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The Meaning, Value and Scope of Lindenbaum-Asser Theorem

J.-Y. Béziau*

Abstract

There are various versions of the Lindenbaum extension lemma, one of which is the Lindenbaum–Asser theorem. The aim of this paper is to show that this theorem is the most fundamental version for proving general completeness. In the case of logics in which there are "excessive theories" that are not maximal, the Lindenbaum–Asser theorem is the only correct tool. This follows from a theorem presented here according to which excessive theories form a minimal semantics. Various versions of the Lindenbaum lemma are studied. At the abstract level, we give a general picture of the relations among them, the completeness theorem, the axiom of choice, and compactness. Finally, we show that the logics in which every excessive theory is maximal are exactly the "maximal" logics.

Remarks on the Origin of the Lindenbaum–Asser Theorem

The subject of the present paper is a theorem known as the Lindenbaum–Asser theorem, or also as the Lindenbaum–Łoś theorem or the Lindenbaum–Łoś–Asser theorem.

What is at issue? Everyone has heard of Lindenbaum's theorem, more often called Lindenbaum's lemma or the Lindenbaum extension lemma. It is called a "lemma" because of its use in proving various completeness theorems.

This result has many versions. We shall present several results explaining why the Lindenbaum–Asser version is the most fundamental one.

Lindenbaum's theorem is due to the Polish logician Adolf Lindenbaum. Little is known about the original presentation of the result or about its author's life and work: even the dates of his birth and death were long uncertain. Information about him was assembled by S. J. Surma [16, 17]. According to Surma [17], Lindenbaum's result was first presented by

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Tarski [19]. It was subsequently used, notably, by Gödel [8] (independently?), Henkin [10], and Łoś [13].

First, it is essential to distinguish the abstract form of the theorem from its various "concrete" versions. By an abstract form we mean one independent of any specific language, formulated at the level of the pure theory of consequence promoted by Tarski [18]. Without a cardinality restriction, the abstract form is equivalent to the axiom of choice, as Dzik [7] showed. Its particular form for classical logic, again without a cardinality restriction, is equivalent to the prime ideal theorem and to the completeness theorem for propositional or first-order logic (see [7]); these principles are, as is well known, strictly weaker than the axiom of choice.

It is difficult to say whether Lindenbaum conceived the most abstract form. In any case, a very general formulation appears only in Asser [2]. The same work also distinguishes two fundamental forms of Lindenbaum's theorem, a distinction that arises both abstractly and in particular cases. Pogorzelski and Wojtylak [14] give numerous results about these theorems.

The present study remains at the purely abstract level. Our terminology differs somewhat from that of "consequence operators." The concept introduced by Asser in the Lindenbaum–Asser theorem is that of a set *complete relative to* something (*vollständig in bezug auf*), translated into English by Polish logicians as "relatively maximal" (see, e.g., [21]). As we shall see, the notion is indeed to be understood in relation to maximal sets. We do not adopt that terminology, however, because it is cumbersome. Since the 1970s the concept has been used in Brazil under the name *saturated set* (see [11, 12, 5]), but this is also potentially confusing because "saturated" has a different established meaning in model theory.

Whatever name is used, the important point is the essential distinction between these sets and their purely maximal counterparts. This is precisely one of our main claims: the notion of a maximal set is generally inadequate, whereas its alter ego takes precedence. Hence the importance of the Lindenbaum–Asser theorem, which governs the latter, while Lindenbaum's theorem concerns only maximal sets.

1 The Lindenbaum–Asser Theorem: A Review

Preliminary concepts

A *logic* is a structure $\mathcal{L} = \langle L, \vdash \rangle$, where $\vdash$ is a relation on $\mathcal{P}(L) \times L$. A *theory* is a subset T of L. A theory is *limited* iff there is an $a \in L$ such that $T \nvdash a$. A theory T is *excessive in* a, or more briefly *a-excessive* (our rendering of *vollständig in bezug auf* a), iff it is a-limited and $T, b \vdash a$ for every $b \notin T$. A theory is *excessive* iff it is a-excessive for some a.

A logic is *normal* iff it satisfies the following three laws:

- self-deducibility $[A]$: $a \vdash a$;
- infinitary monotonicity $[M_\infty]$: $(T \vdash a \text{ and } T \subseteq U) \Rightarrow U \vdash a$;
- infinitary syllogism $[Y_\infty]$: $((T \vdash x \text{ for every } x \in U) \text{ and } U \vdash b) \Rightarrow T \vdash b$.

We also consider:

- deductive compactness $[CD]$: if $T \vdash a$, there is a finite subtheory $T_0 \subseteq T$ such that $T_0 \vdash a$.

Logics satisfying CD are called *deductively compact*.

The theorem

Theorem 1.1 (Lindenbaum–Asser)**.** *In a normal deductively compact logic, every a-limited theory can be extended to an a-excessive theory.*

In fact, only monotonicity and deductive compactness are needed.

Theorem 1.2. *In a monotonic deductively compact logic, every a-limited theory can be extended to an a-excessive theory.*

Proof. Let T be a-limited, and let $\mathrm{EXTLIM}(a)[T]$ be the set of a-limited extensions of T. This set is closed under unions of chains. Indeed, let U be the union of a chain $C \subseteq \mathrm{EXTLIM}(a)[T]$, and suppose $U \vdash a$. By deductive compactness there is a finite subtheory $U_0 \subseteq U$ such that $U_0 \vdash a$. Since C is a chain, some $V \in C$ contains U_0; by monotonicity, $V \vdash a$, contradicting $V \in \mathrm{EXTLIM}(a)[T]$.

By Zorn's lemma, $\mathrm{EXTLIM}(a)[T]$ has a maximal element M. If $b \notin M$ and $M \cup \{b\} \nvdash a$, then $M \cup \{b\}$ belongs to $\mathrm{EXTLIM}(a)[T]$ and is a proper extension of M, contradicting maximality. Hence M is a-excessive. □

Dzik [7] showed that the Lindenbaum–Asser theorem is equivalent to the axiom of choice. Its simplified form is therefore equivalent to it as well.

A corollary: completeness

A *bivalent semantics* for $\mathcal{L} = \langle L, \vdash \rangle$ is a set BIV of functions from L to $\{0, 1\}$. A class of theories can be identified with a bivalent semantics by using characteristic functions. The logic $\langle L, \vdash_{\mathrm{BIV}} \rangle$ induced by BIV is defined by

$$T \vdash_{\mathrm{BIV}} a \quad \Longleftrightarrow \quad \forall \beta \in \mathrm{BIV}\,(\beta(T) = 1 \Rightarrow \beta(a) = 1),$$

where $\beta(T) = 1$ means $\beta(x) = 1$ for every $x \in T$. The semantics is *complete* iff $T \vdash_{\mathrm{BIV}} a$ implies $T \vdash_{\mathcal{L}} a$.

Corollary 1.3 (Completeness of EXC)**.** *For every normal deductively compact logic, the class of excessive theories forms a complete semantics:*

$$\mathcal{L} \textit{ normal and deductively compact} \quad \Rightarrow \quad (T \vdash_{\mathrm{EXC}} a \Rightarrow T \vdash_{\mathcal{L}} a).$$

Proof. Suppose $T \nvdash_{\mathcal{L}} a$, so that T is a-limited. By the Lindenbaum–Asser theorem, T has an excessive extension T' that is a-limited. Its characteristic function assigns 1 to every member of T and 0 to a (if $a \in T'$, self-deducibility would give $T' \vdash a$). Therefore $T \nvdash_{\mathrm{EXC}} a$. □

2 The Minimality Theorem

A semantics BIV is *adequate* for a logic $\mathcal{L}$ if it is both complete and sound, where soundness means $T \vdash_{\mathcal{L}} a \Rightarrow T \vdash_{\mathrm{BIV}} a$. A theory is *closed* when it contains everything deducible from it: $T \vdash_{\mathcal{L}} a \Rightarrow a \in T$. For any normal logic, the class of closed theories is a sound semantics, as is every reduction of it. Since every excessive theory is closed, EXC is sound for every normal logic.

Theorem 2.1 (Minimality of EXC)**.** *Let $\mathcal{L}$ be normal and deductively compact. If a semantics is a proper subset of* EXC*, it is not complete:*

$$\text{BIV} \subsetneq \text{EXC} \quad \Rightarrow \quad \exists T \exists a \, (T \nvdash_{\mathcal{L}} a \text{ and } T \vdash_{\text{BIV}} a).$$

Lemma 2.2. *Let* BIV *be adequate for $\mathcal{L}$. If, for every $\beta \in$* BIV*, there are T and a such that β is the unique member of* BIV *satisfying $\beta(T) = 1$ and $\beta(a) = 0$, then* BIV *is minimal among the complete semantics for $\mathcal{L}$.*

Proof. Let $\text{BIV}_2 \subsetneq \text{BIV}_1 = \text{BIV}$ and choose $\beta \in \text{BIV}_1 \setminus \text{BIV}_2$. Let T, a isolate β as in the hypothesis. Then $T \nvdash_{\text{BIV}_1} a$ but $T \vdash_{\text{BIV}_2} a$. By adequacy, $T \nvdash_{\mathcal{L}} a$, so BIV_2 is incomplete. □

The converse also holds.

Lemma 2.3. *An a-excessive theory has no proper a-excessive extension.*

Proof. Let T be a-excessive and suppose a proper extension T' is also a-excessive. Choose $b \in T' \setminus T$. Then $T, b \vdash a$, and since $T \cup \{b\} \subseteq T'$, monotonicity yields $T' \vdash a$, a contradiction. □

Proof of the minimality theorem. Let β_T be the characteristic function of an a-excessive theory T. Then $\beta_T(T) = 1$ and $\beta_T(a) = 0$. If a distinct characteristic function $\beta_{T'}$ of an excessive theory T' also satisfied $\beta_{T'}(T) = 1$ and $\beta_{T'}(a) = 0$, then T' would be a proper a-excessive extension of T, contrary to the preceding lemma. The first lemma now establishes the minimality of EXC. □

3 Meaning and Consequences of the Minimality Theorem

Lindenbaum's theorem

In classical logic, Lindenbaum's theorem says that every consistent theory has a maximal consistent extension. Abstractly, a theory is *maximal* when it is maximal in the inclusion-ordered class of limited theories; negation plays no role in this definition.

Consider another compactness law, equivalent to the first in classical logic but not in general:

- limiting compactness [CL]: if every finite subtheory of T is limited, then T is limited.

This must not be confused with the contrapositive of CD: if all finite subtheories of T are a-limited, then T is a-limited. A logic satisfying CL is *limitatively compact.*

Theorem 3.1 (Lindenbaum)**.** *In a normal limitatively compact logic, every limited theory can be extended to a maximal theory.*

Again the result can be simplified: in every monotonic limitatively compact logic, each limited theory has a maximal extension.

Proof. Let EXTLIM[T] be the set of limited extensions of a limited theory T. If the union U of a chain $C \subseteq$ EXTLIM[T] were unlimited, CL would yield a finite unlimited $U_0 \subseteq U$. Some $V \in C$ contains U_0; monotonicity would make V unlimited, a contradiction. Thus EXTLIM[T] is closed under unions of chains and has a maximal element by Zorn's lemma. Such an element is a maximal theory, since any proper limited extension would contradict its maximality in EXTLIM[T]. □

Lindenbaum's theorem is also equivalent to the axiom of choice [7]. Thus the Lindenbaum and Lindenbaum–Asser theorems are equivalent as set-theoretic principles, but this must not obscure their very different meanings.

Three essential differences

In a normal logic, every maximal theory is excessive; indeed, a theory is maximal iff it is a-excessive for every a not belonging to it. If some excessive theories are not maximal, the minimality theorem shows that the semantics MAX formed by characteristic functions of maximal theories is incomplete. This happens in many logics, including intuitionistic and relevant logic (the latter observation was communicated to us by R. Sylvan).

Second, the compactness normally available for a logic induced by ordinary finitary rules and proofs is deductive compactness. There is no general guarantee of limiting compactness, and CD cannot in general replace CL in a proof of Lindenbaum's theorem. Thus even when EXC = MAX, the Lindenbaum–Asser theorem is the appropriate result.

Third, suppose exceptionally that Lindenbaum's theorem can be proved from CD and that EXC = MAX. Completeness still requires not merely that T have a maximal extension, but that for every a for which T is a-limited, it have a maximal a-limited extension (an *a-maximal theory*). These requirements differ.

More generally, let SCH be any class of closed theories of $\mathcal{L}$ (call them "Smurf theories"). The following are not equivalent:

[**LES**] If T is a-limited, it has an a-limited Smurf extension.

[**LES#**] If T is limited, it has a limited Smurf extension.

The contrapositive of LES is

$$T \vdash_{\mathcal{L}} a \quad \Longleftarrow \quad a \in \bigcap \mathrm{EXTSCH}[T],$$

so LES expresses exactly the completeness of SCH. In general, $LES\#$ is weaker and does not guarantee completeness. In particular, the equivalence fails when "Smurf" means maximal or excessive.

4 Lindenbaumology

The four Lindenbaum laws

There are four distinct, non-equivalent versions of the Lindenbaum law:

[**LEA#**] Weak Lindenbaum–Asser law: every limited theory can be extended to an excessive theory.

[**LEA**] Lindenbaum–Asser law: every a-limited theory can be extended to an a-excessive theory.

[**LE#**] Lindenbaum law: every limited theory can be extended to a maximal theory.

[**LE**] Strong Lindenbaum law: every a-limited theory can be extended to an a-maximal theory.

A logic satisfying LEA is called a Lindenbaum–Asser logic, and similarly for the other laws. A logic is *finite* $[FIN]$ iff its domain is finite, and *compact* $[C]$ iff it is both deductively and limitatively compact.

The main relations among these notions (for normal logics) are summarized by the following implication diagram; no additional non-transitive arrow is valid:

$$\begin{array}{ccccc} & & CD & \Longrightarrow & LEA \\ & & \Downarrow & & \Downarrow \\ FIN & \Longrightarrow & C & & LEA\# \\ & & \Downarrow & & \\ & & CL & \Longrightarrow & LE\# \\ & & & & \Uparrow \\ & & & & LE \end{array}$$

$LEA \Rightarrow$ completeness of EXC,
$LE \Rightarrow$ completeness of MAX,
the extension theorems use the axiom of choice.

The four central counterresults are

$$CD \not\Rightarrow LE\#, \qquad CL \not\Rightarrow LEA, \qquad LE \not\Rightarrow C, \qquad FIN \not\Rightarrow LE.$$

Before proving them, note that A and M_∞ may be replaced in the definition of normal logic by $[A_\infty]$: if $a \in T$, then $T \vdash a$.

Theorem 4.1. *There are normal deductively compact logics that are not Lindenbaum logics:* $CD \not\Rightarrow LE\#$.

Proof. Let $\mathcal{L}_N = \langle \mathbb{N}, \vdash \rangle$, with

$$N \vdash_{\mathcal{L}_N} n \quad \Longleftrightarrow \quad \exists m \in N \ (m \geq n).$$

This logic is plainly normal and deductively compact. Moreover, N is infinite iff it is unlimited, so no infinite theory is maximal. If N is finite, choose a strict upper bound n of N. Then $N \nvdash n$ and $N \cup \{n\} \nvdash n+1$, so N is not maximal either. Thus there are limited theories but no maximal theories. □

Theorem 4.2. *There are normal limitatively compact logics that are not Lindenbaum–Asser logics:* $CL \not\Rightarrow LEA$.

Proof. Let $\mathcal{L}_P = \langle P, \vdash_P \rangle$, where P is the set consisting of 1 together with the prime numbers, and define, for every theory $X \subseteq P$,

$$X \vdash_P 1 \Longleftrightarrow 1 \in X,$$
$$X \vdash_P p \ (p \neq 1) \Longleftrightarrow (\exists p' \in X \ (p' \text{ divides } p)) \text{ or } X \text{ is infinite.}$$

A_∞ is clear. For Y_∞, suppose $X \vdash_P X'$ and $X' \vdash_P p$. If $p \in X'$, then $X \vdash_P p$. If $1 \in X'$, then $1 \in X$, hence $X \vdash_P p$. If X' is infinite, either X is infinite or $1 \in X$; again $X \vdash_P p$. Thus $\mathcal{L}_P$ is normal.

It is limitatively compact: if X is unlimited then $X \vdash_P 1$, hence $1 \in X$, and the finite theory $\{1\}$ is unlimited. But LEA fails. For example, $\varnothing \nvdash_P 13$, yet there is no 13-excessive theory. If $X \nvdash_P 13$, then X is finite and neither 13 nor 1 belongs to X. Choose a prime $p \notin X$, different from 13; then $X \cup \{p\}$ is finite and $X \cup \{p\} \nvdash_P 13$. □

Theorem 4.3. *There are normal strong Lindenbaum logics that are not compact: $LE \not\Rightarrow C$.*

Proof. Consider an infinitary propositional language L with countably many atoms $\mathrm{ATOM} = \{p_1, \ldots, p_n, \ldots\}$, a unary connective $\neg$, and a countably infinitary connective $\bigwedge$. Let KLI contain the bivaluations satisfying

$$\beta(a) = 0 \iff \beta(\neg a) = 1,$$
$$\beta(\bigwedge_i a_i) = 1 \iff \beta(a_i) = 1 \text{ for every } i.$$

(For simplicity, the order of the arguments of $\bigwedge$ is ignored.) The induced logic is normal.

First, $\mathrm{ATOM} \vdash_{\mathrm{KLI}} \bigwedge_i p_i$, but no finite subset of ATOM entails $\bigwedge_i p_i$; hence deductive compactness fails. Second, $\mathrm{ATOM} \cup \{\neg \bigwedge_i p_i\}$ is unlimited, while none of its finite subtheories is; hence limiting compactness fails.

Nevertheless LE holds. If $T \nvdash_{\mathrm{KLI}} a$, choose β with $\beta(T) = 1$ and $\beta(a) = 0$. The theory $T_\beta = \{b : \beta(b) = 1\}$ is a-maximal. It is a-limited; and if $b \notin T_\beta$, then $\neg b \in T_\beta$, so no bivaluation satisfies $T_\beta \cup \{b\}$. Hence $T_\beta \cup \{b\} \vdash_{\mathrm{KLI}} c$ for every c. □

Theorem 4.4. *There are finite normal logics that are not strong Lindenbaum logics: $FIN \not\Rightarrow LE$.*

Proof. Let $\mathcal{L}_0 = \langle \{a, b\}, \vdash \rangle$, where the only nontrivial instances are

$$\{a\} \vdash a, \quad \{b\} \vdash a, \quad \{b\} \vdash b, \quad \{a, b\} \vdash a, \quad \{a, b\} \vdash b.$$

The logic is normal. There is no a-maximal theory: $\varnothing$ is not maximal because $\{a\} \nvdash b$, while $\{a\}$, $\{b\}$, and $\{a, b\}$ are not a-maximal because they are not a-limited. Since $\varnothing$ is a-limited, LE fails. □

Further remarks

Without monotonicity and the appropriate form of compactness, neither extension theorem can be proved.

Theorem 4.5. *Without monotonicity, the Lindenbaum and Lindenbaum–Asser theorems are false.*

Proof. Let $L = \{1, 2, 3, 4\}$ and define

$$N \vdash n \quad \iff \quad \forall m \in N \ (m \geq n).$$

This finite logic is compact but not monotonic: for example, $\{4\} \vdash 2$ while $\{4, 1\} \nvdash 2$. Moreover, weak Lindenbaum–Asser fails. No theory is excessive. If N is limited and $N \nvdash n$, then $n \neq 1$ and some $m \in N$ satisfies $m < n$. For any j, in particular for a suitable $j \notin N$, one still has $N \cup \{j\} \nvdash n$. Thus N is not n-excessive. □

Theorem 4.6. *There are normal logics that are not weak Lindenbaum–Asser logics.*

Proof. Let L be infinite and define

$$T \vdash a \quad \Longleftrightarrow \quad a \in T \text{ or } T \text{ is infinite.}$$

A_∞ holds by definition. If $T \vdash U$ and $U \vdash a$, then either T is infinite, in which case $T \vdash a$, or T is finite. In the latter case every member of U belongs to T, so U is finite and $a \in U \subseteq T$. Thus Y_∞ holds and the logic is normal.

No theory is excessive. If $T \nvdash a$, then T is finite and $a \notin T$. Choose $b \notin T$, $b \neq a$; then $T \cup \{b\}$ remains finite and does not contain a, so $T \cup \{b\} \nvdash a$. □

It follows immediately that some monotonic logics fail the weak LEA law.

Goldblatt's version

A theory is *finitely limited* iff all its finite subtheories are limited. Consider the following abstract adaptation of Goldblatt's formulation [9]:

[LEG] If a theory is finitely limited, it can be extended to a maximal finitely limited theory.

Theorem 4.7 (Lindenbaum–Goldblatt)**.** *The Lindenbaum–Goldblatt law holds in every logic.*

Proof. Let T be finitely limited and EXTFLI[T] the set of its finitely limited extensions. This set is nonempty. If the union U of a chain $C \subseteq$ EXTFLI[T] were not finitely limited, it would have a finite unlimited subtheory U_0. Some $V \in C$ contains U_0, making V not finitely limited, a contradiction. The rest of the proof proceeds as in Lindenbaum's theorem. □

One might think that Lindenbaum's law follows from *LEG*, but the two are equivalent only if a theory is finitely limited iff it is limited. This equivalence holds for monotonic, limitatively compact logics: one direction is exactly CL, and monotonicity gives the other. Since that other direction need not imply monotonicity, the Lindenbaum–Goldblatt theorem may still be useful for nonmonotonic logics satisfying this equivalence.

5 Characterization of Maximal Logics

We now show that the logics in which every excessive theory is maximal are exactly the "maximal" logics, in the following sense.

Extensions and maximal logics

A logic $\mathcal{L}_2 = \langle L, \vdash_2 \rangle$ is an *extension* of $\mathcal{L}_1 = \langle L, \vdash_1 \rangle$ iff $T \vdash_1 a \Rightarrow T \vdash_2 a$. It is a *proper extension* iff for some T, a, $T \nvdash_1 a$ and $T \vdash_2 a$. A logic is *maximal* iff it has no proper nontrivial extension (a trivial logic has $\vdash = \mathcal{P}(L) \times L$). Here "maximal" refers to the partial order on the class of logics over L.

Some simple facts

Theorem 5.1. *In a Lindenbaum–Asser logic, every theory equals the intersection of its excessive extensions:*

$$T = \bigcap \text{EXTEXC}[T].$$

Proof. Left-to-right inclusion is immediate. Conversely, if $a \notin T$, then $T \nvdash a$. By *LEA*, T has an a-excessive extension T', and $a \notin T'$. Hence $a \notin \bigcap \mathrm{EXTEXC}[T]$. □

Theorem 5.2 (D. W. Miller)**.** *For every Lindenbaum–Asser logic,*

$$\mathrm{MAX} = \mathrm{EXC} \quad \Longleftrightarrow \quad T = \bigcap \mathrm{EXTMAX}[T].$$

Proof. This is an immediate corollary of the preceding theorem. □

Theorem 5.3 (Preservation of maximality)**.** *Maximality is preserved under extension of a logic:*

$$(T \vdash_1 a \Rightarrow T \vdash_2 a) \quad \Rightarrow \quad \mathrm{MAX}_1 \subseteq \mathrm{MAX}_2.$$

Proof. If T is not 2-maximal, some $a \notin T$ and b satisfy $T, a \nvdash_2 b$. Then $T, a \nvdash_1 b$, so T is not 1-maximal. □

Consequently, if $\mathcal{L}_2$ extends $\mathcal{L}_1$, then

$$\mathrm{EXTMAX}_1[T] \subseteq \mathrm{EXTMAX}_2[T].$$

The maximality theorem

Theorem 5.4. *Let $\mathcal{L}$ be a Lindenbaum–Asser logic such that* $\mathrm{MAX} \neq \varnothing$*. Then*

$$\mathcal{L} \textit{ is maximal} \quad \Longleftrightarrow \quad \mathrm{MAX} = \mathrm{EXC}.$$

Proof. Suppose first that $\mathrm{EXC}_1 = \mathrm{MAX}_1$ and that $\mathcal{L}_2$ is a proper nontrivial extension of $\mathcal{L}_1$. Then for some T, a, $T \nvdash_1 a$ and $T \vdash_2 a$. Since $T \nvdash_1 a$, we have $a \notin T$, and therefore

$$\begin{aligned} &a \notin \bigcap \mathrm{EXTMAX}_1[T] && \text{(D. W. Miller's theorem)},\\ &a \notin \bigcap \mathrm{EXTMAX}_2[T] && \text{(preservation of maximality)},\\ &T \nvdash_2 a && \text{(because } \mathrm{MAX}_2 \subseteq \mathrm{CLO}_2\text{)}, \end{aligned}$$

contradicting $T \vdash_2 a$. Hence $\mathcal{L}_1$ is maximal.

Conversely, let $\mathcal{L}$ be a Lindenbaum–Asser logic with maximal theories and suppose $\mathrm{MAX} \neq \mathrm{EXC}$, so $\varnothing \subsetneq \mathrm{MAX} \subsetneq \mathrm{EXC}$. By semantic inversion, the logic induced by MAX extends the logic induced by EXC: $T \vdash_{\mathrm{EXC}} a \Rightarrow T \vdash_{\mathrm{MAX}} a$. If this extension were not proper, MAX would also be a complete semantics for $\mathcal{L}$, contradicting the minimality theorem because some excessive theories are not maximal. Hence the logic induced by MAX is a proper extension of $\mathcal{L}$; it is nontrivial because MAX is nonempty. Thus $\mathcal{L}$ is not maximal. □

6 Prospects: The Connection Between the Abstract and the Concrete

We have remained at the abstract level, but many interesting connections exist between this level and particular logics.

One important question concerns the relation between logical operators and the equation $\mathrm{EXC} = \mathrm{MAX}$. We have proved that the equation holds in the presence of classical negation. D. W. Miller and the present author proved the same for classical implication. A forthcoming

paper, "Negation and Excessivity," will present several further results, including the fact that the equation fails in the presence of a Johansson negation that does not obey Curry's law.

In another paper, "Sequents and Bivaluations" (see [4]), we present an application of the Lindenbaum–Asser theorem to sequent calculus. In brief, the theorem and some easy results show that for a normal deductively compact logic $\mathcal{L}$, a semantics is adequate whenever it contains all excessive theories and is contained in the class of closed theories:

$$\mathrm{EXC} \subseteq \mathrm{BIV} \subseteq \mathrm{CLO} \quad \Rightarrow \quad (T \vdash_{\mathcal{L}} a \Longleftrightarrow T \vdash_{\mathrm{BIV}} a).$$

We proved that excessive theories respect the rules of every structurally standard sequent system. Combined with the Lindenbaum–Asser theorem in the form above, this immediately proves the adequacy of many bivalent semantics for logics induced by sequent-calculus rules. We have already applied the method successfully to a broad family of paraconsistent, paracomplete, and non-alethic logics constructed according to da Costa's methods [3].

broad class of logics. That work was connected with the Polish tradition in logic and would become one of the foundations on which I developed the project of universal logic.

Da Costa had visited Poland in the 1970s. At the beginning of the 1980s, the eminent Polish logician Ryszard Wójcicki spent about a year with him in São Paulo. During this period, Wójcicki wrote *Lectures on Propositional Calculi*, published by Ossolineum in Wrocław in 1984. The book offers a synthesis of the abstract approach to logical systems developed in Poland. Wójcicki later published a substantially expanded treatment, *Theory of Logical Calculi: Basic Theory of Consequence Operations* (Kluwer, Dordrecht, 1988), although I still prefer the earlier book. Andréa told me that she had found nothing in this strand of the Polish School concerning decidability, so she decided—if I may put it that way—to concentrate on precisely that question. She later became deeply interested in programming and computers.

In 1992, I decided—quite logically, perhaps—to go to Poland myself for at least a year and learn more about the Polish School. Shortly before my departure, Andréa sold me a Toshiba T1000, then regarded as one of the first genuine laptop computers. It remained my working machine for years. On it I wrote my doctoral dissertation in mathematics, my doctoral dissertation in philosophy, a master's thesis on cinema, a novel, fifteen short stories, and roughly fifty articles.



The Toshiba T1000, a remarkably productive companion.

I wrote many papers in logic on that computer using ChiWriter, an excellent program for typesetting formal proofs—far better, in my view, than cumbersome LaTeX. There was no Windows, only MS-DOS, which I had learned to use as a student in Paris while taking a course in theoretical computer science. My entire doctoral dissertation in mathematical logic was written with those tools.

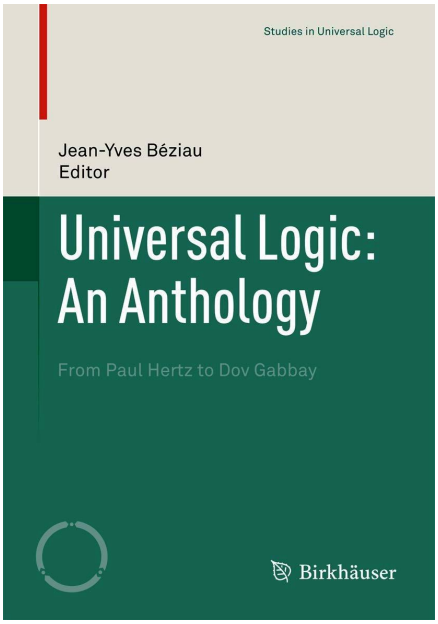
I defended that dissertation in 1995 in the Department of Mathematics at Paris 7, then France's leading centre for logic and one of the world's foremost mathematics departments. Its title was *Recherches sur la logique universelle*. The dissertation contained a genuinely non-trivial theorem—as, after all, a doctorate in mathematics should, rather than merely offering generalisations of distorted versions of existing theorems! To paraphrase a remark attributed to André Weil, a true mathematician is someone who has proved a non-trivial theorem; Weil added that he had proved three.

After the upheavals caused by his refusal of military service during the Second World War, Weil later spent time at USP at the invitation of Edison Farah, who also invited Jean Dieudonné and Alexandre Grothendieck. Farah had supervised Newton da Costa, and da Costa once took me to visit him at home. There Farah told me several fascinating stories from that period.

The main theorem of my dissertation, establishing a connection between bivaluations and sequent rules, was inspired in part by A. Loparic and N. C. A. da Costa's "Paraconsistency, Paracompleteness and Valuations" (*Logique et Analyse* 106, 1984, pp. 119–131). I had studied this paper during my master's work at Paris 7, before coming to Brazil. It presented a general completeness theorem for a wide class of logical systems. Its key notion is that of a "saturated theory," rather than a "maximal theory." I suspect that the terminology reflects its use in intuitionistic logic, where maximal theories do not play

the same role. Andréa was interested in intuitionistic logic; in their joint paper, however, Loparic and da Costa formulated saturation abstractly and independently of any particular language, in the spirit of the general theory of consequence operators.

I later republished their paper in *Universal Logic: An Anthology—From Paul Hertz to Dov Gabbay* (Birkhäuser, 2012). It appeared there alongside an essay of my own introducing the work, "Paralogics and the Theory of Valuation."



Universal Logic: An Anthology (Birkhäuser, 2012).

This led me to investigate more closely the distinction between saturated and maximal theories. I discovered that, in Poland, a saturated theory was called a "relatively maximal theory": it is maximal only with respect to a particular formula. Every proper extension of the theory entails that formula, whereas a maximal theory is relatively maximal with respect to every formula it does not contain. There are consequently two extension theorems: one for maximal theories and another for relatively maximal theories.

I also discovered that the second result is due to Günter Asser (1926–2015). It appeared in his *Einführung in die mathematische Logik. I. Aussagenkalkül* (1959), a book never translated into English, which I found in the excellent library of USP's Department of Mathematics. Asser was a German logician who lived in East Berlin after the Second World War, at a time when East Germany maintained close intellectual ties with Poland. He co-founded the journal *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* and served for many years as its editor-in-chief.

The paper accompanying this reminiscence examines the relationship between the two versions of Lindenbaum's theorem. It shows that, in a logic such as intuitionistic logic, where relatively maximal theories need not be maximal, completeness must be proved using the Lindenbaum–Asser theorem: the original version alone is insufficient.

The original paper was written much earlier but published in 1999, in French, in the Belgian journal *Logique et Analyse*—the same journal in which Newton da Costa and I published "Théorie de la valuation" in 1994, and in which Loparic and da Costa's paper had appeared a decade earlier. Da Costa loved the French language and encouraged me to write in French. I eventually switched to English, however, because nowadays relatively few people understand the language of Brigitte Bardot! I am therefore delighted to have an English version of this paper, prepared with the valuable assistance of Sayantan Roy, currently an assistant editor of *Logica Universalis*, the journal I founded in 2007 with Birkhäuser/Springer.

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Jean-Yves Beziau
Federal University of Rio de Janeiro, Brazil
Email: jyb@ufrj.br