South American Journal of Logic
Vol. 3, n. 2, pp. 415–481, 2017
ISSN: 2446-6719

Squaring the unknown: The generalization of logic according to G. Boole, A. De Morgan, and C. S. Peirce

Cassiano Terra Rodrigues

Abstract

This article shows the development of symbolic mathematical logic in the works of G. Boole, A. De Morgan, and C. S. Peirce. Starting from limitations found in syllogistic, Boole devised a calculus for what he called the algebra of logic. Modifying the interpretation of categorial propositions to make them agree with algebraic equations, Boole was able to show an isomorphism between the calculus of classes and of propositions, being indeed the first to mathematize logic. Having a different purport than Boole's system, De Morgan's is conceived as an improvement on syllogistic and as an instrument for the study of it. With a very unusual system of symbols of his own, De Morgan develops the study of logical relations that are defined by the very operation of signs. Although his logic is not a Boolean algebra of logic, Boole took from De Morgan at least one central notion, namely, the one of a universe of discourse. Peirce critically sets out both from Boole and from De Morgan. Firstly, claiming Boole had exaggeratedly submitted logic to mathematics, Peirce strives to distinguish the nature and the purpose of each discipline. Secondly, identifying De Morgan's limitations in his rigid restraint of logic to the study of relations, Peirce develops compositions of relations with classes. From such criticisms, Peirce not only devises a multiple quantification theory, but also construes a very original and strong conception of logic as a normative science. In the end, some brief considerations about this point will be presented.

Keywords: Boole, De Morgan, Peirce, square of oppositions, algebra of logic, semiotic, normativity.

This paper was awarded the 'Newton da Costa Logic Prize 2016'. This prize was created in Brazil in 2015 in honor of Newton C. A. da Costa, the great Brazilian logician. The 'Newton da Costa Logic Prize' is part of a larger project, 'A Prize of Logic in Every Country'. For further information, see http://www.uni-log.org/logic-prize-world.

1 A little bit of history

One can say modern logic emerged from 1850 to 1880, with G. Boole in England and G. Frege in Germany. But this may be a little hasty narrative. Both thinkers recovered Leibniz idea of a pure language for logic, and each one in a own peculiar way (of course). There is a difference Frege himself remarked: Boole sought to devise a *calculus ratiocinator*, wherein extensionality was important for the calculus of classes. Frege's own concept-script should be understood more as a *lingua characteri(sti)ca*, that is, less like a calculus it was a true *symbolic language*, aiming at the purely logical expression of what Frege called the thought. Frege then gave prominence to intensionality as the basis for his own propositional logic. By extension, logic is considered as a descriptive science, that is, with the use of symbolic language, logicians describe the nature of what is truly logical, that is, what it is to be True and to be False independently of any psychologisms or empirical circumstances, such as mental representations or natural idiomatic expressions. Frege's concept-script would then be the fulfillment of Leibniz dream, to wit, through it the rigorous, exact, and objective expression of thinking in its most abstract formulations would be accomplished, and modern logic thus is born.[1]

This narrative became a sort of commonplace in logic books. The development of logic appears as a linear and progressive history, having Frege as the founding father and everything else as a development of Fregean ideas. The gist of the question is quantification theory: Frege would have been the first and the only one to develop a theory of quantification for bound individual variables. But, as we will see, Peirce also developed such a theory, almost at the same time as Frege. In fact, Peirce published his quantification theory few years after Frege, but independently nonetheless. This is a fact Heijenoort, in a very influential work, solemnly ignored.[2] But apart chronological issues, the fact is that logic did not develop exactly like this. And Peirce himself may not fit so well into this categorization, for multiple different reasons I cannot hope to exhaust here. The development of the logic of relations turns out as a central point in the history of logic, with special attention to A. De Morgan. I hope my general argumentation gives the reader some hints on why there is a lot more to tell. So, let us recover a bit of history.

[1]See G. Frege 1881 and 1882. For this historical summary I have draught from: W. Kneale & M. Kneale 1991, especially Chapter VI; S. Burris and J. Legris 2016; V. Peckhaus 1999 and 2004; L. Haaparanta 2009. Other specific references will be properly quoted where they were used.

[2]See, for instance, Van Heijenoort 1967. I. Anellis 2012 and 2015 provide the best arguments for an informed criticism of Heijenoort. But this is no recent story. See, for instance, H. Putnam 1982; C. Tiercelin 1991; G. Brady 2000; A. Molczanow 2012.

1.1 The square of oppositions and the question of existential import

The square of oppositions, as nowadays usually found in logic books, is absent from all known works by Aristotle, even though anyone could in principle infer it from the Estagirite's descriptions of the relations among the four basic categorical propositions—affirmative universal, negative universal, affirmative particular, and negative particular. These descriptions mainly appear in chapters VI and VII of *Peri Hermeneias*. Instructions to construct the square in a diagram (but maybe not the diagram itself) seem to have been given for the first time by Apuleius Madaurensis, or Lucius Apuleius Platonicus, *c.* 125-171, in his commentary on Aristotle's *Peri Hermeneias*.[3]

The starting point is the definition of affirmations and negations as contradictory assertions: there is a contradiction (*antiphasis*; *De Int.*, 17^a 32-33) whenever opposite assertions of the same property about the same thing are made at the same time. Or, more accurately, opposite assertions cannot be true at the same time. Another basic point is predication. General and singular predications are formally similar. So, for instance, propositions like "Machado de Assis is an animal" and "Humans are animals" are similar, since the universe of discourse is completely denoted by the subject term, that is, it is universally denoted. A whole different thing is to say "Some human is an animal" or "Not every human is an animal". There are two different ways of predicating, then, universally and particularly. Such expressions are to be understood side by side with the distinction between universal terms and particular, or singular, terms. Because they denote classes of items, propositions in Aristotelian logic are said to be *categorical*, that is, they categorize the subjects they refer to.

In declaratory—*apophantic*, in Greek—discourses subjects and predicates are terms capable of being related in affirmations or negations. Terms can in turn be individual like proper names, denoting one item, or universal, in which case terms can stand by multiple items, e.g., "human", "woman", "horse", "white", etc. The word "universal" is said *katholou*, which in a more literal translation means "of a whole", as opposed to *kath'hekaston*, "of a particular" (*De Int.*, 17^a 38-39). Subjects can be both, but predicates are only universal: "Socrates is human", "Carolina is a woman", "Horses are animals", "The wall is white", etc. In apophantic discourses, one can affirm or deny *universally* ("all men are white" or "no man is white", the predicate is affirmed or denied for all items denoted by the subject); *particularly*, or *partially* ("some men are white" or "some man is not white"—the same as "not every man is white," that is, the predicate is affirmed or denied of a part of the items denoted by

[3]On this point, see especially M. Correia 2003 and 2011; also D. Londey and C. Johanson 1984.

the subject); or *indefinitely* ("man is a beast" or "man is not a beast", so there is not a specification of the subject, and the declaration can be interpreted as being either an universal or partial assertion). In any case, just the subject term is quantified, which means only the universe of items denoted by the subject term is capable of specification. From this, let us pass to the following quote:

> I call an affirmation and a negation *contradictory* opposites when what one signifies universally the other signifies not universally, e.g., "every man is white" and "not every man is white", "no man is white". But I call the universal affirmation and the universal negation contrary opposites, e.g. "some man is just" and "no man is just". So these cannot be true together, but their opposites may both be true with respect to the same thing, e.g. "not every man is white" and "some man is white". Of contradictory statements about a universal taken universally it is necessary for one or the other to be true or false; similarly if they are about particulars, e.g., "Socrates is white" and "Socrates is not white". But if they are about a universal not taken universally it is not always the case that one is true and the other false. For it is true to say at the same time that a man is white and that a man is not white, or that a man is noble and a man is not noble [...] (*De Int.*, 17b 17-29).

First, for didactic reasons, let us call as the medievals did universal and particular affirmative propositions respectively by A and I—from *AffIrmo*—and universal and particular negatives by E and O—from *nEgO*. Next, if we try to make a diagram for the text above, we might get something like:

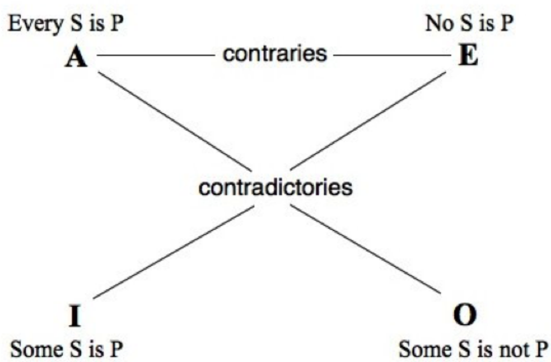


Figure 1: The traditional square of oppositions

This diagram—a possible representation of the square— is constructed only for evidencing the relations between propositional forms. Contradictories cannot be true and false together, as the philosopher says. For instance, if A is true, one infers that O is false, and vice-versa. So, if it is true all cats are mammals, and then it is false to deny some is not. Contraries in turn cannot be both simultaneously true, but they can be simultaneously false: if it is true all cats are mammals, to assert no cat is a mammal is obviously false. But from the falsity of the affirmation that all cats are hedonists one cannot rule out the possibility that no cat is an hedonist is also false, for it is still possible that some cat is an hedonist, or yet that not every cat is an hedonist. From this, several other relations can be inferred. Two are particularly important:

- I and O are *subalterns* of A and E respectively. This means that if the universals are true, their correspondent subalterns also are;
- I and O are *subcontraries*, that is, since they are contradictories of the contraries, I and O cannot be both simultaneously false, but they can be simultaneously true.

These are immediate inferences, that is, they do not need any argumentation to be justified. In other words, some logical equivalences between the propositions can be pointed.[4] From these, more complex reasonings can be made by combining different assertions: if I affirm the truth of the proposition that all men are mortals, then I immediately allow for the conclusion that some man is also a mortal, as well as the falsity that some may not be.

But by being categorical, Aristotelian propositions present some difficulties. A true proposition "is similar to facts" inasmuch as its articulations—its syntax, so to say—denote the articulations of beings themselves. The word "man", for instance, is not like the being man is; but the assertions we make about the subject man make it possible to understand its relations, necessary or not, with other beings, or with other qualities, be they essential to it or not. The act of composition, or combination defining a proposition is what allows us to understand how facts happen in the word. This is not because propositions are like maps or images of worldly facts, but because a proposition reveals factual articulations and distinctions. Words are not like things, but the syntactical relations between words within a true proposition allow us to say without contradiction that the proposition is "similar to facts". Thus, in Aubenque's words, it is in judgments and assertions that "discourse in a sense overcomes itself to go towards things in themselves".[5]

[4]There are others, but not relevant for the discussion here. See *Peri Hermeneias*, chapter X. These relations are also discussed in the *Topics* (e.g., II, 1, 109^a 11ff.; II, 8, 113^b 15–26), as well as other passages in the *Analytics* and the *Sophistical Refutations*.

[5]P. Aubenque 1962, pp. 110-112, our translation.

Aristotelian propositions, then, seem to always assume some existence. In other words, the assertion of "some" would assume *at least one*.[6] For instance, if we say "some horse is fast", the subject term and the predicate term are categorical, that is, they denote classes, groups or sets of items sharing some common character. The proposition is then true because *at least one* fast horse exists. However, how we should consider the following propositions is not clear:

α] All unicorns are single-horned horses.

β] No vampire is not a hematophagus.

The problem is that unicorns and vampires do not exist. The subject terms of both propositions denote empty classes, and then the *at least one* clause is devoid of sense. Does this mean the propositions should be classified as false? Not necessarily. See that *by definition* they are true, so that they are indeed tautologies: all unicorns are unicorns; no vampire is not a vampire. But as those classes are empty—categories "unicorn" and "vampire" do not find to what refer in the world—does it make any difference whether they are true or not?

Now, considering the possibility of building categorical propositions with terms denoting empty classes, [α] and [β] have to be classified evidently as false. But if we abandon the assumption of an existential import, then [α] and [β] can be considered as true by definition, even if there is *not a single* unicorn and *not even one* vampire. In other words, not all categorical propositions describe classes or genera of entities, what in Aristotelian philosophy is impossible.

Lack of existential import leads to abandoning some immediate inferences. Propositions A and E cease to be contraries, as Aristotle thought, for they can be both true with subjects denoting empty classes. And, among others, subalternation also becomes invalid. Take the following propositions for instance:

[A] All cyclopses are stupid.	[I] Some cyclopses are stupid.
[E] No cyclops is stupid.	[O] Some cyclopses are not stupid.

If there are no cyclopses: I and O can both be false and in certain cases not even subcontraries they are. And how can something be simultaneously a cyclops and a stupid? If I is necessarily false, subalternation is impossible, and so forth.

Here, the problem is that it is impossible to decide if propositions are true or false only from a logical-linguistic analysis. Before that, one has to inquiry

[6]But the O form, if translated as "not every S is P", would not. See T. Parsons 2017, for this point.

into the world to find out if there are cyclopses at all! Now, for cyclops, this may not be a big problem. But and for angels? And for God?

Leaving theology aside, there are some strategies to avoid this problem. One of them is to interpret universal categorical propositions as if they were formally equivalent to conditionals. Thus, "all S is P" should be understood as saying that "if there is some S, then it will be a P"; and "no S is P" as "if there is some S, then it will not be a P". Now, to say, "if there is" does not involve a compromise with existence or inexistence of any items, and thus nothing is stated about the propositional truth-content.

This strategy will be Boole's, and, then it will become a sort of commonplace. Besides that, the identity relation between subject and predicate established by the Aristotelian copula will also be modified, mainly by De Morgan and later on by Peirce. To this we pass now.

1.2 The algebra of logic and the calculus of classes

The liberation of modern algebra from the theory of numbers is a major development in mathematics in the 19th century. The general idea that certain symbolic operations can be extended to other elements than positive entire numbers has uncountable and far-reaching consequences. One can say that in logic this is the most important idea for the renewing of the discipline, that is, the very reason for Kant's *dictum* on the completeness of logic since Aristotle's times becoming but a hasty and unfortunate exercise of futurology (notwithstanding Kant could never guess what was to come).[7]

Historians of mathematics usually refer to the pioneering works of G. Peacock (1791-1858) and D. F. Gregory (1813-1844) as the starting points of the liberation of algebra, a liberation taken farther in the seminal works of W. R. Hamilton (1805-1865) and H. G. Grassmann (1809-1877).[8] For our concerns, suffice it to say that Gregory defines "symbolical algebra" as "the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject" [Gregory 1840, 208]. The main idea is here quite clear: the interpretation of mathematical symbols does not need to follow the definition of numbers and numerical operations. The main question is about the very nature of algebra: was it only a theory of equations, having numbers as the solution, or would it be like a formal discipline concerned with the study of symbols used in certain operations under certain well defined rules? Is logic superior to mathematics or dependent upon it? This is Boole's starting point,

[7]There are entire libraries on the subject. For a very brief account of the development of logic specially related to the subject of this article, see N. Houser 1994.

[8]See H. Eves 1990, chap. 13-10 for a succinct account of this liberation.

as well as De Morgan's, as will be clear. Let us begin with Boole.

A self-taught person, Boole became professor at the Queen's College, in Cork, Ireland, without having obtained any academic qualification. This meant nothing in fact, for his knowledge of the then state of art discussion on algebra could hardly be paralleled. Boole formulated his own algebra for expressing logical relations, initiating the inquiry on an algebraic symbolic system for expressing logical reasonings. This is his algebra of logic, or "Boolian algebra", as Peirce nicely used to call it. Boole thought this was the right approach to establish the precise relation between logic and mathematics in general, on the one hand, and between logic and algebra specifically, on the other. A precise definition of the nature of each science would then be reached.

For Boole, the key idea is that extension is more important than intension, that is, classes are more important to logic than propositions. Another point of contact between logic and algebra comes from other two very widespread ideas among English logicians in the first half of the 19$^{\text{th}}$ century, to wit, the *quantification of the predicate*, due to such names as J. Bentham, R. Whately, W. Hamilton and A. De Morgan, and the notion of *universe of discourse* Boole developed from De Morgan. Let us see this latter one first, and then return to the relations between logic and mathematics later.

In the Aristotelian tradition, negation was considered to hold exclusively for the copula, for in syllogistic theory a negative term like "not-human" is considered as an indefinite term. However, if by the *universe of discourse* we understand the domain of *what is talked about*, we have another possibility. The positive term determined, the negative is immediately determined as well, and so the universe is divided into two distinct and complementary classes, symmetrically defined. For instance, if we are talking about "humans" as a specific class of animals, then the universe of discourse is the class of all animals, with the term "not-human" designating the class of non-human animals. This procedure makes it possible to negate the predicate, something totally unpredicted in syllogistic. Now, Aristotelian logicians distinguished between universal, particular and singular propositions by quantifying the subject-term. The idea of a universe of discourse however allows us to apply the same operation to the predicate-term, so that we obtain propositions with both the subject and the predicate quantified, such as "All humans are some mortals" for instance, where there is a universal subject and a particular predicate. Such formulation makes it possible to interpret the relation of predication as one of identity, giving to the proposition the form of an equation: "All humans = some mortals". The same applies to negative propositions.

The idea of negating the predicate is not originally Boole's, it was part of the *Zeitgeist*, if I am allowed to use this expression. For instance, Richard Whately, whose works were studied by G. Boole and years later decisively influenced C. S. Peirce to dedicate his life to the study of logic,[9] considers whether conversion can be made valid for *all* propositions of the square, including O. Aristotle famously considered conversion valid only for propositions A, E, and I. Whately gives the following example of a *conversion by negation*, or "by contra-position": the proposition "Some members of the university are not learned" can be interpreted to mean "Some who are not learned are members of the University". Whately says although the terms are not exactly the same, the propositions are "aequipollent", that is, they convey the same meaning [1849, p. 36]. Now, Boole understood this in the form of equations, and this is exactly what makes his logic a sort of algebra. The way to abandon the propositional form relating one subject to one predicate is therefore wide open.

2 Algebra and logic according to George Boole

In his 1847 book, *The Mathematical Analysis of Logic*, and mainly in his 1854 *The Laws of Thought*, Boole explains his ideas on a "new system" of logic. He radically breaks with undoubted ideas at his times, claiming, "logic forms no part of [philosophy]" and that "we ought no longer to associate Logic and Metaphysics, but Logic and Mathematics" [1847, *Introduction*, p. 10]. By taking what he considers to be essential from algebra, namely, the combination of different symbols apt to receive specific meanings with the purpose to devise a calculus, Boole claims:

> They, who are acquainted with the present state of the theory of Symbolic Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics or optics. [Boole, 1847, p. 3,].

Boole's aim is to show that reasoning obeys the laws of algebra. To show that it is so, Boole applies the algebraic calculus to logic in a purely formal way. In fact, Boole recovers the Leibnizian ideal of a *calculus ratiocinator*, but without a *lingua characterica universalis.* Adding that the study of numbers and

[9] Richard Whately (1787-1863) was an English logician and theologian. The quotes above are from his *Logic* (1849), which became the first volume of the *Encyclopaedia Metropolitana*, edited by Samuel T. Coleridge and projected to compete with *Encyclopaedia Britannica.* On the influence of *Whately's Elements of Logic* (1855) over C. S. Peirce see W 1: xviii-xix.

quantities is not essential to mathematics, he claims something much stronger indeed: there is more than a close analogy between the "operations of the mind in general reasoning and its operations in the science of Algebra", there is indeed "to a considerable extent an exact agreement in the laws by which both the two classes of operations are conducted" [1854, chap. 1, § 6]. This is the reason why Boole comes to think of the ultimate forms and processes of the science of logic as necessarily mathematical. He says:

> That Logic, as a science, is susceptible of very wide applications is admitted; but it is equally certain that its ultimate forms and processes are mathematical. Any objection *à priori* which may therefore be supposed to lie against the adoption of such forms and processes in the discussion of a problem of morals or of general philosophy must be founded upon misapprehension or false analogy. It is not of the essence of mathematics to be conversant with the ideas of number and quantity [1854, chap. 1, § 11].

In the *Mathematical analysis of logic*, chapter 2, Boole quotes [1847: 20] from Whately, when expressing the basic categorical propositions – A, E, I, O – with algebraic symbols. At this time, his own way of understanding those propositions is very innovative: "A logical proposition is, according to the method of this Essay, expressible by an equation the form of which determines the rules of conversion and of transformation, to which the given proposition is subject" [1847: 8]. Boole is already perfectly aware of the revolutionary character of his claims, but at this time he only symbolizes the traditional categorical propositions. The step further will be given only in the 1854 book, when he assumes the task to formulate in a mathematical way the processes of human reasoning. To accomplish this task, Boole adapts a "tool", an "instrument", to wit, a symbolic language conceived *more algebrico*, that is, following the rules of algebra, and with each of its sign arbitrarily defined with one meaning. He expects with this to fulfill something modern philosophers thought to be possible, which is to exhibit the very logical structure of mental operations, that is, the movement of thought itself. From this, Boole expects to be able to express the laws that are common to algebra and to human reasoning *regardless of a human subject* (dismissing the kind of subjectivism or empiricism philosophers like J. S. Mill, for instance, argued for). Those are the famous laws of thought.[10] See:

[10] L. Couturat's statement that Leibniz had already anticipated all of Boole's (and of Schroeder's) logic principles in more or less 150 years, even going beyond Boole, is well known. It is not possible to go deeper into this here. The reader may find it useful to see V. Peckhaus 2012; W. Lenzen 2004.

> All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.: 1st. Literal symbols, as x, y, &c., representing things as subjects of our conceptions. 2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements. 3rd. The sign of identity, $=$. And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra. [1854, chap. 2, §4, p. 27].

How Boole interprets syllogistic using the calculus, thus naming the new formal language thereby derived *algebra of logic* is what we are going to see next.

2.1 A calculus and a logic of classes

Boole's algebra of logic is an *abstract* calculus that should be interpreted in a double way, as he himself reminds us: first, as an algebraic calculus, but also as a calculus of *classes*. According to this latter interpretation the symbols are thus defined: "x", "y", "z" are variables representing propositions; the signs "$+$" and "$\times$" respectively stand for *exclusive disjunction* and *conjunction*; the sign "$-$" corresponds to the negation of the predicate; the sign "$=$" as sign of identity means in fact equality, i.e., to say "$x = y$" means x and y are classes with the same members; and "1" and "0" stand for True and False, which are the only two possible values assignable to propositions. Take then, for instance, the formula $x^2 = x$. It expresses a logical law of the propositions: $x \wedge x = x$ (x symbolizes any proposition p). Then, if x and y respectively refer to two propositions p and q whatsoever, the equation $x(1 - y) + y(1 - x) = 1$ means that either p is true or q is true, that is, only one of them two is true, not both.

The last passage quoted above showed that the Boolean sign "$=$" kept exclusively one function of the Aristotelian copula, that is, the identity function. This gives Boolean logic its extensional nature, meaning that each instance of elementary reasoning is expressed as an equation, just like in algebra. The symbols of the first group can be interpreted in terms of classes – Boole favors this interpretation naturally – and those of the second group correspond to the operations upon classes. For instance, if one writes $x = y$, then it is possible to understand that "xy" represents the class of the members of x that are also the members of y and vice-versa, that is to say, xy represents the intersection of both classes x and y.

This is exactly what makes Boole's logic a calculus.[11] The table below shows some analogies between some *laws* of algebra and some *laws* of logic (of course, not strictly expressed in ordinary language).

Laws of algebra	**Translation in ordinary language**
$xy = yx$	human things = things human
$x + y = y + x$	either violets or roses = either roses or violets
$z(x + y) = zx + zy$	flowers (either violets or roses) = either violets flowers or roses flowers
$z(x–y) = zx–zy$	flowers (violets, but not roses) = violets flowers, but not roses flowers
$(x = y + z) = (x–z = y)$	flowers are roses and violets = flowers, but violets, are roses

Figure 2: Translation of Boolean equations to natural language

There is, however, no perfect analogy between the laws of logic and the laws of algebra, no "exact equivalence", as Boole himself reminds several times. For instance, every time we have in algebra $x^2 = x$, we have a problem in Boole's logic: if x stands for the class of horses, xx, or x^2, standing for the class of horses that are horses also stands for the class of all horses. In algebra, this equation is valid only for numbers, 0 and 1, the solutions for the equation $x^2 = x$, wherein $x(1–x) = 0$. Boole's algebra of logic is, then, a specific algebra, a binary one, wherein the only possible numerical values are 0 and 1 (T and F). In sum, we can truly say that Boole's algebra is characteristically distinguished by the idempotent law: $x^2 = x$.[12] There remains the task to find the laws of such an algebra and an acceptable logical interpretation for 0 and 1.

The first point would lead us too far stray. Suffice it to say that Boole's algebra satisfies the majority of the laws of common algebra, as will be clear soon: commutativity of multiplication and sum, distributivity of multiplication over sum and subtraction, rule over "change of member" in an algebraic equation etc.

On the second point, let us remember how Boole interpreted the categorical propositions of syllogistic, borrowing one of De Morgan's idea he himself

[11] See W. Kneale and M. Kneale 1962, p. 407.

[12] According to C. Polcino Millies and S. Sehgal (2002, p. 127), the very conception of idempotency as used in current ring theory was firstly introduced by Charles S. Peirce's father, Benjamin Peirce, in his *Linear Associative Algebra*, published as litograph in 1870 and then posthumously edited in 1881 by his son Charles himself.

renamed as *universe of discourse*. The Boolean universe of discourse circumscribes the limited domain of what is talked about (so to say):

> Now, whatever may be the extent of the field within which all the objects of our discourse are found, that field may properly be termed the universe of discourse.
>
> Furthermore, this universe of discourse is in the strictest sense the ultimate *subject* of the discourse. The office of any name or descriptive term employed under the limitations supposed is not to raise in the mind the conception of all the beings or objects to which that name or description is applicable, but only of those, which exist within the supposed universe of discourse [1854, chap. III, §§ 4-5, p. 42].[13]

The universe of discourse is denoted by 1 and it is called "the Universe", for it represents "the only class in which are found *all* the individuals that exist in *any* class". This is assumed on the basis that $1 \times y = y$ (or $1y = y$) [1854, chap. III, §13, p. 47]. The symbol 0, for its turn, represents the empty class, called "Nothing", that is, the class talked about that has no members. This is assumed on the basis that $0 \times y = 0$ (or $0y = 0$) [*id.*, *ibid.*]. All of this is of considerable novelty, since it is the first time that extensional logic explicitly considers the empty class. As Boole says, "Nothing and Universe are the two limits of class extension", "the limits of the possible interpretations of general names" [*id.*]. From this, the result is that if x denotes a class of objects that do belong to the universe of discourse, $1 - x$ denotes the class of objects that do not belong to x, that is, its complement.[14] Moreover, Boole is seems to be convinced that human beings are apt to "mentally selecting" [1847, *Introduction*, p. 5] the objects that belong and the ones that do not belong to a given class. For instance, if the universe of discourse is the class of animals and if v represents *vertebrates*, $1 - v$ represents the *invertebrates*, or the animals without a backbone. Therefore, we have: $v + (1 - v) = 1$ (that is, the union of vertebrates and invertebrates is the universe of discourse); and $v \times (1 - v) = 0$ (that is, the classes of vertebrates and of invertebrates are mutually exclusive).

Now, the symbolization Boole offers for the basic categorical propositions of syllogistic logic can be presented in the following table:

[13] Boole is indeed considered the first to ever have used this expression, in the passage just quoted above. See S. Nambiar (2000), p. 230.

[14] As Boole himself states [1847, p. 20], the more precise formulation is that class X is denoted by the symbol x, which selects from the universe of discourse the individuals that belong to X (the individuals that are Xs). But Boole is inconsistent in using this distinction, as can be easily shown by the reading of the very same chapter.

A	All Xs are Ys	a	$x(1-y)=0$	a'	$xy=x$
E	No Xs are Ys	b	$xy=0$	b'	$x=v(1-y)$
I	Some Xs are Ys	c	$v=xy$	c'	$vx=vy$
O	Some Xs are not Ys.	d	$v=x(1-y)$	d'	$vx=v(1-y)$

Figure 3: Categorical propositions in Boolean form

Take proposition A. Boole's reasoning is as follows [1847, p. 21]: since all existing Xs are in class Y, it is evident that to select all the Xs from the Ys and that to select all those Ys from the universe is equal to selecting all the Xs from the universe of discourse at a single time. That is why we should write $xy = x$. And next, from the law of distribution and the law of transposition (not stated in this book), Boole writes $xy = x$ in form (a). For proposition I, the reasoning is as follows: if some Xs are Ys, there are some common terms to classes X and Y. Let those terms be members of a distinct class, the class V, symbolized by v, and, then, $v = xy$. As v includes all the terms that are common to classes X and Y, we can interpret it both as *Some* X or as *Some* Y. With this interpretation, Boole claims all equations expressing particular truths can be understood as necessary deductions from a single general proposition of the form

$$x = y.$$

If we multiply this formula by x, we shall have $x^2 = xy$, therefore $x = xy$. That is truly form (a): $x(1-y) = 0$. Next, he considers this latter formula as an equation, where x should be determined in terms of y, saying that its "general solution" is that $x = vy$, that is, form (a'). According to Boole, this can be verified by a simple replacement by (a). And last but not at all least, by multiplying (a') by v, which is implicitly assumed as a not-empty class, Boole obtains form (c'), which itself implies that "Some Ys are Xs and some Xs are Ys" [*id.*, pp. 22-24].

In the next chapters, Boole employs this calculus to questions of traditional logic, more often than not preferring formulas (a'), (b'), (c'), and (d'). This way of proceeding truly has several weak–or rather problematic points. The most important one for our concerns here is that the replacement of form (a) by (a') alters the existential content of proposition A.[15] This point will be made clearer if we first consider the some aspects of the use of Boolean algebra for complete syllogisms. Let us take two of Boole's formulas that value as laws and follow from the laws he himself formulated.

[15]This is not the only weak point, but it is impossible to discuss all of them here. Some difficulties will nevertheless be presented soon ahead. For a deeper take on the subject, see J. Corcoran and S. Wood 1980.

$$[\alpha]\ x + (1 - x) = 1$$

$$[\beta]\ x \times (1 - x) = 0$$

Formula $[\alpha]$ expresses the *law of the excluded middle* and formula $[\beta]$ *the contradiction law.* Both derive from the single most important innovation in Boole's 1847 book, which is his "index law": "The result of a given act of election performed twice, or any number of times in succession, is the result of the same act performed once" [1847, p. 17]. With "election", Boole seems to mean something like picking up one element of the class. So, the law of the excluded middle says the union of the class of xs with the class of *not-*xs is the universal class, that is, it is completely saturated, if we want to speak in a Peircean jargon [see Boole 1997 (*c.* 1860), p. 141]; and, accordingly, the contradiction law says the intersection of the class of xs with the class of *not-*xs is empty, i.e., its value is *zero*, that is, in a Peircean jargon, completely unsaturated. In other words, the empty class means that it is impossible to be simultaneously x and *not-*x. In this way in Boole's logic of classes, the principle of contradiction follows from the idempotent law. In other words, this law says that choosing property x n times is the same of choosing x once. Or else, the index law is the only peculiarly logical law, the basic law of Boole's system [see, e.g., Boole 1854, p. 35; Pantecki 2000, p. 195]. According to this interpretation, $+$ symbolizes the reunion of two disjunct classes, $\times$ represents their intersection and $=$ their equality. The sign $-$ allows the passage from one class to its complement. In this way, Boolean algebra of logic is ideally suitable to extensional interpretation, according to which its symbols are combined to form formally solvable algebraic equations, that is, equations solvable under its laws and the solutions of which can be thereby interpreted in terms of classes. Such a procedure permits the examination of all possible combinations in a way that we can derive for a syllogism all the conclusions that are also coherent with traditional syllogistic deduction. For instance, the following syllogism:

1. No god is mortal.
2. All men are mortal.
3. So, no god is a man *or* No man is a god.

Next, x represents the class of Xs (gods), y represents the class of Ys (mortals) and z represents the class of Zs (men).

1. $xy = 0$: the class of individuals that are at once Xs and Ys is empty.
2. $z(1-y) = 0$: the class of individuals that are Z without being Y is empty.

3. From (1) and (2), according to the laws of Boole's algebra, we derive by deduction that: $[xy = 0$, thereby $y = 1 - x$, thereby $1 - y = x$, thereby $z(1 - y) = zx = 0] : xz = zx = 0$.

The class of individuals that are Xs and Zs at the same time is empty.

However, Boole's algebra is not valid for all Aristotelian logic, let alone all syllogistic. This is an important point. To avoid the problems of implying an existential content to categorical propositions, Boole interprets the propositions of the logical square in a way that suppresses certain aspects of Aristotelian negation. This obliges him to completely modify the traditional propositions and their mutual relations. For instance, propositions A, such as "All men are mortal" become something like "Nothing exists that it is at the same time a man and not-mortal". The same applies to propositions E, like "No man is immortal", which becomes something like "Nothing exists that it is a man and at the same time it is immortal". As a result, only one opposition of the traditional square, namely, the one of contradiction, can be maintained. All the others – subalternation, contrariety, sub-contrariety – have to be dismissed, as well as the operations on the propositions themselves – conversion, obversion and contraposition.

Let us see more about this. In Boole's calculus, whether possessing existential content or not, each traditional categorical proposition is to be interpreted as the product of the *intersection* of classes of items represented by the subject term and the predicate term. Then if the class of the subject term is empty, that is, if there are no items under it, it does not possess existential content and is represented by the equation $\mathrm{S} = 0$; if it possesses existential content, that is, if it is not empty, then $\mathrm{S} = 1$ represents it. Now, if SP represents the categorical proposition "S is P", we have two possibilities: either (a) $\mathrm{SP} = 0$; or (b) $\mathrm{SP} \neq 0$. Even though Boole himself, as we will shortly see, does not use the symbol $\neq$, that is what he means. If (a), then there is nothing that is member of the class S and at the same time is a member of the class P. If (b), then there is some item that is at the same time a member of class S and a member of the class P. All the traditional categorical propositions of syllogistic can be reformulated thereby. Let us use a more up-to-date symbolism. In the first place, it is clear that E: $(\mathrm{S} \wedge \mathrm{P} = 0)$; and also I: $(\mathrm{S} \wedge \mathrm{P} = 1)$. Besides, a proposition A can no longer merely state that "All Ss are Ps", for it can be that the class is empty and there is no S. It must be rewritten in a negative way, in E form: "There is no S that is a P". Now, as such, this form means that the intersection between S and P is empty. Hence A: $(\mathrm{S} \wedge \overline{P} = 0)$. Finally, the proposition O states that there is at least one item that is S and not-P, hence O: $(\mathrm{S} \wedge \overline{P} = 1)$.

It is clear then that while the existential content of the universal propositions disappear, the particular propositions keep theirs.[16] The consequences of this for the traditional syllogistic theory we have already seen: the so-called immediate inferences of the traditional square are not valid anymore, but for contradiction. Besides, propositions A and E can now be considered as true at the same time, I and O can be false at the same time. As this happens at the cost of a complete reformulation of the propositions, a totally new way of conceiving logic itself arises and needs to be fully developed.

Boole accomplished what Leibniz only slightly grasped, namely, he developed a system where the *isomorphism* between the calculus of classes and the calculus of propositions can be clearly seen. Underlining though that both calculi are contrasted but for the interpretation of the symbols, Boole himself was nevertheless well aware the formal analogy between the two fields is not satisfactory enough when it comes to quantifying the subject and also the predicate of a proposition.

By this way of conceiving logic operations, Boole was obliged to introduce an auxiliary operator, symbolized with v, to avoid the more arithmetical and not algebraic symbol $\neq$ to express "some", for to "some x is a y" would be rendered as $xy \neq 0$ according to his rules. As seen, this v operator can be explained in different ways: the propositions "all Xs are some Ys" (or more simply "all Xs are Ys", etc.), "Some Xs are some Ys", "Some not-Xs are some not-Ys" are possible interpretations of the following equations (respectively): $x = vy, vx = vy$ and $v(1-x) = v(1-y)$. This is imprecise, though, and the elimination of the sign is desired for an improvement in the system. Boole nevertheless does not accomplish this.

2.2 Some difficulties of Boolean logic

From such abstract calculus, Boole derives a logic of classes and finds a lot of problems. For our concerns here, some difficulties related to symbolism are to be examined.

First, let us return to symbol v, which is of an imprecise nature, as just mentioned. Its vagueness allows it to represent that an *indefinite* quantity of individual items pertaining to the class it is attached is retained: if x represents class X, vx represents a subclass containing *one*, *several* or *all* the X individuals. Thus denoting an indeterminate class, or more precisely an "operator", the symbol satisfies the idempotent law. It can be assimilated to a quantifier, but not in the sense we understand now. Attached to a class, it does

[16] A myth in logical theory appears here: since Boole to present days it is believed that there is no assumption of existential import in contemporary logic. But J. Corcoran and H. Masoud 2015 have recently showed that universalized conditional sentences assume existential import in some cases.

not mean "for all" or "there is". Second, the symbol "+" brings problems for propositional logic. The sign is used in Boolean algebra of logic to express *exclusive* disjunction and aptly corresponds to numerical sum, but not to logical sum, for *inclusive* disjunction is the more precise analogue for the conjunction found in logical product. And third, the symbol "−" (minus) imperfectly symbolizes negation. Not only the predicate needs to be negated, but the whole proposition. And finally, there is no logical equivalent to algebraic division. Boole's firm adherence to the latter distances him from logic and prevents the articulation of propositional calculus with predicate calculus, which will be invented simultaneously but independently by Frege and Peirce only about 40 years later.

Such difficulties make a flaw in Boolean logical calculus quite evident: the adequacy between *logic* and *algebra* is imperfect, for propositions are not classes. Boolean algebra of logic does efficiently solve more complex problems than Aristotelian syllogistic, but it seems to be more an instrument *for* logicians than a logic proper in itself. Boole meant to develop a general theory of deductive reasoning, but his algebra does not represent all elementary logic. Boole gives much value to an ideal of symmetry he finds in the algebraic interpretation of logical operations, but a fundamental character of the study of logic does not meet this ideal: while a logician uses formal languages to study *inferences* and make *deductions*, an algebraist solves *equations* by manipulating signs. Later on, Peirce will stress this point, as we shall soon see.[17]

These few critical remarks cannot make us forget Boole's remarkable contribution: he showed syllogistic does not exhaust the whole of logic and invented *mathematical* and *formal* logic indeed, definitely changing the discipline and opening a whole new and vast field of research for the 20th century. Boole's interest was to develop a symbolic universal method highly abstract for working with differential equations, "a *genuine* and *complete* method".[18] He might have not accomplished his goal, but started notwithstanding a revolution in logic. His failure is the measure of his success: by not being able to symbolize in a clear and defined way the very specific character of logical operations, Boole left the way open for others to complete this revolution after him. As any *Google* user nowadays may know, Boole's work is indispensable.

[17] This is just one of the aspects distinguishing Peirce from Frege. It is impossible to tackle this comparison in depth here, but we will briefly see why Peirce is not a logicist, that is, why he considers mathematics cannot be founded upon logical notions. See I. Anellis 2012 for a deeper account of the subject, as well as the editor's introduction to M. Moore 2010, and V. Peckhaus 1999 and 2004, for some very illuminating clarifications and reinterpretations.

[18] Boole's letter to W. Thomson, April 1847, *apud* M. Panteki 2000, p. 202

3 Augustus De Morgan and the birth of the logic of relations

Teaching mathematics at the then recently founded London University (London College University nowadays), De Morgan was interested both in a renewing of algebra as in a renewing of logic. His first work in that direction was published in 1847 entitled *Formal logic: or, the calculus of inference, necessary and probable.* While Boole himself presented his new algebra of logic as groundbreaking, De Morgan's aim was not to construct a revolutionary calculus. His purpose instead is to find a more adequate symbolism for traditional Aristotelian syllogistic (or what he understands it to be). In pursuing his aim, De Morgan in fact discovered some very radical innovations, accomplishing indeed a reformulation of the old formulas of reasoning and amplifying the scope of logic to encompass more reasoning forms than the known syllogistic ones. These innovations culminated in his 1847 *Formal Logic*, where De Morgan presented his full account of the generalization of the copula, recognizing relations other than identity as objects for logical study.

Two crucial points can be singled out. First, the recognition just mentioned leads to considering relations as objects of logical study in their own right, no matter if there is some identity established between terms or not. Rather than that, inferences performed by means of transitive and *convertible* relations (which we call *symmetric* relations, following E. Schroeder's vocabulary), are just as good for logical inquiry as a any sort of syllogistic reasoning. As pointed out by Merrill, "any relational terms will satisfy the syllogistic laws, as long as the 'formal' properties of transitivity and convertibility apply to them" [Merrill 1990, p. 48]. Logic, then, is not anymore just the study of syllogisms, but the study of the ways one can draw consequences from the mere formal properties of relations between terms. For instance, equality can be considered in terms of transitivity, so that the following argument form is valid:

a is a brother of b.
b is a parent of c.
Therefore, a is an uncle of c.

This can be symbolized as: $((a = b) \wedge (b = c)) \vdash (a = c)$. Note that the relation of term a to term c is inferred from the relation of a to b and of b to c, so that it is a *compound* relation. This is the basic assumption that led De Morgan to develop a formal logic of relations.

The second point comes from that. De Morgan came to recognize that not all inferential relations are reducible to syllogistic forms. In fact, it is just the opposite: the whole of syllogistic can be derived from the logic of relations. De Morgan says:

> There is another process which is often necessary, in the formation of the premises of a syllogism, involving transformation which is neither done by syllogism, nor immediately reducible to it. It is the substitution, in a compound phrase, of the name of the genus for that of the species, which the use of the name is particular. For example, "man is animal, therefore the head of a man is the head of an animal" is inference, but not syllogism. And it is not mere substitution of identity, as would be "the head of a man is the head of a *rational animal*" but a substitution of a larger term in a particular sense [De Morgan 1847, p. 114].

By the quote above we understand that De Morgan's concern is with defining some logical rule to work with universal and particular terms: the substitution of a general term – the name of the genus – for a particular one – the name of the species – makes a reasoning, or an inference, as he says. The problem is that he remains with no criteria for defining when a term is used as a genus or as a species, that is, when it is used universally or particularly. In other words, could we reformulate De Morgan's *dictum* in a way to include quantification? For instance, take the following inference

> Every man is an animal, therefore every head of some man is the head of some animal.[19]

Is it valid or not? De Morgan claims it is, and to prove so he resorts to algebra, as early as 1844 claiming the laws of his logical system greatly resemble those of algebra.[20] He did not understand this in the same sense as Boole, though, but only in the general sense that the semantics of the terms is given by the syntax of their combination.[21] This understanding is basic for his attempts to generalize the copula.

Besides, there four advantages for adopting a symbolic notation are stated by De Morgan in the preface:

[19]The example is not exactly what De Morgan writes. We will see later on how Peirce deals with this problem. See D. Merrill 1990, p. 79. De Morgan is of course aware of the problem, but only in his second article on the syllogism (1850) he deals with such quantified expressions without quantifiers, not yet in his 1847 *Formal Logic*.

[20]As De Morgan claims, in a paper read in 1844 for the Cambridge Philosophical Society, the idea of "a distinct system of unit-symbols, and investigating or assigning relations which define their mode of action on each other" first came to him through a paper W. R. Hamilton on quaternions. The difference between Hamilton's and his own system is that his's is triple and not quadruple, "and as preserving, in their laws, a greater resemblance to those of ordinary Algebra" [De Morgan, 1844, p. 241].

[21]Of course, a full analogy between the laws of logic and the laws of algebra does not always work. See Merrill 1990, p. 84 ff. for more on this.

> A simple notation, which includes the common one, gives the means of representing every syllogism by three letters, each accented above or below. By inspection of one of these symbols it is seen immediately, 1. What syllogism is represented, 2. Whether it be valid or invalid, 3. How it is at once to be written down, 4. What axiom the inference contains, or what is the act of the mind when it makes that inference [1847, Preface, p. iv].[22]

De Morgan clearly wants to preserve syllogistic, but nevertheless makes innovative claims that lead far beyond it. He thinks of the symbolism as an instrument, and an adequate auxiliary tool for studying inferences. The subject itself, of course, should not be misidentified with the instrument, but symbolism is an active instrument that modifies the theory and allows for the understanding of inferential relations that are not explicit. This is an important notion De Morgan bequeaths to the next generation of logicians, especially Peirce, as we will see. The quotation above makes it very clear that symbolism has the advantage of exhibiting what otherwise would remain hidden, that is, what kind of reasoning is being performed.

Another important conception developed by De Morgan concerns the already mentioned idea of *universe of discourse*. Its more famous definition says: "universe of a proposition or of a name[23]" . Let us quote for more context: "The whole extent of the matter of thought under consideration I call the *universe*. In common logic, hitherto, the universe has always been the whole universe of possible thought" [1860, §122, p. 39]. "Common logic, hitherto", says De Morgan, meaning logic since Aristotle, through the Middle Ages till the 19th century, the universe of discourse has always been conceived as a the fixed universe of all things conceivable. But De Morgan's definition has an apparent originality, which is to limit the universe to what is under consideration, the domain of what is being talked about so to say, the items upon which it is possible to make assertions. His argument for this restriction is that logicians always thought of the universe not as a universe of discourse, but of everything that there is. For instance, considering the universe of all men, then everything is put under the overarching umbrella of *not-men*, from women to umbrellas, including aliens and "everything imaginable or real" [1847, p. 37]. This is of little or in fact no use at all, since no one uses any language like this. When talking about a class of items, for instance, the class of *vertebrate* animals, I limit myself to consider that part of the general class of animals that have a

[22] A thorough presentation of De Morgan's logical system is out of the scope of this article. The best study on the subject till the day seems to still be D. D. Merrill's (1990).

[23] This quote is from the 1846 paper, "On the syllogism, I: on the structure of the syllogism", which is previous to Boole's work, then. *Apud* Anellis & Abeles 2016, p. 76.

backbone, the contrary of which is the class of animals called *invertebrate*. I do not ponder of qualifying a star, a theorem or a football game as vertebrate or invertebrate. So, the conception of universe means the whole whose parts are being considered. As De Morgan says:

> The universe is the whole sphere of thought within which the matter at hand is contained: usually not the whole possible universe of thought, but a limited part of it [1858, p. 208]. Thus, the universe being mankind, Briton and alien are contraries, as are soldier and civilian, male and female, &.: the universe being animal, man and brute are contraries, &c. [1847, 38].

Boole's notation tries to express this. The use of 1 and 0 shows that the universe of discourse is thought of as a truly universal class, that is, if we are talking about animals, 1 means the universal class of all animals and 0 means the empty class of everything that is not an animal (angels, cabinets, planets, whatever). De Morgan's notation conveys a different idea. By referring to every so-called "positive" notion in the universe with an uppercase letter, and to its complementary with a lowercase letter, it obliterates the distinction between affirmative and negative propositions, thus avoiding Boole's stratospheric universality. As a matter of fact, as he understands them, the categorical propositions of the tradition are to be interpreted in terms of degrees of extension and intension, as it was usual to say in those times [1858, p. 208]; or, in other words, inclusion or exclusion of a class into another. The table below shows how the categorical propositions of the square of oppositions are understood and symbolized by De Morgan in his own"spicular" notation. Being a formalist, De Morgan devises a whole symbolic system made up of letters, dots, parentheses and converted parentheses, brackets and converted brackets, and exponents, calling it *spicular* for that reason, that is, because formed with punctuation signs (from the Latin *spiculum*, point). He explains the name was chosen first in derision, but then decided it worth keeping because better than "parenthetic" [De Morgan 1858, 198]. Then, explicitly stating categorical propositions as expressing relations of inclusion and exclusion between classes, the spicular notation brings:

	Proposition of assertion or denial (Mathematical form)	Arithmetical form	De Morgan's spicular notation rendering
A	Assertion of X contained in Y	Every X is Y	X))Y
O	Denial of X contained in Y	Some Xs are not Ys	X(·(Y
E	Assertion of X excluded from Y	No X is Y	X)·(Y
I	Denial of X excluded from Y	Some Xs are Ys	X( )Y

Figure 4: De Morgan's rendering of the categorical propositions, published in De Morgan 1858

It is evident he conceives of those propositions as expressing relations between a whole and its parts, and this is the meaning of "onymatic". Each term – subject and predicate – represent a distinct class, a different part of the whole, and the copula defines the quality of the relation, whether inclusive or exclusive. As he himself says:

> Taking the mathematical form first, and dichotomizing the universe in two ways, by classes X and not-X (x), and Y and y, the onymatic relations of class to class can be predicated without any explicit dichotomy of class, whether vague or definite. The *relations* are *inclusion* and *exclusion* (inclusion in *contrary*); the judgments, *assertion* and *denial* [1858, p. 195].

In short, this means that, for any universe U, the uppercase letters X and Y represent different delimited classes within U, and x and y represent their complementaries, e.g., respectively not-X and not-Y. The universe can then be considered as equivalent to (X,x), that is, the class of all things that are of a certain class and of those that are not. De Morgan can thereby say the universe is the maximum of extension and the minimum of intension, that is, "in extension, [the universe] contains everything; in intension, it belongs to everything" [1858, 208]. For instance, if the universe considered is that of animals, and if X stands for the notion of *human*, x will stand for the notion of *not-human animal*; if Y stands for the notion of *vertebrate*, y will stand for the complementary notion of *invertebrate*. Then, negation refers to the predicate and is implicit in the symbolism. A negative proposition, such as "No X is a Y", becomes the affirmative one "All X is not-Y". So, in the universe of animals, "No human is not a vertebrate" becomes "All humans are invertebrates" (both false propositions).

XXVII. *Logico-mathematical Reading.* *Terminal Precision*, § XI.

<table>
<tr><th>Proposition of Assertion.</th><th>X subjected to Y as</th><th>Y predicated of X as</th><th>Compounded of</th><th>Notation.</th></tr>
<tr><td>X and more contained in Y</td><td>Subidentical
Deficient Species</td><td>Superidentical
Exient Genus</td><td>X))Y, X)·)Y</td><td>X)o)Y</td></tr>
<tr><td>X contained in and containing Y</td><td>Identical
Species Genus</td><td>Identical
Genus Species</td><td>X))Y, X((Y</td><td>X || Y</td></tr>
<tr><td>X containing Y and more</td><td>Superidentical
Exient Genus</td><td>Subidentical
Deficient Species</td><td>X((Y, X(·(Y</td><td>X(o(Y</td></tr>
<tr><td>X and more contained in y</td><td>Subcontrary
Coinadequate External</td><td>Subcontrary
Coinadequate External</td><td>X)·(Y, X)(Y</td><td>X)o(Y</td></tr>
<tr><td>X contained in and containing y</td><td>Contrary
External Complement</td><td>Contrary
External Complement</td><td>X)·(Y, X(·)Y</td><td>X |·| Y</td></tr>
<tr><td>X containing y and more</td><td>Supercontrary
Copartient Complement</td><td>Supercontrary
Copartient Complement</td><td>X(·)Y, X()Y</td><td>X(o)Y</td></tr>
</table>

Here X)o) Y is a compound of X)) Y and X)·) Y, coexisting. We deny it by denying either X)) Y or X)·) Y, or by asserting *either* X (·(Y *or* X ((Y. Let this aggregate be represented by X (,(Y. Then X)o) Y is denied by X (,(Y, X (o(Y by X),) Y, X)o(Y by Y (,) Y, X (o) Y by X),(Y. And identity, X || Y, is denied by either X (·(Y or X)·) Y; and contrariety, X |·| Y, by either X ( ) Y or X) (Y. When we read in mathematical intension, the prepositions *sub* and *super* change places.

XXVIII. *Logico-metaphysical Reading.* *Terminal Ambiguity*, § XVII. 2.

Proposition of Assertion or Denial.	Y predicated of X as	X subjected to Y as	Reading in arithmetical whole (*by intension only*).	Notation.
Assertion of Y a component of X	Essential	Dependent	Y always in X	X]]Y
Denial of Y a component of X	Inessential	Independent	Y sometimes not in X	X[·[Y
Assertion of Y incompatible with X	Repugnant	Repugnant	Y never in X	X]·[Y
Denial of Y incompatible with X	Irrepugnant	Irrepugnant	Y sometimes in X	X[]Y
Assertion of Y a component of x	Alternative	Alternative	In everything either X or Y	X[·]Y
Denial of Y a component of x	Inalternative	Inalternative	In some things neither X nor Y	X][Y
Assertion of Y incompatible with x	Dependent	Essential	In Y always X	X[[Y
Denial of Y incompatible with x	Independent	Inessential	In Y sometimes not X	X]·]Y

Figure 5: Facsimile of page 211 from A. De Morgan's "On the syllogism, III", published in the *Transactions of the Cambridge Philosophical Society*, vol. X, 1858.

As D. Merrill stresses, this is a crucial point warranting De Morgan a place in the history of logic, that is, "his emphasis on the necessity of specifying the universe of discourse in interpreting any proposition, and requiring that for any X, both ‘X’ and ‘x’ name something” [1990, p. 153]. Thus, it becomes clear De Morgan’s assumption of existential import, breaking with the idea of erasing away all reference to existence. In fact, that is what is currently done in logic: logicians assume existence, they do not deny it or talk about empty classes all the time.

Although De Morgan corrects a limitation of Aristotelian logic here, another of his contributions is of a still higher significance. Namely, the discovery – or *re*discovery – of the laws of *duality* between logical sum and logical product.[24] Calling logical sum “aggregate”, and logical product “compound”, De Morgan expressed these concepts in the following manner:

> The contrary of an aggregate is the compound of the contraries of the aggregants: the contrary of a compound is the aggregate of the contraries of the components. Thus (A, B) and AB have ab and (a,b) for contraries.[25]

Since De Morgan’s spicular notation is quite idiosyncratic, as can be seen by his symbolic rendering of the propositions of the square, the relations he is describing in the quote above can be expressed in a more up-to-date notation like the following:

$$[\alpha]\ \ \overline{A+B} = \overline{A} \times \overline{B}$$

$$[\beta]\ \ \overline{A \times B} = \overline{A} + \overline{B}$$

Or, in a Boolean manner:

$$[\alpha]\ \ 1-(x+y) = (1-x) \times (1-y)$$

$$[\beta]\ \ 1-(x \times y) = (1-x) + (1-y)$$

[24]It may be argued that Peter Abelard (*c.* 1079-1142) had already discovered, or at least laid down the grounds for the discovery, of what we call De Morgan’s laws. See P. Boehner 1951. But the most common view is that those laws were commonplace knowledge in the 13$^{\text{th}}$ century. The classical reference on the subject is E. Moody 1953. See E. Gomes and I. M. L. D’Ottaviano 2017, pp. 223-249, for how they appear in the context of a demonstration of the *ex falso* by the Pseudo-Scotus. El-Rouayehb 2016 shows De Morgan’s laws were known for Arabian logicians.

[25]From the third paper on syllogistic, “On the syllogism, n$^{\text{o}}$ III, and on logic in general”, in: Transactions of the *Cambridge Philosophical Society*, v. 10 (1864), pp. 173-230, this quote p. 208. The date of its presumptive final version is written at the end of the piece, August 3rd, 1857, with final corrections and additions in June 25$^{\text{th}}$, 1858.

The general idea can be defined, then, as following: the complement of the intersection of two classes is the union of the complements of those classes.[26]

In propositional logic, those laws hold good for *weak disjunction* and for *conjunction* as well, the calculus of classes being isomorphic to the calculus of propositions:

[α] the negation of disjunction is equivalent to the conjunction of the propositions negated, that is:

$$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q.$$

[β] the negation of conjunction is equivalent to the disjunction of the propositions negated, that is:

$$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q.$$

The recognition of this duality – holding good exclusively for weak disjunction – is one of the reasons most of later logicians will adopt it (but for Boole). Today, the whole idea may seem very simple, but at that time the expression of these relations in the logic of classes uncovered centuries of forgotten logical work.

Maybe for those same reasons, and maybe mainly for them, more or less 20 years later Peirce will say: "De Morgan was one of the best logicians that ever lived and unquestionably the father of the logic of relatives" . And it is true, De Morgan gets to the conclusion that certain valid and very simple inferences could not be expressed with syllogisms by reflecting on Aristotle's logic, mainly on the difficulties raised by the use of the verb "to be" as the copula. In short, De Morgan identified at least three senses of the copula that are not clearly separated in syllogistic. First, convertibility, that is, "A is B" and "B is A" must have the same meaning, that is, they must be either both true or both false. Second, transitivity, that is, "the connexion is, existing between one term and each of two others, must therefore exist between those two others", such that "A is B" and "A is C" must imply "B is C". And third, contrariety (considered as a synonym for contradiction), that is, "*is* and *is not* are *contradictory alternatives*, one must, both cannot, be true" [1850, p. 103ff]. He indeed claims that syllogistic is but one instance of the "composition of relations" [1850, p. 109ff.; 1860, p. 331], meaning that affirmation or

[26] $\overline{A}$ is the class of individuals of the universe of discourse that are not in A. A set-theoretic notation has also been used to express De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$. For the spicular notation, see F. Cajori 1993, §677.

denial are only special cases of identity between subject and predicate, other relations being possible. De Morgan's interest, then, is to show how identity can be transformed by *conversion* or *transitivity*. In his words, "a relation is *convertible* (though it should rather be said that the subject and predicate are convertible) when it is its own converse", that is, when it is symmetrical, as we say today. And a relation is *transitive* "when a relative is a relative of the same kind" [1860, p. 345-346]. For instance, respectively: "If a man is an animal, then the head of a man is the head of an animal"; and, "if 12 is a multiple of 3, then 3 is a divisor of 12". This latter instance shows that to some relation its *converse* can be associated: here, "to be a divisor of" and "to be a multiple of". Or still: "If John is Mary's brother and Mary is Peter's mother, then John is Peter's uncle". Although De Morgan's findings are not totally new, his conclusions are nevertheless not usual: from the relation between the terms "brother" and "mother", we get "mother's brother", that is, "uncle". In other words, the relations can be composed so we get the *relative product* of the related terms. Once this kind of relation is introduced, innumerable other forms of inference can be made. The instances below show De Morgan's symbolism used to express the kind of relations just mentioned. Uppercase letters X and Y represent definite singular terms, L is for any relation whatsoever, two sequential dots one after the other is for affirmation, a single dot is for negation [1860, p. 336]:

- $X..LY$ means X is a L of Y;
- $X.LY$ means X is not some L of Y (negation);
- $X..lY$ means X is a *not-L* of Y (contrariety, or contradiction).
- $Y..L^{-1}X$ means Y is a *L-converse* of X (conversion, or symmetry).

In this way, the formulas above could correspond in the universe of natural integers that are different from 1 – with L meaning "being a multiple of" – to the following propositions:

- "12 is a multiple of 3".
- "5 is not the multiple of any number".
- "5 is not a multiple of 2".
- "3 is a divisor of 12".

Notwithstanding its small expressiveness, the treatment of syllogisms was elevated to a superior level by De Morgan's highly idiosyncratic system of symbolization. And, most importantly, with the help of this system De Morgan was able to initiate the logic of relations, expressing different kinds of reasoning than the known syllogistic ones. He was quite aware of his accomplishments: "And here the general idea of relation emerges, and for the first time in the history of knowledge, the notions of relation and *relation of relation* are symbolised" [1860, p. 358]. With this, he believes we could see more clearly the differences between "the invented syllogism of the logicians and the natural syllogism of the external world" [1850, p. 111].

So, maybe De Morgan's contribution to the history of logic is the one of an explorer: to open a field of exploration in the study of non-syllogistic forms of inference, the roads it opened leading far beyond the visible horizon of the those times. Hobart and Richards (2008, p. 329), make a bold claim: "The conceptual core of De Morgan's deep conservatism harbored the belief that everything we conceive, think, or imagine in the universe of our logic, sensations, or psychology takes us outside ourselves, into another, parallel universe, that 'natural syllogism' of the outside world". Maybe the opinion of Charles S. Peirce [W 2: 449-450] is not totally overcome till today, and the final judgment of De Morgan's system is yet to come.

4 The birth of quantification, or Peirce between Boole and De Morgan

Besides de algebraic context of the 19th century, another important element to understand C.S. Peirce's logic and philosophy of logic is his father's, Benjamin Peirce, definition of mathematics. For B. Peirce, mathematics is a purely hypothetical activity, the science that draws necessary consequences from hypotheses. This is also influenced by the previous liberation of algebra from arithmetic: if the interpretation of symbols is freed from fixed numerical determinations, algebra itself can be conceived as a sheer deductive-hypothetical discipline. This is a crucial point for C. S. Peirce, as we will see.

Chemist by original formation (the first Harvard *summa cum laude* degree), astronomer and geophysicist by profession, philosopher, mathematician and above all else a passionate logician, Peirce is a polymath of a rare scope. His most important works on logic can be circumscribed to a short period of 5 years, from 1880 to 1885, roughly corresponding to the only period he was hired as lecturer at the then recently inaugurated Johns Hopkins University. However short the period of Peirce's single academic position, his contributions to logic and mathematics were nonetheless numerous and variegated. For diverse reasons though he was not able to develop all his theoretical projects – in any areas – to full achievement; besides this fact, a great part of his writings is practically unedited till today, yet in manuscript form. This situation makes it the more difficult to fairly assess the value of his contributions.[27]

Developing his own logic of relatives, Peirce started from Boole's system, trying then to apply it to De Morgan's logic of relations. Indeed, Peirce's aim is to include the logic of relations into the calculus of algebra using his own system of algebraic signs. As a mathematician, Boole was not concerned with the development of a specific mathematical notation, so the task of devising a sound symbolic notation for the logic of relations was taken up by others, the younger Peirce included. De Morgan's symbolism may be functional, but it is very idiosyncratic, besides lacking the expressive power of Boole's algebra of logic especially when dealing with binary relations. So, Peirce was indeed the first to successfully combine the algebraic calculus of logic – as he calls it – devised by Boole with the logic of relations initiated by De Morgan[28].

Peirce himself told De Morgan what an algebraic calculus à la Boole, being a much more powerful appliance than the spicular notation, could do in logic. In 1870, 31 years old Charles Peirce visited the then 65 years old De Morgan in London, one year before his death in 1871. Carrying a letter from his father Benjamin to introduce him, the younger Peirce hand-delivered to the old English logician a copy of his paper "Description of a notation for the logic of relatives, resulting from an amplification of Boole's calculus of logic", which is the culmination of his early developments in algebraic logic.[29]

Now, Peirce departs both from Boole's and De Morgan's systems in his long and difficult article from 1870, adopting there a meta-theoretical approach clearly stated in the very beginning:

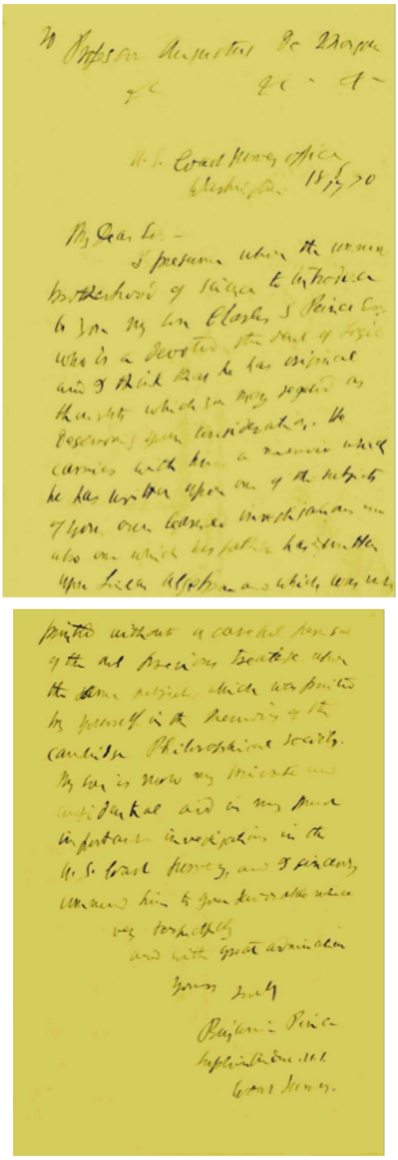
> I think there can be no doubt that a calculus, or art of drawing inferences, based upon the notation I am to describe, would be

[27]It is a hard task even to edit Peirce's writings. For an introduction to the matter, see N. Houser 1992. The article was written when the author was still working at the *Peirce Edition Project (PEP)*, whose main aim was to publish 30 volumes, called Peirce's *Chronological Writings*. Till the day only 7 volumes were published (1-6 and 8). For a more recent assessment on the same history and newer difficulties and challenges, see the *PEP*'s current editor-in-chief A. De Tienne's 2014 paper. The *PEP* website is also very informative: http://peirce.iupui.edu/index.html (access July 20th, 2017).

[28]It is indeed a question to ascertain if Peirce developed his logic of relatives before and independently from De Morgan or not. E. Michael 1974 shows that Peirce's own recollections on when he came to know of De Morgan's work are not reliable, and also that Peirce got interested in the logical study of relations via the distinction between equiparant and disquiparant propositions. For D. Merrill 1990, p. 46, De Morgan is "the father of the logic of relations", and exerted but a minimal influence over Peirce's own development of the subject [Merrill, 1978]. Be it as it may, it seems sure to say that Peirce's conjunction of algebraic calculus and relational logic is his original contribution. For more on this, see I. Grattan-Guinness 1997b, A. Walsh 1999, R. Dipert 2004.

[29]I. Grattan-Guinness 1997b gives a vivid description of the meeting. The transcription of the letter is taken from http://www.unav.es/gep/BPeirce17.06.70En.html, which is slightly different from Grattan-Guinness's. Access date 16/08/2016.

perfectly possible and even practically useful in some difficult cases, and particularly in the investigation of logic. [W 2: 360].



To Professor Augustus De Morgan
etc – etc – etc –
U. S. Coast Survey Office
Washington 17/6/1870
My Dear Sir -
I presume upon the unseen brotherhood of science to introduce to you my son Charles S. Peirce Esq. who is a devoted student of Logic and I think that he has original thoughts which you may regard as deserving your consideration. He carries with him a memoir which he has written upon one of the subjects of your own learned investigations and also one which his father has written upon Linear Algebra and which was not printed without a careful perusal of this [?] and [of a] previous treatise upon the same subject, which was printed by yourself in the Memoirs of the Cambridge Philosophical Society. My son is now my private and confidential aid in my most important investigations in the U. S. Coast Survey, and I sincerely commend him to your favorable notice

Very respectfully
and with great admiration

Yours truly

Benjamin Peirce
Superintendent U. S.
Coast Survey

Figure 6: Letter by Benjamin Peirce to introduce his son Charles S. Peirce to Augustus De Morgan.

Although he regretfully confesses he is not able to perform this task, yet he thinks he has laid the ground for it. And let my reader allow me to stress he says "investigation of logic", in support of the claim for the meta-theoretical approach. Peirce even introduces different letters for different logical terms, in order to make it clear that he understands his system is functional to representing classes.[30] This will be clarified further on.

In what follows, I will first deal with Peirce's own way of generalizing the copula, in connection with his criticisms to Boole's system. Only then I will deal with his theory of quantifiers – a term he seems to have invented – as connected to his criticisms to De Morgan's logic.

4.1 Peirce *contra* Boole, or illation as fundamental

Notwithstanding claiming a place for himself in the Boolean tradition of the algebra of logic, Peirce assumes a critical perspective about Boole, trying to combine the Boolean calculus with the logic of relations. Already in 1867, in a paper called "On an improvement on Boole's calculus of logic", Peirce declares: "[Boole's calculus of logic] consists, essentially, in a system of signs to denote the logical relations of classes" [W 2: 12]. He then criticizes Boole for the insufficiencies of his calculus. We will soon see more specifically what is the issue, but, in general, for Peirce Boole had excessively subjected logic to mathematics. So, Peirce tried to devise a system of signs of his own by improving Boole's notation, in order to clearly distinguish logical from mathematical operations.[31] For Peirce, logical and mathematical operations are distinct: mathematics is for him a sort of *pre-logical* science; its arguments are evident in such a way that it does not have any need to seek help in logic. Besides, Peirce thinks of the calculus of classes only as an instrument for deduction, to be integrated into or even sometimes identified with semiotics, the *quasi*-formal and necessary doctrine of all signs. Peirce's philosophy of logic and its relation to mathematics will reappear in the end of this paper. First, let us see how Peirce developed his own logic.

Peirce identified a serious problem in Boole's algebraic calculus of logic: the subjection of logic to algebra increases the quantity of equations, of equalities. A *conditio sine qua non* for equality is the existence of two relations: two quantities are equal whenever one is less than the other and vice-versa. But in logic the same does not hold: inclusion between classes is previous to identity,

[30] I basically follow A. Walsh 1999 in this understanding. She clearly shows that Peirce does not confuse relative terms with relations, but this confusion is due to his use of absolute terms and relative terms interchangeably.

[31] Let us also note that Peirce's modifications already at that time meet the criticisms made by Frege to Boole's calculus. See H. Putnam 1982; H. Sluga 1987; N. Houser 1994; V. Peckhaus 1999; I. Anellis 2012.

and, in a deeper manner, *implication* is previous to both. This is why Peirce finds it necessary to use a specific sign for the fundamental subsumptive logical operation. Then, replacing the identity sign "=" Boole employed by the sign "$\prec$", Peirce states it stands for *illation*, which is explained by an example of *inclusion* as follows:

> I use the sign $\prec$ in place of $\leqq$. My reasons for not liking the latter sign are that it cannot be written rapidly enough, and that it seems to represent the relation it expresses as being compounded of two others which in reality are complications of this. [...] Now all equality is inclusion in, but the converse is not true; hence inclusion in is a wider concept than equality, and therefore logically a simpler one [W 2: 360 n. 1, 1870].

Peirce then goes to prove that *being as small as, being equal to, being less than*, and *being greater than* all can be defined by means of transitivity and inconvertibility, in his vocabulary, or asymmetry, in today's vocabulary.[32] For instance, to say that $x = y$ is to say that $x \supset y$ and $y \supset x$. Thus the sign "=" is clearly unable to express class inclusion, since it does not express which is more or less, if x or y. In another case, to say that $x < y$ is to say that $x \supset y$ is true and that $y \supset x$ is false. This reveals Peirce's Philonian understanding of the illative relation, which, by the way, is just as nowadays material implication is conceived. In 1896, Peirce elucidates the point in the following manner:

> Now it is to be observed that the illative relation is not simply convertible; that is to say, that "from A necessarily follows B" does not necessarily imply that "from B necessarily follows A." [...] Consequently, the copula of inclusion, which is but the ergo freed from the accident of asserting the truth of its antecedent, is equally inconvertible. For though "men include only mortals," it does not follow that "mortals include only men," but, on the contrary, what follows is "mortals include all men" [CP 3.474].

Peirce holds this Philonian understanding since very early [W 2: 295, 1869; W 2: 349, 1869-1870]. Worth noticing, also the idea of a "leading principle" of any logical argument is defined by Peirce in the same general spirit: the leading principle is a rule of inference warranting transitivity from the truth of the premises to the truth of the conclusion [W 2: 295, 1869; see also W 4:

[32] Today's vocabulary comes from Ernst Schroeder. But Schroeder's preference to use "symmetry" rather than "convertibility" did not pass uncriticized by Peirce, who, following principles of his ethics of terminology, blamed Schroeder of disregarding previous work by elder logicians (presumably his own), and also of confounding convertibility with commutativity. See CP 3.474, n. 1, 1896; see also Houser 1991.

165, 1880].[33] Thereby, the more abstract or, better said, the more general a logical relation, the simpler it is: because implication is not convertible, it is a simpler relation, being thus more general than inclusion, which in its turn is simpler and more general than equality [W 2: 360, 1870].

In 1870, Peirce does not seem to think it necessary to go much further to prove this idea. But even if he does not prove it, what he means and the importance of it is not difficult thing to grasp. To understand this, let us notice that from $a = b$ one can infer that $a\mathrm{R}b$ and also that $b\mathrm{R}a$, but that is valid only for classes with equal quantities, not for anything else. Any other possible relation R could signify cannot be inferred from $a = b$. Now, by identifying the copula with illation, Peirce understands propositions as forms of inferences. Indeed, propositions of identity, e.g., "A is B", can be interpreted as a sort of reasoning, say, *any A is B and any B is A* [W 2: 60, 1867]. This understanding goes along with one of Peirce's most strong positions in logic, namely, that there is no logical difference between categorical and hypothetical propositions [W 1: 337, 1866; W 2: 174, 1868; W 2: 208, 1868; W 2: 257, 1869; W 5: 169-170, 1885]. The demonstration that conditionals are basic forms of more developed inferences becomes equivalent to the demonstration that all propositions are in fact rudimentary forms of inference, so that even terms can be so considered.[34] Indeed, this is the crucial point of Peirce's criticism to Boole's calculus: "That which first led me to seek for the present extension of Boole's logical notation was the consideration that as he left his algebra, neither hypothetical propositions nor particular propositions could be properly expressed" [W 2: 421, 1870]. This will be a point to return soon.[35]

Now, all this shows that the sign "$\prec$" is understood by Peirce as a sort of "subsumption operator", according to Dipert:[36] it subsumes the logical

[33] Two other things are also worth noticing. First, implication can be either material or formal, which distinction Peirce understands as such: a formal consequence states a fact only about the relation between the expressions; a material consequence states a fact about the relation of the very "matters in question", and not merely about their expressions [W 2: 348-349, 1869/1870]. Second, Frege and Russell also did understand implication in Philonian terms, as well known. But Peirce seems to be more influenced by the medieval treatment of conditionals as expressing inferential relations. His use of consequence-vocabulary is not the only indication of such an understanding, as I hope my argumentation shows. See J. Boler 1963, p. 70-71: "consequence [for Peirce] is the prototype of argument". For a history of implication emphasizing Frege's propositional understanding, but acknowledging Peirce's pioneering efforts, see J. Łukasiewicz 1934.

[34] As claimed by P. Castrillo Criado 1988, p. 19. Peirce gives the proof in W 2: 423-424, 1870.

[35] To particular propositions, that is. Dealing with Peirce's understanding of hypothetical propositions would lead me too far beyond the scope of this article. Suffice it to say that Peirce was well aware of the paradoxes of implication, as the 20th century nomenclature says. See W 5: 169-170, 1885. For an account, see J. Zeman 1997.

[36] R. Dipert 2004, p. 288. See also F. Cajori 1993, §§678-683 for a bit of context.

relations of conditional ($\rightarrow$), inclusion ($\subseteq$) and logical consequence ($\vDash$) under a single logical notion. This shows Peirce's interest in using an abstracter calculus for both "propositional logic, the logic of classes, or a metalogical theory of logical consequence".[37] All logical relations can be defined solely upon the formal characters of this fundamental relation.

According to Merrill, although the substitution of illation for identity "may have been primarily dictated by formal considerations, it was an important step on the road to a less algebraic approach to the logic of classes".[38] This subject marks a constant in Peirce's refusal to consider logic as a *lingua characterica*. This is a point never abandoned indeed. Take, for instance, this very clarifying remark from 1880:

> Logic supposes inferences not only to be drawn, but also to be subjected to criticism; and therefore we not only require the form P∴C to express an argument, but also a form, $\mathrm{P}_i \prec \mathrm{C}_i$, to express the truth of its leading principle. Here P_i denotes any one of the class of premises, and C_i the corresponding conclusion. The symbol $\prec$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true [W 4: 166, 1880].

Then, even tough Boole's influence was in 1870 still very strong, an important difference was marked and one can say it was never subsequently abandoned. Such a departure may also be considered an influence of De Morgan's *Formal Logic*, actually quoted in 1870 in this very context [W 2: 367]. But it is also true that the generalization of the copula already concerned Peirce since at least 1867 [W 2: 68-69].[39]

Peirce's studies on implication led him to discover central elements for contemporary logic. The most famous of those discoveries was quantification theory. Now, the specific point Peirce identifies as a failure in Boole's calculus is the v operator, already discussed in section 2 [W 2: 21, 1867]. The 1870 article is at the same time the culmination of his early attempts to surpass Boole's inadequacies as well as the starting point for the subsequent develop-

[37] R. Dipert 1981, p. 580.

[38] D. D. Merrill, "The 1870 logic of relatives memoir", included as part of the introduction to Peirce's W 2: xlii.

[39] It is not certain if he had any knowledge of De Morgan's writings at that time. See references in note 28 above. Also A. J. Iliff 1997, p. 201 shows that Peirce's expertise in algebra was decisive for his logical development, his theory of quantification being much more than a simple generalization of Boole's algebra of logic. Peirce's mathematical background is also emphasized in M. Murphey 1961.

ment of quantifiers.[40] With this, we also find Peirce reservations concerning De Morgan's approach to the logic of relations.

4.2 Peirce contra De Morgan, or quantification as operational

Working with dual relations, even if the generalization of the copula makes this restriction unnecessary, De Morgan defined the notion of a *relative product*, which is just a compound relation: brother + father = uncle, for instance. Now, Peirce devised a way to deal with n-adic expressions – for three or a n number of terms. In fact, more than recognizing there is no need to work within the limits De Morgan restricted himself, Peirce goes further and declares one of his most famous thesis, namely, that all relations of degree higher than three can be constructed out of and are reducible to a triadic relation. This point will reappear in the presentation of his relative terms.[41]

Maybe the most important point is Peirce's concern with the composition of relations with classes, and not the strictly relational composition, that is, the composition of a relation with another. For instance, while De Morgan analyzes expressions like "X is a lover of a servant of Y", Peirce prefers to work with expressions like "lover of every woman", or yet "lover of a servant of a woman", thereby combining both kinds of composition, thus making quantifications explicit.[42] The 1870 article presents several ways of dealing with quantified expressions, but does not have specific signs for the existential and universal quantifiers. The introduction of relational expressions with gaps for the insertion of variables, from one to three gaps, or more, allows for that. These relational expressions are what Peirce calls "logical terms", and this is why he prefers to say logic of *relatives* than logic of *relations*. Logical relative terms can then be classified in three kinds:

1. "Absolute terms" are one-gaped terms, like "a_____": "They regard an object as it is in itself as *such (quale)*; for example, as horse, tree, or man." Absolute terms are, then, like single functional terms, possible expressions of individuals. Absolute terms are symbolized in his calculus by letters of the Roman alphabet, a, b, c etc.

2. "Simple relative terms" are also one-gaped terms, but they do not refer to a single object, rather expressing dual relations, like "father of_____" or "lover of_____": "They regard an object as over against another, that

[40] J. Łukasiewicz 1930, p. 147, attributes the use of the word "quantifier" to Peirce for the first time in the history of logic.

[41] This point is also impossible of deep development here. See J. Brunning 1997.

[42] This can be interpreted as another sign of Boole's influence, of course. Cf. D. Merrill, *op. cit.*, W 2: xliv.

is as relative". Later on, Peirce will say that, if this kind of term includes the copula, it is called in his semiotic vocabulary a *rhema*, that is, a term "somewhat closely analogous to a chemical atom or radicle with unsaturated bonds. A relative rhema is like a multivalent radicle" [CP 3.421, 1892]. The reason he gives for this is that the relation between two things as exclusively expressed by a specific verb can only make sense in certain idioms, so that it is not a logical relation proper, but a grammatical one [EP 2: 308-310, c. 1904]. Relative terms are symbolized in 1870 by italicized letters, *a*, *b*, *c* etc.

3. "Conjugative terms" are at least two-gaped terms, and their distinctive character is that "they regard an object as medium or third between two others, that is, as conjugative". For instance, "giver of_____ to_____", or "buyer of_____ to_____ from_____" [W 2: 365, all quotes above]. Peirce specifies a typeface called Madisonian for these terms: a, b, c etc.

Later on, he identified conjugative terms with *predicates* themselves. Predicates and rhemas are nothing but "blank forms of propositions" [EP 2: 221; 299, 1903;427, 1907]. Because they are mere forms, they are iconic, according to Peirce's semiotic vocabulary, and especially fit for representing relations [EP 2: 20-21, 1895]. A further denomination clarifies an important point:

> Copulants, which neither describe nor denote their Objects, but merely express universally the logical sequence of these latter upon something otherwise referred to. Such, among linguistic signs, as "If _____ then _____", "_____ is _____", "_____ causes _____", "_____ would be _____" , " _____ is relative to _____ for _____" , "Whatever", etc. [EP 2: 484, 1908].

The quote clearly shows Peirce's understanding that the blank forms are schematic patterns for various other possible formulas. This is the basics for Peirce's syntax: unsaturated gaps in logical terms are places to introduce deictics that function as individual variables, that is, individual letters or even other terms to denote the subject within a specific universe of discourse, or, as Peirce said in 1870, individuals of a certain *class* "*about* which alone the whole discourse is understood to run" [W 2: 366]. In Peirce's semiotic jargon, such a deictic, or denotative sign, works like an index.[43] Also notice Peirce prefers—at

[43]Kneale and Kneale praise Peirce for this clarifying way to relate the new logic of relations to the old syllogistic terminology. Kneale and Kneale 1971, p. 432. Peirce's semiotic jargon is not really consistent and underwent several changes as Peirce searched for the best possible words for his conceptualizations. The crucial concepts themselves, however, did not change but for details. For instance, in a letter to *Lady* Victoria Welby dated from Christmas 1908, he uses *designative*, *denotative*, *indicative*, and *denominative* as synonyms for index. Predicates and rhemas as synonyms for copulants is just another example.

least in 1870—expressions like “lover of _____”, or “giver to _____ of _____”, rather than “_____ loves _____” or “_____ gives _____ to _____”, which are expressions with verbs, more common today. So, Peirce is not dismissing the traditional interpretation of propositions as subject-predicate structures, he is just inventing another way of interpreting this structure, not restricted to the predication of only one subject. Peirce’s preferred way of explaining a proposition as a blank-predicate-form, from which the subject-indexical-terms were dropped of, reveals his adherence to the tradition, even after his semiotic conceptions were more developed [EP 2: 172-173, 1903].

So, in 1870 Peirce develops a logic fully functional to represent both the class operations of Boole and the relational operations of De Morgan. Peirce claims conjugative terms “consider an object as a means or a third between two others” [1870: 365]. Although in the 1870 paper the way to use these kind of expressions is not yet fully worked out, this represents a clear advancement, if compared both to De Morgan’s expressions with only two gaps and to Boole’s way of representing quantified expressions.

Now, at that time Peirce did not yet have specific signs for *quantifiers* themselves. And he also does not present any full-fledged theory of *multiple* quantification. The paper in question only presents ways to work with monadic quantification, that is, single quantification for relative terms within the limits of Boolean algebra of logic. As an instance of existential quantification[44], let us see the particularly important operation of multiplication, which is how Peirce combines relations by means of their relative product.

Multiplication is defined as “the application of a relation” [W 2: 369]. Let s be used for “servant of”, and l for “lover of” (they are *simple relative terms*, then); m stands for “men”, just as “w” for “women” (*absolute terms*, then, denoting classes, if we follow Peirce’s vocabulary). So, lw means *whatever is a lover of a woman.* From this,

$$s(\mathrm{m} +, \mathrm{w})$$

means *whatever is servant of anything of the class composed of men and women together.* So that:

$$s(\mathrm{m} +, \mathrm{w}) = s\mathrm{m} +, s\mathrm{w}.$$

Next, $(l +, s)$w is used for *whatever is lover or servant to a woman*, so that

$$(l +, s)\mathrm{w} = l\mathrm{w} +, s\mathrm{w}.$$

[44] I take the following from Merrill in W 2. See also A. J. Iliff 1997, p. 200, quoting exactly the multiplication technique. But see J.-Y Béziau 2012 for the idea that Peirce did not develop a sufficiently abstract mathematical conception of truth-values.

And then:

> (sl)w will denote whatever stands to a woman in the relation of servant of a lover,
> and
>
> $(sl)\mathrm{w} = s(l\mathrm{w})$ [W 2: 370].

This shows the operation of multiplication in Peirce's algebra is associative.[45] Indeed, lw is the product of a simple relative and an absolute term, being itself an absolute term, that is, it can denote a class, so that what we have here is a relative product of a relation and a class: a given individual i can be a member of the class lw iff i loves some woman w [W 2: 369-370]. Peirce further explains that adding a comma after the term extends the arity of the term, so that l, sw means a lover of a woman that *besides that* is a servant of that woman. Accordingly, an absolute term such as "man" can be transformed into a relative, so that "$\mathrm{m},_\infty$" means *some* man, since the infinity symbol is interpreted in the sense of indetermination [W 2: 374].

Peirce understands multiplication in a way that it implicitly contains existential quantification, that is, quantified expressions are the result of a relational operation. For instance, let 1 denote any individual of a possible class, so that $\mathit{1}_\infty$ means "something" and $\mathit{1}_0$ means "anything". Thence, $l\mathit{1}, \mathrm{m}_\infty$ can be interpreted as *something that is a lover of some man*. This is indeed the relative product of the simple relative term "lover of_____" and the absolute term "something". Peirce introduces universal quantification as a relation of *involution*, which is De Morgan's preferred word for exponentiation. The operation is defined as follows:

> I shall take involution in such a sense that x^y will denote everything which is an x for every individual of y. Thus l^w will be a lover of every woman. Then $(\mathrm{s}^l)^w$ will denote whatever stands to every woman in the relation of servant of every lover of hers; and $s^{(lw)}$ will denote whatever is a servant of everything that is lover of a woman. So that
>
> $$(s^l)^w = s^{(lw)} \text{ [W 2: 377].}^{46}$$

So, there is always some term in the formulas working as a quantifier, but there are no specific signs for the quantifiers themselves.

[45] Distributive of addition on both sides, as the formulas show, and non-commutative. See G. Brady 2000, p. 30 for B. Peirce's influence over C. S. Peirce on this point.

[46] G. Brady 2000, p. 40-41 provides a proof of Peirce's claim that $(s^l)^w = s^{(lw)}$.

Peirce was totally aware of the insufficiency of dealing with quantification in this manner. One of his remarks is truly interesting in this respect:

> As for particular propositions, Boole could not accurately express them at all. He did undertake to express them, and wrote
>
> | | |
> |---|---|
> | Some Y's are X's: | $v, y = v, x$ |
> | Some Y's are not X's: | $v, y = v, (1 - x)$. |
>
> The letter v is here used, says Boole, for an "indefinite class symbol." This betrays a radical misapprehension of the nature of a particular proposition. To say that some Y's are X's, is not the same as saying that a logical species of Y's are X's. For the logical species need not be the name of anything existing. It is only a certain description of things fully expressed by a mere definition, and it is a question of fact whether such a thing really exist or not. St. Anselm wished to infer existence from a definition, but that argument has long been exploded. If, then, v is a mere logical species in general, there is not necessarily any such thing, and the equation means nothing. If it is to be a logical species, then, it is necessary to suppose in addition that it exists, and further that some v is y. In short, it is necessary to assume concerning it the truth of a proposition, which, being itself particular, presents the original difficulty in regard to its symbolical expression [W 2: 422, 1870. The commas in the formulas are Peirce's].

Now, what Peirce means is that, in Boole's notation, from $v, y = v, (1 - x)$ one could infer that $v, x = v, (1 - y)$, that is, *some X's (that) are not Y's*, which is not a valid inference (remember that illation is inconvertible). This is a remarkable observation indeed, for it implies that the search for an adequate way for dealing with quantified expressions led to the logic of relatives, and not that the logic of relatives led to quantification. Tackling the problem of existential import, Peirce claims that Boole's system cannot properly express neither hypothetical nor particular propositions. We saw how Peirce dealt with both. In fact, Peirce's claim, here, is just that existential quantification is needed to express Boole's algebra in relative terms. In other words, to properly express hypothetical and particular propositions, an existential condition is needed: a relative term for *case of the existence of* _____; or for *what exists only if there is not* _____; or else *case of the non-existence of* _____; or still *what exists only if there is not*_____ [W 2: 423]. These relatives are required to properly symbolize monadic relative terms, or absolute terms, and allowed Peirce to avoid negating an equation to represent particular propositions, as did

Boole, using equations instead. The operation depends on Peirce's involution, and the crucial passage is the following:

> Particular propositions are expressed by the consideration that they are contradictory of universal propositions. Thus, as $\mathbf{h,(1 — b) = 0}$ means every horse is black, so $\mathbf{O}^{\mathbf{h,(1 — b)}} = \mathbf{0}$ means that some horse is not black; and as $\mathbf{h,b = 0}$ means that no horse is black, so $\mathbf{O}^{\mathbf{h,b}} = \mathbf{0}$ means that some horse is black. We may also write the particular affirmative $\mathbf{1(h,b) = 1}$, and the particular negative $\mathbf{1(h,}\boldsymbol{n}^{\mathbf{b}}\mathbf{) = 1}$ [W 2: 424].[47]

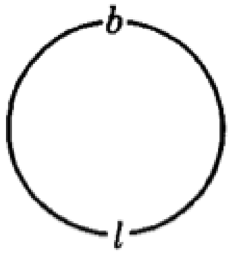
So, superscripts are used to indicate universal quantification. But, again, there is no specific sign for a universal quantifier. Peirce will take a bit longer to separate quantifiers both from Boolean relatives and from predicates, thus completely overcoming Boole's notational difficulties. The introduction of specific signs for the universal and existential quantifiers[48] happened only in 1883. In this year, Peirce organized a volume with contributions from his students at Johns Hopkins and two others of his own, wherein the treatment of quantifiers was developed from his student O. H. Mitchell's contribution to the volume.[49] The very first time Peirce uses a specific notation for the universal and the existential quantifiers is in a letter to Mitchell, indeed. The letter is dated from December 21, 1882, and there it reads:

> The notation of the logic of relatives can be somewhat simplified by spreading the formulae over two dimensions. For instance suppose we write

[47]See D. Merrill 1997, p. 164. Merrill says this idea points to the opposite the direction in the history of logic, maybe thinking of the usual narrative of it, dominant in the 20th century, as stemming almost exclusively from the arithmetical work of Frege.

[48]For a more thorough analysis of the path trodden by Peirce, from the logic of relatives to quantification theory, see: D. Merrill 1997; J. v. Evra 1997; G. Brady 2000. For the historical place of Peirce's theory of quantification, see D. Bonevac 2012.

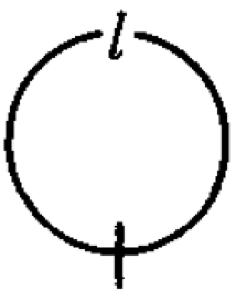
[49]The volume was entitled, *Studies on logic by the members of the Johns Hopkins University.* Peirce's contributions were in fact two final "notes" to the book: "Note A" deals with syllogisms and statistics, without special mention to quantifiers; "Note B" elaborates quantification theory from Mitchell's work. Mitchell introduces subscripts as the universal and existential quantifiers, respectively 1 and u, but does not have a concept of well formed formula, therefore neither of bound or free occurrences of variables. For a more detailed comparison of Peirce's and Mitchell's treatments of quantifiers, see G. Brady 2000, chapter 4; A. Walsh 1999, chapter 5.



to express the proposition that something is at once benefactor and lover of something. That is,

$\sum_x \sum_y b_{xy} l_{xy} > 0.$

[...] In the same way



will mean that everything is a lover of itself
$\prod_x l_{xx} > o$. [W 4: 394; 396].

Apart the graphical representation, which is a subject by itself, Peirce's linear notation is very original, with a strength none of his predecessors was able to develop. As a first distinctive feature, notice the use of juxtaposed subscript letters to the side of the operative sign. These subscripts function as specific deictic signs for indicating the specific items within the universe of discourse should. They are indeed *indexes*, or signs functioning like "a pointing finger" [W 5: 163, 1885]. In this 1883 article, tough without this name, the subscripts reappear and serve to indicate which terms are linked by a certain relation and in which specific order, solving a preoccupation of Peirce since at least 1870 [W 2: 370; W]. For instance, if l denotes the relation of loving, then

l_{ij} signifies "i loves j", with "i" and "j" as indexes for whatever individuals are in this love relation. Indeed, indexes function here as what now would be considered individual variables.

This much simple symbolism allows for the easy expression of some situations: first, the reflexivity of a relation, in which case l_{ii} represents "i loves it/her/him-self". Second, the symbolization of convertibility, in which case we have the following:

$$l_{ij} = l_{ji} : \text{“}j \text{ is loved by } i\text{”}.$$

And finally, the symbolism allows for the visualization of binary or higher-level relations or predicates, as for instance (Peirce's favorite example of a relation):

$$g_{ijk} : \text{“}i \text{ gives } j \text{ to } k\text{”} .$$

But how to symbolize propositions like "Everybody loves Chaplin"; or: "Everyone loves someone"? To deal with propositions like these, Peirce introduces Greek letters $\prod$ – for *product* – and $\sum$ – for *sum* – to designate the *universal quantifier* and the *existential quantifier* respectively. In the letter to Mitchell, Peirce still retains the Boolean algebraic inequality: "> 0" clearly indicates the universe is not empty, and "< 0" presumptively would indicate just the opposite, that is, that there is nothing the given property can be predicated upon. In 1883, the inequality sign and 0 were for the most part abandoned, but Peirce still retained a Boolean numerical coefficient $(l)_{ij}$, indicating the existence of the relative term l,, so that the value of $(l)_{ij}$ "is 1 in case I is a lover of J, and 0 in the opposite case" [W 4: 454; in 464 the inequalities reappear, but only as a sort of explanation]. As the reader might be already guessing, the universal quantifier is defined with the help of conjunction, or logical product, and the second with the help of weak disjunction, or logical sum. For instance, if x denotes whatever property:

1. Be $\prod_i x_i = x_s x_p x_c x_m x_y \ldots$, etc. $\prod_i x_i$ means x is a property of *all* the individuals denoted by i. For instance, "All men are mortals" means that Socrates, *and* Pelé, *and* Chaplin, *and* me, *and* you.... are all mortals.

2. Be $\sum_i x_i = x_s + x_g + x_c + x_y \ldots$, etc. $\sum_i x_i$ means x is a property of *at least one* of the individuals denoted by i. For instance, "Some men are philosophers" means that Socrates, *or* Gramsci, *or* Chaplin, *or* you, ... are philosophers. [W 5: 180, 1885].

The propositions "Everybody loves Chaplin" and "Everyone loves someone" can then be respectively expressed as follows: $\prod_i l_{iC}$ (C as an index for the individual Chaplin) and $\prod_i \sum_j l_{ij}$ (j for *jemand*, German for *someone*). So, the coefficient and the signs for quantifiers are enough to represent all propositions of traditional logic. In fact, much more.

A noteworthy observation is made in the famous paper from 1885, "On the algebra of logic: A contribution to the philosophy of notation". There, Peirce remarks, the symbols $\prod$ and $\sum$ were chosen to make the notation as *iconic* as possible. And the quantifiers are defined in terms of *potentially* infinite conjunctions and disjunctions, with the care not to straightforwardly identify them with sum and product, since "the individuals of the universe may be innumerable" [W 5: 180]. Peirce's symbolism, then, is very useful, for it allows for the representation of very complex quantified relations, elevating the logic of relations to a much superior level. Only one of his examples is enough to show its power:

> Let us express this: one or other of two theories must be admitted, 1st, that no man is at any time unselfish or free, and some men are always hypocritical, and at every time some men are friendly to men to whom they are at other times inimical, or 2 , at each moment all men are alike either angels or fiends. Let
>
> uij mean the man i is unselfish at the time j,
> fij " " " free " ",
> hij " " " hypocritical " ",
> aij " " " an angel " ",
> dij " " " a fiend " ",
> $pijk$ " " " friendly " ",
> to the man k,
> $eijk$ the man i is an enemy at the time j to the man k,
> $1jm$ the two objects j and m are identical.
> Then the proposition is
>
> $$\prod_i \sum_h \prod_j \sum_k \sum_l \sum_m \prod_n \prod_p \prod_q (\bar{u}_{ij}\bar{f}_{ij}h_{hj}p_{kjl}e_{kml}\bar{1}_{jm} + a_{pn} + d_{qn}).$$
>
> [W 5: 180].

Worthy of mention is also that although Peirce isolated quantifiers just a few years after G. Frege, who is known as the inventor of modern quantification theory as we use it, it was an independent achievement. And notwithstanding the historical fact of Frege's precedence, Peirce's system is a complete system for first-order predicate calculus. But there are differences, the most manifest of which is that Peirce wanted to avoid any understanding of his logic as a universal language, "like that of Peano", he says [CP 4.424, c. 1903]. This will be clarified in the next section.

One can say the invention of modern quantification theory in logic has multiple origins, and not only one. There's a whole discussion in the 19th century about problems emerged from different interpretations of syllogisms. Without intending to sound too metaphysical, but running the risk of a freer expression, I hold those problems were part of the mood of the times, part of the *Zeitgeist* – not a single problem, several of them. Be it as it is, that neither Peirce nor Frege knew of each other's work is nowadays widely accepted.[50]

Now, it is necessary to say that the Peircean conception of logic is quite different from the propositional understanding of it that dominated the 20th century till today. Peirce's very original – and reasonable, so to say – understanding of the discipline stands out in its history. Some of this originality will be presented in what follows.

4.3 Peirce's conception of logic: for a quasi-formal activity

Let us from the start remark that Peirce's conception of logic is not as Frege's a linguistic one. For instance, quantification is not conceived in a propositional way, as in contemporary logic. Of course, there is the understanding that propositional truth-values are of central concern. But Peirce's take on the subject may today be regarded as a more heterodox position, for it emphasizes properties in predicate logic instead of the possibility of assigning a definite truth-value to a proposition.[51] Nevertheless, this means no fault of Peirce's logic if compared to more widespread notions. As soon will be clear, Peircean quantifiers, for instance, do range over bound variables, but not only that. Besides working for "first-intentional logic", or first-order logic as we call it, Peirce expects his notation also to work for "second-intentional logic", that

[50] For a similar point-of-view, see A. Molczanow 2012. The hypotheses of Peirce being aware of Frege's work is also plausible, for his student C. Ladd-Franklin quotes Frege in her bibliography for the article on the algebra of logic published in the volume Peirce edited in 1882 while at Johns Hopkins. And Frege also might have had knowledge of Peirce by Schroeder's review of the *Begriffsschrift* published in the *Zeitschrift für Mathematik und Logik* in 1880. But it seems that neither of them had the interest to check out the references. See Appendix A for Ladd-Franklin's bibliography. The most interesting approach to the subject still seems to be the one by I. Anellis 2012. Anellis seeks to show how Peirce's logic satisfies all the criteria commonly assigned to—and are considered to be originally defined by—Frege's works, and, in my opinion, he does this successfully. For the link with Peirce's wider conception of logic as a normative science and as semiotic inquiry, see Tiercelin 1991.

[51] This can be evidenced from his treatment of the law of excluded middle and the law of contradiction. It is impossible to deal with this here, but see R. Lane 1997 and 2001; C. T. Rodrigues 2016. Peirce's approach can arguably be seen as harmonious with more contemporary work in fuzzy logic. On this, see S. Haack 1996, p. 109ff for a critical view of Peirce's logic of vagueness.

is, a second-order logic [W 2: 56, 1867; W 5: 185, 1885]. The point is that he understands his own work in model-theoretic terms: one thing is to make inferences, but we are also capable of interpreting the inferences we make in more than a few ways.[52] Some qualifications are needed in order to properly understand this point.

Peirce's adherence to the algebraic tradition informs his approach to logic from a whole-part perspective rather than the set-theoretical and axiomatic approach that became dominant in the 20th century. It is not my intention to pursue this point much deeper, but it is important at least to stress that neither Boole, nor De Morgan, and not even Peirce were much concerned with axiomatization. Particularly for Peirce, this point is clarified when put in context within the background of his broader conception of logic, which is also a way of understanding the history of discipline. Logic, then, can emerge as less formal than might at first blush seem. It is to this conception we now turn.

For Peirce, logic can be conceived both in a strict and in a broader sense. The strict sense is just formal logic, logic written with a mathematical language, which he calls exact logic following Schroeder. But in the general sense, logic is a normative science, of a higher rank than and not to be confused with *exact logic*, because it is truly the general and *quasi*-necessary doctrine of signs, which Peirce idiosyncratically calls *semeiotic* [e.g., CP 2.227, 1897; EP 2: 79, 1901; 198, 1903; 327ff., 1904]. As such, logic cannot be reduced to mathematics, although being a less general science it uses mathematical principles and methods in its inquiries, and we have just seen how Peirce uses mathematics to build his own algebraic notation. This of course is not a naïf anti-logicism, but a completely original conception of the aims and purposes of different scientific activities.

In Rodrigues (2007) I have provided a deeper presentation of Peirce's philosophy of mathematics, showing how it is important for his conception of philosophy as a science of discovery. Now, suffice it to highlight just the central idea in order to clarify how Peirce thinks of the nature of logical inquiry itself. First, Peirce adopts his father's Benjamin definition of mathematics as the science that draws necessary conclusions from purely hypothetical ideal constructions [CP 4.239, 1902]. This is important, because in contrast to drawing conclusions—which is the business of deductive logic—mathematics seeks the most economic—elegant, so to say—way to reach the conclusions. Mathematics obeys a sort of principle of parsimony: the less steps it takes

[52] The model-theoretic approach of Peirce's logic is most famously argued for by J. Hintikka 1997 and I. Grattan-Guinness 1997, who take the nomenclature from J. v. Heijenoort 1967, but arguing for the universality of the model-theoretical approach, which Heijenoort denied. But, as said, Peirce may not fit so well within this categorization. See J. Legris 2016, for instance.

reach the conclusions, the better. So, the dissection of reasoning in as many different steps as are required to establish a conclusion is not of a mathematician's concern, but a logician's. While the mathematician sees algebra as just a calculus, that is, a mechanism for "unravelling a complicated question", the logician in turn "demands that the algebra shall analyze a reasoning into its last elementary steps" [*id.*].[53]

Second, mathematics is not just a matter of tracing out necessary consequences, but also of framing hypothesis – it is truly "the study of the substance of hypotheses, or mental creations, with a view to the drawing of necessary conclusions" [NEM 4: 268, *On Quantity*, *c.* 1896]. In other words, mathematics is not defined neither by means of the specificity of its objects, nor by the nature of its propositions, nor even by the kinds of truths it may exhibit. Having nothing to say about any truth of fact, mathematics deals with ideally constructed hypothesis in the highest level of abstraction, structures that are truly *entia rationis* [EP 2: 352][54] . What defines mathematics is the way the mathematician works with the abstractions, experimenting with them: mathematical reasoning is "a transformation of our diagrams that characters of one diagram may appear in another as things. A familiar example is where in analysis we treat operations as themselves the subject of operations." [EP 2: 213, 1903]. The mathematician, then, not only reasons from abstract hypothesis, he also constructs the hypothesis from which he reasons. The process of mathematical reasoning can thus be defined as a double process of construction and experimentation upon ideal formal structures.[55]

Given this definition of mathematics, it is possible that even Frege would not fully disagree with Peirce's anti-logicism.[56] Once the degree of generality reached by mathematical reasoning is unattainable by any other science, even logic would be a science under mathematics. But it is also true Peirce never accepted any form of logicism. About his later system of existential graphs, he claims: "this system is not intended to serve as a universal language for mathematicians or other reasoners, like that of Peano" [CP 4.424, 1903]. Indeed, only if strictly conceived as a deductive science logic can be said to be under mathematics. In fact, "it is impossible to reason necessarily concerning anything else than a pure hypothesis" [CP 4.232, 1902]. Logicians then study the very forms of mathematical reasoning mathematicians themselves use, but do not necessarily reflect upon [CP 1.417, 1902].

[53]A. Walsh 1999, pp. 278-279 claims the younger Peirce in fact influenced the elder Peirce on this respect.

[54]On this point, see C. Tiercelin 2010.

[55]In C. T. Rodrigues 2007 Peirce's distinction between theorematic and corollarial kinds of deductions is explained, where the reader can find other references on the subject. But see also D. Campos 2010 for the relation of this distinction with imagination in mathematics.

[56]I take this suggestion from M. Moore 2010.

A passage from Peirce deserves to be quoted in full, taken from J. Baldwin's 1902 *Dictionary of Philosophy and Psychology* entry on logic:

> If symbolic logic be defined as logic – for the present only deductive logic – treated by means of a special system of symbols, either devised for the purpose or extended to logical from other uses, it will be convenient not to confine the symbols used to algebraic symbols, but to include some graphical symbols as well.
>
> The first requisite to understanding this matter is to recognize the purpose of a system of logical symbols. That purpose and end is simply and solely the investigation of the theory of logic, and not at all the construction of a calculus to aid the drawing of inferences. These two purposes are incompatible, for the reason that the system devised for the investigation of logic should be as analytical as possible, breaking up inferences into the greatest possible number of steps, and exhibiting them under the most general categories possible; while a calculus would aim, on the contrary, to reduce the number of processes as much as possible, and to specialize the symbols so as to adapt them to special kinds of inference. It should be recognized as a defect of a system intended for logical study that it has two ways of expressing the same fact, or any superfluity of symbols, although it would not be a serious fault for a calculus to have two ways of expressing a fact [CP 4.372, 473].

Mathematical logic is in fact the strictly formal logical field of inquiry of applying mathematics to logic in order to lend exactness to the latter (CP 3.615, 1901). This is Boole's and De Morgan's logic, as seen. This is not the *whole* of logic, though. The distinctive logical activity is in the study and the analysis of reasoning, and not in its practice. Peirce also spells this theoretical point in his famous distinction—taken from his medieval sources—between *logica utens* and *logica docens* [CP 2.182, 1902; EP 2: 188, 1903][57] . The fundamental difference between the logic of first intentions and the logic of second intentions resurfaces here in another way: a mathematician reasons, a logician studies reasoning as an object, or, better said, analyses forms of reasoning [CP 4.134, 1893]. Then, a logician's task is also to analytically reconstruct the argumentative and semantic framework of reasoning itself, and not only of propositions, for this framework frequently escapes language.[58] But

[57]For the distinction, see A-V. Pietarinen 2005. Bellucci and Pietarinen 2016 also stress this point for Peirce's diagrammatic logic.

[58]Which can indeed be an interesting way of understanding the distinction between saying and showing, a most prominent *topos* in *Wittgenstein's Tractatus Logic-Philosophicus*, as widely known.

we can always compare the way we think with assumed norms of good thinking, and then to try to change our form of thinking. An idea of logic emerges then as a kind of experimentation upon reasoning processes—breaking them up into the greatest possible number of steps, and then comparing them with the rules of our *logica utens*—showing Peirce intended his logics mainly in a metatheoretic fashion. This is the reason why Hintikka calls Peirce's approach as "model-theoretic" in the first place.[59]

This is crucial for the amplification of logic as semiotic. Indeed, De Morgan's influence over Peirce can be measured by how logic can be extended to include not only necessary deductive forms of reasoning, but also induction and *abduction*, central as they are for scientific inquiry.[60] If logical analysis should not be taken for a mere calculus, this means the principle of parsimony—or simplicity, let us say—does not hold here. Rather, logic follows the principle of *complexity*—the passage from the complex to the simple is a logician's task.

5 Conclusion

As rightly remarked by I. Grattan-Guinness, as typical in the 19$^{\text{th}}$ century, Boole, De Morgan and Peirce consider the mathematization of logic useful for the study of probabilities, not possibilities. Necessity then is thought of as the other side of probability, and not of possibility, as nowadays we usually think.[61] The theory of probabilities is considered to provide the best analytic apparatus for studying multiple forms of reasoning. This also throws a light on how mathematics and logic should be conceived.

Boole was the first to advocate for a closer association of logic with mathematics, thus generalizing the discipline's scope. He indeed devised a universal language with algebraic signs capable of expressing the traditional categorical propositions of Aristotelian syllogistic, but his main interest was in providing a functional symbolic language for expressing the universal laws of thought. With this, Boole in fact accomplished to devise a calculus of classes, which is indeed the first algebraic approach to logic. Valid syllogism forms could now be obtained dismissing the complicated mnemonics invented by medieval logicians (Barbara, Celarent, Baroco etc.). Using Boole's algebra, logical reasonings can be performed by employing basic mathematical techniques, in a more efficient

[59] J. Hintikka 1997; see also V. Peckhaus 2004. In fact, Hintikka generalizes the Leibnizian-inspired position by Heijenoort 1967. But De Morgan also stresses that logic is both an art and a science, the art being called *logica utens*. De Morgan 1858, p. 181. For a historical overview of the algebra of logic tradition, including E. Schroeder's contributions in dialogue with Peirce, which are lacking here but I cannot provide now, see N. Houser 1994 and 1991; V. Peckhaus 2009.

[60] I have discussed this subject in detail in C. T. Rodrigues 2011.

[61] I. Grattan-Guinness 1997, p. 35.

and general way. After Boole, logicians were able to abandon the "catalog of syllogisms"[62] , because being in possession of an exact calculus to establish necessary conclusions.

But it was De Morgan the first to perceive that logic could not remain restricted to syllogistic reasonings, and he also trying to devise a logical symbolism without using mathematical signs to express these other forms of reasoning. He lays a strong claim about the need of a proper symbolism for logic: "Every science that has *thriven* has thriven upon its own symbols; logic, the only science which is admitted to have made no improvements in century after century, is the only one which has *grown no symbols"*[De Morgan 1858, p. 184]. Symbols in logic are useful for they are able to exhibit how we think, that is, "the *form* of thought, the law of action of its machinery"[De Morgan, 1858, p. 179]. And this is the reason De Morgan advances for why algebra should be taken as a "branch" of thought, and not vice-versa: algebra is just the visible expression of the logical operations of thinking [*id.*].

If for Boole syllogisms were used as instances to show the functionality of the calculus, De Morgan took them as the main object and starting point of his researches. For Boole, it was a matter of devising a calculus based on algebra, but different from it, as the idempotent law shows. Although agreeing with Boole that human reasoning could be depicted with algebraic signs and operations, De Morgan did not want to devise a formal algebraic calculus. Instead, he was more interested in symbolizing logical relations the traditional syllogistic could not show were valid, in order to make forms of thought that usually pass "under cover" could be exhibited. This is De Morgan's main objective for his spicular notation. Although it is perfectly possible to translate De Morgan's notation into Boolean algebra, as Peirce showed De Morgan himself, it was not the latter's intention to conflate mathematics and logic.

Peirce established a distinction of aims between the two activities of logic and of mathematics elevating the discussion to another level. As said, according to his approach, one thing is to reason, the other is to study reasoning itself. While the distinction between language as a calculus and language as a universal medium may be useful to understand Frege's logicist project, when applied to Peirce a few more grains of salt are needed to take it. Since his early criticism to the confusion of logical and psychological processes, Peirce argues for the thesis that thought is a function of signs, either internal or external. P. Skagestad calls this the thesis of virtuality of mind[63]: mind is what acts

[62] L. Suguitani, P. Viana and I. D'Ottaviano's expression, in Tarski 2016, p. 15.

[63] For instance, see P. Skagestad 2004, p. 249. The author claims Peirce advocates a sort of dispositional conception of thought, such as K. Popper's, and such as echoing the famous Sapir-Whorf hypothesis on the nature of language, that is, thinking is a sort of exosomatic process capable of embodiment in whatever sign vehicles are available.

like a mind, so that mind can be virtually efficient in or outside the brain. In fact, it can be found everywhere there are means for expressing thoughts, or signs, for short. This is an idea Peirce held from his youth. In 1865, the argument ran like this: logic makes no distinction whatsoever between symbols and thoughts, for it examines symbols as objects of some possible thought, and not as actually thought of. The nature of thought is symbolic, and only as such can thought be an object of logical study [W 1: 166, 1865]. Indeed, logical forms can be actualized in different ways, in various meaningful processes, either in representations of an internal and psychological nature, or embodied in material signs of an objective, external nature. So, in the one hand, the form of thought is independent of the sign system used, but, on the other, signs are virtually interpretable as conveyers of thought, so a sign's specific material qualities enable it to function in a specific manner, thus enabling in turn a specific form of reasoning. All thought could then be logically analyzed and interpreted independently of its specific manifestation, as a semiotic manifestation of some form of logical reasoning – without need for any psychological subjective interpreter, since it is a sign functionality that defines a mind[64]. With Peirce, the generalization of the syllogistic copula passes to a whole new level. Logic itself becomes a more general inquiry not restricted to the study of forms of deduction in symbolic languages, but of thought as a semiotic process in the world.[65]

Pursuing Peirce's comparison of logic with mathematics a little further, a rigid distinction between a "pure" and an "applied" science of mathematics makes no sense at all. Mathematics, as the old Benjamin Peirce held, is the science that draws necessary conclusions, and logic, as his son Charles Peirce stresses, is the science of drawing conclusions. And, as we have, this does not mean Peirce adopted any sort of logicism, quite on the contrary. By an extreme simplification of relations, a mathematician frames ideally universal and abstract forms upon which to experiment. Mathematics, then, can be conceived as a sort of formal logic of relations.[66] But the logician's interest is primarily in what the mathematician in a certain way despises, which is the *rhetorical* character of reasoning.[67] Depending on the mathematician's

[64]This is Peirce's central argument against any form of psychologism in logic, setting the question indeed one year before the very word *Psychologismus* was coined by J. E. Erdmann in Germany. This I assume is because Peirce is critical of J. S. Mill. The German logicians who set anti-psychologism agenda were all heavily influenced by Mill. See M. Kush 1995, p. 2-3; p. 101.

[65]See J. Legris 2016 for a similar line of argumentation. Legris argues for contextualizing Peirce's work in a symbolic medieval tradition of logic, and not in the more modern linguistic tradition.

[66]This is more deeply developed in Rodrigues 2007.

[67]If I am allowed this expression. On the point, see V. Colapietro 2007.

interests, logical constructions can even be superfluous, since a calculus however important is purely instrumental, since it is directed to make deductions in the most expedient and safest way.[68] Indeed, when Peirce criticizes Boole and De Morgan (and others), this is because they try to draw philosophical (ontological) conclusions from calculi, which Peirce himself conceives of but as tools for computing purposes. A logician's genuine activity is not to reach the conclusions, but to study how the conclusions are related to the premises, and which are the necessary steps of this link.

Peirce's claim in favor of a logical inquiry into all kinds of signs, "and not only of symbols"—as in his youth he claimed—is unequivocal [EP 2: 387, 1906]. In this respect, logic becomes semiotic.[69] This can also be claimed as a reason why Peirce never bothered with the axiomatization of his logic of relatives. In fact, axiomatization only became a major concern after the works of the Bourbaki group, in the 20$^{\text{th}}$ century[70], emphasizing the need for exactness in pure mathematical analysis. It is false that Peirce did not care for axiomatization at all, but he thought it was possible only for arithmetic, and not for the whole of mathematics.[71] Mathematics does not need any sort of justification, be it in logic, intuition or by constructing completeness proofs. As seen, for Peirce even the very difference between pure and applied mathematics makes not much sense. Two points in this respect are worthy of mention.

First, mathematical reasoning is *par excellence* diagrammatic, or *iconic* [EP 2: 219-220]. But since the most abstract and general form of thinking is mathematical, we can say all reasoning in one sense or another is diagram-

[68] Peirce repeats this point more times than countable, but he was not always properly understood. See, for instance, CP 3.364, 1885, and CP 4.553, 1906, as he argues for the same idea in different times.

[69] A minimally informed analysis of Peircean semiotic is impossible to be given here. For a systematic overview, any of the following is excellent: L. Santaella 1995; L. F. B. da Silveira 2007; T. L. Short 2007; D. Savan 1956. The same goes for Peirce's logical graphs, so see: A-V. Pietarinen 2007; D. D. Roberts 1973; C. Legg 2013; I. Anellis and F. Abeles 2016; J. Queiroz and L. Moraes (eds.) 2013; from the same authors, see 2001 and 2004.

[70] See T. Roque 2012, p. 374ff. The author claims the excessive emphasis on Cantor's work also has to be taken into account in the contemporary preference for the story of a purely abstract mathematics.

[71] In 1881, Peirce provided the first successful axiomatization for natural numbers in a paper entitled "On the logic of number". Peirce eventually sent Dedekind a copy of that paper, even believing—erroneously—that Dedekind's axiomatic system was developed from his own, since the latter's "Was sind und was sollen die Zahlen?" was published only in 1888. But Peirce did not know Dedekind was already working in his conclusions since 1872. For a full comparison of Peirce's and Dedekind's theories of arithmetic, see P. Shields 1981. See R. Maddux 1991 for a critical overview. For the reconstruction of the philosophical context in Germany in contrast to the mathematical English context, see V. Peckhaus 1999. Of course Peirce's disagreements with Dedekind have to be put against a richer reconstruction of the international intellectual context, but that is impossible to give here.

matic. Although Peirce more usually discusses diagrams and diagrammatic reasoning in the context of deductive mathematical reasoning, this restriction can in principle be softened. The point is that by obeying logical rules for constructing and experimenting upon diagrams, one reasons from contingency and casualness to necessity [NEM 4: 105, *c.* 1886]. Diagrams of all sorts are more than just auxiliary devices for the analysis of logical languages; in fact they exhibit the very logical movement of thought:

> Suppose a man to reason as follows: The Bible says that Enoch and Elijah were caught up into heaven; then, either the Bible errs, or else it is not strictly true that all men are mortal. What the Bible is, and what the historic world of men is, to which this reasoning relates, must be shown by indices. The reasoner makes some sort of mental diagram by which he sees that his alternative conclusion must be true, if the premise is so; and this diagram is an *icon* or likeness. The rest is symbols; and the whole may be considered as a modified symbol. It is not a dead thing, *but carries the mind from one point to another.* The art of reasoning is the art of marshaling such signs, and of finding out the truth. [EP 2: 10, 1894].

Peirce's move towards a diagrammatic logic can then be conceived as a sort of generalization of his early work in algebraic notation: instead of remaining restricted to linear notations or propositional languages, logicians should also inquiry into ways of constructing diagrams. The letter to Mitchell quoted before plainly shows this concern, and even tough Peirce's full system of logical graphs is not yet there, it is a clear illustration of how to combine a linear notation with a diagrammatic representation of relations.

Second, logic is for Peirce a normative science. Without going deep into this subject, suffice it by now to say to reason is to act in a specific way. As a way of acting, reasoning is nothing but self-controlled and deliberate thought, the only one capable of being criticized, and, therefore, corrected:

> Reasoning essentially involves *self-control*; so that the *logica utens* is a particular species of morality. Logical goodness and badness, which we shall find is simply the distinction of *Truth* and *Falsity* in general, amounts, in the last analysis, to nothing but a particular application of the real general distinction of Moral Goodness and Badness, or Righteousness and Wickedness" [EP 2: 188, 1903].

Logic emerges as the science of the ends deliberate and self-controlled thought should attain. A symbolic language is useful for logical inquiry in that signs are instrumental for the ideally best representation of logical goodness and badness, or validity and invalidity, as we would say today [CP 5.534, 1905]. As a *quasi*-necessary science, its purpose is to say what *ought to be* the characters of signs so they can be experimentally used for our learning. The development of a calculus, then, is only one of task of semiotic, which in itself should not be confounded with a formal science of calculus, for the exactness logic employs in calculations is mathematical. Calculi are useful "in the investigation of logic", as initially quoted, to study reasoning, surveying signs of all kinds as embodiments and conveyers of thought. As all sciences less general than mathematics, *quasi*-formal logic is also fallible: the necessity of the conclusion is just as much necessity there is in the premises. Any further necessity is, of course, of an unknown nature.

Appendix A

Bibliographical references from Christine Ladd-Franklin. On the Algebra of Logic. In: *Studies in Logic by members of the Johns Hopkins University.* [Org.] by C. S. Peirce. Boston: Little, Brown, and Company, 1883, pp. 70-71.

---

70 ON THE ALGEBRA OF LOGIC.

BIBLIOGRAPHY.

I. George Boole : The Mathematical Analysis of Logic, being an essay towards the Calculus of Deductive Reasoning. 1847.

II. Boole : The Calculus of Logic. *Cambridge and Dublin Math. Jour.*, Vol. III., 1848.

III. Boole : An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities. 1854.

IV. Hermann Grassmann : Lehrbuch der Arithmetik. Berlin, 1861.

V. W. S. Jevons : Pure Logic, or the Logic of Quality apart from Quantity. London and New York, 1864.

VI. C. S. Peirce : On an Improvement in Boole's Calculus of Logic. *Proc. Am. Acad. of Sciences*, Vol. VI., 1867.

VII. W. S. Jevons : The Substitution of Similars, the true Principle of Reasoning, derived from a modification of Aristotle's dictum. London, 1869.

VIII. C. S. Peirce : Description of a Notation for the Logic of Relatives. *Memoirs of the Am. Acad. of Sciences*, Vol. IX., 1870.

IX. Robert Grassmann : Die Formenlehre oder Mathematik. Zweites Buch : Die Begriffslehre oder Logik. Stettin, 1872.

X. A. J. Ellis : On the Algebraical Analogues of Logical Relations. *Proc. of the Royal Society of London*, 1872–73.

XI. Arthur Cayley : Note on the Calculus of Logic. *Quart. Jour. of Math.*, Vol. XI.

XII. Ernst Schröder : Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende. I. Bd. : Die sieben algebraischen Operationen. Leipzig, 1873.

XIII. W. S. Jevons : Principles of Science. London, 1874. Third edition, 1879.

XIV. Ernst Schröder : Der Operationskreis des Logikkalkuls. Leipzig, 1877.

XV. J. Delboeuf : Logique Algorithmique. Liège et Bruxelles, 1877.

XVI. Hugh McColl : The Calculus of Equivalent Statements, and Integration Limits. *Proc. London Math. Soc.*, Vol. IX., 1877–78. Vol. X., 1878, Vol. XI., 1880.

XVII. Gottlob Frege : Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. Halle, 1879. Re-

ON THE ALGEBRA OF LOGIC. 71

viewed by Ernst Schröder in *Zeitschrift für Mathematik und Physik*, 1880.

XVIII. Wilhelm Wundt : Logik, eine Untersuchung der Principien der Erkenntniss und der Methoden wissenschaftlicher Forschung. I. Bd. : Erkenntnisslehre. Stuttgart, 1880.

XIX. J. Venn : On the Diagrammatic and Mechanical Representation of Propositions and Reasoning. *Phil. Mag.*, July, 1880.

XX. J. Venn : Symbolic Reasoning. *Mind*, July, 1880.

XXI. C. S. Peirce : On the Algebra of Logic. *Am. Jour. of Math.*, Vol. III., September, 1880.

XXII. W. S. Jevons : Studies in Deductive Logic. London, 1880.

XXIII. John Venn : Symbolic Logic. London, 1881.

NOTE.—In the foregoing article "combination" has been used as synonymous with "multiplication." In the following article, "combination" is used as including both multiplication and addition.

Acknowledgements

I have to acknowledge the help I have received from other persons. Especially, I am thankful to Jean-Yves Béziau for his support and unparalleled encouragement. I would not have written this article without his insistence. I want to acknowledge the generous and precise proof-reading from Marcelo E. Coniglio, as well as his editorial patience at the very last moments. I am also thankful to Nathan Houser, for kindly sending me copies of his works, as well as for all the conversations we have whenever possible. And last but not at all least to Irving Anellis – *in memoriam* – for the generosity and immense knowledge on the subject shared through some valuable emails.

A note on the Internet

Over recent years, the Internet has transformed scholarly research. Not only books became easier to find (and buy, as far as money goes), but mainly the amount of known bibliography on almost *any* subject-matter has become practically impossible to master by a single person as almost everything can be found online, either at subscription services or at open-access databases. This was the case with the majority of works I have used and quoted here: almost everything I could find online. And as the amount of available bibliography increases incessantly day by day, this of course put a limit to my work, and accordingly the bibliography is not at all exhaustive.

But this is only part of the story. In the case of Peirce, I resorted to the modern scholarly editions of his writings, but in the case of the other authors all of the 19$^{\text{th}}$ century works I used are available online, either at their original publication form or another. I am especially indebted to Archive.org, wherein I could find all of the *Transactions of the Cambridge Philosophical Society* volumes with De Morgan's papers on the syllogism as well as all the other works by the British logicians I quote. So, even if I do not provide a DOI for everything I quote, anyone can check my references with a quick Internet search on the sources.

As this paper is the result of a research that did not received any kind of funding from any agency at all, I thought it would be worth mentioning the fact. This is the cultural feature of the Web that seems as one of the most interesting. Of course sometimes I felt as if I was in J. L. Borges' labyrinth – but that is part of the fun, isn't it? I hope others may think so as well.

References

The author-data system is preferred, but for Peirce, whose works are quoted according to the generally accepted system of Peircean scholarship. Abbreviations are explained in the entries. Whenever the date of first edition was different from the date of edition used, the first was indicated as main entry identification.

Anellis, Irving H. and Francine F. Abeles. 2016. The historical sources of tree graphs and the tree method in the work of Peirce and Gentzen. In: Francine F. Abeles & Mark E. Fuller (eds.). *Modern Logic 1850-1950, East and West.* Basil, SWI: Birkhäuser; Springer International Publishing Switzerland, 2016, 35-97.

Anellis, Irving H. Tarski's development of Peirce's logic of relations. 1997. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 271-303.

________. 2012. How Peircean was the "Fregean revolution" in logic? In: *Logicheskie issledovaniya*, n°. 18, 239-272.

________. 2015. Peirce's Role in the History of Logic: Lingua Universalis and Calculus Ratiocinator. In: Arnold Koslow & Arthur Buchsbaum (eds.). *The road to universal logic. Festschrift for the 50th birthday of Jean-Yves Béziau*, vol. II. Cham *et al.*: Birkhäuser; Springer International Publishing Switzerland, 2015, 135-169.

Aristotle. 1963. *Categories and De Interpretatione.* Translated with notes by J. L. Ackrill. Oxford: Clarendon Press, 1963.

Aubenque, Pierre. 1962. *Le problème de l'être chez Aristote.* Paris: PUF.

Beatty, Richard. 1969. Peirce's development of quantifiers and of predicate logic. In: *Notre Dame Journal of Formal Logic*, vol. X, n°. 1 (1969), 64-76.

Bellucci, Francesco, and Ahti-Veikko Pietarinen. 2016. Existential Graphs as an Instrument for Logical Analysis. Part 1: Alpha. In: *Review of Symbolic Logic*, vol. 9, n°. 2 (2016), 209-237.
DOI: https://doi.org/10.1017/S1755020315000362

Béziau, Jean-Yves. 2012. A history of truth-values. In: *Handbook of the history of logic.* Vol. 11: *Logic: A history of its central concepts.* Ed. by Dov M. Gabbay et al. (eds.). Amsterdam et al.: Elsevier; North-Holland, 2012, 235-307.

Boehner, Philotheus. 1951. Bemerkungen zur Geschichte der De Morganschen Gesetze in der Scholastik. In: *Archiv für Philosophie*, vol. 4, n°. 2, (September 1951), 113-46.

Boler, John F. 1963. *Charles Peirce and scholastic realism: Study of Peirce's relation to John Duns Scotus.* Seattle, WA: University of Washington Press.

Bonevac, Daniel. 2012. A history of quantification. In: *Handbook of the history of logic.* Vol. 11: *Logic: A history of its central concepts.* Ed. by Dov M. Gabbay *et al.* (eds.). Amsterdam et al.: Elsevier; North-Holland, 2012, 63-126.

Boole, George. 1847. *The Mathematical Analysis of Logic: Being an essay towards a calculus of deductive reasoning.* Cambridge, UK: Macmillan, Barclay & Macmillan; London: George Bell.

________. 1854. *An Investigation of the Laws of Thought, on which are founded the mathematical theories of logic and probabilities.* London: Walton and Maberly; Cambridge, UK: Macmillan and Co.

________. 1997. *Selected Manuscripts on Logic and its Philosophy.* Editors: Ivor Grattan-Guinness; Gerard Bornet. Basel; Boston; Berlin: Birkhauser.

Brady, Geraldine. 2000. *From Peirce to Skolem: A neglected chapter in the history of mathematical logic.* Amsterdam: Elsevier Science B.V.

Brunning, Jacqueline. 1997. Genuine triads and teridentity. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 252-263.

Burch, Robert. 1997. Peirce's reduction thesis. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 235-251.

Burris, Stanley and Legris, Javier. The Algebra of Logic Tradition. In: *The Stanford Encyclopedia of Philosophy* (Fall 2016 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/fall2016/entries/algebra-logic-tradition/>. Access date 20/10/2016.

Cajori, Florian. 1993. *A history of mathematical notations.* 1^{st} ed. 1928-1929. New York: Dover Publications, Inc., (two volumes bound as one).

Campos, Daniel G. The imagination and hypothesis-making in mathematics: a Peircean account. In: *New Essays on Peirce's Mathematical Philosophy.* Edited by Matthew E. Moore. Chicago: Open Court, 2010, 123-145.

Castrillo Criado, Pilar. 1988. Introducción. In: *Charles S. Peirce: Escritos lógicos.* Introducción, selección y traducción de Pilar Castrillo Criado. Madrid: Alianza Editorial, 1988.

Church, Alonzo. 1956. *Introduction to mathematical logic.* Vol. I. Princeton, NJ: Princeton University Press.

Colapietro, Vincent. 2007. C. S. Peirce's rhetorical turn. In: *Transactions of the Charles Sanders Peirce Society*, vol. 43, nº. 1 (2007), 16-52.

Corcoran, John and Hassan Masoud. 2015. Existential import today: new metatheorems; historical, philosophical, and pedagogical misconceptions. In: History and Philosophy of Logic, vol. 36, issue 1 (2015), 39-61.

Corcoran, John and Susan Wood. 1980. Boole's criteria for validity and invalidity. In: *Notre Dame Journal of Formal Logic*, vol. 21, nº. 4, 609-638.

Correia, Manuel. 2003. *La lógica de Aristóteles. Lecciones sobre el origen del pensamiento lógico en la antigüedad.* Santiago de Chile: Ediciones Universidad Católica de Chile.

________. 2011. La similitud de los tratados silogísticos de Boecio y Apuleyo. In: Teología y Vida, v. 52, nº. 1-2 (2011), 291-306.

De Morgan, Augustus. 1844. On the foundation of algebra, nº. IV, on triple algebra. In: *Transactions of the Cambridge Philosophical Society*, vol. 8 (1849), 241-254.

________. 1847. *Formal Logic: Or, the calculus of inference, necessary and probable.* London: Taylor and Walton.

________. 1850. On the symbols of logic, the theory of the syllogism, and in particular of the copula, and the application of the theory of probabilities to some questions of evidence. In: *Transactions of the Cambridge Philosophical Society*, vol. 9 (1856), 79-127.

________. 1858. On the syllogism, nº III, and on logic in general. In: *Transactions of the Cambridge Philosophical Society*, vol. 10 (1864), 173-230.

________. 1860. *Syllabus of a proposed system of logic.* London: Walton and Maberly.

________. 1860. On the syllogism, nº IV, and on the logic of relations. In: *Transactions of the Cambridge Philosophical Society*, vol. 10 (1864), 331-358.

De Morgan, Sophia E. 2010. *Memoir of Augustus De Morgan, with selections from his letters.* Cambridge: Cambridge University Press.

De Tienne, André. 2014. 1914-2014: One hundred years of editing and publishing Peirce. Available at <http://www.commens.org/papers/paper/de-tienne-andr%C3%A9-2014-1914%E2%80%932014-one-hundred-years-editing-and-publishing-peirce>. Access date: July 20th, 2017.

Dipert, Randall. 2004. Peirce's deductive logic: Its development, influence, and philosophical significance. In: Cheryl Misak (ed.). *The Cambridge Companion to Peirce.* Cambridge, UK: Cambridge University Press, 2004, 287-324.

———. 1981. Peirce's propositional logic. In: *The Review of Metaphysics*, vol. 34, n^{o}. 3 (Mar., 1981), 569-595.

———. 1984. Peirce, Frege, the logic of relations, and Church's theorem. In: *History and Philosophy of Logic*, vol. 5, n^{o}. 1 (1984), 49-66.

Dvorák, Petr. 2008. Relational logic of Juan Caramel. In: *Handbook of the history of logic.* Vol. 2: *Mediaeval and Renaissance logic.* Dov M. Gabbay and John Woods (eds.). Amsterdam et al.: Elsevier; North-Holland, 2008, 645-665.

Eisele, Carolyn. 1979. *Studies in the scientific and mathematical philosophy of Charles S. Peirce: essays by Carolyn Eisele.* Edited by Richard M. Martin. The Hague; Paris; New York: Mouton Publishers.

El-Rouayheb, Khaled. 2016. Arabic logic after Avicenna. In: Catarina Dutilh Novaes and Stephen Read (ed.). *The Cambridge companion to medieval logic.* Cambridge: Cambridge University Press, 2016, 67-93.

Eves, Howard. 1990. *An introduction to the history of mathematics.* 6th ed. With cultural connections by James H. Eves. 6th ed. Philadelphia et al.: Saunders College Publishing.

Evra, James van. 2000. The development of logic as reflected in the fate of the syllogism 1600-1900. In: *History and Philosophy of Logic*, vol. 21 (2000), 115-134.

Frege, Gottlob. 1881. Boole's logical calculus and the concept-script. In: *Posthumous writings.* Edited by Hans Hermes, Friedrich Kambartel, Friedrich Kaulbach with the assistance of Gottfried Gabriel and Walburga Rodding. Translated by Peter Long, Roger White, with the assistance of Raymond Hargreaves. London: Basil Blackwell, 1979, 9-46.

———. 1882. Boole's logical formula-language and my concept-script. In: *Posthumous writings.* Edited by Hans Hermes, Friedrich Kambartel, Friedrich Kaulbach with the assistance of Gottfried Gabriel and Walburga Rodding. Translated by Peter Long, Roger White, with the assistance of Raymond Hargreaves. London: Basil Blackwell, 1979, 47-52.

Gasser, James (ed.). 2000. *A Boole Anthology. Recent and classical studies in the logic of George Boole.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Gomes, Evandro and Ítala M. Loffredo D'Ottaviano. 2017. *Para além das colunas de Hércules: uma história da paraconsistência – De Heráclito a Newton da Costa.* Campinas, SP: Ed. UNICAMP.

Grattan-Guinness, Ivor. 2000. *The search for mathematical roots, 1870-1940: logics, set theories and the foundations of mathematics from Cantor through Russell to Gödel.* Princeton, NJ: Princeton University Press.

________. 1997a. Benjamin Peirce's *Linear Associative Algebra* (1870): New light on its preparation and 'publication'. In: *Annals of Science*, 54 (1997), 597-606.

________. 1997b. Peirce between logic and mathematics. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 23-42.

Gregory, Duncan Farquharson. 1840. On the real nature of symbolic algebra. In: *Transactions of the Royal Society of Edinburgh.* Vol. XIV, Part I (1840), 208-216.

Haack, Susan. 1996. *Deviant logic, fuzzy logic: beyond the formalism.* Chicago, IL: The University of Chicago Press.

Haaparanta, Leila (ed.). 2009. *The development of modern logic.* Oxford: Oxford University Press.

Hintikka, Jaakko. 1997. The place of C. S. Peirce in the history of logical theory. In: Jacqueline Brunning and Paul Forster (eds.). *The Rule of Reason: The philosophy of Charles Sanders Peirce.* University of Toronto Press, 1997, 13-33.

Hobart, Michael E. and Joan L. Richards. 2008. De Morgan's Logic. In: *Handbook of the history of logic.* Vol. 4: *British logic in the nineteenth century.* Dov M. Gabbay and John Woods (eds.). Amsterdam et al.: Elsevier; North-Holland, 2008, 283-329.

Houser, Nathan *et al.* (ed.). 1997. *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997.

________. 1994. Algebraic logic from Boole to Schröder. In: *Companion encyclopedia of the history and philosophy of the mathematical sciences.* Volumes 1 & 2. Ed. by I. Grattan-Guinness. London and New York: Rutledge, 1994, 600-616.

________. 1992. The fortunes and misfortunes of the Peirce papers. In: *Signs of Humanity*, vol. 3. Edited by Michel Balat and Janice Deledalle-Rhodes. General editor: Gérard Deledalle. Berlin: Mouton de Gruyter, 1992, 1259-1268.

________. 1991. The Schröder-Peirce correspondence. In: *The Review of Modern Logic*, vol. 1, n°. 2-3 (1991), 206-236.

Iliff, Alan J. 1997. The matrix representation in the development of Peirce's quantifiers. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 193-205.

Jacquette, Dale. 2008. Boole's logic. In: *Handbook of the history of logic.* Volume 4: *British logic in the nineteenth century.* Dov M. Gabbay and John Woods (ed.). Amsterdam et al.: Elsevier; North-Holland, 2008, 331-379.

Kneale, William and Martha Kneale. 1962. *The development of logic.* Oxford: Clarendon Press, 1985 (reprint).

Kretzman, Norman et al. 1982. *The Cambridge history of later medieval philosophy – from the rediscovery of Aristotle to the disintegration of Scholasticism.* Editors Norman Kretzman, Anthony Kenny, Jan Pinborg. Associate editor Eleonore Stump. Cambridge: Cambridge University Press, 1997 reprint.

Kush, Martin. 1995. *Psychologism: A case study in the sociology of philosophical knowledge.* London: Routledge.

Lane, Robert. 2001. Principles of Excluded Middle and Contradiction. In: *The Digital Encyclopedia of Charles S. Peirce.* Available at: <http://www.digitalpeirce.fee.unicamp.br>. Access date: 23/02/2005.

________. 1997. Peirce's "entanglement" with the principles of excluded middle and contradiction. In: *Transactions of the Charles S. Peirce Society*, vol. 33, n°. 3 (Summer, 1997), 680-703.

Legg, Catherine. 2013. What is a logical diagram? In: Sun-Joon Shin and Amirouche Moktefi (eds.) *Visual reasoning with diagrams.* Dordrecht; Boston; New York: Springer, 1-18.

Legris, Javier. 2016. Los gráficos existenciales en la historia de la lógica simbólica. In: *Charles S. Peirce: Ciencia, filosofía y verdad.* Coordinación general de Catalina Hynes y Jaime Nubiola. San Miguel de Tucumán, Ar: La Monteagudo Ediciones, 2016, 59-68.

Lenzen, Wolfgang. 2004. Leibniz's Logic. In: Dov Gabbay and John Woods (ed.). *Handbook of the History of Logic*, vol. 3: *The Rise of Modern Logic: From Leibniz to Frege.* Amsterdam et al.: Elsevier North Holland, 2004, 1-83.

Londey, David and Carmen Johanson. 1984. Apuleius and the square of opposition. In: *Phronesis*, vol. 29 (1984), 165–173.

Łukasiewicz, Jan. 1930. Investigations into the sentential calculus. Translated from the German version by S. McCall. In: *Selected Works.* Edited by L. Borkowski. Amsterdam; London: North Holland Publishing Company; Warszawa: PWN – Polish Scientific Publishers, 1970, 131-152.

________. 1934. On the history of the logic of propositions. Translated from the German version by S. McCall. In: *Selected Works.* Edited by L. Borkowski. Amsterdam; London: North Holland Publishing Company; Warszawa: PWN – Polish Scientific Publishers, 1970, 197-217.

Maddux, Roger D. 1991. The origin of relation algebras in the development and axiomatization of the calculus of relations. In: *Studia Logica: An international journal for symbolic logic*, vol. 50, nº 3/4 (1991), 421-455. DOI: 10.1007/BF00370681.

Merrill, Daniel D. 1990. *Augustus De Morgan and the logic of relations.* Dordrecht et al.: Kluwer Academic Publishers.

________. 1997. Relations and quantification in Peirce's logic. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 158-172.

Michael, Emily. 1979. An examination of the influence of Boole's algebra on Peirce's developments in logic. In: *Notre Dame Journal of Formal Logic*, vol. XX, n, 4, (Oct., 1979), 801-806.

———. Peirce's early study of the logic of relations, 1865-1867. In: *Transactions of the Charles S. Peirce Society*, vol. 10, n. 2 (Spring, 1974), 63-75.

Michael, Frederick and Emily Michael. 1979. Peirce on the nature of logic. In: *Notre Dame Journal of Formal Logic*, vol. XX, n. 1 (Jan., 1979), 84-88.

Molczanow, Aleksy. 2012. *Quantification: Transcending beyond Frege's boundaries. A case study in transcendental-metaphysical logic.* Leiden; Boston: Brill.

Moody, Ernest A. 1953. *Truth and consequence in mediaeval logic.* Amsterdam: North-Holland Publishing Co.

Moore, Matthew E. (ed.). 2010. *New Essays on Peirce's Mathematical Philosophy.* Chicago: Open Court.

Murphey, Murray G. 1961. *The Development of Peirce's Philosophy.* Indianapolis; Cambridge, MA: Hackett Publishing Company, Inc. 1993.

Nambiar, Sriram. 2000. The influence of Aristotelian logic on Boole's philosophy of logic: the reduction of hypotheticals to categoricals. In: *A Boole Anthology. Recent and classical studies in the logic of George Boole.* Edited by James Gasser. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2000, 217-239.

Pantecki, Maria. 2000. The mathematical background of George Boole's Mathematical Analysis of Logic (1847). In: *A Boole Anthology. Recent and classical studies in the logic of George Boole.* Edited by James Gasser. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2000, 167-212.

Parsons, Terence. 2017. The traditional square of opposition. In: *The Stanford Encyclopedia of Philosophy* (Fall 2016 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/entries/square/>. Access date 20/06/2017.

Peacock, George. 1830. *A treatise on algebra.* Cambridge, UK: J. & J. J. Deighton; London: C.J.G. & F. Rivington; Whittaker, Treacher & Arnot.

Peckhaus, Volker. 2012. The reception of Leibniz's logic in 19th century German philosophy. In: R. Krömer and Y.-C. Drian (ed.). *New Essays* *in Leibniz Reception: In Science and Philosophy of Science* 1800-2000. DOI 10.1007/978-3-0346-0503-8_2

________. 2009. The mathematical origins of nineteenth-century algebra of logic. In: *The development of modern logic.* Edited by Leila Haaparanta. Oxford: Oxford University Press, 2009, 159-195.

________. 2004. Calculus ratiocinator versus characteristica universalis? The two traditions in logic, revisited. In: *History and Philosophy of Logic*, vol. 25, n. 1 (February 2004), 3-14. Available at: http://dx.doi.org/10.1080/01445340310001609315. Access date: 25/07/2017.

________. 1999. 19th century logic between philosophy and mathematics. In: *The Bulletin of Symbolic Logic*, vol. 5, nº. 4 (Dec. 1999), 433-450.

Peirce, Charles S. (org.). 1883. *Studies in logic by members of the Johns Hopkins University.* [Org.] by C. S. Peirce. Boston: Little, Brown, and Company.

________. *Collected Papers of Charles Sanders Peirce.* Ed. by: C. Hartshorne & P. Weiss (vols. 1-6); A. Burks (vols. 7-8). Cambridge, MA: Harvard University Press, 1931-1958. 8 vols. Quoted as CP, followed by paragraph number.

________. *The Essential Peirce:* Selected Philosophical Writings. Ed. by: N. Houser & C. Kloesel (vol. 1: 1867-1893); "Peirce Edition Project" (vol. 2: 1893-1913). Bloomington and Indianapolis: Indiana University Press, 1992; 1998. Quoted as **EP**, followed by volume and page numbers.

________. *The New Elements of Mathematics.* Ed. by Carolyn Eisele. The Hague; Paris: Mouton Publishers; Atlantic Highlands, NJ: Humanities Press, 1976, 4 vols. in 5. Quoted as **NEM**, followed by volume and page numbers.

________. *Writings of Charles Sanders Peirce:* A Chronological Edition. Ed. by "The Peirce Edition Project". Bloomington; Indianapolis: Indiana University Press, 1982-2000. 7 (1-6; 8) vols. Quoted as **W**, followed by volume and page numbers.

Pietarinen, Ahti-Veikko. 2005. Cultivating habits of reason: Peirce and the logica utens versus logica docens distinction. In: *History of philosophy quarterly*, vol. 22, nº. 4 (Oct., 2005), 357-372.

________. 2007. *Peirce's magic lantern of logic: moving pictures of thought.* In: [http://www.helsinki.fi/science/commens/papers/magiclantern.pdf]. Access date 02/08/2016.

Polcino Milies, César, and Sudarshan Sehgal. 2002. *An Introduction to Group Rings.* Dordrecht, The Netherlands: Springer Netherlands.

Putnam, Hilary. 1982. Peirce the logician. In: *Historia Mathematica 9*, pp. 290-301.

Queiroz, João and Lafayette de Moraes (eds.). 2013. *A lógica de diagramas de Charles Sanders Peirce: Implicações em ciência cognitiva, lógica e semiótica.* Juiz de Fora, MG: UFJF.

________. 2001. Grafos existenciais de C. S. Peirce: uma introdução ao sistema alfa. In: *Cognitio: Revista de Filosofia*, nº 2, pp. 112-133.

________. 2004. Introdução ao sistema beta dos grafos existenciais de C. S. Peirce. In: *Cognitio: Revista de Filosofia*, vol. 5, nº 1, pp. 28-43

Roberts, Don D. 1973. *The existential graphs of Charles S. Peirce.* The Hague: Mouton Publishers.

Rodrigues, Cassiano T. 2007. Matemática como ciência mais geral: forma da experiência e categorias. In: *Cognitio-estudos: revista eletrônica de filosofia*, vol. 4, nº. 1 (2007), 37-59.

________. 2011. The method of scientific discovery in Peirce's philosophy: deduction, induction, and abduction. In: *Logica Universalis*, vol. 5, nº. 1, 127-164.

________. 2017. The Vagueness of the Muse: Peirce's Humble Argument for the Reality of God. In: *Sophia: International Journal of Philosophy and Traditions*, vol. 56, nº. 2 (2017), 163-182.

Roque, Tatiana. 2012. *História da matemática: uma visão crítica, desfazendo mitos e lendas.* Rio de Janeiro: Zahar.

Santaella, Lucia. 1995. *Teoria geral dos signos: semiose e autogeração.* São Paulo: Ática.

Savan, David. 1988. *An introduction to Peirce's full system of semeiotic.* 2nd ed. Toronto: Toronto University Press.

Shields, Paul. 1981. *Charles S. Peirce on the logic of number.* Boston, MA: Docent Press, 2012.

Short, Thomas L. 2007. *Peirce's Theory of Signs.* Cambridge, MA: Cambridge University Press.

Silveira, Lauro Frederico B. da. 2007. *Curso de semiótica geral.* São Paulo: Quartier Latin.

Skagestad, Peter. 2004. Peirce's semeiotic model of the mind. In: *The Cambridge companion to Peirce.* Edited by Cheryl Misak. Cambridge, UK: The Press Syndicate of the Cambridge University Press, 2004, 241-256.

Sluga, Hans. Frege against the Booleans. 1987. In: *Notre Dame Journal of Formal Logic*, vol. 28, nº 1, Jan. 1987, pp. 80-98.

Tarski, Alfred. 1975. *Conferências na UNICAMP em 1975 / Lectures at UNICAMP in 1975.* Transcrição e organização por Leandro Suguitani, Jorge Petrucio Viana, Ítala M. Loffredo D'Ottaviano. Campinas, SP: Editora da UNICAMP, 2016.

Tiercelin, Claudine. 1991. Peirce's semiotic version of the semantic tradition in formal logic. In: Neil Cooper and Pascal Engel (ed.). *New Inquiries into Meaning and Truth.* Brighton, UK: Harvest Wheatsheaf; New York: St. Martin's Press, 1991, 187–210.

________. 2010. Peirce on mathematical objects and mathematical objectivity. In: *New Essays on Peirce's Mathematical Philosophy.* Edited by Matthew E. Moore. Chicago: Open Court, 2010, 81-122.

Van Heijenoort, Jean. 1967. Logic as calculus and logic as language. In: *Synthese*, vol. 17, nº. 3 (1967), 324-330.

Walsh, Alison. 1999. *Relations between logic and mathematics in the work of Benjamin and Charles S. Peirce.* Boston, MA: Docent Press, 2012.

Whately, Richard. 1849. *Logic.* 2^{nd} ed., revised. Londres: John Joseph Griffin & Co.; Glasgow: Richard Griffin & Co.

Zeman, J. Jay. 1986. Peirce's philosophy of logic. In: *Transactions of the Charles S. Peirce Society*, vol. 22, nº 1 (Winter, 1986), 1-22.

________. 1997. Peirce and Philo. In: *Studies in the Logic of Charles Sanders Peirce.* Edited by Nathan Houser, Don D. Roberts, and James van Evra. Bloomington and Indianapolis: Indiana University Press, 1997, 402-417.

Cassiano Terra Rodrigues
Department of Humanities – Division of Fundamental Sciences (IEFH)
Technological Institute of Aeronautics (ITA), Brazil
E-mail: casster@ita.br