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The 'Math Orgy' and the Philosophy of Science

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Dedicated to Francisco Miró Quesada Cantuarias,
a Peruvian philosopher who contributed significantly
to the development of Logic in Latin America.

Abstract

Francisco Miró Quesada Cantuarias has enlighten several 'philosophical problems' that appeared in Logic after the 'mathematizing orgy' that happened in this field in the nineteenth and twentieth centuries. Here we take his arguments as a motivation and show that similar problems can be attributed to the empirical theories, mainly after the development of Hilbert's sixth mathematical problem.

Keywords: Hilbert's sixth mathematical problem, matematization, development of logic, naturalism, empirical sciences.

1 The Math Orgy in Logic

First of all, let me say that I prefer to write 'Logic', with an initial capital letter, to name the discipline, while 'logic', in lower case, will designate a particular logical system. I also consider a logic as a mechanism of inferences, usually deductive, but I think we should also admit that inferences may be of other nature, such as inductive or of other kinds. Thus, a logic is a mechanism of inferences, while the discipline Logic is something rather different and should not be associated (no more!) with the simple study of inferences or with some 'art of thinking'. Logic, today, is a vast field of knowledge whose topics can be seen in the entry Mathematical Logic and Foundations of the MSC Classification, involving fields that are far removed from the mere study of inferences [2].

The 'mathematization' in Logic, as far as historians tell us, was envisaged by Leibniz, who had noticed that the logic of the Aristotelian tradition was not suitable for schematizing most of the mathematical inferences. But the real move started in the nineteenth century with Boole, De Morgan, Schröder, and many others. De Morgan, for instance, argued that Aristotelian Logic (AL) was not able to formalize a simple argument such as 'A horse is an animal. Hence, the head of horse is the head of an animal' [30]. Several similar examples can be shown, since AL does not deal with relations, which are essential in mathematics. The 'Boolean tradition' was concentrated in the algebraic aspects of logic, and was developed in the twentieth century by people like Tarski and Halmos.

But the real move toward mathematical methods started with Gottlob Frege in 1879, when he presented in his *Begriffsschrift* [see 29] an axiomatization of what today we call second order logic; the Fregean tradition was a move toward different bases, ideas, and methods. Anyway, these developments, later pushed by Bertrand Russell, Alfred N. Whitehead, Giuseppe Peano, and many others, brought what Francisco Miró Quesada called the 'mathematizing orgy' [21, p. 13] in Logic.

Miró Quesada looked for a *rationale* in Logic [21]. By defending a rationalistic position regarding Logic, he looked for the relationships between logic and reason. By acknowledging that it is common to regard Logic as the *locus classicus* of rationality, the proliferation of different logics, especially *heterodox* systems, poses a dilemma to the philosopher: can we justify rationality in the face of possible different 'logicalities'? Then he tries to find necessary and sufficient conditions for 'logicality' in classical logic, and then in the heterodox systems. His conclusion is that it is possible to find a certain unity in the logicality of the different systems so that, although reason does not function in the heterodox systems as in the classical case, being more flexible, the heterodox ways of reasoning still maintain some of the 'classical' patterns, so that one can detect an invariant core of principles that are preserved in all these ways of reasoning [see also 1].

2 What about Empirical Theories?

These developments in Logic and in mathematics—for instance, in sedimenting set theory and type theory as the *loci* where mathematics could be developed—immediately suggested that a similar move could be made in the empirical theories, where the use of mathematical resources was providential. We can consider 1900 as a turning point, mainly due to Hilbert's sixth problem of his list of twenty-three mathematical problems devised to be pursued in the century which was starting [3, pp. 1–34]. Specifically, the problem suggests the axiomatization of the theories of the empirical sciences in the direction pointed by those of mathematics (algebra, geometry, analysis, topology). During the century, much was done in this direction, and we got axiomatized versions of several theories, such as classical particle mechanics [19, 26], continuum mechanics [22], the theory of evolution [17], learning [26], quantum mechanics [for instance, 13], general relativity (GR) [4],[1] among others.

How can we see these developments? At a first glance, we can see them as an attempt to pursue in the empirical sciences the notion of rigor achieved in mathematics when mathematicians like Bolzano, Cauchy, Weierstrass, Dedekind, and many others started a movement termed (by historians) the Arithmetization of Analysis, which brought us precise definitions of real numbers, mappings (functions), the dependence of Analysis to Arithmetic, and so on [20, ch. 22].

But we should also consider the methodological aspects of the move mentioned above. According to A. I. Arruda, the Russian logician Nikolai A. Vasiliev considered that a logical system has two parts: (1) *Metalogic*, congregating a fixed and indispensable core of laws related to thought, and (2) a cluster of laws that depend on the properties of the objects being analyzed [1]. We can say similar things of any well formulated (that is, at least axiomatized) empirical theory, but with some adaptations. Firstly, instead of metalogic, we prefer to speak of *metaphysics*, meaning a world-view that directs the core of the investigations. In classical physics, for instance, the world is supposed to be formed by individual substances that, although they can be aggregated to form compounds, they can also be regarded as isolated *individuals*. Euclidean geometry was also taken as the description of our surrounding reality, and classical logic (that is, Aristotelian logic) was once considered as copying the ways of reasoning. All of this can be subsumed in a metaphysical account of reality. The second part, of course, depends on the particular theory.

After the axiomatization of a theory, we can exchange part of the metaphysics by a formal semantics, but the 'mathematization' and the development of Logic has shown that such a *formal* semantics will depend on the metamathematics we use to express it. We shall turn to this point soon.

3 The Consequences of the Math Orgy in the Empirical Sciences

Anyway, the mathematical and logical analyses of scientific theories brought consequences. Inspired by Miró Quesada, we can enlighten some (until then) unsuspected implications, such as the following ones.

[1] Bunge's was perhaps the first attempt to axiomatize GR. Currently, there are several other axiomatizations that can be easily found through a web search.

First, physical theories were almost always proposed with a purpose, namely, the mathematical treatment of some part of reality. In logic terms, we could say that they were born already interpreted. Our metaphysical credos, in the sense delineated above, provide the main directions to (at least) the *first interpretation.* Accordingly, classical particle mechanics, in the direction pointed out by McKinsey, Sugar, and Suppes [19], aimed at to deal with systems of 'particles' without collision and deformation. These last notions required a more sophisticated approach, developed in the sense of *continuum mechanics* by Walter Noll, among others. But the axiomatization, due to abstraction, makes the theory, or the axiomatized version of the theory, *autonomous* with respect to the initial intensional model, since it can admit several other possible interpretations or 'semantics'. In Hilbert's terms, we go from a *concrete axiomatization* to an *abstract axiomatization* [see also 15]. In principle, any axiomatized theory admits infinitely many interpretations or models. However, some theories are *categorical*, in the sense that all its models are isomorphic, the models varying only with respect to the nature of the elements involved. So, this is the first consequence of this math orgy: the realization that a theory becomes free from its initial motivation and may admit different and alternative models, not making reference to just one domain.

It was precisely the fact that a scientific theory may have several models (as groups, vector spaces, etc. have in mathematics) what made possible the development of the *semantic approach* to scientific theories, which, roughly speaking, characterizes a theory as a class of models. As van Fraassen describes:

> To present a theory is to specify a family of structures, its *models*; and secondly, to specify certain parts of those models (the *empirical substructures*) as candidates for the direct representation of observable phenomena. The structures which can be described in experimental and measurement reports we can call *appearances*: the theory is empirically adequate if it has some model such that all appearances are isomorphic to empirical substructures of that model. [28, p. 64]

The *place* were these models are constructed, or at least assumed to exist, is also a fundamental point in the researches in the philosophy of science, and shall be considered below.

Second, the philosophical consequences of a theory having several different models are huge. Today we tend to put aside the armchair metaphysics and agree that, in order to know something about the world or at least of some parcel of it, we need to look to our best scientific theories. This *naturalism* [23], notwithstanding, needs to be seen with care. In fact, it has been stablished beyond any doubt that, speaking generally, physics (by means of the physical theories) does not tell us how the world is. With a Kantian flavor, we can say that *reality*, whatever it is, remains unknown; for some, it lies behind a veil [9], while for others it simply cannot be know. I assume the latter position; in simpler words, I assume our ability to know depends on the theories we formulate.

Whatever point of view we adopt, though, we need to acknowledge that we can associate mutually different and even incompatible interpretations to the same physical theory. A case in point comes from orthodox quantum mechanics, where the same formalism (say, via Hilbert spaces) is compatible with (at least) two incompatible metaphysics, one which sees quantum basic entities as *individuals* on a pair with their 'classical' counterparts, and another which sees them as *non-individuals*, that is, as entities to which the standard notion of identity does not apply.[2] The first one imposes some limitations on the states the systems can be in, and this restriction to accessible states enables us to keep assuming that quantum particles can be regarded as individuals [see 10, 16]. The non-individuals view, on the contrary, regards these entities as devoid of individuality (and identity! [see 10, § 4.1.2]). More importantly, the two approaches are possible with the same mathematical formalism, and we cannot decide between them except by pragmatic criteria; we do not know what elementary quantum systems are out of our theories.

Hence, the third consequence of this mathematization was that metaphysics remains underdetermined by physics. The most we can do with our theories is to use them in a pragmatic way, insofar as they produce the (to the moment) best results and, as van Fraassen uses to say, the theories "save the appearances" [28].

A fourth interesting consequence, yet to be explored in full by philosophers, would be the following one. Today, precisely due to the mathematization, in particular, to independence proofs in set theory, we recognize that there are different and non-equivalent mathematics.[3] Let me call 'classical' the mathematics which can be developed in a 'standard' ('Cantorian') set theory such as ZFC or NBG or even KM. This is essentially the mathematics you find in the standard books. For instance, in constructing real numbers (say, by Dedekind cuts), one relies on the axiom of *completeness* of the reals, which roughly says that whatever non-empty set of real numbers with an upper bound has a supremum, that is, an upper bound that is less than any other upper bound. The *rational line* (that is, the line composed by rational numbers only) does not have this property, being full of *gaps* that are fulfilled by irrational numbers. If we reject such a non-constructive postulate, we will be confined to a different

[2]The whole discussion can be seen in [10].
[3]For a general reading, see [5].

mathematics, say *intuitionist mathematics* or some other form of constructivity [see 6], or at least *predicativity* [31].

Fifth, there are mathematics, as that developed by Robert Solovay (also known as 'Solovay's model' for set theory) [25] where every set of real numbers is Lebesgue-measurable, contrary to classical mathematics, where, with the help of the full axiom of choice, we can prove that there are non-measurable sets. Can this math be useful for some domain of the empirical sciences?

Sixth, in Solovay's model, we also have that all linear operators over a Hilbert space are bounded [18]. But the standard formulation of quantum mechanics makes use of unbounded operators, e.g., those which express position and momentum. Thus, we can ask whether it is possible to develop a quantum theory in a framework such as Solovay's, or (what would be the same) to construct a model for QM in such a set theory. Of course, you can say this is trivially answered, for we could say that this like asking if we can develop the theory of linear spaces grounded only in the structure of groups. That is, simply choosing the metamathematics that enables the construction of the right model. But the situation is different; what we have learnt from the 'mathematization' was precisely that we need to look to the metamathematics, something never dreamt before; i.e., we should not take the 'mathematics we need' as granted from the start. We need to look at the concepts we use and see if the metamathematics is consonant with them. In logical terms, the semantics of a logical system should be consonant with the logic itself [7].

This idea applies also to the formulations of the theories of the empirical sciences, and we can, at first glance, read 'semantics' as 'interpretation', although *formal semantics* is an important topic in the philosophy of science as well [see 8]. Or at least it should be so. For instance, it has been said that if we adopt a metaphysics of non-individuals regarding the foundations of quantum mechanics, we should also adopt a theory of *quasi-sets* for expressing such a semantics [10, 16].

Finally, without the Axiom of Choice, we can construct set-theoretical models (Läuchli's permutation models) [14] where linear spaces can lack basis or have a basis of different cardinalities. Now imagine someone trying to develop a version of quantum mechanics via Hilbert spaces in one of these models (or in the corresponding set theory). It is essential for the quantum formalism that the relevant Hilbert spaces do have bases and, of course, the different bases should have the same cardinality, which by the way define the dimension of the space. Of course, we could be in trouble in considering such a situation in the metamathematics.

We see the metamathematics cannot be arbitrarily chosen, and the perception of this fact is a result of the math orgy.

4 Conclusion

It is important to realize that it was not just the emphasis on axiomatization (or formalization) what has brought such conclusions. Euclid's geometry remained 'axiomatized' (yet with well-known deficiencies) and *truly about the world* until the raise of non-Euclidean geometries in the beginnings of the nineteenth century. More was needed, the raise of abstract structures, which may be said to be *the* characteristic ones of modern mathematics, presenting structured domains which, although motivated by some particular field, could be also applied to other different domains. The same happened with Logic; different logical systems enable different interpretations.

Just an example: so called *deontic logics* formalize modal notions such as 'obligatory', 'permitted', 'prohibited', among others. The interesting fact is that we can interpret these operators in at least two senses: the *legal* and the *moral* ones. So, $\mathsf{O_m}A$ would say that A is obligatory in the moral sense, while $\mathsf{P_j}A$ says that A is permitted in the legal sense. Then we can discuss the relationships among these notions, for instance, by postulating that $\mathsf{O_m}A \to \mathsf{P_j}A$ and $\mathsf{O_j}A \to \mathsf{P_m}A$, the first one expressing what is called 'Kant's axiom', since it says that 'moral duties are a part, or are included, in the domain of liberty' [24]. Mixed systems (paraconsistent deontic logics) can consider *deontic dilemmas*, such as $\mathsf{O}A \wedge \mathsf{O}\neg A \to \mathsf{O}B$, without *trivialization*, that is, without turning obligatory every well-formed formula.

But with regard to the empirical sciences, the way is still being traced, although it is currently considering the mathematics to be used in the metalevel. Such metalevel, by the way, is where the *models* of the theories will be constructed, for instance, in the *semantic approach* [26, 28], or in the structural ontological approaches [11].

Last but not least, it is necessary to mention category theory, which has been looked as an alternative metamathematics for expressing also physical theories [12], as well as the renewed versions of type theory, such as *homotopy type theory* (HoTT), which is an expansion of Martin Löf's intuitionistic type theory, considered by many as a good way for computation [27].

Philosophically, we see that all these moves have a common core, despite the differences in the ways of reasoning they presuppose. As Miró Quesada would put it, they have *invariants* that enable us to say that all these different approaches and theories bring us alternative but justified ways of being rational about science and the scientific enterprise.

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