e. the presence of only two CL PCDs. Notice, to conclude, that although six of the seven duality patterns from Section 3 were illustrated using bitstrings of length four, and the seventh pattern with bitstrings of length five, it takes bitstrings of length *six* in order for all seven of the patterns to be instantiated simultaneously.

5 Conclusion and Prospects

The central aim of this paper has been to establish a typology of duality patterns exhibited by fragments of four formulas which are closed under negation, i.e. which consist of two pairs of contradictory formulas (PCDs). Section 2 stressed the need to draw a clear conceptual distinction between duality relations such as internal and external negation on the one hand, and Aristotelian relations such as contradiction and (sub)contrariety on the other. Although both sets of relations give rise to so-called classical and degenerate squares, the absence of irreflexivity may yield so-called *collapsed* squares with the duality relations but not with the Aristotelian ones.

Section 3 established the typology of duality patterns, which were computed in three steps. First of all, each of the PCDs in the four-formula fragment $\mathcal{F}$ generates its own intermediate duality square — one for $\delta^*(\pi_\alpha)$ and one for $\delta^*(\pi_\beta)$ —, which may be either classical or collapsed. Secondly, these two intermediate squares are superimposed onto one another, resulting in an extended, eight-formula fragment $\mathcal{F}^*$. The latter then determines the classification of the original fragment $\mathcal{F}$ as either a classical or a degenerate duality square. Among the degenerate duality patterns, six subtypes were distinguished: the *simple* degenerate pattern CLCL2, the two *partially* self-dual/-internal patterns SDCL and SICL, and the three *fully* self-dual/-internal patterns SISD, SDSD and SISI.

Section 4 then investigated the relation between the seven duality patterns emerging from the typology and the complexity of the semantic representation assigned to the four formulas in the fragment, as expressed in terms of the length of the bitstring encodings. The Boolean algebras with an *odd* bitstring length, namely $\mathbb{B}_3$ and $\mathbb{B}_5$, are systematically lacking the three duality patterns involving self-duality, namely SDCL, SISD and SDSD. Furthermore, with $\mathbb{B}_3$ and $\mathbb{B}_4$ the SISI pattern is missing, because they only contain a single SI PCD. It is only when we get to bitstrings of length six (i.e. to $\mathbb{B}_6$), that all seven of the duality patterns turn out to be instantiated simultaneously.

A first topic for further investigation is still situated on the level of 2-PCD-fragments, and concerns the interaction between the seven-way typology of duality patterns established in the present paper, and the two-way distinction in the realm of the Aristotelian patterns between the classical and the degenerate square. In theory, this cross-cutting of both partitions should lead to $7 \times 2 = 14$ joint (duality/Aristotelian) types, but it remains to be explored how many of these are actually possible, and if so, what bitstring length is required. Secondly, in terms of generalising the typology of duality patterns to fragments of more than four formulas, the next natural step is not so much 3-PCD-fragments but rather 4-PCD-fragments. Sets of *eight* formulas have been shown to give rise to much richer duality constellations — the so-called 'duality cubes' [7, 10] — which either involve an intermediate MNEG-operator in between ENEG and INEG, or else involve two internal negations INEG1 and INEG2 instead of just one. That this track is likely to lead to a much more complex typology, may be inferred from the fact that, when we move from 2-PCD-fragments to 4-PCD-fragments in the domain of Aristotelian diagrams, we jump from just two types of squares to *eighteen* types of octagons.

References

[1] J. Ackrill. *Aristotle's Categories and De Interpretatione.* Clarendon Press, Oxford, 1961.

[2] J. Barwise and R. Cooper. Generalized quantifiers and natural language. *Linguistics and Philosophy*, 4:159–219, 1981.

[3] J.-Y. Béziau and G. Payette. Preface. In J.-Y. Béziau and G. Payette, editors, *The Square of Opposition. A General Framework for Cognition*, pages 9–22. Peter Lang, 2012.

[4] B. P. Copenhaver, C. G. Normore, and T. Parsons, editors. *Peter of Spain, Summaries of Logic. Text, Translation, Introduction and Notes.* Oxford University Press, 2014.

[5] D. D'Alfonso. The square of opposition and generalized quantifiers. In J.-Y. Béziau and G. Payette, editors, *Around and Beyond the Square of Opposition*, pages 219–227. Springer, Basel, 2012.

[6] L. M. de Rijk, editor. *Peter of Spain, Tractatus, called afterwards Summule Logicales.* Van Gorcum, Assen, 1972.

[7] L. Demey. Algebraic aspects of duality diagrams. In B. P. Philip T. Cox and P. Rodgers, editors, *Diagrammatic Representation and Inference*, Lecture Notes in Computer Science (LNCS) 7352, pages 300 – 302. Springer, Berlin, 2012.

[8] L. Demey. Interactively illustrating the context-sensitivity of Aristotelian diagrams. In H. Christiansen, I. Stojanovic, and G. Papadopoulos, editors, *Modeling and Using Context*, LNCS 9405, pages 331–345. Springer, 2015.

[9] L. Demey. The logical geometry of Russell's theory of definite descriptions. *Submitted*, 2017.

[10] L. Demey and H. Smessaert. Duality in logic and language. In J. Fieser and B. Dowden, editors, *Internet Encyclopedia of Philosophy.* University of Tennessee, Martin, TN, 2016.

[11] L. Demey and H. Smessaert. Metalogical decorations of logical diagrams. *Logica Universalis*, 10:233–292, 2016.

[12] L. Demey and H. Smessaert. Combinatorial bitstring semantics for arbitrary logical fragments. *Journal of Philosophical Logic*, 2017.

[13] M. H. Dziewicki, editor. *Johannis Wyclif, Tractatus de Logica, Vol. 1.* Trübner, London, 1893.

[14] H. Freudenthal. *Lincos. Design of a Language for Cosmic Intercourse.* North-Holland, Amsterdam, 1960.

[15] S. Givant and P. Halmos. *Introduction to Boolean Algebras.* Springer, New York, NY, 2009.

[16] W. H. Gottschalk. The theory of quaternality. *Journal of Symbolic Logic*, 18:193–196, 1953.

[17] T. Gowers, editor. *The Princeton Companion to Mathematics.* Princeton University Press, Princeton, NJ, 2008.

[18] E. A. Hacker. The octagon of opposition. *Notre Dame Journal of Formal Logic*, 16:352–353, 1975.

[19] L. R. Horn. *A Natural History of Negation.* University of Chicago Press, Chicago, IL, 1989.

[20] G. E. Hughes. The modal logic of John Buridan. In G. Corsi, C. Mangione, and M. Mugnai, editors, *Atti del convegno internazionale di storia della logica, le teorie delle modalità*, pages 93–111. CLUEB, 1987.

[21] D. Jaspers. *Operators in the Lexicon. On the Negative Logic of Natural Language.* LOT Publications, Utrecht, 2005.

[22] W. Johnson. *Logic. Part I.* Cambridge University Press, Cambridge, 1921.

[23] F. A. Kabakov, A. S. Parkhomenko, M. I. Voitsekhovskii, and T. S. Fofanova. Duality principle. In *Encyclopedia of Mathematics.* Springer, 2014.

[24] J. N. Keynes. *Studies and Exercises in Formal Logic.* MacMillan, London, 1884.

[25] G. Klima, editor. *John Buridan, Summulae de Dialectica.* Yale University Press, New Haven, CT, 2001.

[26] E. König. Concessive relations as the dual of causal relations. In D. Zaefferer, editor, *Semantic Universals and Universal Semantics*, volume 12 of *Groningen-Amsterdam Studies in Semantics*, pages 190–209. Foris, Berlin, 1991.

[27] N. Kretzmann. *William of Sherwood's Introduction to Logic.* Minnesota Archive Editions, Minneapolis, MN, 1966.

[28] S. Löbner. German *schon – erst – noch*: an integrated analysis. *Linguistics and Philosophy*, 12:167–212, 1989.

[29] S. Löbner. *Wahr neben Falsch. Duale Operatoren als die Quantoren natürlicher Sprache.* Max Niemeyer Verlag, Tübingen, 1990.

[30] S. Löbner. Why German *schon* and *noch* are still duals: a reply to van der Auwera. *Linguistics and Philosophy*, 22:45–107, 1999.

[31] S. Löbner. Dual oppositions in lexical meaning. In C. Maienborn, K. von Heusinger, and P. Portner, editors, *Semantics: An International Handbook of Natural Language Meaning*, volume I, pages 479–506. de Gruyter Mouton, Berlin, 2011.

[32] B. Mélès. No group of opposition for constructive logics: The intuitionistic and linear cases. In J.-Y. Béziau and G. Payette, editors, *Around and Beyond the Square of Opposition*, pages 201–217. Springer, Basel, 2012.

[33] T. Parsons. The traditional square of opposition. In E. N. Zalta, editor, *Stanford Encyclopedia of Philosophy (Summer 2012 edition)*. CSLI, 2006.

[34] S. Peters and D. Westerståhl. *Quantifiers in Language and Logic.* Oxford University Press, Oxford, 2006.

[35] J. Piaget. *Traité de logique. Essai de logistique opératoire.* Colin/Dunod, Paris, 1949.

[36] S. Read. John Buridan's theory of consequence and his octagons of opposition. In J.-Y. Béziau and D. Jacquette, editors, *Around and Beyond the Square of Opposition*, pages 93–110. Springer, Basel, 2012.

[37] H. Reichenbach. The syllogism revised. *Philosophy of Science*, 19:1–16, 1952.

[38] A. Schumann. On two squares of opposition: the Leśniewski's style formalization of synthetic propositions. *Acta Analytica*, 28:71–93, 2013.

[39] P. Seuren. *The Logic of Language. Language from Within, Volume II.* Oxford University Press, Oxford, 2010.

[40] P. Seuren and D. Jaspers. Logico-cognitive structure in the lexicon. *Language*, 90:607–643, 2014.

[41] H. Smessaert. The classical Aristotelian hexagon versus the modern duality hexagon. *Logica Universalis*, 6:171–199, 2012.

[42] H. Smessaert and L. Demey. Logical and geometrical complementarities between Aristotelian diagrams. In T. Dwyer, H. Purchase, and A. Delaney, editors, *Diagrammatic Representation and Inference*, LNCS 8578, pages 246–260. Springer, Berlin, 2014.

[43] H. Smessaert and L. Demey. Logical geometries and information in the square of oppositions. *Journal of Logic, Language and Information*, 23:527–565, 2014.

[44] H. Smessaert and L. Demey. La géométrie logique du dodécaèdre rhombique des oppositions. In S. Chatti, editor, *Le carré et ses extensions: Approches théoriques, pratiques et historiques*, pages 127–157. Université de Tunis, Tunis, 2015.

[45] H. Smessaert and L. Demey. The unreasonable effectiveness of bitstrings in logical geometry. In J.-Y. Béziau and G. Basti, editors, *The Square of Opposition: A Cornerstone of Thought*, pages 197–214. Springer, Basel, 2017.

[46] H. Smessaert and A. ter Meulen. Temporal reasoning with aspectual adverbs. *Linguistics and Philosophy*, 27:209–261, 2004.

[47] J. van Benthem. Linguistic universals in logical semantics. In D. Zaefferer, editor, *Semantic Universals and Universal Semantics*, volume 12 of *Groningen-Amsterdam Studies in Semantics*, pages 17–36. Foris, Berlin, 1991.

[48] J. van der Auwera. ‘Already’ and ‘still’: beyond duality. *Linguistics and Philosophy*, 16:613–653, 1993.

[49] D. Westerståhl. Classical vs. modern squares of opposition, and beyond. In J.-Y. Béziau and G. Payette, editors, *The Square of Opposition. A General Framework for Cognition*, pages 195–229. Peter Lang, Bern, 2012.

Hans Smessaert
Department of Linguistics
KU Leuven
Blijde Inkomststraat 21, 3000 Leuven, Belgium
E-mail: hans.smessaert@kuleuven.be

Lorenz Demey
Center for Logic and Analytic Philosophy
KU Leuven
Kardinaal Mercierplein 2, 3000 Leuven, Belgium
E-mail: lorenz.demey@kuleuven.be