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Possibility, Contingency and the Hexagon of Modalities

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Abstract

During many centuries there was a confusion between two modal concepts: possibility and contingency. The situation was clarified in the mid XXth century by the hexagon of opposition developed by Robert Blanché making a clear distinction between something which is not impossible and something which is not impossible but not necessary. This hexagon is a refinement of the theory of opposition initiated by Aristotle fixed in a square diagram by Apuleius and then Boethius, a theory based on three notions of oppositions: contrariety, subcontrariety and contradiction. Modern logic has focused on two modalities symbolically represented by "$\Box$" and "$\Diamond$", having some invariant features in the many systems of modal logic. "$\Box$" can easily be interpreted as necessity. "$\Diamond$" is usually interpreted as possibility; however this interpretation is not so obvious from a philosophical perspective. We will discuss this question from the point of view of the hexagon of opposition and explain how this hexagon interestingly puts forward the notion of contingency generally neglected by modern logicians.

Keywords: Possibility, Contingency, Hexagon of Opposition, Modal Logic

Pendant que Candide, le baron, Pangloss, Martin et Cacambo contaient leurs aventures, qu'ils raisonnaient sur les événements contingents ou non contingents de cet univers, qu'ils disputaient sur les effets et les causes, sur le mal moral et sur le mal physique, sur la liberté et la nécessité, sur les consolations que l'on peut éprouver lorsqu'on est aux galères en Turquie, ils abordèrent sur le rivage de la Propontide à la maison du prince de Transylvanie. Les premiers objets qui se présentèrent furent Cunègonde et la vieille, qui étendaient des serviettes sur des ficelles, pour les faire sécher.

Candide, Voltaire

1 Possibility and $\diamond$

The aim of this paper is to discuss the relation between the notion of possibility and the operator denoted by "$\diamond$" in systems of modal logic. We are inquiring here in which sense this operator gives a satisfactory account of the notion of possibility.

Possibility is a central notion of modal logic. Nowadays the expression "modal logic" is used to refer to a family of systems of logic. A book entitled *modal logic* is a book presenting such systems, most of the time with very few, if no, philosophical discussions – see for example the textbook of Brian Chellas [23]. Despite the absence of philosophical discussions the book is rather written for philosophers if we consider the way the technicalities are presented. The author made efforts to go in much details that can be helpful for someone with few mathematical training; on the other hand for a mathematician many things in this book appear as quite trivial.[1]

Modal logic has been, and often still is, considered as part of *philosophical logic* because it is motivated by, or/and dealing with, philosophical notions, possibility being one. The expression "philosophical logic", which became famous in particular through *The Journal of Philosophical Logic*, has evolved in a way that the field it denotes includes systems of logic that formalize reasoning involving, or about, some philosophical notions. It contrasts with *mathematical logic* dealing with mathematical reasoning.

But "mathematical logic" is an expression that can be interpreted in two different ways: (α) the study of mathematical reasoning; (β) the mathematical study of reasoning.[2] The two meanings are not incompatible: one can study in a mathematical way mathematical reasoning. But the two don't necessarily go together: one can mathematically study a reasoning which is not especially related to mathematics and mathematical reasoning can be studied in a not so mathematical way. Contemporary modal logic can be considered as *mathematical logic* in the β-sense. This is nowadays clearly admitted in particular with the development of the Amsterdam school of modal logic (van Benthem *et al.*).

Before that, many mathematical logicians were considering modal logic rather as *philosophical logic*. The expression "philosophical logic" can also be interpreted in two different manners: (α) the study of philosophical reasoning; (β) the philosophical study of reasoning. For mathematicians modal logic was *philosophical logic* not only in this α-sense, but also in this β-sense, such β-sense being rather pejorative: a study which was pseudo-mathematical, not

[1] Compare with textbooks about propositional logic presenting truth-tables like tables of multiplication and addition, and with no philosophical discussion about truth.

[2] This was already pointed out by Zermelo in 1908 as noted by Mancosu ([40], p.7)

really serious, rather trivial despite the use of some symbolism. It is indeed not sufficient to use a symbol like "$\diamond$" instead of the word "possibility" to claim that one is doing mathematics.

Now that modal logic has been recognized as mathematical, the problem appears the other way round: is it still philosophical, in a α-sense? Is it really connected, giving an good account, of modal notions, in particular possibility? With the mathematization of modal logic people have naturally jumped into clouds in the sky of abstractions. And this is good both from an intrinsic and extrinsic perspectives. Abstraction is connected to generalization and the scope of modal logic has enlarged, opening new applications in particular to computer science, linguistics, economics, etc.

But it is also good to come back to the starting point, to the down-to-earth initial philosophical notions it has been inspired by, it has grown from. Maybe we can by so doing, on the one hand have a better understanding of the philosophical notions, on the other hand bounce back to new mathematical constructions. It is in this spirit that we are examining the relation between the notion of possibility and the operator denoted by $\diamond$.

2 The diamond of possibility

In contemporary modal logic the sign "$\diamond$" is called "possibility" (or "possible") because it is intended to represent the notion of possibility. But in spoken language the modal logicians also called it "diamond", a terminology reflecting its shape. Such a terminology can be considered as superficial in a not necessarily negative sense. By superficial we don't want to say that there is less ontological commitment, because the question is not here to argue that the notion of possibility exists or not. What we want to say is that "diamond" is more neutral, it gives a distance between the *mathematical operator* denoted by "$\diamond$" and the *philosophical notion* of possibility.

The mathematical operator we are talking about here is a unary connective, which can be considered both from a proof-theoretical or a model-theoretical point of view, or from a strict syntactical point of view, as a function forming a formula $\diamond\varphi$ from a formula φ. The reader may think we have forgotten here the quotation marks, that we should have written: forming a formula "$\diamond\varphi$" from a formula "φ". But we are not following a formalist ideology, for us "$\diamond\varphi$" is not a formula but the name of a formula, which is an object of a mathematical structure, the absolutely free algebra of formulas constituting the so-called "syntax" of the logical system (see [1]).

We consider "$\diamond$" as a sign denoting this unary connective. Therefore when we are talking about $\diamond$ (no quotation marks), we are talking about the connective itself, in the same way that when we are talking about possibility, we are talking about the notion of possibility denoted by the word “possibility”.

It is important to stress that $\diamond$ and possibility are not the same. Even in the best case, the best of all possible worlds, the best of all systems of modal logic, $\diamond$ will never exactly correspond to the notion of possibility. $\diamond$ can mirror possibility, but will never be possibility itself. It is good to remember the painting of René Magritte entitled *La Trahison des Images* (in English *The Treachery of Images*), a representation of a pipe with the inscription “Ceci n’est pas une pipe” (“This is not a pipe”) below. Giving the name “possibility” to the operator $\diamond$ may generate the kind of pipe confusion denounced by Magritte. On the other hand calling “$\diamond$” the operator diamond may help us to avoid this confusion. However with this second option there is a double danger: giving the impression that we are interested in the sign itself (then we would be wrongly superficial); forgetting that we are concerned with possibility. It is good to keep in mind that this diamond is the diamond of possibility.

The situation here is different from Magritte’s treachery because an image of a pipe may correspond to a concrete pipe having a rather clear objective and definite reality. What is the *reality* of the notion of possibility? There are different ways to conceive what possibility is and the reality of possibility is intimately connected to these conceptualizations. For this reason we can say that possibility is a *concept*, or speaking more loosely, more lightly, an *idea*.

As for other notions, a conceptualization can be more or less descriptive, more or less normative. By contrast to a pipe, for a notion it is very difficult to be purely descriptive unless we choose to make an enumeration of what has been thought, but then this will not give us an idea of what the notion is, in the present case, what possibility is. So we have to disentangled the confusion of the chaotic variety of meanings by being normative, but not too much because then it would make no sense to still use the word denoting the notion, here “possibility”.

Tarski was facing this kind of situation in a more dramatic way dealing with the notion of truth, one of the most heavily loaded notions. In his seminal 1944 paper [46] he wrote the following:.

> I hope nothing which is said here will be interpreted as a claim that the semantic conception of truth is the “right” or indeed the “only possible” one. I do not have the slightest intention to contribute in any way to those endless, often violent discussions on the subject: “What is the right conception of truth?”
>
> Disputes of this type are by no means restricted to the notion of truth. They occur in all domains where – instead of an exact, scientific terminology – common language with its vagueness and ambiguity is used; and they are always meaningless, and therefore in vain.
>
> It seems to me obvious that the only rational approach to such prob-

> lems would be the following: We should reconcile ourselves with the fact that we are confronted, not with one concept, but with several different concepts which are denoted by one word; we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way); to avoid further confusions, we should agree to use different terms for different concepts; and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations.

In traditional modal logic, there were two notions: "possibility" and "contingency" (see [27] for a good historical account). Was this just a wordy difference? One may think so in view of contemporary modal logic where the notion of contingency has almost disappeared. $\diamond$ is never considered as the diamond of contingency. Possibility does not stand by itself in the systems of modal logic, it has a faithful companion, but it is not contingency, it is necessity, which is represented by "$\Box$". Possibility and necessity are the two main characters of contemporary modal logic, the only two having emblematic symbols associated to them: "$\diamond$" and "$\Box$". Is there no place for contingency?

3 $\diamond$ and her big brother $\Box$

When inquiring a notion like possibility, one may focus on the notion itself, the thing which is beyond the word, if any, digging for the meaning. By contrast to this vertical procedure, there is a horizontal one, consisting of studying the relations between this notion and other notions. This horizontal procedure can been qualified as "structural" considering that a network of relations form a structure. The structural approach has been developed by Saussure in linguistics, by Bourbaki in mathematics, by Lévi-Strauss in anthropology (see [37], [49], [50] about the connection among the three and [28] about the general story). Here we are talking of yet another kind of structuralism that can be called "conceptual structuralism" that has been developed in particular by Robert Blanché in his seminal book *Structures intellectuelles* [21].

When developing this structural approach it is important to make a good choice, to choose the right package, it is a question of quantity and quality. In contemporary modal logic possibility is mainly related to necessity. There are other options, for example it is possible to relate possibility to conceptualization and imagination – see[11], another trichotomy relates possibility to reality and virtuality – see [12]. At the beginning of the XXth century possibility was related to truth and falsity, Łukasiewicz considering it has a third truth value (see [38], [6]). And Wittgenstein was relating it to contradiction and tautology, considering that a possible proposition is a proposition that can be true and that can be false (see [10]). Wittgenstein also used the word "possibility" for what we now called valuations. This leads via Carnap to the famous use of the expression “possible world” promoted by Kripke (see [5]).

The operator corresponding to necessity is denoted by “$\Box$”, and in the same way that “$\diamond$” is alternatively called “diamond” instead of “possibility”, “$\Box$” is alternatively called “box” instead of “necessity” (or “necessary”). In both cases this alternative way of speaking reflects the shape of the sign used to denote the operator. The terminology “box” and “diamond” is a kind of informal slang. Referring more mathematically to their shapes, one could speak about “square” and “lozenge”. But this level of precision would be misleading because there are no connections between the properties of these geometrical objects and the meanings and properties of the modal operators they are denoting.

Geometrically speaking squares and lozenges are part of the same family, both are rhombuses. A rhombus is a polygon with four sides of the same length. A square has 45^0 angles, by contrast a lozenge has angles less than 45^0. The name “diamond” is used as a nickname for a lozenge, but “box” is not used in mathematics as a nickname for a square. Moreover the sign “$\diamond$” in LaTeX is not a lozenge, but just a square, of a size smaller than “$\Box$” and with a different position. It is maybe weird to call a square a diamond just because of its position, but the position of “$\diamond$” is the position of one of the most famous lozenges nicknamed “diamond”, the lozenge of playing cards. The idea to represent possibility and necessity by two signs of exactly the same geometrical shape is not necessarily a problem. It establishes a close connection between the two, the difference being only in the position. The fact that “$\diamond$” is smaller than “$\Box$” can be justified by the desire to have the same height (the diagonal of a square being famously longer than the sides). However the adjustment has not been properly done in LaTex, the two signs not even being at the same level: $\Diamond$ $\Box$

Anyway, independently of these small variations, the two signs “$\Box$” and “$\diamond$” form a nice pair. Nicer than “$\forall$” and “$\exists$”, to which they are connected (cf. Wajsberg theorem [48]). Another famous dual pair of symbols in logic is “$\wedge$” and “$\vee$”. It works on a similar turning principle as “$\Box$” and “$\diamond$”.[3] There is no such connection between “$\forall$” and “$\exists$”. “$\exists$” is not a turned “$\forall$”. The sign “$\exists$” is a horizontally inverted “E”, the initial capital letter of “Existenz” in German, which happens to be the same as the initial capital letter of “Exist” in English. Gentzen decided also to use an inverting procedure for the symbolization of the universal quantifier, inverting the initial capital letter of “Alles” (which is also the same as the one of its English cousin “All”). But since horizontal inversion leads to the same sign, he chose vertical inversion. This was the birth of “$\forall$”

[3]The operators of conjunction and disjunction are directly connected with the two quantifiers, so the three pairs of symbols correspond to three pairs of operators having close connections.

(see [26]).

For the modal operators of necessity and possibility one could have chosen just their inverted initials. Something more trivial in fact was done, like with truth and falsity, their initials were chosen: "N" and "M". "N" for "Necessary" or "Notwendig", "M" for "Möglichkeit" (see [25], [30], [31]).

Despite the differences between a geometrical shape like "$\diamond$" and a letter like "M", there is something in common from the semiotical point of view. It is the fact that only one sign is used. This is a common tendency of mathematical written language: $\infty, +, \times$; including mathematical logic: $\vdash, \in, \neg$. Mathematical language is in this sense closer to ideogrammatic and pictogrammatic languages than alphabetical languages.

Mathematical signs have also sometimes a real symbolic dimension, in the original sense of the word "symbol": the fact that there is a connection between the sign and its meaning. This is typically the case of one of the most famous signs of mathematics, the sign of identity/equality "=" attributed to Robert Recorde (1557). The meaning of this sign is mirrored by the sign itself. Moreover, as we have pointed in [9], this is a double symbolization, the second aspect of symbolization being to represent an idea by a typical particular specimen, a prototype. Here identity represented by the identity of two parallel lines, like when representing justice by a balance.

Is there any relation between the notion of possibility" and the shape of "$\diamond$" ? This is not clear at all. The lozenge is traditionally a symbol of feminity and fertility:

PICTURE 1
WOMEN US AIRFORCE SERVICE PILOTS BADGE

In which sense possibility is a fertile notion? At least we can say that it has generated many systems of logics ... If possibility is female, we can consider that necessity is male, that "$\Box$" is the big brother of "$\diamond$"

4 Squaring possibility

In contemporary modal logic necessity is not the only companion going hand to hand with possibility. There is also impossibility. Impossibility does not necessarily come to our mind as a third modality, because it is a kind of double of possibility and moreover in systems of modal logic there is generally no symbol for it.

Possibility and impossibility: we have here a dichotomy. Something cannot both be possible and impossible, and there is no other choice (no other "possibility" ...). Such a dichotomy is perfectly represented by the use of classical negation: $\diamond$ and $\neg\diamond$. Classical negation is a unary operator forming dichotomical pairs.

Let us emphasize that by "classical negation" we mean here the operator generally represented by "$\neg$" and defined by the well-known truth-table. We are not Quinean, we don't think that $\neg$ corresponds to the "right" notion of negation. We think that it mirrors one of the important meanings of the notion of negation, being necessarily normative but disentangling the ambiguity of a notion.

Formally, "impossible" is translated as "$\neg\diamond$". A direct translation back to natural language would be "not possible". With "impossible" we have an incorporation of the negation to form a new word, a new substantive. This is a lexicalization, but a quite artificial one. In formal language it is not lexicalized, there is no primitive sign to denote the operator of impossibility, the compound sign "$\neg\diamond$" is used. It would not be difficult to introduce the use of a new sign in the formal language of modal logic to denote impossibility, let's say "$\circ$". Another option for the impossible would be to use a symbol closer to the construction of "impossible" from "possible" in natural language, a diamond with a cross in it:

$$\diamondplus$$

In a formal language the introduction of a new sign is less artificial than in natural language since a formal language is right at the start more artificial. So we may wonder why no sign for impossibility has been introduced in modal logic. This is symptomatic of the fact that the focus is on the operator $\diamond$, not on the operator $\neg\diamond$.

In most systems of modal logic, it is possible to define necessity from possibility as "$\neg \diamond \neg$", nevertheless the operator $\Box$ is generally introduced, because we want to think about necessity. And this turns things easier to write and easier to understand, as illustrated below:

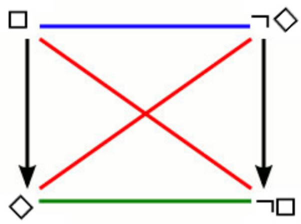
$\neg \diamond \neg\varphi \to \diamond\varphi$

$\Box\varphi \to \diamond\varphi$

Let us examine the situation of the quantificational cousin of possibility, namely the existential quantifier. We also have a dichotomical formal pair: $\exists$ and $\neg\exists$. And $\neg\exists$ is also not lexicalized at the level of the formal language. Is there a lexicalization at the level of natural language? Not really: the negation of "There exists" is "There exists no".

Let us now see what happens with diamond's brother. Formally speaking we naturally have "$\Box$" and "$\neg\Box$". The compound sign "$\neg\Box$" can literally be translated as "not necessary". Can this be lexicalized as "unnecessary"? This is a possible lexicalization based on prefixation similar to “impossible” although the prefix is not the same. The variation of prefixes is based on phonetic.

Someone may think that “contingent” is a good translation in natural language of “$\neg\Box$”. That would be a wonderful lexicalization, without prefixation. But unfortunately this does not properly work. To understand the situation, let us examine the following square of modalities:



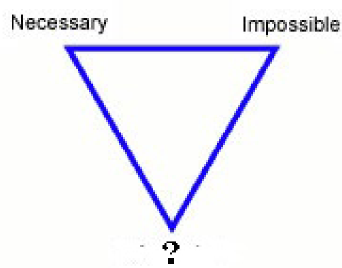
PICTURE 2
SQUARE OF MODALITIES

This square is based on three notions of oppositions (contradiction in red, contrariety in blue, subcontrariety in green) and the notion of subalternation in black. It is a basic framework more general and more fundamental than systems of logic. Łukasiewicz used it as a starting point, a kind of compass, for developing systems of modal logic (see [39]).

Subalternation is a strong implication in the sense that if something is necessary it is possible but something can be possible and not necessary (this is the meaning of the left arrow of subalternation).

The notion of *Contradiction* corresponds to dichotomy: something cannot be at the time possible and impossible and there is not other choice.

The notion of *Contrariety* corresponds to an incompatibility leaving space for a third option. The top blue line therefore expresses the fact that something cannot be at the same necessary and impossible – this seems quite natural – and also the fact that something can be neither necessary, nor impossible, this also seems quite natural. We then have the following triangle of contrariety:



PICTURE 3
CONTRARIETY TRIANGLE OF MODALITIES

What is at the bottom corner? What the name for it and how to formalize it? It cannot be possible, understood as the classical negation of impossibility, $\neg\circ$, because classical negation forms dichotomies and leaves no space for a third notion, here necessity. A possible candidate for this position is contingency. If we decide to put contingency in this location, then contingency is different from possibility understood as $\neg\circ$ and different also from the negation of necessity, $\neg\Box$. So this solution is an argument against interpreting $\neg\Box$ as contingency and the related lexicalization, i.e. to call $\neg\Box$, "contingency".

And there is another argument against the interpretation of $\neg\Box$ as contingency. The right arrow of subalternation of the square of modalities says that if something is impossible it is is not necessary, this makes sense.[4] If we now consider that not necessary is a notion equivalent to contingency, then we have: if something is impossible, it is contingent. This seems absurd even if we don't have a very precise idea of what is contingency.

The notion of *Subcontrariety* corresponds to exhaustion without exclusion of overlap. The bottom green line of the square therefore expresses the fact that something can be at the same time possible and not necessary – this seems quite natural – and also the fact that something cannot be neither possible, nor not necessary, there is no third situation. Let us test the validity of this latter fact by checking possible candidates for such a third position.

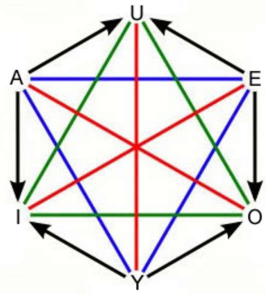
Necessary? It is possible, so this candidate cannot be approved. Impossible? It is not necessary, this candidate should also be rejected. Contingent? If we consider that something which is contingent is possible and not necessary, this candidate also does not fit into a third position.

But this does not invalidate this definition of contingency. It is in fact the way that contingent has been defined in contemporary modal logic, in particular by Montgomery and Routley. They have introduced the symbol "$\bigtriangledown$" for it and the symbol "$\bigtriangleup$" for the contradictory notion, non-contingency (see [42]). Montgomery and Routley have developed some specific systems of modal logic, but the general picture giving the invariant features of the relation between the six modalities is described by a hexagon of modalities that we will study in the next section.

[4]Some people may think that impossibility is a kind of necessity, and they are right in the sense that it corresponds to $\Box\neg$, however the meaning of "necessary not" is certainly different from the meaning of "necessary".

5 Hexagon of Modalities

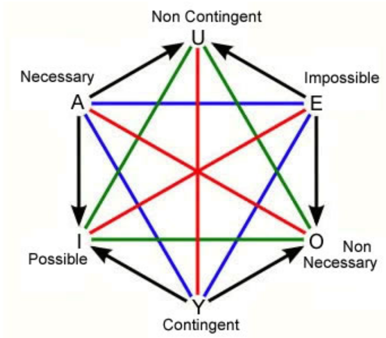
The Hexagon of Modalities has been developed more or less at the same time but independently of the logic of contingency of Montgomery and Routley. It is a particular case of the Hexagon of Opposition put forward by Robert Blanché in the 1950's.[5] The Hexagon of Opposition is a refinement of the Square of Opposition. The general structure is as follows:



PICTURE 4
ABSTRACT HEXAGON OF OPPOSITION

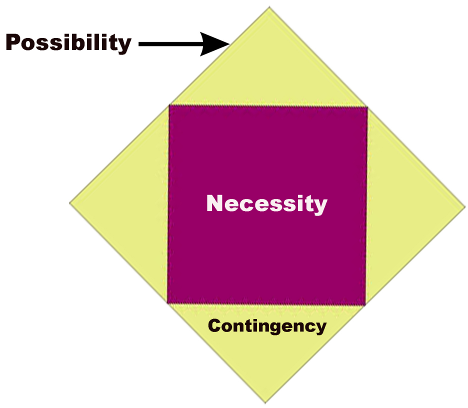
This hexagon is based on the same three notions of oppositions and subalternation of the square of opposition, and we find back the square of opposition in it with its four traditional corners: A, E, I, O. It can be seen as a reconstruction of the square of opposition based on the assemblage of two triangles, a blue triangle of contrariety, a green triangle of subcontrariety. These two triangles are bound together with three axes of contradictions generating a Star of David. A hexagon then appears by putting arrows of subalternations describing furthermore the "external" relations between the six modalities. In the hexagon there are two further corners: U and Y (these names were given by Blanché). U corresponds to the disjunction of the top A and E corners and Y as the conjunction of the bottom I and O corners. This hexagonal structure applied to many concepts, in particular quantifiers and modalities, but also many other non logical concepts, for example colours (see Jaspers's paper, *Logic and colour* [35]). Blanché was led to the hexagon by the study of deontic modalities (cf. a paper he published in English in *Mind* in 1952 [17], criticizing von Wright's conceptualization of deontic logic. There he introduced what would be the bottom corner of the hexagon, already using the letter Y). Here is the hexagon for alethic modalities:

[5]The main book of Blanché on the hexagon [21], originally published in French in 1966, was out of print during many years and has never been translated in English - neither were his two original papers (the first published in the Swedish journal *Theoria* in 1953 [18], the second published in *Revue Philosophique* in 1957 [19]).



PICTURE 5
HEXAGON OF MODALITIES

Due to the structure of the hexagon, the Y position corresponds to things which are possible but not necessary. It has been named "contingent" by Blanché and is contrasting with possible. Something which is contingent is possible, but there are possible things which are not contingent: necessary things. Possibility is an extension of contingency including necessity. This can be represented by the following diagram:



PICTURE 6
NECESSITY, POSSIBILITY, CONTINGENCY

This diagram must be read as follows: all what is within the diamond is possible, all what is within the square is necessary. Contingency is what is within the diamond but not in the square: the four triangles. Funny enough it has the shape of triangles, the form used for contingency by Montgomery and Routley as a sign for the operator of contingency (but they make a difference between an up triangle for contingency and a down triangle for non-contingency, which is not the case here, all triangles corresponding to contingency).

The Y position is fine for contingency. But is the I position suitable for possibility? It is more or less satisfactory. The good point is that in the hexagon of modalities, possibility is the negation of impossibility. But in natural language "possible" is generally used as excluding "necessary". If we say "it is possible that it will rain" we are leaving open the possibility of not raining. We don't want to say "it will necessary rain", we want to say "it may rain or not". We use "possible" as meaning contingency as specified by the hexagon of opposition.

But considering that possibility is also understood in natural language as contradictory of impossible, what we can say is that its meaning is oscillating between the I and the Y corner. This oscillation was a source of much headache since the time of Aristotle, as explained by Gardies [27], because the Stagirite wanted to draw a parallel between quantifiers and modalities, and modalities were appearing rather as a triangle (the triangle presented in PICTURE 3 where the question mark is interpreted as possible/contingent, the two being interpreted as synonymous) and quantifiers rather as a square.[6]

In fact Blanché's hexagon provides a solution to this problem, showing that a square of opposition is neither a good solution for the quantifiers (see [8]), nor for the modalities. One may want to substitute triangles of contrariety, but then things are missing. The hexagon presents a complete picture with including the triangle of contrariety and the square of opposition.

For the notion of possibility we have exactly the situation described by Tarksi [46]: "we are confronted, not with one concept, but with several different concepts which are denoted by one word". There are two concepts whose meaning is perfectly defined by the hexagon of modalities, corresponding to the I and Y corners and denoted respectively by the symbols "$\diamond$" and "$\bigtriangledown$" in formal language.

Now we are back to our initial question: should we call the $\diamond$ operator "possibility"? Let us go on listening to Alfred: "...we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way)". This is what has been done with the hexagon that clearly defines and distinguishes two concepts: the one on the I corner, the one on the Y corner, using the logical structure of the hexagon. Alfred adds " ... to avoid further confusions, we should agree to use different terms for different concepts". This is what has been done in the formal language, there are two terms: "$\diamond$" for the I corner, "$\bigtriangledown$" for the Y corner. But this is not exactly what is done when giving natural names to these two symbols. It is true that they are called by two different names: "possible" and "contingent"; but "possible" is not a good name because its meaning in natural language does not correspond to the I corner, it is oscillating between the I and the Y

[6]What we are saying about Aristotle here has to be interpreted more or less metaphorically, because he didn't draw squares and triangles, but Horn [29] pointed out that Aristotle had the square in mind.

corners.

Is this a tragedy? Not so many dead. But the confusion is reigning if logicians think they are describing the meaning of the Y corner with the $\diamond$ operator. Is this the case? It is difficult to have a clear-cut answer to this question. But it is strange that so few attention has been given to the Y corner and the correlated operator. Montgomery and Routley put emphasis on it, baptizing it with "$\bigtriangledown$" and axiomatizing it. But this does not appear in the mainstream of modal logic. Richard Routley has contributed in many ways to modern logic, in particular in relevant and paraconsistent logic, but for many he is a strange Australopithecus. He changed his name to "Sylvan", wrote about cannibalism, etc. (Other people from this remote part of the world have worked on this topic, see [33], [32], [34], [36]).

The operator $\bigtriangledown$ can be defined in S5 as $\diamond \wedge \neg\Box$. So one may think that we have already a theory about it, everything has been said. This is a reductionist extensionalist point of view. One same thing can be thought in different ways. If we focus on the "Sinn", not on the "Bedeutung", the difference between two different ways of thinking is important and this can formally be described. In a previous work we have given an axiomatization of S5 [2] completely different from the usual one: using as primive connectives only $\rightarrow$, $\wedge$, $\vee$ and the operator corresponding to the O corner which is generally presented as $\neg\Box$. We considered it as a primitive connective $\sim$ because it has some properties of a paraconsistent negation. This is a new look at S5. One can even claim, not being extensionalist, that this is a new logic. We in fact gave a new name for it: the logic Z [4].

From an intensionalist point of view it is not erroneous to say that the logic of contingency with the operator $\bigtriangledown$ as the main character is a different logic from S5. Developing this logic we focus on the property of the operator $\bigtriangledown$, formalizing one of the two fundamental meanings of "possibility" in natural language, the one we are using when saying "It is possible that it will rain".

Moreover we don't necessarily have to restrict the study of the operator $\bigtriangledown$ from the point of view of S5, we may study it from the point of view of another existing system of modal logic or create a new system of modal logic motivated by the meaning of contingency. It is in the line of what we were saying in our introduction: (1) "going back to the down-to-earth initial philosophical notions it has been inspired by", the down-to-earth notion being the possibility of rain; (2) "bouncing back to new mathematical constructions", such constructions being the construction of new modal systems, that nobody has yet even dreamt of. Routley already started the job, but there are still many things to be done and explore. The logic of contingency is an open field.

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