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Valuation Semantics for Some Classical Modal logics

Cezar A. Mortari

Abstract

This paper illustrates an application, to some systems of classical modal logic, of the valuation semantics technique proposed by A. M. Loparić for the basic normal modal logic **K**.

Keywords: valuation semantics, classical modal logics.

Introduction

I first came in contact with valuation semantics in 1980 while studying for my master's degree at the University of Campinas (Unicamp). Andréa Loparić was my dissertation's adviser, and she has just presented valuation semantics for the basic normal modal logic **K** (see [2]). Given my interest at the time in temporal logics, we discussed the application of valuation semantics to these logics. We adapted Loparić's valuation semantics to the minimal tense logic $\mathbf{K_t}$ (see [3]) and, in my master's dissertation ([5]), I extended this result to some other temporal logics. I touched the subject of valuation semantics again in part of my PhD thesis ([6]), working with normal modal logics and solving some problems left open in my master's dissertation. Finally, Loparić and I returned to valuation semantics in [4], defining valuations for **S4**, a problem also left open in my dissertation, a result that can be easily adapted to some other normal logics like **K5** and **KB4**.

In all these works, we considered *normal* modal and temporal logics. In what follows, I will show how to provide a valuation semantics for some *classical* modal logics, modifying and extending some results presented in [6].

We will work in a basic modal language consisting of a countable set of propositional variables (for which we will use p, q etc.), the propositional constant $\top$, and the primitive operators $\neg$, $\rightarrow$ and $\Box$. Formulas and subformulas are defined in the usual way, and so are the standard operators $\vee$, $\wedge$, $\leftrightarrow$, $\Diamond$, and the constant $\bot$. We will also use A, B etc. as meta-variables for formulas,

and capital Greek letters for sets of formulas. '$\mathcal{F}$' will stand for the set of all formulas.

All logics considered in this paper will be extensions of classical propositional logic, so the postulates (axiom schemes and rules) include the set PL of all tautologies, as well as modus ponens (MP).

Proof and theorem are defined in the usual way. Where L is a logic, we write '$\vdash_L A$' to indicate that A is a theorem of L, or just '$\vdash A$' if the context is clear. We will also use the following definition of syntactic consequence:

Definition 0.1 $\Gamma \vdash_L A$ *(*A *is a* syntactic consequence *of the set* Γ *in* L*) iff for some finite subset* $\{B_1, \ldots, B_n\}$ *of* Γ, $\vdash_L (B_1 \wedge \ldots \wedge B_n) \rightarrow A$.

A modal logic is said to be *classical* if it has the following rule:

$$\text{RE.} \quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

The minimal classical modal logic, known as **E**, can be axiomatized by

PL. A, if A is an instance of a tautology,

and the inference rules MP (modus ponens) and RE. As we know, **E** has no theorems of the form $\Box A$; as examples of **E**-theorems we have $\Box p \leftrightarrow \neg\neg\Box p$ (which is an instance of a tautology) and $\Box\Box p \leftrightarrow \Box\neg\neg\Box p$ (which is obtained from the previous theorem by applying RE).

We can extend **E** by means of several different axioms schemes. Three well-known ones (see, for instance, [1]), are the following:

M. $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

C. $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

N. $\Box\top$

Combinations of these schemes give us the eight usual systems of classical modal logic, depicted in Fig. 1 below, where **E**, on the bottom, is the weakest one, and **EMCN**, on the top, the strongest — in fact, **EMCN** is minimal normal modal logic **K**.

The role played by axiom schemes M, C, and N can also be played by inference rules. So we have:

$$\text{RM.} \quad \frac{A \rightarrow B}{\Box A \rightarrow \Box B}$$

$$\text{RC.} \quad \frac{(A \wedge B) \leftrightarrow C}{(\Box A \wedge \Box B) \rightarrow \Box C}$$

$$\text{RN.} \quad \frac{A}{\Box A}$$

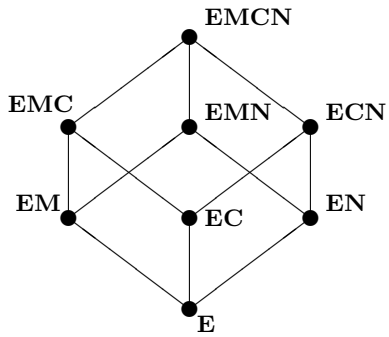


Figure 1: Some classical modal logics

$$\text{RR.}\quad \frac{(A \wedge B) \to C}{(\Box A \wedge \Box B) \to \Box C}$$

$$\text{RK.}\quad \frac{(A_1 \wedge \ldots \wedge A_n) \to B}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box B}, \text{ for } n \geq 0.$$

(Regarding RK, if $n = 0$ then $(A_1 \wedge \ldots \wedge A_n) \to B$ is just B, and RK just RN.)

We can thus show the following equivalences.

Theorem 0.2 *Let L be a logic. Then:*

(a) L has RM *iff it has* RE *and* M*;*

(b) L has RC *iff it has* RE *and* C*;*

(c) L has RN *iff it has* RE *and* N*;*

(d) L has RR *iff it has* RE*,* M*, and* C*;*

(e) L has RK *iff it has* RE*,* M*,* C *and* N*.*

Proof. We show (b) as an illustration; proof of the other results can be found in [1]. So suppose L has RC. Since $(A \wedge B) \leftrightarrow (A \wedge B)$ is a tautology and so a theorem, one application of RC gives us $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, so we have C. To show that L has RE, suppose that $A \leftrightarrow B$ is a theorem. So both $(A \wedge A) \leftrightarrow B$ and $(B \wedge B) \leftrightarrow A$ follow by propositional logic. Two uses of RC give us $(\Box A \wedge \Box A) \to \Box B$ and $(\Box B \wedge \Box B) \to \Box A$, which are equivalent, respectively, to $\Box A \to \Box B$ and $\Box B \to \Box A$. Therefore, by propositional logic, $\Box A \leftrightarrow \Box B$ is a theorem.

For the other direction, suppose L has RE and C as postulates, and that $(A \wedge B) \leftrightarrow C$ is a theorem. Applying RE we obtain $\Box(A \wedge B) \leftrightarrow \Box C$ From this and C, that is, $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, we obtain $(\Box A \wedge \Box B) \to \Box C$ by

propositional logic. ■

1 Defining valuations

Following Loparić's strategy, we will start defining valuations for so-called Σ-sequences of formulas, then show how to extend them to longer Σ-sequences, and finally define valuations. Doing things this way ensures us that we will easily obtain a decision method for the logics in question based on *valuation tables* (also known as generalized truth-tables).

Definition 1.1 *$A_1, \ldots, A_n$ is a* Σ-sequence *if, for $1 \leq i \leq n$, (a) if B is a subformula of A_i, there is a $j \leq i$ such that $B = A_j$; (b) for $1 \leq i \leq j$, if $A_i = A_j$ then $i = j$.*

Definition 1.2 *Let Γ be a set of formulas and f a function from a set Δ, containing Γ, into $\{0, 1\}$:*

(a) $\Gamma^{\Box} =_{df} \{A \in \Gamma :$ *for some* $B, A = \Box B\}$*;*

(b) $\varepsilon(\Gamma^{\Box}) =_{df} \{A : \Box A \in \Gamma\}$*;*

(c) $f \Vdash \Gamma =_{df}$ *for every* $A \in \Gamma$, $f(A) = 1$*;*

(d) $\Gamma_{f,1} =_{df} \{A \in \Gamma : f(A) = 1\}$.

Definition 1.3 *A semi-valuation s is a function from $\mathcal{F}$ into $\{0, 1\}$ such that:*

(a) $s(\top) = 1$*;*

(b) $s(\neg A) = 1$ *iff* $s(A) = 0$*;*

(c) $s(A \to B) = 1$ *iff* $s(A) = 0$ *or* $s(B) = 1$.

The reader may have noticed that semi-valuations are precisely valuations for the classical propositional logic **PL**. What we need to do is to add clauses dealing with the modal operator $\Box$.

In order to render our notation more concise, let us introduce some conventions. Where $A_1, \ldots, A_k$ ia a Σ-sequence, let us abbreviate this by writing $\vec{A}_k$. Hence, $\vec{A}_{n-1}$ abbreviates $A_1, \ldots, A_{n-1}$; $\vec{B}_j$ abbreviates $B_1, \ldots, B_j$, and so on. By $\{\vec{A}_k\}$ we will understand the set consisting of all formulas in the sequence $A_1, \ldots, A_k$.

To define a valuation for a Σ-sequence $\vec{A}_n$ — an $\vec{A}_n$-valuation — we need to specify the conditions governing the behavior of formulas whose main operator is a $\Box$. Let us take **E** as an initial example, and say $\vec{A}_n$ is some Σ-sequence such that $A_n = \Box A_m$, for some $m < n$. Suppose we have defined a valuation

up to $n-1$. What must happen if $v(A_n) = 0$? Well, if $v(\Box A_m) = 0$, then there cannot be any $\Box A_k$ occurring previously to A_n in the sequence, such that $v(\Box A_k) = 1$ and A_k and A_m have the same value in all A_{n-1} valuations.

For example, let $A_n = \Box\neg\neg p$, and suppose, for some $k < n$, that $A_k = \Box p$. Then, if $v(\Box\neg\neg p) = 0$, we cannot have $v(\Box p) = 1$, because then RE would fail to preserve validity. Thus, in general, if $v(\Box A_m) = 0$, the set of all formulas A equivalent up to $n-1$ to A_m, and such that $v(\Box A) = 1$, must be empty.

But let us state all this more precisely. It will simplify things if we can use some abbreviations in defining $\vec{A}_n$-valuations. Definitions will be inductive, so for any Σ-sequence $\vec{A}_n$, and $k \leq n$, $\vec{A}_{k-1}$-valuations will be already defined by the time we define an $\vec{A}_k$-valuation.

Definition 1.4 *Let $\vec{A}_n$ be a Σ-sequence. For $1 \leq k \leq n$, and formulas A and B occurring in $\{\vec{A}_k\}$, and $\Gamma \cup \{A, B\} \subseteq \{\vec{A}_k\}$:*

(a) $A \Leftrightarrow_k B$ *iff for every $\vec{A}_k$-valuation v, $v(A) = v(B)$;*

(b) $A \Rightarrow_k B$ *iff for every $\vec{A}_k$-valuation v, if $v(A) = 1$ then $v(B) = 1$;*

(c) $\Gamma \equiv_k A$ *iff for every $\vec{A}_k$-valuation v, $v \Vdash \Gamma$ sse $v(A) = 1$;*

(d) $\Gamma \Rrightarrow_k A$ *iff for every $\vec{A}_k$-valuation v, if $v \Vdash \Gamma$ then $v(A) = 1$.*

Definition 1.5 *Let $\vec{A}_n$ be a Σ-sequence. Then, for $1 \leq k \leq n$, formulas A and B occurring in $\{\vec{A}_k\}$, and $\Gamma \subseteq \{\vec{A}_k\}$:*

(a) for **E**, **EN**: $\chi^k[A, \Gamma] =_{df} \{B \in \Gamma : B \Leftrightarrow_k A\}$;

(b) for **EM**, **EMN**: $\chi^k[A, \Gamma] =_{df} \{B \in \Gamma : B \Rightarrow_k A\}$;

(c) for **EC**, **ECN**: $\chi^k[A, \Gamma] =_{df} \{\Theta \subseteq \Gamma : \Theta \equiv_k A\}$;

(d) for **EMC**, **EMCN**: $\chi^k[A, \Gamma] =_{df} \{\Theta \subseteq \Gamma : \Theta \Rrightarrow_k A\}$.

Notice that '$\chi^k[A, \Gamma]$' has a different meaning depending on the logic in question. For logics without axiom schema C, $\chi^k[A, \Gamma]$ is a set of formulas, whereas for logics with C, $\chi^k[A, \Gamma]$ is a set of sets of formulas.

We are now ready to define $\vec{A}_n$-valuations. In what follows we will tacitly understand by "a logic L" any of the logics in Fig. 1.

Definition 1.6 *v is an $\vec{A}_n$-valuation for L if $\vec{A}_n$ is a Σ-sequence and:*

(1) $n = 1$ and v is a semi-valuation;

(2) $n > 1$, v is an $\vec{A}_{n-1}$-valuation and if, for some $m < n$, $A_n = A_m$,

(I) if $v(A_n) = 0$, then $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$ [for logics with N: and there is an $\vec{A}_{n-1}$-valuation v_n such that $v_n(A_m) = 0$];

(II) if $v(A_n) = 1$ *then for every* p, *every* q, $q < p < n$ *such that* $A_p = \Box A_q$ *and* $v(A_p) = 0$, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}] = \emptyset$ *[*for logics with *N: and there is an* $\vec{A}_{n-1}$*-valuation* v_p *such that* $v_p(A_q) = 0$*].*

Let us examine this definition, taking **E** as an example. Clause (I) states that, if $v(\Box A_m) = 0$, the set $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})]$ must be empty. This is the set of all formulas B in $\{\vec{A}_{n-1}\}$ such that $v(\Box B) = 1$ and $B \Leftrightarrow_{n-1} A_m$, which means that, for every $\vec{A}_{n-1}$-valuation v', $v'(B) = v'(A_m)$, so $\Box B$ and $\Box A_m$ cannot have different values in v, or RE will fail.

Clause (II), where $v(\Box A_m) = 1$, requires that v keeps a good behavior up to n: we will not find a formula $\Box A_q$ in the sequence such that $v(\Box A_q) = 0$ and $A_q \Leftrightarrow_{n-1} A_m$. If there were some such formula, we could not have $v(\Box A_m) = 1$.

Similar considerations apply for **EM**, **EC** and **EMC**. For the other logics, those having N, we add the requirement "and there is an $\vec{A}_{n-1}$-valuation v_n such that $v_n(A_m) = 0$". This ensures that, if some formula A is valid, $\Box A$ will not receive the value 0.

Definition 1.7 *Let* L *be a logic.*

(a) A function v *from* $\mathcal{F}$ *into* $\{0, 1\}$ *is a* valuation *for* L *iff for every* Σ*-sequence* $\vec{A}_n$, v *is an* $\vec{A}_n$*-valuation for* L*;*

(b) $\models_L A$ *(*A *is* L-valid*) iff, for every valuation* v *for* L, $v(A) = 1$*;*

(c) $\Gamma \models_L A$ *(*A *is a* logical consequence *of the set* Γ *in* L*) iff, for every valuation* v *for* L, *if* $v \Vdash \Gamma$, *then* $v(A) = 1$.

Having defined $\vec{A}_n$-valuations, we must see how to extend them to longer sequences.

Definition 1.8 *Let* v *be an* $\vec{A}_{n-1}$*-valuation and* $\vec{A}_n$ *a* Σ*-sequence. We say that* $\overline{v}$ *is the* canonical extension *of* v *to* $\vec{A}_n$ *if*

A) for every $m < n$, $A_n \neq \Box A_m$ *and* $\overline{v} = v$*;*

B) for some $m < n$, $A_n = \Box A_m$ *and* $\overline{v}$ *is a function from* $\mathcal{F}$ *into* $\{0, 1\}$ *such that, for every formula* A*,*

1) if A_n *is not a subformula of* A, *then* $\overline{v}(A) = v(A)$*;*

2) if A_n *is a subformula of* A, *then:*

a) for $A = A_n$, $\overline{v}(A) = 0$ *iff* $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$ *[*for logics with *N: and there is an* $\vec{A}_{n-1}$*-valuation* v_n *such that* $v_n(A_m) = 0$*];*

b) for $A = \neg B$, $\overline{v}(A) = 1$ *iff* $\overline{v}(B) = 0$*;*

c) for $A = B \rightarrow C$, $\overline{v}(A) = 1$ *iff* $\overline{v}(B) = 0$ *or* $\overline{v}(C) = 1$*;*

d) for $A = \Box B$, $\overline{v}(A) = v(A)$.

Definition 1.9 *Let* v *be an* $\vec{A}_n$*-valuation. For* $1 \leq k \leq n$*, we say that* v *is* $\Box_0$-$\vec{A}_k$-normal *if, for every* p*, every* q*,* $q < p \leq k$ *such that* $A_p = \Box A_q$ *and* $v(A_p) = 0$*,* $\chi^k[A_q, \varepsilon(\{\vec{A}_k\}^{\Box}_{v,1})] = \emptyset$ *[for logics with N: and there is an* $\vec{A}_k$*-valuation* v_p *such that* $v_p(A_q) = 0$*].*

We now show that the canonical extension of an $\vec{A}_{n-1}$-valuation v is an $\vec{A}_n$-valuation, if v is $\Box_0$-$\vec{A}_{n-1}$-normal. And then we prove that $\vec{A}_n$-valuations are $\Box_0$-$\vec{A}_n$-normal.

Lemma 1.10 *If* v *is an* $\vec{A}_{n-1}$*-valuation and* $\overline{v}$ *is the canonical extension from* v *to* $\vec{A}_n$*, then* $\overline{v}$ *is a semi-valuation.*

Proof. Follows immediately from definition 1.8. ■

Proposition 1.11 *Let* $\vec{A}_n$ *be a* Σ*-sequence,* v *an* $\vec{A}_{n-1}$*-valuation and* $\overline{v}$ *the canonical extension of* v *to* $\vec{A}_n$*. Let us suppose that* v *is* $\Box_0$-$\vec{A}_{n-1}$*-normal. In this case,* $\overline{v}$ *is an* $\vec{A}_n$*-valuation.*

Proof. First of all, $\overline{v}$ is an $\vec{A}_{n-1}$-valuation, because it is a semi-valuation and, by construction, $\overline{v}$ and v agree for $i < n$, that is, for $1 \leq i < n$, $\overline{v}(A_i) = v(A_i)$. Now if, for every $m < n$, $A \neq \Box A_m$, $\overline{v}$ fulfills every condition of Definition 1.6, so it is an $\vec{A}_n$-valuation. Suppose, then, that for some $m < n$, $A_n = \Box A_m$. We have two main cases, and some subcases.

(I) $\overline{v}(A_n) = 0$.

By 1.8.B.2.a, $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$. (For logics with N, additionally, there is an $\vec{A}_{n-1}$-valuation v_n such that $v_n(A_m) = 0$.) Since v and $\overline{v}$ agree for $i < n$, it follows that $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] = \emptyset$. So $\overline{v}$ is an $\vec{A}_n$-valuation.

(II) $\overline{v}(A_n) = 1$. We must consider the different logics separately:

(A) for **E**, **EM**, **EN**, and **EMN**:

By 1.8.B.2.a, we have:

(1) $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] \neq \emptyset$ [in **EN**, **EMN**: either this or, for every $\vec{A}_{n-1}$-valuation v^+, $v^+(A_m) = 1$].

Suppose first that there is $q < p < n$ such that $A_p = \Box A_q$ and $\overline{v}(A_p) = 0$. Then $v(A_p) = 0$ and, since v is $\Box_0$-$\vec{A}_{n-1}$-normal, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$ [**EN**, **EMN**: and there is an $\vec{A}_{n-1}$-valuation v_p such that $v_p(A_q) = 0$]. Since v and $\overline{v}$ agree for $i < n$, we have:

(2) $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] = \emptyset$.

We must now show that $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$. Let us suppose that this set is not empty. From (2) it follows that

(3) $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \{A_m\}$.

(a) In **E** and **EN**, from (3) we have $A_q \Leftrightarrow_{n-1} A_m$, and then we see, from (2) again, that $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$, against (1). In **EN**, since $A_q \Leftrightarrow_{n-1} A_m$ and there is an $\vec{A}_{n-1}$-valuation v_p such that $v_p(A_q) = 0$, we also have (ii) $v_p(A_m) = 0$, and this contradicts the second half of the disjunction in (1), so, in both logics, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$.

(b) In **EM** and **EMN**, from (3) we have $A_m \Rightarrow_{n-1} A_q$. Let us consider $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] \neq \emptyset$ first. So there must be some $\Box A_j$, $j < n$, such that $v(\Box A_j) = 1$ and $A_j \Rightarrow_{n-1} A_m$. It follows, since $A_m \Rightarrow_{n-1} A_q$, that $A_j \Rightarrow_{n-1} A_q$. So $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] \neq \emptyset$, against (2). Now consider, in **EMN**, the second half of the disjunction in (1): for every $\vec{A}_{n-1}$-valuation v^+, $v^+(A_m) = 1$. Since $A_m \Rightarrow_{n-1} A_q$, it follows that for every $\vec{A}_{n-1}$-valuation v^+, $v^+(A_q) = 1$. So $v_p(A_q) = 1$, what cannot be. Hence, in both logics, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$.

Therefore, in case (A), that $\overline{v}$ is an $\vec{A}_n$-valuation.

(B) for **EC**, **ECN**, **EMC**, **EMCN**:

By D12.B.2.a, we have:

(1) $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] \neq \emptyset$ [in **ECN** and **EMCN**: or, for every $\vec{A}_{n-1}$-valuation v^+, $v^+(A_m) = 1$].

Suppose first there is $q < p < n$ such that $A_p = \Box A_q$ and $\overline{v}(A_p) = 0$. Then $v(A_p) = 0$ and, since v is $\Box_0$-$\vec{A}_{n-1}$-normal, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_m\}^{\Box}_{v,1})] = \emptyset$ [**ECN**, **EMCN**: and there is an $\vec{A}_{n-1}$-valuation v_p such that $v_p(A_q) = 0$]. Since v and $\overline{v}$ agree for $i < n$, we have:

(2) $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] = \emptyset$.

We have now to show that $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$. Let us suppose that this set is not empty. From (2) it follows that:

(3) for some $\Gamma \subseteq \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})$, $\Gamma \cup \{A_m\} \in \chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})]$.

By definition, for every $\vec{A}_{n-1}$-valuation v',

(4a) **EC**, **ECN**: $v' \Vdash \Gamma \cup \{A_m\}$ iff $v'(A_q) = 1$,

(4b) **EMC**, **EMCN**: if $v' \Vdash \Gamma \cup \{A_m\}$ then $v'(A_q) = 1$.

And we can rewrite this as:

(5a) **EC**, **ECN**: $v'(A_q) = 1$ iff $v' \Vdash \Gamma$ and $v'(A_m) = 1$

(5b) **EMC**, **EMCN**: if $v' \Vdash \Gamma$ and $v'(A_m) = 1$ then $v'(A_q) = 1$.

Now suppose, for **ECN** and **EMCN**, that the left half of the disjunction in (1) obtains, that is:

(*) $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})] \neq \emptyset$.

In **EC** and **EMC**, of course, this is th only alternative. It follows that:

(6) for some $\Theta \subseteq \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})$, $\Theta \in \chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})]$.

By definition, we have, for every $\vec{A}_{n-1}$-valuation v':

(7a) **EC**, **ECN**: $v' \Vdash \Theta$ iff $v'(A_m) = 1$,

(7b) **EMC**, **EMCN**: if $v' \Vdash \Theta$ then $v'(A_m) = 1$.

From (5a), (7a), (5b), and (7b), we get, for every $\vec{A}_{n-1}$-valuation v':

(8a) **EC**, **ECN**: $v'(A_q) = 1$ iff $v' \Vdash \Gamma \cup \Theta$;

(8b) **EMC**, **EMCN**: if $v' \Vdash \Gamma \cup \Theta$ then $v'(A_q) = 1$.

In both cases it follows that there is $\Gamma \cup \Theta \in \chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})]$, what goes against (*). Thus $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$.

In **ECN** and **EMCN** we still have to consider the other half of the disjunction in (1). So suppose that for every $\vec{A}_{n-1}$-valuation v', $v'(A_m) = 1$. From (5a) and (5b) we have, for every $\vec{A}_{n-1}$-valuation v':

(9a) **ECN**: $v' \Vdash \Gamma$ iff $v'(A_q) = 1$;

(9b) **EMCN**: if $v' \Vdash \Gamma$ then $v'(A_q) = 1$.

But then there is Γ in $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1})]$, against (2). Therefore, also in this case, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{\overline{v},1}) \cup \{A_m\}] = \emptyset$.

Thus, also in case (B), $\overline{v}$ is an $\vec{A}_n$-valuation. ■

Lemma 1.12 *Let v be an $\vec{A}_n$-valuation. Then v is $\Box_0$-$\vec{A}_n$-normal.*

Proof. By induction on n. For $n = 1$ the property holds trivially, so let $n > 1$ and let us suppose that every $\vec{A}_{n-1}$-valuation is $\Box_0$-$\vec{A}_{n-1}$-normal. It follows from 1.11 that

(†) The canonical extensions of $\vec{A}_{n-1}$-valuations are $\vec{A}_n$-valuations.

From this, and from the fact that every $\vec{A}_n$-valuation is an $\vec{A}_{n-1}$-valuation, it follows that, for $i < n$, and $\Gamma \subseteq \{\vec{A}_{n-1}\}$,

(1) $\chi^{n-1}[A_i, \Gamma] = \chi^n[A_i, \Gamma]$.

Let us show this for **EM** as an example. Suppose that for some $B \in \Gamma$, $B \in \chi^{n-1}[A_i, \Gamma]$. Then $B \Rightarrow_{n-1} A_i$; that is, for every $\vec{A}_{n-1}$-valuation v^+, if $v^+(B) = 1$ then $v^+(A_i) = 1$. Now, since every $\vec{A}_n$-valuation is an $\vec{A}_{n-1}$-valuation, for every $\vec{A}_n$-valuation v^+, if $v^+(B) = 1$ then $v^+(A_i) = 1$. Thus $B \Rightarrow_n A_i$, and $B \in \chi^n[A_i, \Gamma]$.

Suppose then that for some $B \in \Gamma$, $B \notin \chi^{n-1}[A_i, \Gamma]$. Then it is not the case that $B \Rightarrow_{n-1} A_i$, so there is some $\vec{A}_{n-1}$-valuation v^* such that $v^*(B) = 1$ and $v^*(A_i) = 0$. Let $\overline{v}^*$ be the canonical extension of v^* to $\vec{A}_n$. From (†), $\overline{v}^*$ is an $\vec{A}_n$-valuation. Given that v^* and $\overline{v}^*$ agree on formulas up to $n-1$, and that $\Gamma \cup \{A_i\} \subseteq \{\vec{A}_{n-1}\}$, it is not the case that $B \Rightarrow_n A_i$, so $B \notin \chi^n[A_i, \Gamma]$.

A similar reasoning establishes (1) for the other logics.

Now, if for every $m < n$, $A_n \neq \Box A_m$, there is nothing more to prove, so let us suppose that, for some $m < n$, $A_n = \Box A_m$.

(I) Let $v(A_n) = 0$. We have:

(2) $\varepsilon(\{\vec{A}_n\}^{\Box}_{v,1}) = \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$.

From the induction hypothesis, v is $\Box_0$-$\vec{A}_{n-1}$-normal, so we have:

(3) for every p, every q, $q < p < n$ such that $A_p = \Box A_q$ and $v(A_p) = 0$, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$; moreover, for logics with N, there is an $\vec{A}_{n-1}$-valuation v_p such that $v_p(A_q) = 0$.

From (1) and (2), we get:

(4) $\chi^n[A_q, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] = \emptyset$.

In **EN**, **EMN**, **ECN**, **EMCN**, for each p, let $\overline{v}_p$ be the canonical extension of v_p to $\vec{A}_n$. Obviously $\overline{v}_p(A_q) = v_p(A_q)$ and, from (†), $\overline{v}_p$ is an $\vec{A}_n$-valuation. From this and (4), then:

(5) for every p, every q, $q < p < n$ such that $A_p = \Box A_q$ and $v(Ap) = 0$, $\chi^n[A_q, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] = \emptyset$; for logics with N, there is an $\vec{A}_n$-valuation $\overline{v}_p$ such that $\overline{v}_p(A_q) = 0$.

On the other hand, since v is an $\vec{A}_n$-valuation, we have by definition:

(6) $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$ and, for logics with N, there is an $\vec{A}_{n-1}$-valuation v_n such that $v_n(A_m) = 0$.

From (1), (2) and the fact that $A_m \in \{\vec{A}_{n-1}\}$, we get:

(7) $\chi^{n}[A_m, \varepsilon(\{\vec{A}_{n}\}^{\Box}_{v,1})] = \emptyset$.

In **EN**, **EMN**, **ECN**, **EMCN**, let $\overline{v}_n$ be the be the canonical extension from v_n to $\vec{A}_n$. Obviously $\overline{v}_p(A_m) = v_p(A_m)$ and, from (†), $\overline{v}_n$ is an $\vec{A}_n$-valuation. Thus we have

(8) for $p = n$, $q = m$, $A_p = \Box A_q$ and $v(A_p) = 0$, $\chi^{n}[A_m, \varepsilon(\{\vec{A}_{q}\}^{\Box}_{v,1})] = \emptyset$; moreover, for logics with N, there is an $\vec{A}_n$-valuation $\overline{v}_p$ such that $\overline{v}_p(A_q) = 0$.

From this, together with (5), v is $\Box_0$-$\vec{A}_n$-normal.

(II) Let $v(A_n) = 1$. we then have

(9) $\varepsilon(\{\vec{A}_{n}\}^{\Box}_{v,1}) = \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}$.

Since $v(A_n) = 1$, we have from 1.6 that:

(10) for every p, every $q < p \leq n$ such that $A_p = \Box A_q$ and $v(A_p) = 0$, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}] = \emptyset$; moreover, for logics with N, there is an $\vec{A}_{n-1}$-valuation v_p such that $v_p(A_q) = 0$.

In **EN**, **EMN**, **ECN**, **EMCN**, for each p, let $\overline{v}_p$ be the canonical extension from v_p to $\vec{A}_n$. Obviously $\overline{v}_p(A_q) = v_p(A_q)$ and, from (†), $\overline{v}_p$ is an $\vec{A}_n$-valuation. From (1), (9) and (10) and from the fact that $A_m \in \{\vec{A}_{n-1}\}$, we get

(11) for every p, every $q < p \leq n$ such that $A_p = \Box A_q$ and $v(A_p) = 0$, $\chi^{n}[A_q, \varepsilon(\{\vec{A}_{n}\}^{\Box}_{v,1}) \cup \{A_m\}] = \emptyset$;

Moreover, in **EN**, **EMN**, **ECN**, **EMCN**, there is an $\vec{A}_n$-valuation $\overline{v}_p$ such that $\overline{v}_p(A_q) = 0$. That is, v is $\Box_0$-$\vec{A}_n$-normal. ■

Corollary 1.13 *Let $\vec{A}_n$ be a Σ-sequence, v an $\vec{A}_{n-1}$-valuation and $\overline{v}$ the canonical extension of v to $\vec{A}_n$. Then $\overline{v}$ is an $\vec{A}_n$-valuation and for $1 \leq i < n$, $\overline{v}(A_i) = v(A_i)$.*

Lemma 1.14 *Let $\vec{A}_n$ be a Σ-sequence, A and B formulas, and Γ a set of formulas such that $\Gamma \cup \{A, B\} \subseteq \{\vec{A}_{n-1}\}$. Then:*

(a) if $A \Leftrightarrow_n B$ *then* $A \Leftrightarrow_{n-1} B$*;*

(b) if $A \Rightarrow_n B$ *then* $A \Rightarrow_{n-1} B$*;*

(c) if $\Gamma \equiv_n A$ *then* $\Gamma \equiv_{n-1} A$*;*

(d) if $\Gamma \Rrightarrow_n A$ *then* $\Gamma \Rrightarrow_{n-1} A$*.*

Proof. We show case (d) as an example. Suppose it is not the case that $\Gamma \Rrightarrow_{n-1} A$. Then there must be an $\vec{A}_{n-1}$-valuation v such that $v \Vdash \Gamma$ but $v(A) = 0$. Let $\overline{v}$ be the canonical extension of v to $\vec{A}_n$. By 1.12 v is $\Box_0$-$\vec{A}_{n-1}$-normal and, by 1.11, $\overline{v}$ is an $\vec{A}_n$-valuation. Since v and $\overline{v}$ agree on the value of formulas in $\vec{A}_{n-1}$, $\overline{v} \Vdash \Gamma$ but $\overline{v}(A) = 0$. Thus it is not the case that $\Gamma \Rrightarrow_n A$. ■

Theorem 1.15 *v is an $\vec{A}_n$-valuation iff: (1) $\vec{A}_n$ is a Σ-sequence; (2) v is a semi-valuation; (3) v is $\Box_0$-$\vec{A}_n$-normal.*

Proof. If v is an $\vec{A}_n$-valuation, the proposition immediately follows from 1.6 and 1.12.

For the other direction, let $\vec{A}_n$ be a Σ-sequence and v a function from $\mathcal{F}$ into $\{0, 1\}$ satisfying (2) and (3). We prove the theorem by induction on n. If $n = 1$, it follows from (2) that v is an A_1-valuation. Let $n > 1$ and suppose that every function from $\mathcal{F}$ into $\{0, 1\}$ that satisfies (2) and is $\Box_0$-$\vec{A}_{n-1}$-normal is an $\vec{A}_{n-1}$-valuation. Since v is $\Box_0$-$\vec{A}_n$-normal, v is $\Box_0$-$\vec{A}_{n-1}$-normal; since v satisfies (2), v is an $\vec{A}_{n-1}$-valuation. Now we have two cases:

1) If, for every $m < n$, $A_n \neq \Box A_m$, there is nothing to prove.

2) For some $m < n$, $A_n = \Box A_m$.

I) $v(A_n) = 0$. Since v is $\Box_0$-$\vec{A}_n$-normal, $\chi^n[A_m, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] = \emptyset$ [and, for logics with N, there is an $\vec{A}_n$-valuation — hence an $\vec{A}_{n-1}$-valuation — v_n such that $v_n(A_m) = 0$].

Given that $v(A_n) = 0$, $\varepsilon(\{\vec{A}_n\}^{\Box}_{v,1}) = \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$, so $\chi^n[A_m, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] = \chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})]$.

It follows that v is an $\vec{A}_n$-valuation.

II) $v(A_n) = 1$. Since v is $\Box_0$-$\vec{A}_n$-normal, for every p, every q, $q < p < n$ such that $A_p = \Box A_q$ and $v(A_p) = 0$, $\chi^n[A_q, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] = \emptyset$ [and, for logics with N, and there is an $\vec{A}_n$-valuation — hence an $\vec{A}_{n-1}$-valuation — v_p such that $v_p(A_q) = 0$].

We need to show that $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}] = \emptyset$. Given that $A_n = \Box A_m$ and $v(A_n) = 1$, $\varepsilon(\{\vec{A}_n\}^{\Box}_{v,1}) = \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}$. So $\chi^n[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup$

$\{A_m\}] = \emptyset$. Since $\{A_q, A_m\} \subseteq \{\vec{A}_{n-1}\}$, from lemma 1.14 it follows that

$$\chi^n[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}] = \chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}].$$

Hence v is an $\vec{A}_n$-valuation. ■

2 Correctness

Lemma 2.1 *Let L be a logic, and v an $\vec{A}_n$-valuation for L. Then, for $1 \leq i \leq n$, if A_i is an axiom of L, then $v(A_i) = 1$.*

Proof. If A is a tautology then it follows from the fact that v is a semi-valuation that $v(A_i) = 1$, so let us consider the case in which A_i is one of the modal axioms.

(a) $A_i = \Box\top$. For logics with n, if $v(\Box\top) = 0$, then there is some $\vec{A}_n$-valuation v' such that $v'(\top) = 0$, what cannot be.

(b) $A_i = (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$. For logics with C, if $v(A_i) = 0$ then $v(\Box A) = v(\Box B) = 1$ and $v(\Box(A \wedge B)) = 0$. Let us take the set $\{A, B\}$. Obviously $\{A, B\} \subseteq \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})$. Now, for every $\vec{A}_n$-valuation v',

EC, **ECN**: $v' \Vdash \{A, B\}$ iff $v'(A \wedge B) = 1$, that is, $\{A, B\} \equiv_n A \wedge B$;

EMC, **EMCN**: if $v' \Vdash \{A, B\}$ then $v'(A \wedge B) = 1$, that is $\{A, B\} \Rrightarrow_n A$.

In both cases, it follows that $\{A, B\} \in \chi^n[A \wedge B, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})]$, but this set should be empty, if $v(\Box(A \wedge B)) = 0$. So $v(A_i) = 1$.

(c) $A_i = \Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$. For logics with M, suppose $v(\Box(A \wedge B)) = 1$. We consider the logics separately.

EM, **EMN**: Clearly $A \wedge B \Rightarrow_n A$ and $A \wedge B \Rightarrow_n B$. Since $A \wedge B \in \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})$, we have that $\chi^n[A, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] \neq \emptyset$ and $\chi^n[B, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})] \neq \emptyset$, so $v(\Box A) = v(\Box B) = 1$. Hence $v(A_i) = 1$.

EMC, **EMCN**: Let us take $\{A \wedge B\}$. Clearly $A \wedge B \in \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})$ and, for every $\vec{A}_n$-valuation v', if $v' \Vdash \{A \wedge B\}$ then $v'(A) = v'(B) = 1$, that is $\{A, B\} \Rrightarrow_n A$, $\{A, B\} \Rrightarrow_n B$. Hence $\{A \wedge B\} \in \chi^n[A, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})]$ and $\{A \wedge B\} \in \chi^n[B, \varepsilon(\{\vec{A}_n\}^{\Box}_{v,1})]$, so $v(\Box A) = v(\Box B) = 1$, and $v(A_i) = 1$. ■

Lemma 2.2 *For every n, every i, $1 \leq i \leq n$, if $\vec{A}_n$ is a Σ-sequence, v is an $\vec{A}_n$-valuation, and $\vdash_L A_i$, then $v(A_i) = 1$.*

Proof. By induction on the number r of lines of a proof of A in L.

A) $r = 1$. Then A_i is an axiom and the property follows from 2.1.

B) $r > 1$. If A_i is an axiom, the property follows from 2.1; else:

(a) If A_i was obtained by MP, the proof is immediate, for every valuation is a semi-valuation.

(b) $A_i = \Box B \leftrightarrow \Box C$ and was obtained by RE from $B \leftrightarrow C$. We have $\vdash_L B \leftrightarrow C$. Obviously, for every Σ-sequence $\vec{A}_m$ in which $\Box B \leftrightarrow \Box C$ occurs, B, C, $\Box B$, and $\Box C$ occur, too, so they occur in $\vec{A}_m$. Let v_m be an $\vec{A}_m$-valuation. Now, if $B \leftrightarrow C$ also occurs in $\vec{A}_m$, let $\vec{A}_n = \vec{A}_m$ and $v_n = v$. If not, let $\vec{A}_n = A_1, \ldots, A_m, A_{m+1}$, where $A_{m+1} = B \leftrightarrow C$. Clearly $\vec{A}_n$ is a Σ-sequence and, in this case, let v_n be the canonical extension of v_m to $\vec{A}_n$. v_n is thus an $\vec{A}_n$-valuation. Since $\vdash_L B \leftrightarrow C$, we have by the induction hypothesis that for every n, if v is an $\vec{A}_n$-valuation, $v(B \leftrightarrow C) = 1$ and so $v(B) = v(C)$. Hence, $v_n(B \leftrightarrow C) = 1$.

Suppose now that $v_n(\Box B \leftrightarrow \Box C) = 0$. Without loss of generality, suppose that $v_n(\Box B) = 1$ and $v_n(\Box C) = 0$. From 1.12, we have:

(†) $\chi^n[C, \varepsilon(\{\vec{A}_n\}^{\Box}_{v_n,1})] = \emptyset$.

However, given that $v(B) = v(C)$ for every $\vec{A}_n$-valuation, we have, for the different logics:

E, **EN**:	$B \Leftrightarrow_n C$	**EM**, **EMN**:	$B \Rightarrow_n C$
EC, **ECN**:	$\{B\} \equiv_n C$	**EMC**, **EMCN**:	$\{B\} \Rrightarrow_n C$

Since $v_n(\Box B) = 1$, it follows, for every logic L, that $\chi^n[C, \varepsilon(\{\vec{A}_{n+1}\}^{\Box}_{v',1})] \neq \emptyset$, against (†). Hence, $v_n(\Box C) = 1$, and so $v_n(\Box B \leftrightarrow \Box C) = 1$. If $v_n = v_m$, we are done. If v_n is the canonical extension of v_m, then $v_m(\Box B \leftrightarrow \Box C) = 1$. ■

Theorem 2.3 (Correctness) *If* $\vdash_L A$ *then* $\models_L A$.

Proof. Suppose $\vdash_L A$, and v be a valuation. Let $\vec{A}_n$ be any Σ-sequence in which, for some $i \leq n$, $A = A_i$. By definition, v is an $\vec{A}_n$-valuation, so $v(A) = 1$ from 2.2, and thus $\models_L A$. ■

Corollary 2.4 *If* $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof. Follows immediately from 0.1 and 2.3. ■

3 Completeness

Completeness is proved using saturated sets.

Definition 3.1 Δ *is an* A-saturated set, *for some formula* A *and logic* L, *if* $\Delta \nvdash_L A$ *and for every formula* B *such that* $B \notin \Delta$, $\Delta \cup \{B\} \vdash_L A$. *We say simply that a set* Δ *is* saturated *(for some logic* L*) if, for some* A, Δ *is* A*-saturated.*

Lemma 3.2 *If* Δ *is a saturated set,*

(a) $A \in \Delta$ *iff* $\Delta \vdash A$;
(b) $\neg A \in \Delta$ *iff* $A \notin \Delta$;
(c) $A \to B \in \Delta$ *iff* $A \notin \Delta$ *or* $B \in \Delta$.

Lemma 3.3 *If* $\Gamma \nvdash A$, *then there is an* A*-saturated set* Δ *such that* $\Gamma \subseteq \Delta$.

Proof. By a standard Lindenbaum argument. ■

Definition 3.4 *Let* A *and* B *be formulas, and* Θ *a finite set of formulas.*

(a) $A \Leftrightarrow_S B$ *iff for every saturated set* Δ, $A \in \Delta$ *iff* $B \in \Delta$;
(b) $A \Rightarrow_S B$ *iff for every saturated set* Δ, *if* $A \in \Delta$ *then* $B \in \Delta$;
(c) $\Theta \equiv_S A$ *iff for every saturated set* Δ, $\Theta \subseteq \Delta$ *iff* $A \in \Delta$;
(d) $\Theta \Rrightarrow_S A$ *iff for every saturated set* Δ, *if* $\Theta \subseteq \Delta$ *then* $A \in \Delta$;

Definition 3.5 *Where* Γ *is a set of formulas, and* Θ *a finite set of formulas, we define:*

(a) for ***E***, ***EN***: $\chi[A, \Gamma] =_{df} \{B \in \Gamma : B \Leftrightarrow_S A\}$;
(b) for ***EM***, ***EMN***: $\chi[A, \Gamma] =_{df} \{B \in \Gamma : B \Rightarrow_S A\}$;
(c) for ***EC***, ***ECN***: $\chi[A, \Gamma] =_{df} \{\Theta \subseteq \Gamma : \Theta \equiv_S A\}$;
(d) for ***EMC***, ***EMCN***: $\chi[A, \Gamma] =_{df} \{\Theta \subseteq \Gamma : \Theta \Rrightarrow_S A\}$.

Lemma 3.6 *Let* Δ *be a saturated set. If* $\Delta \nvdash \Box A$ *then:*

(i) $\chi[A, \varepsilon(\Delta^{\Box})] = \emptyset$;
(ii) for logics with N*: there is an* A*-saturated set* Δ'.

Proof. (i) Suppose $\chi[A, \varepsilon(\Delta^{\Box})] \neq \emptyset$.

E, **EN**: Let $B \in \chi[A, \varepsilon(\Delta^{\Box})]$. So $\Box B \in \Delta$ and, by definition 3.5.(a), $B \Leftrightarrow_S A$. Thus $B \leftrightarrow A$ is a member of every saturated set, $B \leftrightarrow A \in \Delta$, and $\Box B \leftrightarrow \Box A \in \Delta$ by RE. It follows that $\Delta \vdash \Box A$, against the hypothesis of the lemma.

EM, **EMN**: Let $B \in \chi[A, \varepsilon(\Delta^{\Box})]$. So $\Box B \in \Delta$ and, by definition 3.5.(b), $B \Rightarrow_S A$. Thus $B \rightarrow A$ is a member of every saturated set, $B \rightarrow A \in \Delta$, and $\Box B \rightarrow \Box A \in \Delta$ by RM. It follows that $\Delta \vdash \Box A$, against the hypothesis of the lemma.

EC, **ECN**: Let $\Theta = \{\theta_1, \ldots, \theta_k\}$ be some finite set of formulas such that $\Theta \in \chi[A, \varepsilon(\Delta^{\Box})]$. By definition 3.5.(c), $\Theta \equiv_S A$. And since $\Theta \subseteq \varepsilon(\Delta^{\Box})$, $\{\Box\theta_1, \ldots, \Box\theta_k\} \subseteq \Delta$. It follows that $\Box\theta_1 \wedge \ldots \wedge \Box\theta_k \in \Delta$, and that $\Box(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta$. Now, since $\Theta \equiv_S A$, for every saturated set Δ', $A \in \Delta'$ iff $\Theta \subseteq \Delta'$. That is, for every saturated set Δ', $A \in \Delta'$ iff $\{\theta_1, \ldots, \theta_k\} \subseteq \Delta'$. It follows that for every saturated set Δ', $A \in \Delta'$ iff $\theta_1 \wedge \ldots \wedge \theta_k \in \Delta'$. Thus, for every saturated set Δ', $A \leftrightarrow \theta_1 \wedge \ldots \wedge \theta_k \in \Delta'$. By RE, $\Box A \leftrightarrow \Box(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta'$; $\Box A \leftrightarrow \Box(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta$. It follows that $\Delta \vdash \Box A$, against the hypothesis of the lemma.

EMC, **EMCN**: Let $\Theta = \{\theta_1, \ldots, \theta_k\}$ be some finite set of formulas such that $\Theta \in \chi[A, \varepsilon(\Delta^{\Box})]$. By definition 3.5.(d), $\Theta \Rrightarrow_S A$. And since $\Theta \subseteq \varepsilon(\Delta^{\Box})$, $\{\Box\theta_1, \ldots, \Box\theta_k\} \subseteq \Delta$. It follows that $\Box\theta_1 \wedge \ldots \wedge \Box\theta_k \in \Delta$, and that $\Box(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta$. Now, since $\Theta \Rrightarrow_S A$, for every saturated set Δ', if $\Theta \subseteq \Delta'$ then $A \in \Delta'$. That is, for every saturated set Δ', if $\{\theta_1, \ldots, \theta_k\} \subseteq \Delta'$ then $A \in \Delta'$ It follows that for every saturated set Δ', if $\theta_1 \wedge \ldots \wedge \theta_k \in \Delta'$ then $A \in \Delta'$. Thus, for every saturated set Δ', $\theta_1 \wedge \ldots \wedge \theta_k \rightarrow A \in \Delta'$. By RM, $\Box(\theta_1 \wedge \ldots \wedge \theta_k) \rightarrow \Box A \in \Delta'$; $\Box(\theta_1 \wedge \ldots \wedge \theta_k) \rightarrow \Box A \in \Delta$. It follows that $\Delta \vdash \Box A$, against the hypothesis of the lemma.

(iii) By the hypothesis of the lemma, $\Delta \nvdash \Box A$. Suppose there is no A-saturated set Δ'. So, for every Δ', $A \in \Delta'$. Now $\top$ is a theorem; so, for every saturated set Δ', $\Delta' \vdash \top$, $\top \in \Delta'$. Hence $\top \equiv_s A$, and so $\top \leftrightarrow A$, and by RE $\Box\top \leftrightarrow \Box A$, belong to every saturated set Δ'. Further, since $\vdash \Box\top$ for logics with N, for every saturated set Δ', $\Box\top \in \Delta'$. But then $\Box\top \in \Delta$, $\Box A \in \Delta$ and $\Delta \vdash \Box A$, against the hypothesis. Therefore, there is an A-saturated set Δ'. ■

Theorem 3.7 *For every A-saturated set Δ and every Σ-sequence $\vec{A}_n$, the characteristic function f of Δ is an $\vec{A}_n$-valuation.*

Proof. First of all, it is easy to prove by 3.2 that

(†) the characteristic function f of Δ is a semi-valuation.

We now prove the theorem by induction on n. If $n = 1$, the property follows from (†) above. Let us suppose that $n > 1$.

(1) If, for every $m < n$, $A_n \neq \Box A_m$, f is trivially an $\vec{A}_n$-valuation.

(2) For some $m < n$, $A_n = \Box A_m$.

Induction hypothesis: for every saturated set Δ' the characteristic function f' of Δ' is an $\vec{A}_{n-1}$-valuation. So f is an $\vec{A}_{n-1}$-valuation.

(I) $f(A_n) = 0$. Then $A_n \notin \Delta$, $\Delta \nvdash \Box A_m$.

From 3.6 we now have the following, for every logic L:

(i) $\chi[A_m, \varepsilon(\Delta^{\Box})] = \emptyset$.

We have that $\{\vec{A}_{n-1}\}^{\Box}_{f,1} \subseteq \Delta^{\Box}$, thus:

(ii) $\chi[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1})] = \emptyset$.

We must now show, for every logic L, that $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1})] = \emptyset$. Let us take **EC** as an example; for the other logics, the proof is analogous.

From (ii) we have, by definition, that:

(iii) $\{\Theta \subseteq \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1}) : \Theta \equiv_S A_m\} = \emptyset$.

Since Θ is a finite set, let $\Theta = \{\theta_1, \ldots, \theta_k\}$. By definition, $\{\theta_1, \ldots, \theta_k\} \equiv_S A_m$ iff for every saturated set Δ', $\{\theta_1, \ldots, \theta_k\} \subseteq \Delta'$ iff $A \in \Delta'$. And since $\{\theta_1, \ldots, \theta_k\} \subseteq \Delta'$ iff $(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta'$, we have, for every saturated set Δ', that $(\theta_1 \wedge \ldots \wedge \theta_k) \in \Delta'$ iff $A_m \in \Delta'$. But then $(\theta_1 \wedge \ldots \wedge \theta_k) \leftrightarrow A_m$ belongs to every saturated set and is a theorem. Now, since every θ_i and A_m occur in $\vec{A}_{n-1}$, it follows from 2.2 that, for every $\vec{A}_{n-1}$-valuation v', $v'(\theta_1 \wedge \ldots \wedge \theta_k) = 1$ iff $v'(A_m) = 1$, that is, $v' \vDash \Theta$ iff $v'(A_m) = 1$. But then $\Theta \equiv_{n-1} A_m$, and thus:

(iii) $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1})] = \emptyset$.

So f is an $\vec{A}_n$-valuation for **EC**.

The proof is similar for the other logics without N. For logics with N, from 3.6 there is an A_m-saturated set Δ'. By the induction hypothesis, the characteristic function f' of Δ' is an $\vec{A}_{n-1}$-valuation. So there is an $\vec{A}_{n-1}$-valuation f' such that $f'(A_m) = 0$. Hence f is an $\vec{A}_n$-valuation.

(II) $f(A_n) = 1$. So $\Box A_m \in \Delta$. Let us suppose that there is some p, some q, $q < p < n$ such that $A_p = \Box A_q$ and $f(A_p) = 0$.

From 3.6 we now have the following, for every logic L:

(i) $\chi[A_q, \varepsilon(\Delta^{\Box})] = \emptyset$.

By the induction hypothesis, f is an $\vec{A}_{n-1}$-valuation, so it is $\Box_0$-$\vec{A}_{n-1}$-normal. Thus, for every p, every q, $q < p < n$, such that $A_p = \Box A_q$ and $f(A_p) = 0$,

(ii) $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1})] = \emptyset$.

Now, since $f(A_n) = 1$, $\varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1}) \cup \{A_m\} = \varepsilon(\{\vec{A}_n\}^{\Box}_{f,1})$; and since $\{\vec{A}_n\}^{\Box}_{f,1} \subseteq \Delta^{\Box}$,

(ii) $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{f,1}) \cup \{A_m\}] = \emptyset$.

In logics with N, from 3.6 there is an A_q-saturated set Δ' and, by the induction hypothesis, the characteristic function f' of Δ' is an $\vec{A}_{n-1}$-valuation. So there is an $\vec{A}_{n-1}$-valuation f' such that $f'(A_q) = 0$. Hence, f is an $\vec{A}_n$-valuation. ■

Corollary 3.8 *v is a valuation iff v is the characteristic function of some saturated set Δ.*

Proof. (a) Suppose that v is a valuation. Let $[v]_1 = \{A : v(A) = 1\}$, and $[v]_0 = \{B : v(B) = 0\}$. Let $C \in [v]_0$; then $C \notin [v]_1$, and we easily see that $[v]_1 \nvdash C$. Let D be a wff such that $D \notin [v]_1$. Then $v(D) = 0$, $v(\neg D) = 1$ and $\neg D \in [v]_1$. But $v(\neg D \to (D \to C)) = 1$; so $\neg D \to (D \to C) \in [v]_1$. Thus $[v]_1 \vdash D \to C$; hence, $[v]_1 \cup \{D\} \vdash C$. Let $\Delta = [v]_1$. Then Δ is a C-saturated set and, by construction of Δ, v is the characteristic function of Δ.

(b) Suppose that, for some saturated set Δ, v is the characteristic function of Δ. Let $\vec{A}_n$ be any normal sequence: then (by 3.7) v is an $\vec{A}_n$-valuation. Since $\vec{A}_n$ is any normal sequence, v is a valuation. ■

Theorem 3.9 (Completeness) *If $\Gamma \models_L A$ then $\Gamma \vdash_L A$.*

Proof. Suppose $\Gamma \nvdash_L A$. Then $\Gamma \cup \{\neg A\}$ is L-consistent. By 3.3 there is a saturated set Δ such that $\Gamma \cup \{\neg A\} \subseteq \Delta$. By 3.8, f_Δ is a valuation for L. But then $f_\Delta \Vdash_L \Gamma$ and $f_\Delta(A) = 0$, so $\Gamma \not\models_L A$. ■

4 Valuation tables

As mentioned before, defining valuations for Σ-sequences easily yields a decision method based on valuation tables. In what follows we will just sketch the method by means of two examples.

Let us start with table for the Σ-sequence $\vec{A}_5 = p, \neg p, \neg\neg p, \Box p, \Box\neg\neg p$. We will arrive at the following table $T[\vec{A}_5]$:

	A_1	A_2	A_3	A_4	$A5$
	p	$\neg p$	$\neg\neg p$	$\Box p$	$\Box\neg\neg p$
v_1	1	0	1	1	1
v_2	0	1	0	1	1
v_3	1	0	1	0	0
v_4	0	1	0	0	0

The construction $T[\vec{A}_3]$ is just a usual truth-table for classical propositional logic. Now to extend the table to $T[\vec{A}_4]$, we consider each line in its turn. Let us take v_1. Since there are no formulas of the form $\Box A$ in $\vec{A}_3$, $\chi^3[p, \varepsilon(\{\vec{A}_3\}^{\Box}_{v_1,1})] = \emptyset$, so $v_1(\Box p)$ can receive 0. It can also receive 1 since condition II of the definition of an $\vec{A}_n$-valuation is vacuously satisfied. Thus we set $v_1(\Box p) = 1$ and create line v_3, which copies v_1 but assigns 0 to $\Box p$. In a similar way, we create line v_4.

Now let us extend this valuation table to $T[\vec{A}_5]$. Consider line v_1. Could we have $v_1(\Box\neg\neg p) = 0$? Only if $\chi^4[\neg\neg p, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1})] = \emptyset$ — but this set is not empty. Notice that $p \Leftrightarrow_4 \neg\neg p$ (p and $\neg\neg p$ are equivalent with regard to lines 1 to 4) and $v_1(\Box p) = 1$. Thus $\chi^4[\neg\neg p, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1})] = \{p\}$, so $v_1(\Box\neg\neg p)$ must be 1. (Condition II is vacuously satisfied because there are no formulas of the form $\Box A$ such that $v_1(\Box A) = 0$.)

For the same reasons, $v_2(\Box\neg\neg p) = 1$.

Let us now consider v_3. Since $v_3(\Box p) = 0$,$\varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1}) = \emptyset$, so $\chi^4[\neg\neg p, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1})]$ is empty, too. Hence, we can have $v_3(\Box\neg\neg p) = 0$. Could it also have been 1? Since $v_3(\Box p) = 0$, by condition II the set $\chi^4[p, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1}) \cup \{\neg\neg p\}]$ should be empty — but it is actually $\{\neg\neg p\}$. So $v_3(\Box\neg\neg p)$ cannot be 1. For the same reasons, $v_4(\Box\neg\neg p) = 0$.

Notice that $\Box p$ and $\Box\neg\neg p$ are logically equivalent, so $\Box p \leftrightarrow \Box\neg\neg p$ will be valid by the valuation table. Which is as desired, given that $p \leftrightarrow \neg\neg p$ is a theorem and $\Box p \leftrightarrow \Box\neg\neg p$ results from it by RE.

For another example, this time considering **EC**, let us show a table for the

Σ-sequence $p, q, p \wedge q, \Box p, \Box q, \Box(p \wedge q)$.

	A_1	A_2	A_3	A_4	A_5	A_6
	p	q	$p \wedge q$	$\Box p$	$\Box q$	$\Box(p \wedge q)$
v_1	1	1	1	1	1	1
v_2	0	1	0	1	1	1
v_3	1	0	0	1	1	1
v_4	0	0	0	1	1	1
v_5	1	1	1	0	1	1
v_6	0	1	0	0	1	1
v_7	1	0	0	0	1	1
v_8	0	0	0	0	1	1
v_9	1	1	1	1	0	1
v_{10}	0	1	0	1	0	1
v_{11}	1	0	0	1	0	1
v_{12}	0	0	0	1	0	1
v_{13}	1	1	1	0	0	1
v_{14}	0	1	0	0	0	1
v_{15}	1	0	0	0	0	1
v_{16}	0	0	0	0	0	1
v_{17}	1	1	1	0	1	0
v_{18}	0	1	0	0	1	0
v_{19}	1	0	0	0	1	0
v_{20}	0	0	0	0	1	0
v_{21}	1	1	1	1	0	0
v_{22}	0	1	0	1	0	0
v_{23}	1	0	0	1	0	0
v_{24}	0	0	0	1	0	0
v_{25}	1	1	1	0	0	0
v_{26}	0	1	0	0	0	0
v_{27}	1	0	0	0	0	0
v_{28}	0	0	0	0	0	0

$T[\vec{A}_3]$ is a table for classical propositional logic. Now for $T[\vec{A}_4]$, again, since there are no formulas of the form $\Box A$ in $\vec{A}_3$, $\chi^3[p, \varepsilon(\{\vec{A}_3\}^{\Box}_{v_1,1})] = \emptyset$, so $v_1(\Box p)$ can be 0. But $\Box p$ can also receive 1 since condition II of the definition of an $\vec{A}_n$-valuation is vacuously satisfied. Thus $T[\vec{A}_4]$ has eight lines.

Now for A_5. On the first four lines we have a true necessity ($\Box p$). Let us take v_1 as an example. Since it is not the case that $\{p\} \equiv_4 q$, we have $\chi^4[q, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1})] = \emptyset$, so $v_1(\Box q)$ can be 0. Vacuosly, $v_1(\Box q)$ can also be 0. And so for the other lines where $\Box p$ is true, and we extend these lines with 1 and also with 0 (adding v_9–v_{12}).

For the last four lines, in which $\Box p$ has 0, again given that it is not the case that $\{p\} \equiv_4 q$, both $\chi^4[q, \varepsilon(\{\vec{A}_4\}^{\Box}_{v_1,1})]$ and $\chi^4[q, \varepsilon(\{\vec{A}_4 \cup \{q\}\}^{\Box}_{v_1,1})]$ are empty, hence the new lines v_{13}–v_{16}.

Finally, let us consider $\Box(p \wedge q)$, starting with line v_1. Notice that $\varepsilon(\{\vec{A}_5\}^{\Box}_{v_1,1}) = \{p, q\}$, and since $\{p, q\} \equiv_5 p \wedge q$, $\chi^5[q, \varepsilon(\{\vec{A}_5\}^{\Box}_{v_1,1})] \neq \emptyset$, so $\Box(p \wedge q)$ cannot

have 0 on line v_1 (and neither on v_2–v_4). But $v_1(\Box(p \wedge q))$ can be 1, since condition II is vacuously satisfied.

For line v_5, however, $\varepsilon(\{\vec{A}_5\}^{\Box}_{v_1,1}) = \{q\}$, and since $\{q\} \not\equiv_5 p \wedge q$, $\chi^5[q, \varepsilon(\{\vec{A}_5\}^{\Box}_{v_1,1})] = \emptyset$, so this line can be extended giving 0 to $\Box(p \wedge q)$. On the other hand, $\varepsilon(\{\vec{A}_5\}^{\Box}_{v_1,1}) \cup \{p \wedge q\} = \{q, p \wedge q\}$, and since $\{q, p \wedge q\} \not\equiv_5 p$, $\chi^5[p, \varepsilon(\{\vec{A}_5 \cup \{p \wedge q\}\}^{\Box}_{v_5,1})] = \emptyset$, so $v_5(\Box(p \wedge q))$ can also have 1. We accordingly duplicate the lines, allowing both possibilities.

If we now extend this table adding $\Box p \wedge \Box q$, we will see that axiom C turns out valid: in every line in which $\Box p$ and $\Box p$ are true, so is $\Box(p \wedge q)$. On the other hand, axiom M is not valid: in line v_5, for instance, $\Box(p \wedge q)$ has 1, but $\Box p \wedge \Box q$ would receive 0.

5 Some other logics

In the previous sections we saw valuation semantics for some classical modal logics. A few remarks about this. First of all, **EMCN** is the same logic as **K**, the minimal normal modal logic, and for this logic Andréa Loparić had already presented a valuation semantics. So our definitions of $\vec{A}_n$-valuations should be equivalent.

Let $\vec{A}_n$ be a Σ-sequence such that $A_n = \Box A_m$, and v an $\vec{A}_{n-1}$-valuation. Loparić's original definition read as follows:

I) if $v(A_n) = 0$ then there is some $\vec{A}_{n-1}$ valuation v' such that $v' \Vdash \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ and $v'(A_m) = 0$.

Here we have, for **EMCN**:

(I) if $v(A_n) = 0$, then $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$ and there is an $\vec{A}_{n-1}$-valuation v_n such that $v_n(A_m) = 0$.

Of course, if there is some $\vec{A}_{n-1}$-valuation v' such that $v' \Vdash \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ and $v'(A_m) = 0$, then there cannot be any subset Γ of $\varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ such that, for every $\vec{A}_{n-1}$ valuation v'', $v'' \Vdash \Gamma$ iff $v''(A_m) = 1$. Hence, $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$, and there is an $\vec{A}_{n-1}$-valuation v' such that $v'(A_m) = 0$.

For the other direction, supposet that $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$, but there is no $\vec{A}_{n-1}$-valuation v' such that $v'(A_m) = 0$ and $v' \Vdash \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$. Thus, for every $\vec{A}_{n-1}$-valuation v', if $v' \Vdash \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ then $v'(A_m) = 1$. Now this means that $\varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \Rrightarrow_{n-1} A_m$, and so $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] \neq \emptyset$, against the hypothesis.

In the case where $v(A_n) = 1$, we can show in a similar way that the two definitions lead to the same results.

We can also adapt the semantics presented here for other classical modal logics. One group of logics can be obtained by restricting inference rules in a way similar to Lemmon's **S0.5**°, which is a subsystem of **K** in which the necessitation rule $A/\Box A$ is restricted to the cases in which A is a tautology. We could thus have the following restricted versions of our inference rules:

RET. $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$, if $A \leftrightarrow B$ is a tautology

RMT. $\dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$, if $A \rightarrow B$ is a tautology

RCT. $\dfrac{(A \wedge B) \leftrightarrow C}{(\Box A \wedge \Box B) \rightarrow \Box C}$, if $(A \wedge B) \leftrightarrow C$ is a tautology

RRT. $\dfrac{(A \wedge B) \rightarrow C}{(\Box A \wedge \Box B) \rightarrow \Box C}$, if $(A \wedge B) \rightarrow C$ is a tautology

RKT. $\dfrac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box B}$, for $n \geq 0$, if $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$ is a tautology.

If we now redefine $A \Leftrightarrow_k B$ etc. as follows:

(a) $A \Leftrightarrow_k B$ iff for every semi-valuation v, $v(A) = v(B)$;

(b) $A \Rightarrow_k B$ iff for every semi-valuation v, if $v(A) = 1$ then $v(B) = 1$;

(c) $\Gamma \equiv_k A$ iff for every semi-valuation v, $v \Vdash \Gamma$ sse $v(A) = 1$;

(d) $\Gamma \Rrightarrow_k A$ iff for every semi-valuation v, if $v \Vdash \Gamma$ then $v(A) = 1$.

we will obtain valuation semantics for $\mathbf{E}_T$, which is like **E** but having RET instead of RE. And analogouosly for the other logics.

Moreover, valuation semantics for some simple extensions of **E**, **EM,** etc. are easily obtained. Consider the axioms:

D. $\Box A \rightarrow \neg\Box\neg A$

T. $\Box A \rightarrow A$

The following definition of an $\vec{A}_n$-valuation will be adequate for **ET**, and can be suitably modified for D- and T-extensions of the other logics as well.

Definition 5.1 *v is an $\vec{A}_n$-valuation if $\vec{A}_n$ is a Σ-sequence and:*

(1) $n = 1$ and v is a semi-valuation;

(2) $n > 1$, v is an $\vec{A}_{n-1}$-valuation and if, for some $m < n$, $A_n = A_m$,

(I) if $v(A_n) = 0$, then $\chi^{n-1}[A_m, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})] = \emptyset$;

(II) *if* $v(A_n) = 1$ *then* $v(A_m) = 1$ *and, for every* p, *every* q, $q < p < n$ *such that* $A_p = \Box A_q$ *and* $v(A_p) = 0$, $\chi^{n-1}[A_q, \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1}) \cup \{A_m\}] = \emptyset$.

For logics with D instead of T, clause (II) of the definition would read:

(II) if $v(A_n) = 1$ then for some $\vec{A}_{n-1}$-valuation v', $v' \Vdash \varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ and, for every p, every q, ...

This will ensure that the set $\varepsilon(\{\vec{A}_{n-1}\}^{\Box}_{v,1})$ is satisfiable, so there cannot be any valuation in which both $\Box A$ and $\Box\neg A$ are true.

Finally, there are many other classical modal logics for which it would be interesting to provide valuation semantics, systems like **EK** or **EX**, strict systems like **S0.9** and **S1**, or extensions of **E** using iterative axioms such as 4 ($\Box A \rightarrow \Box\Box A$) or 5 ($\neg\Box A \rightarrow \Box\neg\Box A$). We have obtained a few preliminary results, and hope to say more about this topic in future work.

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Cezar A. Mortari
Department of Philosophy
Federal University of Santa Catarina (UFSC)
Campus Universitário, Trindade, CEP 88040-970, Florianópolis, SC, Brazil
E-mail: c.mortari@ufsc.br