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Minimal Logics: A Foray into Non-Tarskian Galaxies

Hilan Bensusan, Agnes Calado, Gregory Carneiro, and Emanuel Paiva

Abstract

We present minimal logics, non-classical and non-Tarskian systems. Although most often paraconsistent and displaying other non-standard properties, we show that minimal systems still have some familiar features if some additional constraints are introduced. Drawing on considerations concerning the galaxies of non-Tarskian logics and minimal logics in particular, we shed light into the way in which sets of possible worlds are compatible both with a logic and its minimized counterparts.

Keywords: non-Tarskian logics, antimonotonicity, addition, minimal logics, galaxies, heterodox logics.

Introduction

Heterodox logics have explored reasoning under non-standard, divergent and unfamiliar constraints. They have paved the way to understand the role that principles such as those of the excluded middle and of non contradiction play in the routes taken by thinking. Issues such as negation and conditionality were much illuminated by these alternative systems. Universal logic [3] has developed frameworks to study the friction between these logics and classical systems in general. In this exploration of different logics, nevertheless, far more attention has been given to more conventional Tarskian systems. In this work we attempt to remedy this by looking at minimal logics, strange and surprising non-Tarskian systems that can be obtained by minimizing the arguments of any logic, including Tarskian ones. Considering sets of possible worlds, it is possible to show some interesting features of the relation between Tarskian and non-Tarskian systems.

1 Non-Tarskian Logics and their Galaxies

Tarskian logics are those in which the consequence relation is reflexive, transitive and monotonic. Tarskian logics are those which satisfy the properties of the logical consequence stated by Tarski [see 10]. To be sure, as Jean-Yves Béziau points out [4], Tarski posits further constraints in the operators in his subsequent works in the 30s [see 12–14], but we consider here the earlier and broader conception formulated by Tarski. This is a remarkably general image of what an appropriate logical consequence should look like.

Taskian logics are often taken to draw an outer boundary for alternative, non-classical logics for they keep most expected tenets of good behavior in place. While Tarskian non-classical logics could be divergent, unusual and filled with unexpected results, they still belong to a framework that is familiar. As a consequence, non-Tarskian systems are far less explored; yet much knowledge about logics in general can be gained from considering these very unconventional domain. This work is a small foray into the strange non-Tarskian lands. It will also indicate interesting ways in which non-Tarskian systems are associated to Tarskian ones.[1]

We take a logic to be a consequence relation over F, that is, a set of arguments (Γ, φ) where $\Gamma \subseteq F$ and $\varphi \in F$. As usual, we take a world w to be a subset of F. A world w is compatible to a logic l if and only if $\Gamma \subseteq w$ entails that $\varphi \in w$ for every argument (Γ, φ) of l. The compatibility between a logic l and a world w is referred as $w \models l$. Alongside these concepts, we introduce the following definition:

Definition 1.1 (Signature logic) *The signature logic of a set of worlds c (call c a constellation) is the greater set of arguments (Γ, φ) such that for every $w \in c$, if $\Gamma \in w$ then $\varphi \in w$.*

The signature logic is the greater logic compatible with every w in c. As defined by Bensusan, Costa-Leite, and de Souza [2], a galaxy of a logic l is defined by:

Definition 1.2 (Galaxy) *A galaxy $G(l) = \{w : w \models l\}$ of a logic l is the set of worlds compatible with l.*

Notice that:

Proposition 1.3 *If $l \subseteq l'$, then $G(l') \subseteq G(l)$.*

[1] Incidentally, Francisco Miró Quesada [8] had a quite broad notion of heterodox logics. Non-Tarskian logics broaden this scope of heterodoxy even further by not satisfying structural properties such as monotonicity. Notice that most non-classical logics, including some paraconsistent and paracomplete ones, can still be Tarskian.

Proof. Suppose that $l \subseteq l'$. Let it be $w \in G(l')$. By hypothesis, if $(\Gamma, \varphi) \in l$, then $(\Gamma, \varphi) \in l'$. Considering that $w \in G(l')$, we know that $\Gamma \subseteq w$ implies that $\varphi \in w$. This shows that $w \models l$. ■

Proposition 1.4 *If $c \subseteq c'$, then $L(c') \subseteq L(c)$*

Proof. Suppose that $c \subseteq c'$. Let it be $(\Gamma, \varphi) \in L(c')$. By hypothesis, if $w \in c$, then $w \in c'$. Considering that $(\Gamma, \varphi) \in L(c')$, we know that $\Gamma \subseteq w$. This is enough to show that for all $w \in c$, if $\Gamma \subseteq w$, then $\varphi \in w$. ■

Any logic has a galaxy, and every galaxy is a constellation. We can now say that $L_s = L(G(l))$. We can specify a constellation by presenting a logic; a logic is a way to define a set of possible worlds. Interestingly, though, we cannot specify a logic by pointing at a constellation, for not every constellation is a galaxy. A constellation c is a galaxy if and only if there is a logic l such that $c = G(l)$. However, we will see that the signature logic of a constellation is always Tarskian. We can show that the operators G and L form a Galois connection:

Proposition 1.5 *For any c and l, $l \subseteq L(c)$ if and only if $c \subseteq G(l)$*

Proof. If $c = G(L(c)))$, there is a logic $l = L(c)$ such that $c = G(l)$. If there is an l such that $c = G(l)$, $G(L(c)) = G(L(G(l)))$. By the definition of Galois connection [see 7], $G(l) \subseteq G(L(G(l)))$ if and only if $L(G(l)) \subseteq L(G(l))$. Then $G(l) \subseteq G(L(G(l)))$. Also, $G(L(G(l))) \subseteq G(l)$ if and only if $l \subseteq L(G(L(G(l))))$. As L and G are both decreasing, $L \circ G$ is increasing. As $l \subseteq L(G(l))$, $L(G(l)) \subseteq L(G(L(G(l))))$. It follows that $l \subseteq L(G(L(G(l))))$ and $G(L(G(l))) \subseteq G(l)$. By what we showed so far and if $c = G(l)$, we can conclude that $c = G(L(c))$. ■

From this, it follows that:

Proposition 1.6 *For a logic l the three following claims are equivalent:*
(a) There is a constellation c such that $l = L(c)$.
(b) $l = L(G(l))$.
(c) l is Tarskian.

Proof. It follows from proposition 1.5 that (a) is equivalent to (b).

To show that (c) follows from (b), suppose that $l = L(G(l))$. If Γ is a set of formulas and $\varphi \in \Gamma$, then $(\Gamma, \varphi) \in l$. Indeed, $(\Gamma, \varphi) \in l$ if and only if for all $w \in G(l)$, $\Gamma \subseteq w$ implies $\varphi \in w$, which is the case since $\varphi \in \Gamma$. Hence, l is reflexive.

Now suppose that $(\Delta, \varphi) \in l$ and $(\Gamma, \psi) \in l$ for each $\psi \in \Delta$. Considering these conditions, $(\Gamma, \varphi) \in l$. In order to show this is sufficiente to verify that for all $w \in G(l)$, $\Gamma \subseteq w$ imples that $\varphi \in w$. If $\Gamma \subseteq w$, then $\Delta \subseteq w$, since $(\Gamma, \psi) \in l = L(G(l))$ for all $\psi \in \Delta$. From $(\Delta, \varphi) \in l$ follows tha $\varphi \in w$. Therefore, $\Gamma \subseteq w$ implies that $\varphi \in w$ and $(\Gamma, \varphi) \in l$. l is transitive.

Finally, if $\Gamma \subseteq \Delta$, then $(\Gamma, \varphi) \in l$ implies that $(\Delta, \varphi) \in l$. To show this suffice to observe that if $\Gamma \subseteq \Delta$, then for all $w \in G(l)$, if $\Delta \subseteq w$, then $\Gamma \subseteq w$.

Now assuming that $(\Gamma, \varphi) \in l$, for all $w \in G(l)$, if $\Delta \subseteq w$, then $\varphi \in w$, since $\Delta \in w$ implies that $\Gamma \subseteq w$ and $\Gamma \subseteq w$ implies $\varphi \in w$. l is also monotonic.

In order to prove that (c) implies (b) suppose that l is Tarskian. By definition of galois connection, we know that $l \subseteq L(G(l))$ for any l. $(\Gamma, \varphi) \in L(G(l))$ if and only if or all $w \in G(l)$, se $\Gamma \subseteq w$, then $\varphi \in w$. Besides that, $w \in G(l)$ if and only if $w \vdash l$.

Suppose now that $(\Gamma, \varphi) \notin l$. Be w the set $\{\varphi : (\Gamma, \varphi) \in l\}$. Considering that l is Tarskian, $\Gamma \subseteq w$ and $\varphi \notin w$. Consequently, if $w \vdash l$, then $(\Gamma, \varphi) \notin L(G(l))$.

To conclude is enough to show that $w \vdash l$. If $(\Delta, \psi) \in l$ and $\Delta \subseteq w$, then, by idempotence, $(\Gamma, \psi) \in l$ and $\psi \in w$, by definition of w. This shows that $w \vdash l$ and $(\Gamma, \varphi) \notin L(G(l))$. ■

Only Tarskian logics are signature of their galaxies. That is, even if there are non-Tarskian logics compatible with a galaxy, they cannot be its signature logic. If there is a signature logic for a constellation, it is a Tarskian one. The remarkable fact about Tarskian logics that emerge from the notion of galaxies is that they provide a signature for a constellation that could be a galaxy for other, non-Tarskian logics. The framework of galaxies—which is similar to the one of valuation spaces developed by Hardegree [7]—indicates that the properties that makes a logic Tarskian are not arbitrary in the sense that they demarcate a kind of logic that can be a signature for a set of possible worlds. Doing that, they also shed some light into the less well-known non-Tarskian domain.

An interesting observation concerning non-Tarskian systems has to do with their diverging nature. In order to study galaxies and their relation to logics, Bensusan, Costa-Leite, and de Souza [2] have studied the notion of antilogical systems. The *antilogic* of a given logic l, denoted by $\bar{l}$.

Definition 1.7 (Antilogic) $\Gamma \models_{\bar{l}} \varphi$ *if and only if it is not the case that* $\Gamma \models_{l} \varphi$.

The antilogic of a given logic always interesting. For example, it is clear that the antilogic of the classical propositional logic is, by definition, paraconsistent, since there is no instance of the *ex contradictione quodlibet* is valid.

Furthermore, Béziau and Buchsbaum [5] have shown that this anticlassical logic, $\bar{k}$, is non-Tarskian (it is not reflexive, neither transitive, nor monotonic). As a consequence, there is a Tarskian logic t such that $G(t) = G(\bar{k})$. In other words, the constellation of a non-Tarskian logic even as non-standard as the anticlassical logic can be captured by a Tarskian one. As we have shown, if a constellation is a galaxy, its signature is Tarskian.

2 Antimonotonicity and Minimality

We will take a closer look into the monotonicity Tarskian clause. Monotonicity is about the addition of premises. A monotonic addition is one that doesn't ruin a valid argument. If we consider a set of premises A and an added premise a, A can be sufficient or insufficient to infer the conclusion φ in a given logic. If A is not sufficent to infer φ and the addition of a makes it sufficient to infer, we can say that it *complements* A. Otherwise, it is a neutral addition. If, on the other hand, A is sufficent to infer φ and the addition of a makes it insufficient to infer, we can say that it *supplements* A. Otherwise, again, it is a neutral addition. The distinction between complement and supplement is inspired by Derrida [6], who introduced the idea of a possible logic of the supplement that places restriction on reflexivity [see also 1]. We can then distinguish the following four types of addition:

	A	$A+a$	Type of addition
1	insufficient	sufficient	Complementing
2	insufficient	insufficient	Non-complementing
3	sufficient	insufficient	Suplementing
4	sufficient	sufficient	Non-suplementing

Notice that 2 and 4 are neutral additions and monotonicity is present in all but 3. As a consequence, only 3 is a non-Tarskian addition. A supplementing addition is one that turns a previously sufficient set of premises uncapable of inferring the conclusion.

An argument (Γ, φ) is monotonic if and only if for any γ, if $\Gamma \vdash \varphi$, then $\Gamma \cup \{\gamma\} \vdash \varphi$. That is, no γ is a supplement and any addition is non-supplementing. An argument is non-monotonic if and only if there is a γ such that $\Gamma \vdash \varphi$ and $\Gamma \cup \{\gamma\} \nvdash \varphi$. Here some γ is a supplement. We can further define an argument (Γ, φ) as antimonotonic if and only if for any γ, if $\Gamma \vdash \varphi$, then $\Gamma \cup \{\gamma\} \nvdash \varphi$. In an antimonotonic argument, no addition of premises is non-supplementing. The three types of argument suggest a triangle of oppositions that can be developed into a square if we consider non-antimonotonic arguments where some addition is non-supplementing. (These and other corresponding squares of oppositions were studied by Bensusan and Carneiro [1].)

Definition 2.1 (Antimonotonic or minimal logic) *A antimonotonic (or minimal) logic is composed only by antimonotonic arguments. In an antimonotonic logic* $\Gamma \vdash \varphi \Longrightarrow \Gamma' \nvdash \varphi$ *for any* $\Gamma' \subset \Gamma$.

That is, by arguments that cease to be valid if any extra premise is added. These arguments are in this sense minimal. It is reasonable to understand a minimal argument as one where if $\Gamma \vdash \varphi$, so $\Gamma \cup \{\gamma\} \nvdash \varphi$. An antimonotonic (or minimal) logic is one where there is no unnecessary premises in any of its arguments. Any addition to the premises of the arguments is a supplementing.

We can now define an operator that minimizes any argument (Γ, φ):

Definition 2.2 (Minimization operator) $Min(\Gamma, \varphi) = (\Gamma', \varphi)$ *where* Γ' *is the smallest set of* Γ *such that* $\Gamma' \vdash \varphi$.

This minimization (or antimonotonicization) operator can be applied to all arguments of a logic l; the resulting logic, $\widehat{l}$, is one where all arguments are minimal. If l is formed by a set of arguments (Γ, φ), $\widehat{l}$ is formed by a set of arguments (Γ, φ). Notice that for each l there could be more than one $\widehat{l}$ for there could be, for example, an argument $\{\alpha, \beta, \delta\} \vdash \varphi$ in l such that both $\{\alpha, \beta\} \vdash \varphi$ and $\{\alpha, \delta\} \vdash \varphi$. In any case, the operator generates *a* minimal logic. It will prove important to see that for any $\widehat{l}$ of l, we can show that $\widehat{l} \subseteq l$.

Minimal logics are paraconsistent systems. To see this, it is enough to notice that if $\alpha \in \Gamma$, then $\Gamma \cup \{\neg\alpha\} \nvdash \varphi$; it is then not the case that *ex contradictione quodlibet*. To minimize a non-paraconsistent logic is therefore to paraconsistentize it [9]. Some minimal logics could be antimonotonic (as much as monotonic) by vacuity; if there is no premise to add to a set, the addition can be both supplementing and non-supplementing; for instance, consider a logic with a single formula α and the consequence relation $\vdash \alpha$ that is its own minimization. However, any logic with the power to express a contradiction can generate a paraconsistent counterpart through minimization.

Notice, however, that antimonotonic systems display some familiar features. If l is compact, $\widehat{l}$ is also compact. This follows from the fact that $\widehat{l}$ is a sublogic of l. Consider now, to see more clearly, classical propositional logic, k and a minimalized counterpart $\widehat{l}$. It is easy to see that the connectives in $\widehat{l}$ are not truth-functional. If it is the case that $\alpha \vdash \beta$, then $\alpha \to \beta, \alpha \nvdash \beta$ and Modus Ponens is not valid. In $(\widehat{k})$, nonetheless, it is the case that if $\alpha \to \beta$ and $\nvdash \beta$ then $\alpha \vdash \beta$. And, surely, if $\alpha \vdash \beta$ then $\emptyset \vdash (\alpha \to \beta)$. A weaker version of the theorem of deduction is therefore valid in a $(\widehat{k})$.

Minimal logics can be developed in different directions. If we look at supplementing addition, it would be interesting to consider not only additions to sets of premises but also supplementing additions within a premise. We can then define an hyperminimization operator for logics with formulas and subformulas:

Definition 2.3 (Hyperminimization operator) *$Hypermin(\Gamma, \varphi) = (\gamma', \varphi)$ where γ' is the smallest subformula of all $\gamma \in \Gamma'$ such that $(\Gamma', \varphi) = Min(\Gamma, \varphi)$ and $\gamma' \vdash \varphi$.*

This operator can be applied to minimized arguments in a logic and the resulting logic is a hyperminimized one. The operator can be used to further prune an already minimized logic. A hyperminimized logic is one where the premises are all in its shortest possible formulation. This resulting logic—which can be called a hyperantimonotinicized logic—can make explicit how negation symbols, for example, could be seen as additions supplemented to a premise otherwise sufficient to infer the conclusion. Thus, in a classic logic $k' \subseteq k$, consider $\neg ... \neg\alpha \subseteq \Gamma$ in an argument of k', (Γ, φ). $Hypermin(\Gamma, \varphi)$ is then either $(\{\alpha\}, \varphi)$ or $(\{\neg\alpha\}, \varphi)$ once one of these two premises would be enough to (classically) infer φ. There is at least one hyperantimonotonized counterpart of k' that is therefore paracomplete (that is, the principle of excluded middle is not valid). This indicates that negation itself can be seen as a kind of addition. This also shows how relevant it may be to consider supplementing additions in an argument and, in general, in non-Tarskian logical systems. In those systems, paracompleteness and paraconsistency—and possibly other features that are at face value understood as effects of negation—result from breaking with monotonicity and taking every addition as being supplementing.

3 Minimization and Galaxies

Minimal (and hyperminimal) systems show how addition can be behind paraconsistent and paracomplete behavior in a logic. Within the Tarskian realms, adding anything to premises or set of premises can at most complete an otherwise insufficient base to infer the conclusion. Minimal (and hyperminimal) systems enables an analysis of the impact of supplementing addition in the premises of an argument. We know (Proposition 1.6 above) that Tarskian systems provide signature logics to galaxies of non-Tarskian logics, including minimal ones. Further, we can prove that that the galaxy of a logic is the same as that of all its minimized counterparts.

Proposition 3.1 *$G(l) = G(Min(l))$.*

Proof. Let l be a logic and $Min(l)$ its minimization. Let w be a world such that $w \models Min(l)$. Suppose $(\Gamma, \varphi) \in l$. In this case, there is $\Gamma' \subseteq \Gamma$ such that $\Gamma', \varphi \in Min(l)$. Now, if $\Gamma \subseteq w$, then $\Gamma' \subseteq w$, because $\Gamma' \subseteq \Gamma$. As $w \models Min(l)$, we have that $\varphi \in w$. Therefore, w is compatible with (Γ, φ). As w is any world in $G(l)$, we have $G(min(l)) \subseteq G(l)$. Therefore, we have $G(min(l)) = G(l)$ ■

If the minimization of the signature logic of a constellation is not antimonotonic (and non-Tarskian) by vacuity, there is a genuine non-Tarskian logic sharing the same galaxy. Not all non-Tarskian systems can be obtained by minimization, but it is reasonable to conjecture that if the are genuine non-Tarskian logics ompatible with the constellation, the signature logic can be minimized into a logic that is not antimonotonic (or monotonic) by vacuity. To give a simple example, consider a propositional logic formed by $F = \{p, q\}$ and the consequence relation $\emptyset \models p$, $\{p\} \models p$, $\{q\} \models q$, $\{p, q\} \models p$, $\{p, q\} \models q$. A non-reflexive logic formed by F and the consequence relation $\models p$ is compatible with the same constellation (composed by worlds where p is the case). We can see that, in this case, the minimization of a is also a proper sublogic of a. Minimal logics seem to indicate where there are non-Tarskian logics compatible with a constellation.

4 Final Remarks

There is much to be explored in non-Tarskian systems in general and in minimization (and hyperminimization) in particular. In fact, there is an abstract relation between negation and addition that we are just beginning to unveil. What is decisive here is that Tarskian logics provide a signature to the galaxies of non-Tarskian systems and minimization is proving to be an interesting instrument to explore this equivalences of galaxy. In fact, minimization seems to be a way to study how (supplementing) additions have an effect on logical systems. In any case, it is clear that there is an important distinction between Tarskian and non-Tarskian systems.

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Hilan Bensusan
Department of Philosophy
University of Brasilia
E-mail: hilantra@gmail.com

Agnes Calado
Department of Philosophy
University of Brasilia
E-mail: acaladoc@gmail.com

Gregory Carneiro
Department of Philosophy
University of Brasilia
E-mail: gregorycarneiro@outlook.com

Emanuel Paiva
Department of Philosophy
University of Brasilia
E-mail: emanuelhenrique665@gmail.com