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A Simplified Completeness Proof for the Paraconsistent Logic C_1

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Abstract

This paper develops a new completeness proof of da Costa's paraconsistent logic C_1, w.r.t. the class of F_1-*structures*, whose original definition is indebted to M. Fidel in [7]. The novelty of the proof here presented is grounded on the fact that such class can be characterized in a simpler way (cf. [11]), together with a simpler reformulation of the F_1-*valuations* involved in Fidel's original result. By the way, it is shown that the proof of *decidability* of C_1 can be also simplified. Moreover, this new demonstration shows the evident connection between the *canonical* F_1-*structure* and the *quasi-matrix semantics*, originally proposed in [6].

Keywords: Paraconsistent Logics, F-structures, Quasi-matrix Semantics.

1 Introduction and preliminaries

One of the first semantics for the paraconsistent calculi C_n, with $1 \leq n \leq \omega$ (see [5]), was provided by M. Fidel by means of the (today called) F-structures, in [7]. Briefly, this kind of semantics, which will be presented later, consists of a class of algebraic-relational structures together with a family of *non homomorphical interpretation functions*. So, for every C_n-calculus (which is defined determining a consequence relation $\vdash_{C_n}$), its respective class $\mathbb{F}_n$ of F_n-*structures* determines a relation $\models_{C_n}$ in such a way that $\Gamma \vdash_{C_n} \alpha$ iff $\Gamma \models_{C_n} \alpha$, for every $\Gamma \cup \{\alpha\} \subseteq Fm$. Moreover, the class $\mathbb{F}_n$ determines the *decidability* of every C_n-calculus (actually, this is the main result of such paper).

However, it is not so easy to work with F_n-structures in a general way, due to the complexity of its definition. An alternative, simpler characterization of F_1-structures was provided in [11], as a (partial) solution to this operative problem. So, it is natural (taking into account this new characterization) to simplify the definition of the F_1-*bivaluations* involved in the relation $\models_{C_n}$. Therefore, using this alternative definition, the completeness proof relating

$\vdash_{C_1}$ with $\models_{C_1}$ can be formulated in a very simple way. This paper shows such simplifications, indeed. Moreover, it will easily obtained the decidability of C_1 *based on* F_1*-structures*, by means of the *canonical* F_1*-structure* $\langle B_2, \bar{\mathfrak{f}}\rangle$, to be defined later. In addition, it will be discussed the following point: it is well-known that *another semantics* for the C_n-calculi was presented in [6] by means of the so-called *quasi-matrices* (determining the relations $\models_{Q_n}$). So, we will show additionally that, in the case of the quasi-matrix for C_1, it can be understood as *the own canonical* F_1*-structure* $\langle B_2, \bar{\mathfrak{f}}\rangle$, explained in a different way. Actually, this result will follow easily from the simplified semantics for C_1. Finally, we will discuss briefly the relations of F_1-structures (such as they are presented here) with respect to previous works.

The formalism used in this paper will be as simple as possible. We will fix the set Fm of formulas as the least set containing a countable set Var of *atomic formulas*, and being closed by the applications of the connectives $\neg$, $\&$, $\vee$ and $\supset$ (with the usual arities) and with the help of parenteheses behaving as punctuation symbols. Recall here that Fm can be understood as the absolutely free algebra generated by the set of operations $\{\neg, \vee, \&, \supset\}$ over Var. Some other standard definitions and notions from algebraic logic (such as homomorphism, quotient algebras, products, projections, subalgebras, Boolean algebras and so on) will be used, following the formalism of [1] or [13]. In this context, greek (capital) letters are metavariables over (sets of) formulas, meanwhile Roman letters will be used to denote elements/sets of the involved algebras. If necessary, some additional notions will be added along this paper.

2 F_n-structures: a brief overview

The starting point of our paper will be the definition of the C_n-logics, by means of a Hilbert-style axiomatization (even when we will work specifically with C_1). For that, recall the following well-known abbreviations:

$$\alpha^{\circ} := \neg\,(\alpha\&\neg\alpha)$$
$$\alpha^{(1)} := \alpha^{\circ}$$
$$\alpha^{(n)} := \alpha^{(n-1)}\&\left(\alpha^{(n-1)}\right)^{\circ}$$

Definition 2.1 Let n be a natural number. The **logic C_n** is given by means of a Hilbert-style axiomatics (defining the relation $\vdash_n \subseteq \wp(Fm)\times Fm$ as usual), such as it was presented in [5].

A1) $\alpha \supset (\beta \supset \alpha)$
A2) $(\alpha \supset \beta) \supset ((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \gamma))$
A3) $(\alpha \supset \gamma) \supset ((\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma))$
A4) $\alpha \& \beta \supset \alpha$
A5) $\alpha \& \beta \supset \beta$
A6) $\alpha \supset (\beta \supset \alpha \& \beta)$
A7) $\alpha \supset \alpha \vee \beta$
A8) $\beta \supset \alpha \vee \beta$
A9) $\alpha \vee \neg\alpha$
A10) $\neg\neg\alpha \supset \alpha.$
A11) $\beta^{(n)} \supset ((\alpha \supset \beta) \supset ((\alpha \supset \neg\beta) \supset \neg\alpha))$
A12) $\alpha^{(n)} \;\&\; \beta^{(n)} \supset (\alpha \;\&\; \beta)^{(n)},$
A13) $\alpha^{(n)} \;\&\; \beta^{(n)} \supset (\alpha \vee \beta)^{(n)}$
A14) $\alpha^{(n)} \;\&\; \beta^{(n)} \supset (\alpha \supset \beta)^{(n)}.$

The only rule of inference used here is Modus Ponens: $\dfrac{\alpha \supset \beta,\ \alpha}{\beta}.$

Remark 2.2 In the case of the logic C_1, the axioms $A11$)-$A14$) are simply:
A11) $\beta^\circ \supset ((\alpha \supset \beta) \supset ((\alpha \supset \neg\beta) \supset \neg\alpha))$
A12) $\alpha^\circ \;\&\; \beta^\circ \supset (\alpha \;\&\; \beta)^\circ$
A13) $\alpha^\circ \;\&\; \beta^\circ \supset (\alpha \vee \beta)^\circ$
A14) $\alpha^\circ \;\&\; \beta^\circ \supset (\alpha \supset \beta)^\circ.$

The following result lists some well-known properties of $\vdash_{C_n}$ (and, in particular, of $\vdash_{C_1}$) that will be used along this paper. We omit their proof, since they are already known in the specialized literature.

Proposition 2.3 For every $1 \leq n \leq \omega$ and $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$, the relation $\vdash_{C_n}$ verifies:
(1) (Syntactic Deduction Theorem): $\Gamma, \alpha \vdash_{C_n} \beta$ iff $\Gamma \vdash_{C_n} \alpha \supset \beta$.
(2) $\vdash_{C_n} \alpha \supset \alpha$
(3) $\alpha \supset \beta, \beta \supset \gamma \vdash_{C_n} \alpha \supset \gamma$
(4) For every connective $\# \in \{\vee, \&\}$, if $\Gamma \vdash_{C_n} \alpha \supset \gamma$ and $\Gamma \vdash_{C_n} \beta \supset \delta$, then $\Gamma \vdash_{C_n} \alpha \# \beta \supset \gamma \# \delta$.
(5) If $\Gamma \vdash_{C_n} \alpha \supset \beta$ and $\Gamma \vdash_{C_n} \gamma \supset \delta$, then $\Gamma \vdash_{C_n} (\beta \supset \gamma) \supset (\alpha \supset \delta)$.
(6) $\Gamma, \alpha^\circ, \alpha, \neg\alpha \vdash_{C_n} \beta$.
(7) $\vdash_{C_n} \alpha^\circ \vee \alpha$
(8) $\neg(\alpha^\circ \& \neg\alpha) \vdash_{C_n} \alpha$

With respect to the semantics for the C_n-logics, it was defined in [7] a model for them, by means of the class $\mathbb{F}_n$ of F_n-structures, as we said. We will recall the basics about these semantics. For that, let us fix some algebraic notation: every Boolean algebra will be denoted usually as an algebra $\mathbf{L} = (L, \vee, \wedge, \to, -, 0, 1)$, carrying this simbology the usual meaning. To avoid unnecessary notation, often we will denote every Boolean algebra **L** simply by L (its support). All the operations in Boolean algebras will be indexed when necessary.

Definition 2.4 An ***F*-structure for C_n** (or, simply, an F_n-structure) is a system:

$$\left\langle L, \{N_x\}_{x\in L}, \{N_x^{(n)}\}_{x\in L}\right\rangle$$

being L a bounded classical implicative lattice[1] of the form $(L, \vee, \wedge, \neg, \to, 0, 1)$, and the families $\{N_x\}_{x\in L}$ and $\{N_x^{(n)}\}_{x\in L}$ verify:

(F-1) For every $x \in L$, $\emptyset \neq N_x \subseteq L$ and:

(a) If $x' \in N_x$, then $x \vee x' = 1$.

(b) For every $x' \in N_x$ exists $x'' \in N_{x'}$, such that $x'' \leq x$.

(F-2) $\{N_x^{(n)}\}_{x\in L}$ is a family of non-void subsets of L.

(F-3) If $x' \in N_x$ and $y' \in N_y$, then exists $(x \wedge y)' \in N_{x\wedge y}$ such that $(x \wedge y)' \leq x' \vee y'$.

(F-4) If $x^{(n)} \in N_x^{(n)}$ and $y^{(n)} \in N_y^{(n)}$, then exist $(x\vee y)^{(n)}$ in $N_{x\vee y}^{(n)}$ and $(x \to y)^{(n)}$ in $N_{(x\to y)}^{(n)}$, such that $x^{(n)} \wedge y^{(n)} \leq (x \vee y)^{(n)}$ and $x^{(n)} \wedge y^{(n)} \leq (x \to y)^{(n)}$.

(F-5) For every $x^{(n)} \in N_x^{(n)}$ exist $x' \in N_x$, $x'' \in N_{x'}$, $x^1 \in N_{x\wedge x'}$, $(x^1)' \in N_{x^1}$, $(x^1)'' \in N_{(x^1)'}$, $x^2 \in N_{x^1\wedge(x^1)'}$, $(x^2)' \in N_{x^2}$, $(x^2)'' \in N_{(x^2)'}$,..., $x^n \in N_{x^{n-1}\wedge(x^{n-1})'}$, such that:

(a) $(x^k)'' \leq x^k$ (with $k = 0, \ldots, n-1$; $x^0 = x$);[2]

(b) $x^k \leq (x^{k-1})' \vee (x^{k-1})''$ (with $k = 1, \ldots, n$);

(c) $(x^k)' \leq x^{k-1} \wedge (x^{k-1})'$ (with $k = 1, \ldots, n-1$);

(d) $x^{(n)} = x^1 \wedge x^2 \wedge \cdots \wedge x^n$;

(e) $x \wedge x' \wedge x^{(n)} = 0$.

(F-6) For every $x' \in N_x$, there are $x^{(n)} \in N_x^{(n)}$, $x'' \in N_{x'}$, $x^1 \in N_{x\wedge x'}$, $(x^1)' \in N_{x^1}$, $(x^1)'' \in N_{(x^1)'}$, $x^2 \in N_{x^1\wedge(x^1)'}$,..., $x^n \in N_{x^{n-1}\wedge(x^{n-1})'}$, such that conditions (F-5) (a)-(e) are satisfied.

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[1]Given a bounded classical implicative lattice $(L, \vee, \wedge, , \to, 0, 1)$, we can define a Boolean negation "$-$" by $-x := x \to 0$, for all $x \in L$. In turn, if $(L, \vee, \wedge, -, 0, 1)$ is a Boolean algebra, then can define a relative pseudocomplement "$\to$" by $x \to y := -x \vee y$, for all x, $y \in L$.

[2]Please, distinguish between x° (x is well-behaved) and x^0 ($=x$).

Remarks 2.5 The previous definition is textually transcribed from [7]. Note that in item (c) of condition (F-5) there is some misunderstanding with the superscripts: we suppose here that this item means, simply, that $(x^1)' \leq x \wedge x'$ (since $x^0 = x$, as it was already remarked). Therefore, it would be necessary to ask for the existence of $(x^1)' \in N_{x^1}$, additionally. It is possible to prove (see [11] and [10]) that this element always exists, anyway. This fact will justify the "missing" of (c) (for the case of C_1) in the next definition, as we shall see.

On the other hand, in [7] there is not formulated any condition similar to the given one in Definition 2.4 (F-4), for the case of $\wedge$. Anyway, in [10] it is shown that (at least when $n = 1$) the function $\wedge$ verifies such kind of properties, indeed. By the way, it should be remarked that this is not an obvious result.

Turning back to Definition 2.4, it can be defined (taking into account the previous remarks):

Definition 2.6 An ***F*₁-structure** is a system $\langle L, \{N_x\}_{x \in L}, \{N_x^\circ\}_{x \in L}\rangle$ such that:

(F-1) For every $x \in L$, $\emptyset \neq N_x \subseteq L$ and:

(a) If $x' \in N_x$, then $x \vee x' = 1$.

(b) For every $x' \in N_x$ exists $x'' \in N_{x'}$, such that $x'' \leq x$.

(F-2) $\{N_x^\circ\}_{x \in L}$ is a family of non-void subsets of L.

(F-3) For every x, $y \in L$, and for every $x' \in N_x$, $y' \in N_y$, there is $(x \wedge y)' \in N_{x \wedge y}$ such that $(x \wedge y)' \leq x' \vee y'$.

(F-4) For every x, $y \in L$, if $x^\circ \in N_x^\circ$ and $y^\circ \in N_y^\circ$, then there are $(x \vee y)^\circ$ in $N_{x \vee y}^\circ$ and $(x \rightarrow y)^\circ$ in $N_{(x \rightarrow y)}^\circ$, satisfying $x^\circ \wedge y^\circ \leq (x \vee y)^\circ$ and $x^\circ \wedge y^\circ \leq (x \rightarrow y)^\circ$.

(F-5) For every $x \in L$ and $x^\circ \in N_x^\circ$ there are $x' \in N_x$, $x'' \in N_{x'}$ and $x^1 \in N_{x \wedge x'}$ such that:

(a) $x'' \leq x$;

(b) $x^1 \leq x' \vee x''$

(c) $x^\circ = x^1$;

(d) $x \wedge x' \wedge x^\circ = 0$.

(F-6) For every $x \in L$, $x' \in N_x$, there are $x^\circ \in N_x^\circ$, $x'' \in N_{x'}$ and $x^1 \in N_{x \wedge x'}$ satisfying conditions (a)-(d) of (F-5).

Remark 2.7 The previous formalism can be cleaned-up as follows: for every $x \in L$, let us define $\uparrow x$ ($\downarrow x$) as being the up-set (down-set) generated for x. In addition, let us define the *para-annihilator of* x [3] in the following way: $x^{\intercal}{:=}\{y \in L : x \vee y = 1\}$. Using this notions, an F_1-structure can be understood as a system $\langle L, \mathfrak{f}, \mathfrak{F}\rangle$ such that:

(f-1) $\mathfrak{f}$ is a function ($\mathfrak{f} : L \to \wp(L) \setminus \{\emptyset\}$), verifying:

(a) $\mathfrak{f}(x) \subseteq x^{\intercal}$;

(b) $\mathfrak{f}(y) \cap \downarrow x \neq \emptyset$ (for every $y \in \mathfrak{f}(x)$).

(f-2) $\mathfrak{F}$ is a function ($\mathfrak{F} : L \to \wp(L) \setminus \{\emptyset\}$).

(f-3) For every x, $y \in L$, for every $z \in \mathfrak{f}(x)$, for every $w \in \mathfrak{f}(y)$, it holds that $\mathfrak{f}(x \wedge y) \cap \big\downarrow (z \vee w) \neq \emptyset$.

(f-4) For every x, $y \in L$, for every $z \in \mathfrak{F}(x)$, for every $w \in \mathfrak{F}(y)$, it is satisfied:

(a) $\mathfrak{F}(x \vee y) \cap \big\uparrow (z \wedge w) \neq \emptyset$.

(b) $\mathfrak{F}(x \to y) \cap \big\uparrow (z \wedge w) \neq \emptyset$.

(f-5) For every $x \in L$ and $z \in \mathfrak{F}(x)$, there are $y \in \mathfrak{f}(x)$, $u \in \mathfrak{f}(y)$ and $v \in \mathfrak{f}(z)$, verifying:

(a) $u \leq x$;

(b) $z \leq y \vee u$;

(c) $v \leq x \wedge y$;

(d) $z \in f(x \wedge y)$;

(e) $x \wedge y \wedge z = 0$.

(f-6) For every $x \in L$, for every $y \in \mathfrak{f}(x)$, there are $z \in \mathfrak{F}(x)$, $u \in \mathfrak{f}(y)$, $v \in \mathfrak{f}(z)$, such that conditions (a)-(e) are satisfied.

The main result of [11] establishes that the function $\mathfrak{F}$ can be defined in terms of $\mathfrak{f}$ (in Fidel's formalism, this means that the families $N_x^{(1)} = N_x^{\circ}$ are obtained from the families N_x). Moreover:

[3]This notion is dual to the definition of *annihilator* established in [9] and used frequently in Universal Algebra.

Theorem 2.8 ([11], Theorems 3.7 and 3.8) Let $\langle L, \mathfrak{f}, \mathfrak{F}\rangle$ be an F_1-structure. Then:
1) for every $x \in L$, $\mathfrak{F}(x) := \{-x \vee -a : a \in \mathfrak{f}(x)\}$, for every $x \in L$.
2) Every F_1-structure is, simply, *a pair* $\langle L, \mathfrak{f}\rangle$ verifying:
(F_1-1) L is a Boolean algebra.
(F_1-2) $\mathfrak{f} : L \to \wp(L)$ verifies, for every $x \in L$, $-x \in \mathfrak{f}(x) \subseteq x^{\mathsf{T}}$.

The previous result is essential: the characterization of F_1-structures presented in it will be the standard one to be used along this paper. Some illustrative examples of F_1-structures following this presentation of F_1-structures are shown in the sequel. First of all, the *canonical F_1-structure* (which will be used later) is presented as follows:

Definition 2.9 The **canonical F_1-structure** is $\langle B_2, \bar{\mathfrak{f}}\rangle$, being B_2 the standard two-valued Boolean algebra with support $\{0, 1\}$, $\bar{\mathfrak{f}}(0) = \{1\}$ and $\bar{\mathfrak{f}}(1) = \{0, 1\}$ (by the way: $\overline{\mathfrak{F}}(0) = \overline{\mathfrak{F}}(1) = \{1\}$).

Besides that, other examples of non-canonical F_1-structures are the following:

Example 2.10 Let B_4 the "Boolean algebra with support set $B_4 = \{a, b, 0, 1\}$ (see Figure 1).

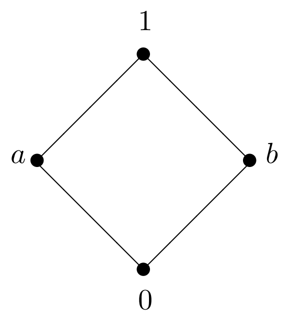

Figure 1: Hasse's Diagram of B_4

If we define $f : B_4 \to \wp(B_4)$ as follows: $\mathfrak{f}(0) = \{1\}$; $\mathfrak{f}(a) = \{b\}$; $\mathfrak{f}(b)= \{a, 1\}$ and $\mathfrak{f}(1)= \{0, a\}$, then $\langle B_4, \mathfrak{f}\rangle$ is an F_1-structure.

We shall see other interesting examples of F_1-structures along this paper. We will conclude this section enumerating some basic properties of the F_1-structures, to be used later. Their proof is very easy.

Proposition 2.11 For every F_1-structure $\langle L, \mathfrak{f}\rangle$, for every x, $y \in L$, the following properties are valid:
(1) If $a \in \mathfrak{f}(x)$, then $-x \leq a$.
(2) If $a \in \mathfrak{f}(x)$, then $\mathfrak{f}(a) \cap \downarrow x \neq \emptyset$.
(3) If $z \in \mathfrak{f}(x)$ and $w \in \mathfrak{f}(y)$, then $\mathfrak{f}\,(x \wedge y) \cap \big\downarrow (z \vee w) \neq \emptyset$.
(4) If $0 \in \mathfrak{f}(x)$, then $x = 1$.
(5) If $x = 0$, then $\mathfrak{f}(x) = \{1\}$.

3 Simplification of C_1-Completeness

Despite their algebraic-relational properties, let us recall that F_1-structures were defined with the aim of giving a Completeness Theorem for C_n (and for C_1, in particular). For that, M. Fidel defines in [7] some functions that interpret the formulas of Fm in F_1-structures. We will proceed in the same way (but taking into account the new definition of F_1-structures, cf. Theorem 2.8). This will allow us to obtain the simplified Completeness Theorem. This process will be developed in the sequel.

Definition 3.1 Let $\langle L, \mathfrak{f}\rangle$ be an F_1-structure. An F_1**-valuation** into $\langle L, \mathfrak{f}\rangle$ is a map $v : Fm \to L$, verifying:
(a) v behaves homomorphically w. r. t. $\vee$, & and $\supset$. That is, $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$; $v(\alpha \& \beta) = v(\alpha) \wedge v(\beta)$ and $v(\alpha \supset \beta) = -v(\alpha) \vee v(\beta) = v(\alpha) \to v(\beta)$.
(b) $v(\alpha^\circ) \wedge v(\alpha) \wedge v\,(\neg\alpha) = 0$
(c) With respect to $\neg$, v verifies (for every $\alpha \in Fm$):
  (c.1) $v(\neg\alpha) \in \mathfrak{f}(v(\alpha))$
  (c.2) $-v(\neg\neg\alpha) \in \mathfrak{f}(v(\alpha))$
(d) Finally, v verifies $-v\,(\beta^\circ) \vee -v\,(\gamma^\circ) \in \mathfrak{f}(v(\alpha^\circ))$, for every α of the form $\beta \vee \gamma$, or $\beta \,\&\, \gamma$, or $\beta \supset \gamma$.

If there is no risk of confussion, the F_1-valuations into $\langle L, \mathfrak{f}\rangle$ will be mentioned as F_1-valuations into L, or even as L-valuations, simply. It is easy to see that, given an arbitrary F_1-structure $\langle L, \mathfrak{f}\rangle$, any standard Boolean homomorphism $h : Fm \to L$ is, obviously, a F_1-valuation into L (which verifies, aditionally, $v(\neg\alpha) = -v(\alpha)$ for every $\alpha \in Fm$). We shall see other (specific) F_1-valuations in the next two examples. For that, let us use the following notation: $\neg^k\alpha$ denotes $\underbrace{\neg\neg\ldots\neg}_{k \text{ times}}\alpha$, being k a natural number. In addition, the set of *literals of* Fm is the set Lit:=$\{\neg^k p : p \in Var, \textbf{with } 0 \leq k \leq 1\ \}$. Besides, the notion of *Boolean extension* deserves to be highlighted:

Definition 3.2 For every Boolean algebra L, every subset $A \subseteq Fm$, every map $h : A \to L$ and every map $w : Var \to L$, the **Boolean extension**

determined by $\mathbb{A} := (A, w, h)$ is the function $v_{\mathbb{A}} : Fm \to B_2$ recursively defined in this way:
(1) Given $\alpha \in Var$: if $\alpha \in A$, then $v_{\mathbb{A}}(\alpha){:=}h(\alpha)$. If not, then $v_{\mathbb{A}}(\alpha) := w(\alpha)$.
(2) If $\alpha = \neg\beta$. Then: if $\alpha \in A$, $v_{\mathbb{A}}(\alpha){:=}h(\alpha)$. If not, $v_{\mathbb{A}}(\alpha) := v_{\mathbb{A}}(\neg\beta){:=}{-}v_{\mathbb{A}}(\beta)$.
(3) If $\alpha = \beta\#\gamma$, with $\# \in \{\vee, \&, \supset\}$: if $\alpha \in A$, $v_{\mathbb{A}}(\alpha){:=}h(\alpha)$. Otherwise, $v_{\mathbb{A}}(\alpha)$ behaves homomorphically. That is, $v_{\mathbb{A}}(\beta \vee \gamma){:=}\ v_{\mathbb{A}}(\beta) \vee v_{\mathbb{A}}(\gamma)$ (and in a similar way when $\#$ is & or $\supset$).

It is obvious that $v_{\mathbb{A}}$ is a well-defined map that extends h, but it does not extend w. Besides that, the expression "Boolean extension" comes from the fact that $v_{\mathbb{A}}$ behaves as a Boolean homomorphism "outside A". This notion will help us to give the following examples:

Example 3.3 Let $\langle L, \mathfrak{f}\rangle$ be any F_1-structure such that $1 \in \mathfrak{f}(1)$ and $A := Lit$. We define the map $h_1 : A \to L$ as $h_1(\alpha) = 1$ for every $\alpha \in A$. For every function $w : Fm \to L$ [4], the Boolean extension $v_{\mathbb{B}}$ is an F_1-valuation into L, with $\mathbb{B} := (A, w, h_1)$. In fact:
Condition **(a)** of Def. 3.1 is verified from (3). In addition, given $\alpha \in Fm$, applying (2) and (4), we get $v_{\mathbb{B}}(\alpha^\circ) = v_{\mathbb{B}}(\neg(\alpha\&\neg\alpha)) = -v_{\mathbb{B}}(\alpha\&\neg\alpha) = -v_{\mathbb{B}}(\alpha) \vee -v_{\mathbb{B}}(\neg\alpha)$. From this, consider the following cases: If $\alpha \in Var$, then $v_{\mathbb{B}}(\alpha^\circ) = -h_1(\alpha) \vee -h_1(\neg\alpha) = 0$. Otherwise, $\alpha \notin Var$ (and then $\neg\alpha \notin Lit$), so, in this case, $v_{\mathbb{B}}(\neg\alpha) = -v_{\mathbb{B}}(\alpha)$. Moreover, $v_{\mathbb{B}}(\alpha^\circ) = -v_{\mathbb{B}}(\alpha) \vee --v_{\mathbb{B}}(\alpha) = -v_{\mathbb{B}}(\alpha) \vee v_{\mathbb{B}}(\alpha) = 1$. From all these results, we have: if $\alpha \in Var$, then $v_{\mathbb{B}}(\alpha^\circ) \wedge v_{\mathbb{B}}(\alpha) \wedge v_{\mathbb{B}}(\neg\alpha) = 0 \wedge v_{\mathbb{B}}(\alpha) \wedge v_{\mathbb{B}}(\neg\alpha) = 0$. On the other hand, if $\alpha \notin Var$, then $v_{\mathbb{B}}(\alpha^\circ) \wedge v_{\mathbb{B}}(\alpha) \wedge v_{\mathbb{B}}(\neg\alpha) = 1 \wedge v_{\mathbb{B}}(\alpha) \wedge -v_{\mathbb{B}}(\alpha) = 0$. From all this, **(b)** is satisfied.
Condition **(c)** can be verified in a similar way, as follows: if $\alpha \in Var$, then $v_{\mathbb{B}}(\alpha) = v_{\mathbb{B}}(\neg\alpha) = 1$, and $v_{\mathbb{B}}(\neg\neg\alpha) = 0$. Moreover, $-v_{\mathbb{B}}(\neg\neg\alpha) = 1$, too. Thus, $\{v_{\mathbb{B}}(\neg\alpha), -v_{\mathbb{B}}(\neg\neg\alpha)\} = \{1\} \subseteq \mathfrak{f}(v_{\mathbb{B}}(\alpha))$, by hypothesis. On the other hand, if $\alpha \notin Var$, then we have $v_{\mathbb{B}}(\neg\alpha) = -v_{\mathbb{B}}(\neg\neg\alpha) = -v_{\mathbb{B}}(\alpha) \in \mathfrak{f}(v_{\mathbb{B}}(\alpha))$, from condition ($\mathrm{F}_1$-2) of Theorem 2.8. Note finally that, for every $\alpha \in Fm$ it holds:

$$v_{\mathbb{B}}(\alpha^\circ) = \begin{cases} 0 & \text{if } \alpha \in Var \\ 1 & \text{otherwise} \end{cases} \qquad (*)$$

In particular, for every $\alpha = \beta\#\gamma$, $v_{\mathbb{B}}(\alpha^\circ) = 1$. From this, Theorem 2.8 (F_1-2) and hypothesis, we have $\{0, 1\} \subseteq \mathfrak{f}(v_{\mathbb{B}}(\alpha^\circ))$. Besides that, $(*)$ implies additionally that $-v_{\mathbb{B}}(\beta^\circ) \vee -v_{\mathbb{B}}(\gamma^\circ) \subseteq \{0, 1\}$, always. Thus, condition **(d)** of Definition 3.1 is valid, too.

[4] In this case, the map w is not important, because $Var \subseteq A$. By the way, the same situation happens in every $h_1 : A \to Fm$ such that $Var \subseteq A$.

Note that the F_1-valuation of the previous example verifies a special condition: for every $\alpha \in Fm$, the value $v_{\mathbb{B}}(\alpha^\circ)$ only can be 0 or 1. This property *is not valid in general terms*, neither for every F_1-structure nor for every F_1-valuation. The following example shows such a kind of a "not so good" F_1-valuation:

Example 3.4 Let $\langle B_4, \mathfrak{f}\rangle$ be the F_1-structure given in Example 2.10, $A := Lit$ and the function $h_2 : A \to B_4$ by: for every $\alpha \in A$, $h_2(\alpha) := \begin{cases} 1 \text{ if } \alpha \in Var \\ a \text{ if } \alpha \notin Var \end{cases}$.
It can be proved (proceeding as in Example 3.3) that if $\mathbb{C} := (A, w, h_2)$, then the Boolean extension $v_{\mathbb{C}}$ is an F_1-valuation, too (for any map $w : Var \to B_4$). By the way, it is easy to check that, for every $\alpha \in Var$, $v_{\mathbb{C}}(\alpha^\circ) = b$ (and, if $\alpha \notin Var$, then $v_{\mathbb{C}}(\alpha^\circ) = 1$), as it was claimed above.

Some basic properties of the F_1-valuations, to be used along this paper, are:

Proposition 3.5 Every F_1-valuation v on an arbitrary F_1-structure $\langle L, \mathfrak{f}\rangle$ verifies (for every $\alpha \in Fm$):
(1) $v(\alpha^\circ) = -v(\alpha) \vee -v(\neg\alpha)$
(2) $v(\alpha) \vee v(\alpha^\circ) = 1$
(3) $v(\alpha) \vee v(\neg\alpha) = 1$
(4) $v(\neg\neg\alpha) \leq v(\alpha)$
(5) $v(\neg\alpha) \vee v(\neg\neg\alpha) = 1$
(6) $v(\alpha) = -v(\alpha^\circ) \vee -v(\neg\alpha)$
(7) $v(\alpha \& \beta) = 1$ if, and only if, $v(\alpha) = 1$ and $v(\beta) = 1$.
(8) If $v(\alpha) = 0$, then $v(\neg\alpha) = v(\alpha^\circ) = 1$ and $v(\neg\neg\alpha) = 0$.
(9) If $v(\alpha) = 1$, then $v(\alpha^\circ) = -v(\neg\alpha)$.
(10) If $v(\neg\alpha) = 0$, then $v(\alpha) = v(\neg\neg\alpha) = 1$
(11) If $v(\neg\alpha) = 1$, then $-v(\alpha) = v(\alpha^\circ)$.
(12) If $v(\alpha^\circ) = 0$, then $v(\alpha) = v(\neg\alpha) = 1$.
(13) If $v(\alpha^\circ) = 1$, then $-v(\alpha) = v(\neg\alpha)$.
(14) If $v(\neg\neg\alpha) = 1$, then $v(\alpha) = 1$ and $v(\alpha^\circ) = v((\neg\alpha)^\circ)$.
(15) If $v(\beta^\circ) = v(\alpha \supset \beta) = v(\alpha \supset \neg\beta) = 1$, then $v(\alpha) = 0$.
(16) If $v(\alpha^\circ) = v(\beta^\circ) = 1$, then $v(\gamma^\circ) = 1$, for every $\gamma = \alpha \# \beta$ (with $\# \in \{\vee, \&, \supset\}$).
(17) If $v(\alpha) = v(\beta) = 0$ and $\gamma = \alpha \# \beta$ (with $\# \in \{\vee, \&, \supset\}$), then $v(\gamma^\circ) = 1$.

Proof. First of all, let $\langle L, \mathfrak{f}\rangle$ be an F_1-structure, v an F_1-valuation into L, and $\alpha \in Fm$: from (**c**), $v(\alpha^\circ) = v(\neg(\alpha \& \neg\alpha)) \in \mathfrak{f}(v(\alpha \& \neg\alpha)) \subseteq v(\alpha \& \neg\alpha)^{\mathsf{T}}$ (recalling Theorem 2.8). Hence, considering (**a**), $1 = v(\alpha^\circ) \vee v(\alpha \& \neg\alpha) = v(\alpha^\circ) \vee (v(\alpha) \wedge v(\neg\alpha))$ $(*)$. Now, since L is a Boolean algebra and taking into account (**b**), we have $v(\alpha^\circ) = -(v(\alpha) \wedge v(\neg\alpha)) = -v(\alpha) \vee -v(\neg\alpha)$. Moreover, since $v(\alpha) \wedge v(\neg\alpha) \leq v(\alpha)$, from $(*)$ we have $v(\alpha) \vee v(\alpha^\circ) = 1$. Thus, (1) and (2) are satisfied. Besides, from (**c**) we have $v(\neg\alpha) \in \mathfrak{f}(v(\alpha))$; $-v(\neg\neg\alpha) \in \mathfrak{f}(v(\alpha))$. So, by Theorem 2.8, it holds $v(\alpha) \vee v(\neg\alpha) = 1$. In addition, from Prop. 2.11(1), it holds $-v(\alpha) \leq -v(\neg\neg\alpha)$. So, (3) and (4) are valid, too. Property (5) is also valid, since $v(\neg\neg\alpha) \in \mathfrak{f}(v(\neg\alpha)) \subseteq v(\neg\alpha)^{\intercal}$. With respect to (6), let us note that (from (2) and (3)) $v(\alpha) \vee (v(\alpha^\circ) \wedge v(\neg\alpha)) = 1$. This fact, together with (**b**), implies $v(\alpha) = -(v(\alpha^\circ) \wedge v(\neg\alpha)) = -v(\alpha^\circ) \vee -v(\neg\alpha)$. Note that (7) is trivial from Definition 3.1. Supposing now that $v(\alpha) = 0$, we have $v(\alpha^\circ) \wedge v(\neg\alpha) = -v(\alpha) = 1$, from (6). So, $v(\neg\alpha) = v(\alpha^\circ) = 1$. And, since $-v(\neg\neg\alpha) \in \mathfrak{f}(v(\alpha)) = \{1\}$ (by Prop. 2.11(5)), we have that $v(\neg\neg\alpha) = 0$. So, (8) is valid. To prove (9), suppose that $v(\alpha) = 1$ and apply (1). For (10): if $v(\neg\alpha) = 0$, then (by (6) again), $v(\alpha) = 1$. So, from (8) and (9), $v(\neg\neg\alpha) = v(\alpha^\circ) = 1$. Now, supposing $v(\neg\alpha) = 1$ and applying (6) one more time, (11) is proved. Suppose now that $v(\alpha^\circ) = v(\neg(\alpha \& \neg\alpha)) = 0$. From (10), it holds $v(\alpha \& \neg\alpha) = 1$, and then $v(\alpha) = v(\neg\alpha) = 1$ (by (7)). Thus, (12) holds, too. Item (13) is valid supposing $v(\alpha^\circ) = 1$ and applying (6). To prove (14), if $v(\neg\neg\alpha) = 1$, then $0 = -v(\neg\neg\alpha) \in \mathfrak{f}(v(\alpha))$, from by Definition 3.1**(c.2)**. But this only can be valid if $v(\alpha) = 1$ (Prop. 2.11(4)). From this, (9) and (11), we have $v(\alpha^\circ) = -v(\neg\alpha) = v((\neg\alpha)^\circ)$, as it is expected. To prove (15), let us suppose that $v(\beta^\circ) = v(\alpha \supset \beta) = v(\alpha \supset \neg\beta) = 1$. From this and Definition 3.1**(a)** are valid: $-v(\alpha) \vee v(\beta) = 1$ $(*)$ and $-v(\alpha) \vee v(\neg\beta) = 1$ $(**)$. Now, from (13) and $(**)$, it holds $-v(\alpha) \vee -v(\beta) = 1$. This and $(*)$ together imply $-v(\alpha) = 1$. That is, $v(\alpha) = 0$. To prove (16), suppose $v(\alpha^\circ) = v(\beta^\circ) = 1$ and $\gamma = \alpha \# \beta$, with $\# \in \{\vee, \wedge, \supset\}$. Since $-v(\alpha^\circ) \vee -(\beta^\circ) \in \mathfrak{f}(v(\gamma^\circ))$ (by Def. 3.1**(d)**), we have $0 \in \mathfrak{f}(v(\gamma^\circ))$ and so, $v(\gamma^\circ) = 1$ from Prop. 2.11(4). Suppose now that $v(\alpha) = v(\beta) = 0$, and let γ be as in the previous item. From (8) and (16), we have $v(\gamma^\circ) = 1$. Therefore, it holds (17). This concludes the proof. ■

Turning back to completeness, the F_1-structures together with its F_1-valuations define several consequence relations on Fm:

Definition 3.6 For every arbitrary F_1-structure $\langle L, \mathfrak{f} \rangle$, the **consequence relation** $\models_{\mathbb{F}_1}^{\langle L, \mathfrak{f} \rangle}$ is defined in the following way: $\Gamma \models_{\mathbb{F}_1}^{\langle L, \mathfrak{f} \rangle} \alpha$ iff, for every L-valuation $v : Fm \to L$, $v(\Gamma) \subseteq \{1_L\}$ implies $v(\alpha) = 1_L$ (if there is no risk of confussion, the relation $\models_{\mathbb{F}_1}^{\langle L, \mathfrak{f} \rangle}$ will be indicated as $\models_{\mathbb{F}_1}^{L}$). In addition, the **consequence relation** $\models_{\mathbb{F}_1}$ **determined by the class** $\mathbb{F}_1$ is defined as usual: $\Gamma \models_{\mathbb{F}_1} \alpha$ iff $\Gamma \models_{\mathbb{F}_1}^{L} \alpha$ for every F_1-structure $\langle L, \mathfrak{f} \rangle$.

It is easy to see that $\models_{\mathbb{F}_1}$ is a *Tarskian consequence relation* (cf. the formalization of [2]). That is, it satisfies extensiveness, idempotency and monotonicity. Moreover, it verifies:

Proposition 3.7 For every F_1-valuation v on $\langle L, \mathfrak{f}\rangle$, $v(\alpha) = 1$ and $v(\alpha \supset \beta) = 1$, implies $v(\beta) = 1$.

Theorem 3.8 [Semantic Deduction Theorem] For every $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$, $\Gamma \cup \{\alpha\} \models_{\mathbb{F}_1} \beta$ iff $\Gamma \models_{\mathbb{F}_1} \alpha \supset \beta$.

Both results are valid because the F_1-valuations are homomorphic w.r.t. $\supset$, and L is a Boolean algebra.

We will prove soundness, now. For that, we need this result, previously:

Proposition 3.9 For every C_1-axiom α, for every F_1-valuation $v : Fm \to L$ (with $\langle L, \mathfrak{f}\rangle$ an arbitrary F_1-structure), it holds that $v(\alpha) = 1$.

Proof. Taking into account that v is homomorphic w.r.t. $\vee$, & and $\supset$, we have that $v(\alpha) = 1$, for every instance α of axioms $A1$) - $A8$) of Def. 2.1. Suppose now that α is an instance of $A9$) (that is, $\alpha = \beta \vee \neg\beta$). From Def. 3.1**(c.1)** and Theorem 2.8, we have $v(\neg\beta) \in \mathfrak{f}(v(\beta)) \subseteq v(\beta)^{\mathsf{T}}$. Thus, $v(\alpha) = v(\beta \vee \neg\beta) = v(\beta) \vee v(\neg\beta) = 1$. In addition, if α is an instance of $A10$) (i.e. $\alpha = \neg\neg\beta \supset \beta$) we have that $v(\neg\neg\beta \supset \beta) = -v(\neg\neg\beta) \vee v(\beta) = 1$, by Def. 3.1**(c.2)** and Theorem 2.8 again. Consider now when $\alpha := \beta^\circ \supset ((\gamma \supset \beta) \supset ((\gamma \supset \neg\beta) \supset \neg\gamma))$ (that is, α is an instance of $A11$)). Applying (**a**) and (**b**) now, and recalling that L is a Boolean algebra, we have $v(\alpha) = ((-v(\beta^\circ) \vee v(\gamma)) \wedge -v(\beta^\circ \& \beta \& \neg\beta)) \vee v(\neg\gamma) = ((-v(\beta^\circ) \vee v(\gamma)) \wedge 1) \vee v(\neg\gamma) = -v(\beta^\circ) \vee v(\gamma) \vee v(\neg\gamma) = 1$ (from Prop. 3.5(3)).

Finally, consider α as being an instance of the C_1-axioms of the form $A12$)-$A14$) (which have the common structure $\alpha := (\beta^\circ \& \gamma^\circ) \supset (\beta \# \gamma)^\circ$, with $\# \in \{\vee, \&, \supset\}$). Since L is a Boolean algebra, and applying Def. 3.1((**a**) and (**d**)), and Theorem 2.8, it results: $-v(\beta^\circ \& \gamma^\circ) = -[v(\beta^\circ) \wedge v(\gamma^\circ)] = -v(\beta^\circ) \vee -v(\gamma^\circ) \in \mathfrak{f}(v((\beta\#\gamma)^\circ)) \subseteq (v((\beta\#\gamma)^\circ))^{\mathsf{T}}$. From all this, $1 = -v(\beta^\circ \& \gamma^\circ) \vee v((\beta\#\gamma)^\circ) = v((\beta^\circ \& \gamma^\circ) \supset (\beta\#\gamma)^\circ)$. The analysis of these last axioms concludes the proof. ∎

Theorem 3.10 For every $\alpha \in Fm$, $\Gamma \vdash_{C_1} \alpha$ implies $\Gamma \models_{\mathbb{F}_1} \alpha$.

Proof. First of all, it can be proved a weak version of this result (specifically, $\vdash_{C_1} \alpha$ implies $\models_{\mathbb{F}_1} \alpha$), applying Propositions 3.9 and 3.7. From this, apply finitariness of $\vdash_{C_1}$, Propositions 2.3(1) and 3.8, and monotonicity of $\models_{\mathbb{F}_1}$. ∎

With respect to completeness, it will be proved by means of an "algebraic-relational" adaptation of the well-known Lindenbaum-Tarski process. For that, we will need the following definitions:

Definition 3.11 Given the F_1-structures $\langle L_i, \mathfrak{f}_i \rangle$ $(i = 1, 2)$, and $h : L_1 \to L_2$, we say that:
- h is an F_1**-homomorphism** if it satisfies:
  - h is a Boolean homomorphism.
  - For every $x \in L_1$, $h(\mathfrak{f}_1(x)) \subseteq \mathfrak{f}_2(h(x))$.
- h is an F_1**-bimorphism** if, in addition, $h(\mathfrak{f}_1(x)) = \mathfrak{f}_2(h(x))$ (for every $x \in L_1$).
- Finally, h is called an F_1**-isomorphism** iff it is a bijective bimorphism.

Referring to the previous definitions, this easy result (which is presented without proof) will be very useful later.

Proposition 3.12 Let $\langle L, \mathfrak{f} \rangle$ be an F_1-structure. If, for every Boolean isomorphism $h : L \to L'$, we define the map $\mathfrak{f}' : L' \to \wp(L')$ by: $\mathfrak{f}'(h(x)) := h(\mathfrak{f}(x))$, then the system $\langle L', \mathfrak{f}' \rangle$ is an F_1-structure which is isomorphic (in the sense of Def. 3.11) to $\langle L, \mathfrak{f} \rangle$.

In addition, we have this obvious result:

Proposition 3.13 If $\langle L_1, \mathfrak{f}_1 \rangle$ and $\langle L_2, \mathfrak{f}_2 \rangle$ are F_1-isomorphic, Then $\Gamma \models^{L_1}_{\mathbb{F}_1} \alpha$ iff $\Gamma \models^{L_2}_{\mathbb{F}_1} \alpha$.

Besides that, the equivalence relation that will determine the Lindenbaum-structure is the same as the one used in Classical Logic (from now on we will be focused on *non-trivial theories* Γ; that is, sets Γ whose set of C_1-consequences is different from the own set Fm):

Proposition 3.14 For every $\Gamma \subseteq Fm$, the relation $\simeq_\Gamma \subseteq Fm \times Fm$ defined by:

$$\alpha \simeq_\Gamma \beta \text{ iff } \Gamma \vdash_{C_1} \alpha \supset \beta \;\text{ and }\; \Gamma \vdash_{C_1} \beta \supset \alpha$$

is an equivalence relation. In addition, it is compatible with respect to $\vee$, & and $\supset$.

Proof. Use results (2), (3), (4) and (5) of Proposition 2.3. ■

notation 3.15 When $\Gamma = \emptyset$ we will denote $\simeq_\Gamma$ simply by $\simeq$. The equivalence class of any formula $\alpha \in Fm$ will be denoted by $\|\alpha\|_\Gamma$. In addition, the quotient set determined by $\simeq_\Gamma$ will be denoted by $Fm/{\simeq_\Gamma}$.

Let us note that $\simeq_\Gamma$ is not compatible with $\neg$, in general terms. For instance (taking $\Gamma = \emptyset$ and $\alpha \in Var$) it is obvious that $\alpha \simeq \alpha \vee \alpha$, because we are dealing with formulas without negation. Now, considering the F_1-valuation of

Example 3.3 it is easy to see that $\not\models_{\mathbb{F}_1} \neg\alpha \supset \neg(\alpha \vee \alpha)$. So, by soundness, $\nvdash_{C_1} \neg\alpha \supset \neg(\alpha \vee \alpha)$. Therefore, $\neg\alpha \not\simeq \neg(\alpha \vee \alpha)$, if $\alpha \in Var$.

Besides that, defining naturally (by Prop. 3.14) the operations $\vee$, $\wedge$ and $\rightarrow$, we have that we can define an order relation in $Fm/\simeq_\Gamma$:

Proposition 3.16 For every $\Gamma \subseteq Fm$, the relation $\leq_\Gamma$ defined on $Fm/\simeq_\Gamma$ as follows: $\|\alpha\|_\Gamma \leq_\Gamma \|\beta\|_\Gamma$ iff $\|\alpha\|_\Gamma \wedge \|\beta\|_\Gamma = \|\alpha\|_\Gamma$ (iff $\|\alpha\|_\Gamma \vee \|\beta\|_\Gamma = \|\beta\|_\Gamma$) is a partial order. Moreover, $\leq_\Gamma$ can be defined in an alternative way by:

$$\|\alpha\|_\Gamma \leq_\Gamma \|\beta\|_\Gamma \text{ iff } \Gamma \vdash_{C_1} \alpha \supset \beta$$

Finally, with this definition, $(Fm/\simeq_\Gamma, \vee, \wedge, \rightarrow, 1_\Gamma)$ is a *classical implicative lattice* (CIL) with greatest element 1_Γ (cf. [4])[5], where 1_Γ (the greatest element of $Fm/\simeq_\Gamma$) verifies: $1_\Gamma = \|\alpha\|$ iff $\Gamma \vdash_{C_1} \alpha$.

Proof. It is a straightforward adaptation of the "weak version" (when $\Gamma = \emptyset$), proved in [7]. ■

In the next result, we will show that $Fm/\simeq_\Gamma$ can be "extended" to a Boolean algebra, in a non-standard way:

Lemma 3.17 Let $Fm/\simeq_\Gamma$, with Γ non-trivial. If we define the element $0_\Gamma := \|\alpha \& \alpha^\circ \& \neg\alpha\|_\Gamma$ (being $\alpha \in Fm$), and the map $- : Fm/\simeq_\Gamma \longrightarrow Fm/\simeq_\Gamma$ by: $-\|\alpha\|_\Gamma := \|\alpha^\circ \& \neg\alpha\|_\Gamma$, then $(Fm/\simeq_\Gamma, \vee, \wedge, -, 0_\Gamma, 1_\Gamma)$ is a Boolean algebra.

Proof. From Proposition 3.16, We only need to prove that:
(a) 0_Γ is a well-defined 0-ary operation, and it is the first element of $Fm/\simeq_\Gamma$: indeed, for every α, $\beta \in Fm$, $\alpha \& \alpha^\circ \& \neg\alpha \simeq_\Gamma \beta \& \beta^\circ \& \neg\beta$, by Prop. 2.3(6). In addition, for any $\alpha \in Fm$, $0_\Gamma = \|\alpha\|_\Gamma \wedge \|\alpha^\circ\|_\Gamma \wedge \neg\|\alpha\|_\Gamma \leq \|\beta\|_\Gamma$.
(b) The map $-$ is well defined: suppose $\|\alpha\|_\Gamma = \|\beta\|_\Gamma$: from (*a*), $\|\beta^\circ \& \neg\beta\|_\Gamma = 0_\Gamma \vee \|\beta^\circ \& \neg\beta\|_\Gamma$ $(\|\alpha\|_\Gamma \wedge \|\alpha^\circ \& \neg\alpha\|_\Gamma) \vee \|\beta^\circ \& \neg\beta\|_\Gamma = (\|\beta\|_\Gamma \wedge \|\alpha^\circ \& \neg\alpha\|_\Gamma) \vee \|\beta^\circ \& \neg\beta\|_\Gamma$. Since $\|\beta\|_\Gamma \vee \|\beta^\circ \& \neg\beta\|_\Gamma = (\|\beta\|_\Gamma \vee \|\beta^\circ\|_\Gamma) \wedge (\|\beta\|_\Gamma \vee \|\neg\beta\|_\Gamma) = 1_\Gamma$ (by Prop. 2.3(7) and *A*9), we have $\|\beta^\circ \& \neg\beta\|_\Gamma = 1_\Gamma \wedge (\|\alpha^\circ \& \neg\alpha\|_\Gamma \vee \|\beta^\circ \& \neg\beta\|_\Gamma) = \|\alpha^\circ \& \neg\alpha\|_\Gamma \vee \|\beta^\circ \& \neg\beta\|_\Gamma$, which implies $\|\alpha^\circ \& \neg\alpha\|_\Gamma \leq \|\beta^\circ \& \neg\beta\|_\Gamma$. The other inequality is similar.
(c) For every $\alpha \in Fm$, $\|\alpha\|_\Gamma \vee -\|\alpha\|_\Gamma = 1_\Gamma$: given α, we have $\|\alpha \vee \alpha^\circ\| = 1_\Gamma$ and $\|\alpha \vee \neg\alpha\|_\Gamma = 1_\Gamma$, by *A*9) and Prop. 2.3(7). Hence [6], $\|\alpha\|_\Gamma \vee -\|\alpha\|_\Gamma = \|\alpha\|_\Gamma \vee (\|\alpha^\circ\|_\Gamma \wedge \|\neg\alpha\|_\Gamma) = 1_\Gamma$.

[5]Roughly speaking, a CIL L is the $\{\vee; \wedge; \rightarrow\}$-reduct of a Boolean algebra. Therefore, it have greatest element $1_L = x \rightarrow x$, with $x \in L$.

[6]Even when it is an obvious warning, we prevent to the reader that $-\|\alpha\|_\Gamma$ *is not* $\|\neg\alpha\|_\Gamma$, usually.

(d) For every $\alpha \in Fm$, $\|\alpha\|_\Gamma \wedge -\|\alpha\|_\Gamma = 0_\Gamma$. It is obvious, from the Definition of 0_Γ and of $-\|\alpha\|_\Gamma$. This concludes the proof. ∎

Corollary 3.18 For every $\alpha, \beta \in Fm/\simeq_\Gamma$, it holds:
(1) $\|\alpha\|_\Gamma \to \|\beta\|_\Gamma = -\|\alpha\|_\Gamma \vee \|\beta\|_\Gamma$.
(2) $-\|\alpha\|_\Gamma = \|\alpha\|_\Gamma \to 0_\Gamma$.

The following additional property of $Fm/\simeq_\Gamma$ will be useful along all this paper:

Proposition 3.19 For every set $\Gamma \cup \{\alpha\} \subseteq Fm/\simeq_\Gamma$, $-\|\alpha^\circ\|_\Gamma = \|\alpha \& \neg\alpha\|_\Gamma$.

Proof. Consider $\Gamma \cup \{\alpha\} \subseteq Fm$: since $\neg(\alpha^\circ) = \neg\neg(\alpha\&\neg\alpha) \vdash_{C_1} \alpha\&\neg\alpha$ (by $A10$) of Def. 2.1), we have $\alpha^\circ \vee \neg\alpha^\circ \vdash_{C_1} \alpha^\circ \vee (\alpha\&\neg\alpha)$, applying Prop. 2.3(4). Therefore, $1_\Gamma = \|\alpha^\circ \vee (\alpha\&\neg\alpha)\|_\Gamma = \|\alpha^\circ\|_\Gamma \vee \|\alpha\&\neg\alpha\|_\Gamma$. Moreover, $0_\Gamma = \|\alpha\&\alpha^\circ\&\neg\alpha\|_\Gamma = \|\alpha^\circ\|_\Gamma \wedge \|\alpha\&\neg\alpha\|_\Gamma$. So, $-\|\alpha^\circ\|_\Gamma = \|\alpha\&\neg\alpha\|_\Gamma$, again by uniqueness of the Boolean complements. ∎

Definition 3.20 The **F_1-Lindenbaum structure relative to** Γ ($\Gamma \subseteq Fm$) is the system $\mathbb{L}_\Gamma := \langle Fm/\simeq_\Gamma, \mathfrak{f}_{\simeq_\Gamma}\rangle$, where $\langle Fm/\simeq_\Gamma, \vee, \wedge, -, 0_\Gamma, 1_\Gamma\rangle$ is the Boolean algebra determined in Lemma 3.17, and (for any $\|\alpha\|_\Gamma \in Fm/\simeq_\Gamma$), $f_{\simeq_\Gamma}(\|\alpha\|_\Gamma) := \{\|\lambda\|_\Gamma : \Gamma, \neg\lambda \vdash_1 \alpha\}$.

Theorem 3.21 The system $\mathbb{L}_\Gamma$ is an F_1-structure.

Proof. Taking into account Theorem 2.8, we only need to prove:
(a) $\mathfrak{f}_\simeq : Fm/\simeq_\Gamma \to \wp(Fm/\simeq_\Gamma)$ is well-defined: assuming $\alpha \simeq_\Gamma \beta$, let us prove that $\mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma) = \mathfrak{f}_{\simeq_\Gamma}(\|\beta\|_\Gamma)$, with $\mathfrak{f}_{\simeq_\Gamma}$ given as above. If $\|\lambda\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma)$, then $\|\neg\lambda\|_\Gamma \leq \|\alpha\|_\Gamma = \|\beta\|_\Gamma$. So, $\Gamma \vdash_{C_1} \neg\lambda \supset \beta$ (by Prop. 3.16). That is, $\Gamma, \neg\lambda \vdash_{C_1} \beta$. Then, by definition of $\mathfrak{f}_{\simeq_\Gamma}$, $\|\lambda\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\beta\|_\Gamma)$. The other inclusion is similar.
(b) for every $\alpha \in Fm$, $-\|\alpha\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma)$: from Prop. 2.3(8), it is valid $\Gamma \vdash_{C_1} \neg(\alpha^\circ\&\neg\alpha) \supset \alpha$. So, $\|\neg(\alpha^\circ\&\neg\alpha)\|_\Gamma \leq \|\alpha\|_\Gamma$, by Prop. 3.16. Therefore $-\|\alpha\|_\Gamma = \|\alpha^\circ\&\neg\alpha\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma)$.
(c) $\mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma) \subseteq \|\alpha\|_\Gamma{}^\intercal$: suppose $\|\lambda\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma)$ (that is, $\|\neg\lambda\|_\Gamma \leq \|\alpha\|_\Gamma$, from Prop. 3.16). Now, $1_\Gamma = \|\lambda \vee \neg\lambda\|_\Gamma = \|\lambda\|_\Gamma \vee \|\neg\lambda\|_\Gamma \leq \|\lambda\|_\Gamma \vee \|\alpha\|_\Gamma$ (by A9)). Thus, $\|\lambda\|_\Gamma \vee \|\alpha\|_\Gamma = 1_\Gamma$, which means $\|\lambda\|_\Gamma \in \|\alpha\|_\Gamma^\intercal$. ∎

Proposition 3.22 For every $\Gamma \subseteq Fm$, the map $q_\Gamma : Fm \to Fm/\simeq_\Gamma$ defined by $q_\Gamma(\alpha) := \|\alpha\|_\Gamma$ is an F_1-valuation into the F_1-structure $\mathbb{L}_\Gamma$ (which will be called the *canonical* $\mathbb{L}/\Gamma$*-valuation*).

Proof. Let q_Γ be defined as above. Taking into account Proposition 3.16, it is clear that q_Γ satisfies **(a)** of Def. 3.1. On the other hand, condition **(b)** is satisfied by the definition of 0_Γ, included in Lemma 3.17.
Let us prove condition **(c)**, now. That is (for every $\alpha \in Fm/\simeq_\Gamma$):
(c.1) $q_\Gamma(\neg\alpha) \in \mathfrak{f}(q_\Gamma(\alpha))$: this is valid from axiom A10) and Definition 3.20.
(c.2) $-(q_\Gamma(\neg\neg\alpha)) \in \mathfrak{f}_\Gamma(q_\Gamma(\alpha))$: we have $\neg((\neg\neg\alpha)^\circ \& \neg(\neg\neg\alpha)) \vdash_{C_1} \neg\neg\alpha$ and $\neg\neg\alpha \vdash_{C_1} \alpha$, from Prop. 2.3(8) and A10). So, $\Gamma \vdash_{C_1} \neg((\neg\neg\alpha)^\circ \& \neg(\neg\neg\alpha)) \supset \alpha$. Thus, by the definition of $\mathfrak{f}_{\simeq_\Gamma}$, we get $\|(\neg\neg\alpha)^\circ \& \neg(\neg\neg\alpha)\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|\alpha\|_\Gamma) = \mathfrak{f}_{\simeq_\Gamma}(q_\Gamma(\alpha))$. Besides that, by definition of $-$, it holds $-q_\Gamma(\neg\neg\alpha) = -\|\neg\neg\alpha\|_\Gamma = \|(\neg\neg\alpha)^\circ \& \neg(\neg\neg\alpha)\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(q_\Gamma(\alpha))$ too, as it was expected.
Finally, let us prove **(d)**: $-q_\Gamma(\beta^\circ) \vee -q_\Gamma(\gamma^\circ) \in \mathfrak{f}_{\simeq_\Gamma}(q_\Gamma((\beta\#\gamma)^\circ))$, for every $\{\beta,\gamma\} \subseteq Fm/\simeq_\Gamma$ (with $\# \in \{\vee, \&, \supset\}$): using Prop. 2.3(8) again (and applying axioms A12)-A14)), we have $\neg((\beta^\circ\&\gamma^\circ)^\circ \& \neg(\beta^\circ\&\gamma^\circ)) \vdash_{C_1} \beta^\circ\&\gamma^\circ$ and $\beta^\circ\&\gamma^\circ \vdash_{C_1} (\beta\#\gamma)^\circ$. Thus, $\Gamma \vdash_{C_1} \neg((\beta^\circ\&\gamma^\circ)^\circ \wedge \neg(\beta^\circ\&\gamma^\circ)) \supset (\beta\#\gamma)^\circ$, by Prop. 2.3(3). So, it holds $\|(\beta^\circ\&\gamma^\circ)^\circ \wedge \neg(\beta^\circ\&\gamma^\circ)\|_\Gamma \in \mathfrak{f}_{\simeq_\Gamma}(\|(\beta\#\gamma)^\circ\|_\Gamma) = \mathfrak{f}_{\simeq_\Gamma}(q_\Gamma((\beta\#\gamma)^\circ))$, from Definition 3.20. In addition, we have $-\|\beta^\circ\&\gamma^\circ\|_\Gamma = \|(\beta^\circ\&\gamma^\circ)^\circ \wedge \neg(\beta^\circ\&\gamma^\circ)\|_\Gamma$, by Lemma 3.17. Summarizing all this, $-q_\Gamma(\beta^\circ\&\gamma^\circ)$ belongs to $\mathfrak{f}_{\simeq_\Gamma}(q_\Gamma((\beta\#\gamma)^\circ))$. Thus, q_Γ is an F_1-valuation. ■

Corollary 3.23 For every $\Gamma \subseteq Fm$, for every $\alpha \in Fm$, it holds that $q_\Gamma(\alpha) = 1_\Gamma$ if, and only if, $\Gamma \vdash_{C_1} \alpha$.

Proof. Immeditate, from Propositions 3.16 and 3.22. ■

Corollary 3.24 For every Γ, $q_\Gamma(\Gamma) \subseteq \{1_\Gamma\}$.

Proof. By the previous corollary and extensiveness of $\vdash_{C_1}$. ■

Finally, we get completeness:

Theorem 3.25 For every $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{\mathbb{F}_1} \alpha$ implies $\Gamma \vdash_{C_1} \alpha$.

Proof. By contrapositive: consider $\Gamma \cup \{\alpha\} \subseteq Fm$ such that $\Gamma \nvdash_{C_1} \alpha$. It is possible to define the F_1-structure $\mathbb{L}_\Gamma$, according Definition 3.20. Moreover, the map q_Γ defined following Proposition 3.22 is an F_1-valuation, verifying $q_\Gamma(\Gamma) \subseteq \{1_\Gamma\}$ and $q_\Gamma(\alpha) \neq 1_\Gamma$ (by Corollaries 3.24 and 3.23). Hence, $\Gamma \not\models_{\mathbb{F}_1}^{\mathbb{L}_\Gamma} \alpha$, and then $\Gamma \not\models_{\mathbb{F}_1} \alpha$. ■

4 Decidability of C_1, simplified

As it was previously remarked, the definition of F_n-structures (besides its use in completeness) had as main motivation the proof of decidability of the calculi

C_n. In this context we say that a given logic is *decidable* if and only if it set of valid formulas (the set of C_1-theorems, in this case) is decidable. So, we will not work with arbitrary theories Γ, but we will consider that $\Gamma = \emptyset$, simply. Turning back to Fidel's proof of decidability of C_1, it is possible to simplify it, too. For this, we will adapt Birkhoff's Theorem for subdirectly irreducible Boolean algebras. This result will be proved from the notion of F_1-homomorphism (already given), together with the definitions of F_1-substructure and F_1-product, that will be provided along this section.

Definition 4.1 An F_1-structure $\langle S, \mathfrak{g}\rangle$ is ***F_1*-substructure** of $\langle L, \mathfrak{f}\rangle$ iff:
(1) S is a Boolean subalgebra of L.
(2) For every $x \in S$, $\mathfrak{g}(x) \subseteq \mathfrak{f}(x)$.

It is worth to note here that, if the definition given above were expressed according the formalism of Model Theory (as it was done in [3] for the case of the logic **mbC**), it would not correspond to the standard notion of substructure: in the model-theoretic definition of substructure it is necessary that $\mathfrak{g}(x) = \mathfrak{f}(x) \cap S$. Actually, Def. 4.1 is referred to the notion known as *weak substructure* in the literature [7]. This weakening will be necessary to relate any F_1-structure with *its saturated version*, to be defined later.

Proposition 4.2 For every injective F_1-homomorphism h between $\langle L_1, \mathfrak{f}_1\rangle$ and $\langle L_2, \mathfrak{f}_2\rangle$, for every Boolean subalgebra S of L_1, if we define (for every $x \in S$) $\mathfrak{g}(h(x)) := h(\mathfrak{f}_1(x)) \cap h(S)$, then the system $\langle h(S), \mathfrak{g}\rangle$ is an F_1-substructure of $\langle L_2, \mathfrak{f}_2\rangle$.

Proof. Considering the hypotheses above, define $\mathfrak{g}(h(x)) := h(\mathfrak{f}_1(x)) \cap h(S)$. Obviously it holds $\mathfrak{g} : h(S) \to \wp(h(S))$, being $h(S)$ a Boolean subalgebra of L_2. In addition, for every $x \in S$, $\mathfrak{g}(h(x)) \subseteq \mathfrak{f}_2(h(x)) \cap h(S)$, from Def. 4.1. Besides that, $-x \in \mathfrak{f}_1(x) \cap S$ and then (since h is Boolean homomorphism): $-h(x) = h(-x) \in h(\mathfrak{f}_1(x)) \cap h(S) = \mathfrak{g}(h(x))$. From Def. 4.1, $\langle h(S), \mathfrak{g}\rangle$ is a substructure of $\langle L_2, \mathfrak{f}_2\rangle$. ■

Definition 4.3 The ***F_1*-product** of the family $\{(L_i, \mathfrak{f}_i)\}_{i\in I}$ of F_1-structures is the following system:

$$\boldsymbol{L_\pi} := \left\langle \prod_{i\in I} L_i, \mathfrak{f}_\pi \right\rangle \text{ (where, for every } \overline{x} \in \prod_{i\in I} L_i,\ \mathfrak{f}_\pi(\overline{x}) := \prod_{i\in I} \mathfrak{f}_i(\pi_i(\overline{x}))).$$

[7] By the way, the mentioned paper is the first one, to our knowledge, that distinguish the two model-teorethic notions compared here, in the context of F-structures.

Proposition 4.4 The F_1-product of F_1-structures is also an F_1-structure. In addition, for every $i \in I$, the projection map $\pi_i : \prod_{i \in I} L_i \to L_i$ is a surjective F_1-bimorphism.

Proof. Given a family $\{\langle L_i, \mathfrak{f}_i \rangle\}_{i \in I}$ of F_1-structures and considering that $\prod_{i \in I} L_i$ is the Boolean product of $\{L_i\}_{i \in I}$, it is easy to see that $\boldsymbol{L_\pi}$ satisfies the conditions established in Theorem 2.8. Besides that, given any $i \in I$, by Def. 4.3, $\pi_i(\mathfrak{f}_\pi(\overline{x}))= \pi_i\left(\prod_{i \in I} \mathfrak{f}_i(\pi_i(\overline{x}))\right) = \mathfrak{f}_i(\pi_i(\overline{x}))$. From this, π_i is a surjective F_1-bimorphism, since π_i is a Boolean epimorphism, for every $i \in I$. ■

We will prove now some basic properties of the F_1-valuations (recall Definition 3.1), related to all the notions given above. By the way, the items (**a**), (**b**), (**c**) and (**d**) that will be mentioned all along this section are always referred to the mentioned definition:

Proposition 4.5 Let h be an F_1-homomorphism from $\langle L_1, \mathfrak{f}_1 \rangle$ to $\langle L_2, \mathfrak{f}_2 \rangle$. For every F_1-valuation $v : Fm \to L_1$, the composition $h \circ v$ is an F_1-valuation into L_2.

Proof. From the hypothesis above indicated, we have that $h \circ v$ behaves homomorphically w.r.t. $\vee$, & and $\supset$, verifying (**a**), consequently. Condition (**b**) is valid because h is a Boolean homomorphism. Suppose $\alpha \in Fm$, now. Since v satisfies (**c**), $\{v(\neg\alpha), -v(\neg\neg\alpha)\} \subseteq \mathfrak{f}_1(v(\alpha))$. In addition, from Definition 3.11, it holds $h(\mathfrak{f}_1(v(\alpha))) \subseteq \mathfrak{f}_2((h \circ v)(\alpha))$. From all this, $(h \circ v)(\neg\alpha) = h(v(\neg\alpha)) \in \mathfrak{f}_2((h \circ v)(\alpha))$. Moreover, it is valid that $-(h \circ v)(\neg\neg\alpha) = h(-v(\neg\neg\alpha)) \in \mathfrak{f}_2((h \circ v)(\alpha))$. That is, $h \circ v$ verifies (**c**). Finally, consider $\beta, \gamma \in Fm$ and $\# \in \{\vee, \&, \supset\}$. Since v verifies Definition 3.1 (**d**), it holds $-v(\beta^\circ) \vee -v(\gamma^\circ) \in \mathfrak{f}_1(v((\beta\#\gamma)^\circ))$. Besides, since $h(\mathfrak{f}_1(v((\beta\#\gamma)^\circ))) \subseteq \mathfrak{f}_2(h(v((\beta\#\gamma)^\circ)))$, we have $h(-v(\beta^\circ) \vee -v(\gamma^\circ)) \in \mathfrak{f}_2(h(v((\beta\#\gamma)^\circ)))$. Then, $-(h \circ v)(\beta^\circ) \vee -(h \circ v)(\gamma^\circ) \in \mathfrak{f}_2((h \circ v)((\beta\#\gamma)^\circ))$ (since h and v are homomorphic w.r.t. $\vee$, & and $\supset$). Thus, $h \circ v$ verifies (**d**), too. ■

From this result it is easy to demonstrate:

Proposition 4.6 Let $\langle S, \mathfrak{g} \rangle$ be an F_1-substructure of $\langle L, \mathfrak{f} \rangle$. Every F_1-valuation into S is an F_1-valuation into L.

Proof. Consider $\langle L, \mathfrak{f} \rangle$ and $\langle S, \mathfrak{g} \rangle$ as defined above. Let v be an arbitrary F_1-valuation into S , and let $\alpha \in Fm$: from Def. 3.1 (**c**) and Def. 4.1, we have $\{v(\neg\alpha), -v(\neg\neg\alpha)\} \subseteq \mathfrak{g}\,(v(\alpha)) \subseteq \mathfrak{f}\,(v(\alpha))$. Now, let $\{\alpha, \beta\}$ be any pair of formulas. By (**d**), it holds: $-v\,(\alpha^\circ) \vee -v\,(\beta^\circ) \in \mathfrak{g}\,(v((\alpha\#\beta)^\circ)) \subseteq \mathfrak{f}\,(v((\alpha\#\beta)^\circ))$. The rest of the conditions are obviously valid. ■

From the previous result and Def. 3.6 it follows easily:

Proposition 4.7 If $\langle S, \mathfrak{g}\rangle$ is an F_1-substructure of $\langle L, \mathfrak{f}\rangle$, then $\models^{L}_{\mathbb{F}_1} \alpha$ implies $\models^{S}_{\mathbb{F}_1} \alpha$.

Concerning the relation between F_1-products and F_1-valuations note that, by its own definition, the "product of F_1-structures" induces a *product of valuations*, as we shall see in the sequel.

Definition 4.8 Let $\{\langle L_i, \mathfrak{f}_i\rangle\}_{i\in I}$ be a family of F_1-structures and let $\{v_i\}_{i\in I}$ a family of F_1-valuations $v_i : Fm \to L_i$. We define the map $v_\pi : Fm \to \prod_{i\in I} L_i$ by $v_\pi(\alpha) := (v_i(\alpha))_{i\in I}$, for every $\alpha \in Fm$.

Proposition 4.9 Let $\{\langle L_i, \mathfrak{f}_i\rangle\}_{i\in I}$ be a family of F_1-structures and consider the F_1-product $\boldsymbol{L_\pi}$. Then, it holds:
(1) The map $v_\pi : Fm \to \prod_{i\in I} L_i$ is an F_1-valuation into $\boldsymbol{L_\pi}$.
(2) If $v : Fm \to \prod_{i\in I} L_i$ is any F_1-valuation into $\boldsymbol{L_\pi}$, then $\pi_i \circ v : Fm \to L_i$ is an F_1-valuation into L_i, for every $i \in I$.

Proof. Item (1) follows straightforward, since the operations on the Boolean algebra $\prod_{i\in I} L_i$ are defined componentwise. On the other hand, every projection map $\pi_i : \prod_{i\in I} L_i \to L_i$ is an F_1-homomorphism, cf. Prop. 4.4. Applying Prop. 4.5 now, it follows (2). ■

Proposition 4.10 Let $\{\langle L_i, \mathfrak{f}_i\rangle\}_{i\in I}$ be a family of F_1-structures and let $\boldsymbol{L_\pi}$ be its F_1-product associated. For every $\alpha \in Fm$, it holds: if $\models^{L_i}_{\mathbb{F}_1} \alpha$ for every $i \in I$, then $\models^{\boldsymbol{L_\pi}}_{\mathbb{F}_1} \alpha$.

Proof. Let $\{\langle L_i, \mathfrak{f}_i\rangle : i \in I\}$ be a family of F_1-structures and let $\boldsymbol{L_\pi}$ be its product associated where, for every $\overline{x} \in \prod_{i\in I} L_i$, $\mathfrak{f}_\pi\,(\overline{x}) = \prod_{i\in I} \mathfrak{f}_i\,(\pi_i\,(\overline{x}))$ (from Def. 4.3). Let α be in Fm such that, for every $i \in I$, it holds $\models^{L_i}_{1} \alpha$, and let

$v : Fm \to \prod_{i\in I} L_i$ be any F_1-valuation into $\boldsymbol{L_\pi}$. From Prop. 4.9(2), we have that (for every $i \in I$) $\pi_i \circ v$ is an F_1-valuation into L_i. So, for every $i \in I$ it holds $\pi_i(v(\alpha)) = (\pi_i \circ v)(\alpha) = 1_i$, by our hypothesis. Thus, $v(\alpha) = (1_i)_{i\in I} = 1_\pi$. From this, $\models^{\boldsymbol{L_\pi}}_{\mathbb{F}_1} \alpha$. ■

We will define now a special kind of F_1-structures, essential to our proof of decidability:

Definition 4.11 For every Boolean algebra L, the **saturated F_1-structure determined by** L is the F_1-structure of the form: $\langle L, \overline{\mathfrak{f}_L}\rangle$, where $\overline{\mathfrak{f}_L}(x) := \{y \in L : x \vee y = 1\} = x^\intercal$.

It is easy to see that $\langle L, \overline{\mathfrak{f}_L}\rangle$ is an F_1-structure, indeed. It is also obvious that every Boolean algebra L can determine several F_1-structures, depending of the choice of the map $\mathfrak{f}$ involved. However, L can determine *only one saturated F_1-structure*. The following example shows this:

Example 4.12 The systems $\langle B_2, \mathfrak{f}\rangle$ and $\left\langle B_2, \overline{\mathfrak{f}_{B_2}}\right\rangle$ are all the possible F_1-structures that can be defined on B_2, where:

$$\mathfrak{f}(0) = \{1\} \qquad \overline{\mathfrak{f}_{B_2}}(0) = \{1\}$$
$$\mathfrak{f}(1) = \{0\} \qquad \overline{\mathfrak{f}_{B_2}}(1) = \{0, 1\}$$

In addition, $\left\langle B_2, \overline{\mathfrak{f}_{B_2}}\right\rangle$ is the saturated F_1-structure determined by B_2 (see Definition 4.11). By the way, it is the canonical F_1-structure, already presented in Example 2.9.

Example 4.13 The following is an example of a *non-saturated F_1-structure*. Let us consider the eight-element Boolean algebra B_8 (see Figure 2):

According to that figure, the complement of every element of B_8 is indicated below:

$\boldsymbol{x}$	0	a	b	c	d	e	f	1
$\boldsymbol{-x}$	1	e	d	f	b	a	c	0

Now, defining for every $x \in B_8$:

$$\mathfrak{f}(0) = \{1\} \qquad \mathfrak{f}(d) = \{b, e, 1\} \quad \mathfrak{f}(a) = \{e\} \qquad \mathfrak{f}(e) = \{a, d, f\}$$
$$\mathfrak{f}(b) = \{d, 1\} \qquad \mathfrak{f}(f) = \{c, d\} \quad \mathfrak{f}(c) = \{f\} \qquad \mathfrak{f}(1) = \{0, d\}$$

it is not difficult to check that $\langle B_8, \mathfrak{f}\rangle$ is an F_1-structure, indeed. Moreover, it is obviously a non-saturated one.

The following result is obvious.

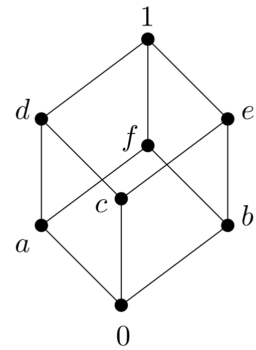

Figure 2: Hasse's Diagram of B_8

Proposition 4.14 For every F_1-structure $\langle L, \mathfrak{f}\rangle$ it holds:
(1) $\langle L, \mathfrak{f}\rangle$ is substructure of the saturated structure $\langle L, \overline{\mathfrak{f}_L}\rangle$.
(2) If S is a subalgebra of L, then $\langle S, \overline{\mathfrak{f}_S}\rangle$ is an F_1-substructure of $\langle L, \overline{\mathfrak{f}_L}\rangle$.

On the other hand, to symbolize the saturated F_1-structure of the Boolean product of a family of algebras, we write $\overline{\boldsymbol{L_\pi}} = \left\langle \prod_{i\in I} L_i, \overline{\mathfrak{f}_\pi} \right\rangle$.

Proposition 4.15 Let $\{\langle L_i, \mathfrak{f}_{L_i}\rangle : i \in I\}$ be a family of F_1-structures, and let $\boldsymbol{L_\pi}$ be its F_1-product. Then $\boldsymbol{L_\pi} = \overline{\boldsymbol{L_\pi}}$ iff, for every $i \in I$, $\mathfrak{f}_{L_i} = \overline{\mathfrak{f}_{L_i}}$.

Proof. It follows from Def. 4.3 and this fact: for every $\overline{x} = (x_i)_{i\in I} \in \prod_{i\in I} L_i$, it holds that $\overline{x}^{\mathsf{T}} = \prod_{i\in I}(\pi_i(\overline{x}))^{\mathsf{T}} = \prod_{i\in I}(x_i)^{\mathsf{T}}$. ■

All the previous definitions and results allow us to obtain the following Representation Theorem for F_1-structures:

Theorem 4.16 Every F_1-structure $\langle L, \mathfrak{f}\rangle$ is F_1-isomorphic (in the sense of Def. 3.11) to a substructure of the saturated F_1-structure determined by the Boolean product of a family $\{L_i\}_{i\in I}$, where $L_i = B_2$, for every $i \in I$.

Proof. Given $\langle L, \mathfrak{f}\rangle$, it is well-known that L is representable as a subdirect product of a family $\{L_i\}_{i\in I}$, where $L_i = B_2$, for every $i \in I$ (see [1]). So, there is L' verifying:
(a) There exists an (algebraic) isomorphism $h : L \to L'$.
(b) L' is subalgebra of $\prod_{i\in I} L_i$.
Now, from Prop. 3.12 and (a), $\langle L', \mathfrak{f}'\rangle$ is an F_1-structure, where $\mathfrak{f}'(h(x)) := h(\mathfrak{f}(x))$, for every $x \in L$. Moreover, h is an F_1-isomorphism. Besides that,

from Prop. 4.14(1), $\langle L', \mathfrak{f}' \rangle$ is an F_1-substructure of $\left\langle L', \overline{\mathfrak{f}_{L'}} \right\rangle$, the saturated F_1-structure of L', which, in turn, is a substructure of the F_1-saturated product $\overline{\boldsymbol{L_\pi}}$ (apply Prop. 4.14(2) and (b), now). This completes the proof. ■

Theorem 4.17 For every $\alpha \in Fm$ the following conditions are equivalent:
(i) $\models_{\mathbb{F}_1} \alpha$
(ii) $\models_{\mathbb{F}_1}^{\langle L, \overline{\mathfrak{f}_L} \rangle} \alpha$ for every saturated F_1-structure $\langle L, \overline{\mathfrak{f}_L} \rangle$.
(iii) $\models_{\mathbb{F}_1}^{\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle} \alpha$.

Proof. It is obvious that (i) implies (ii) and (ii) implies (iii). To prove that (iii) implies (i), let us suppose $\models_{\mathbb{F}_1}^{\langle B_2, \bar{\mathfrak{f}} \rangle} \alpha$, and let us consider any F_1-structure $\langle L, \mathfrak{f} \rangle$. Then, by Theorem 4.16, there is an F_1-isomorphism $h : L \to L'$, being $\langle L', \mathfrak{f}' \rangle$ an F_1-substructure of a F_1-saturated structure $\overline{\boldsymbol{L_\pi}} = \left\langle \prod_{i \in I} L_i, \overline{\mathfrak{f}_\pi} \right\rangle$ where, for every $i \in I$, $L_i = B_2$. Now, according Prop. 4.15, $\overline{\boldsymbol{L_\pi}}$ is the product of the saturated F_1-structures $\left\langle L_i, \overline{\mathfrak{f}_{L_i}} \right\rangle$. From this, our hypothesis and Prop. 4.10, we have $\models_{\mathbb{F}_1}^{\overline{\boldsymbol{L_\pi}}} \alpha$. Hence, from Prop. 4.7, it holds $\models_{\mathbb{F}_1}^{\langle L', \mathfrak{f}' \rangle} \alpha$. Thus, by Prop. 3.13, $\models_{\mathbb{F}_1}^{\langle L, \mathfrak{f} \rangle} \alpha$. Since $\langle L, \mathfrak{f} \rangle$ is arbitrary, $\models_{\mathbb{F}_1} \alpha$. ■

Corollary 4.18 The logic C_1 is decidable.

Proof. The algebra B_2 is finite. Besides, for every $\alpha \in Fm$, there is a finite number, m_α, of F_1-valuations from Fm to $\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle$, necessary to test if $\models_{\mathbb{F}_1}^{\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle} \alpha$. Actually, m_α is bounded by 2^{k_α}, being k_α the cardinal of the set of all the subformulas of α. ■

As it was pointed out in the previous result, the number m_α of F_1-valuations needed to test the validity of a given formula $\alpha \in Fm$ is finite. However, it is not clear exactly how to obtain m_α, and which are the expressions that determine its obtention. This is an interesting problem, to be discussed later.

5 F_1-structures and Quasi-matrices, compared

In [6] it was proposed a semantics for the C_n-logics (and, in particular, for C_1) based on the so-called *quasi-matrices*. It was one of the more "algorithmic" approaches for this family of logics [8] and it was, *a priori*, different from the

[8]Of course, neither F-structures nor quasi-matrices are the only semantic approaches for the C_n-logics. It deserve to be mentioned the Possible-Translation Semantics, Tableaux Methods and Kripke-style semantics, among others. We refer the interested reader to [2], for a very updated survey of this an other related topics.

F_1-structures-based semantics. We will show in the sequel that, even with different motivations, the quasi-matrix semantics and the semantics focused on the canonical F_1-structure $\langle B_2, \overline{\mathfrak{f}_{B_2}}\rangle$ are, essentially, the same. We will begin this analysis with a simple proof of some properties of F_1-valuations that are specific for $\langle B_2, \overline{\mathfrak{f}_{B_2}}\rangle$ (which will be called simply as *canonical F_1-valuations*, from now on):

Proposition 5.1 Every canonical F_1-valuation v verifies (for any α, $\beta \in Fm$):
(1) $v\,(\alpha \vee \beta) = 1$ if, and only if, $v\,(\alpha) = 1$ or $v\,(\beta) = 1$.
(2) $v\,(\alpha \supset \beta) = 1$ if, and only if, $v\,(\alpha) = 0$ or $v\,(\beta) = 1$.
(3) If $\gamma = \alpha\#\beta$, with $\# \in \{\vee, \&, \supset\}$, and $v\,(\gamma^\circ) = 0$, then $v\,(\alpha) = 1$ or $v\,(\beta) = 1$.

Proof. Consider any F_1-valuation v on $\langle B_2, \overline{\mathfrak{f}_{B_2}}\rangle$. From Def. 3.1**(a)**, (1) and (2) are trivial in this structure. To prove (3), suppose that $\gamma = \alpha\#\beta$ ($\# \in \{\vee, \wedge, \supset\}$) with $v\,(\gamma^\circ)$ =0. Then, applying Definition 3.1 one more time, we have $-v\,(\alpha^\circ) \vee -v\,(\beta^\circ) = -v\,(\alpha^\circ \& \beta^\circ) \in \overline{\mathfrak{f}_{B_2}}\,(v\,(\gamma^\circ)) = \overline{\mathfrak{f}_{B_2}}\,(0) = \{1\}$ (by Proposition 2.11(5)). From (1), $-v\,(\alpha^\circ) = 1$ or $-v\,(\beta^\circ) = 1$. That is, $v\,(\alpha^\circ) = 0$ or $v\,(\beta^\circ) = 0$. So, from Proposition 3.5(12), $v\,(\alpha) = v\,(\neg\alpha) = 1$ or $v\,(\beta) = v\,(\neg\beta)$= 1. ■

Definition 5.2 ([6], Definition 5) A QM_1**-valuation** is a map $q : Fm \to \{0, 1\}$ verifying:
$QM1$): $v(\alpha) = 0$ implies $v(\neg\alpha) = 1$
$QM2$): $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$
$QM3$): $v(\beta^\circ) = v(\alpha \supset \beta) = v(\alpha \supset \neg\beta) = 1$ implies $v(\alpha) = 0$
$QM4$): $v(\alpha \supset \beta) = 1$ if and only if $v(\alpha) = 0$ or $v(\beta) = 1$
$QM5$): $v(\alpha\&\beta) = 1$ if and only if $v(\alpha) = v(\beta) = 1$.
$QM6$): $v(\alpha \vee \beta) = 1$ if and only if $v(\alpha) = 1$ or $v(\beta) = 1$.
$QM7$): $v(\alpha^\circ) = v(\beta^\circ) = 1$ implies $v\,((\alpha\#\beta)^\circ) = 1$ (with $\# \in \{\vee, \&, \supset\}$).

Definition 5.3 The **consequence relation** $\models_{QM_1} \subseteq \wp(Fm) \times Fm$ is defined as follows: $\Gamma \models_{QM_1} \alpha$ iff, for every QM_1-valuation q such that $q(\Gamma) \subseteq \{1\}$, it holds that $q(\alpha) = 1$.

It is worth noting that, given α, the set of all the QM_1-valuations needed to check the validity of α according the definition of $\models_{QM_1}$ can be spreaded out in a kind of truth-table for α (which we will mention as the "quasi-truth table for α"). Besides this comment, the main result of this section is:

Proposition 5.4 For every map $v : Fm \to \{0, 1\}$, are equivalent:
i) v is a canonical F_1-valuation.
ii) v is a QM_1-valuation.

Proof. Note first that every F_1-valuation v to $\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle$ satisfies all the conditions $QM1$)-$QM7$) of Def. 5.2, as they form part of Prop. 3.5 and Prop. 5.1. On the other hand, if v is a QM_1-valuation, then it verifies Def. 3.1 **(a)** from $QM4$)-$QM6$) (and since we are focused on B_2). With respect to **(b)**: if $v(\alpha^\circ) = 0$ or $v(\alpha) = 0$, then it is satisfied. If not, then $v(\alpha^\circ) = v(\alpha) = 1$. Note now that $v(\alpha \supset \alpha) = 1$, because of $QM4$). From all this and $QM3$), $v(\alpha \supset \neg\alpha) = 0$, and so $v(\neg\alpha) = 0$ (by $QM4$) again). Condition **(b)** is also valid, consequently. To prove **c.1)**: suppose that $v(\alpha) = 0$. From $QM1$), $v(\neg\alpha) = 1 \in \overline{\mathfrak{f}_{B_2}}(v(\alpha))$. And, if $v(\alpha) = 1$, $\overline{\mathfrak{f}_{B_2}}(v(\alpha)) = \{0, 1\}$. So, $v(\neg\alpha) \in \overline{\mathfrak{f}_{B_2}}(v(\alpha))$ always. To prove **c.2)**: suppose first that $v(\alpha) = 1$. In this case, $-v(\neg\neg\alpha) \in \overline{\mathfrak{f}_{B_2}}(\alpha) = \{0, 1\}$, trivially. If, on the contrary, $v(\alpha)$ =0, then $v(\neg\neg\alpha) = 0$, from $QM2$), and so $-v(\neg\neg\alpha)$= $1 \in \overline{\mathfrak{f}_{B_2}}(\neg\alpha)$, too. Let us prove **(d)** finally, considering $\gamma = \alpha\#\beta$, with $\#$ any binary connective. This condition is trivially valid if $v(\gamma^\circ) = 1$, since in this case $\overline{\mathfrak{f}_{B_2}}(\gamma^\circ) = \{0, 1\}$. If not, then $v(\alpha^\circ) = 0$ or $v(\beta^\circ) = 0$, because $QM7$). Then $-v(\alpha^\circ) \vee -v(\beta^\circ) = 1 \in \overline{\mathfrak{f}_{B_2}}(\gamma^\circ)$ always, proving **(d)**, consequently. Summarizing, v verifies every condition of Def. 3.1. ■

Corollary 5.5 For every $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{\mathbb{F}_1}^{\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle} \alpha$ iff $\Gamma \models_{QM_1} \alpha$.

It will be noted in addition that, once the F_1-valuations into $\langle B_2, \overline{\mathfrak{f}_{B_2}} \rangle$ have been identified as being QM_1-valuations, the problem of the "number m_α", of F_1-valuations needed to check if $\vdash_{C_1} \alpha$ (commented at the end of Section 4), is reduced to the number or QM_1-lines of the quasi-matrix in the "quasi-truth-table" of α. This question is very interesting, as we said. Moreover, it is related with the existence of a *set of generators* of all the canonical F_1-valuations (or, alternatively, the QM_1-valuations).

6 Historical Remarks

This section intends to provide a little comparison between the F_1-structures and the early works about semantic models for C_1, as it was done with the case of quasi-matrices in the previous section. In addition, we will compare the former presentation of F_1-semantics of Fidel with the one shown in this paper. First of all, we would wish to mention a (somewhat forgotten) work of A. Sette (see [14]): it is one of the first attempts of provide an algebraic interpretation of a da Costa logic (specifically, of C_ω). With this aim, it is defined there the notion of C_ω-*algebra* as being a system $(L, \vee, \wedge, \to, ')$ wherein $(L, \vee, \wedge, \to)$ is an RPL (that is, it is a relatively pseudo-complemented lattice, cf. [13]), and additionally $' : L \to L$ verifies $a \vee a' = 1$ and $a \leq a''$. Note that this algebraic structure is cleary related with some of the requirements established in Def. 2.4, indebted to M. Fidel, useful for every C_n, with $1 \leq n \leq \omega$.

However, even when the F-structures directly associated with C_ω (that is, F_ω-structures, cf. [12]) are based on RPL (as in the case of C_ω-algebras), the essential difference of both approaches is that the element a' in F_ω-structures *does not need to be unique*. This last fact is strongly related with the fact that F-structures (in a general way) define *algebraic-relational semantics*, and not exclusively algebraic ones, as it was already commented. Indeed, algebraic-relational semantics seems to be more natural to the C_n-logics, because they are not algebraizable, cf. [8].

On the other hand, we already have seen the strong connection between quasi-matrices and F-structures, focusing our analysis on the formalism for F_1-structures presented here. This study suggests a similar comparison between *the original formalism and definitions* of F-structures given in [7] and the ones here presented. At this respect, it is possible to see that:

• The formalism given here does not modify the essence of the F_1-structures in the way that they were originally defined.

• However, the F_1-valuations used here are not the same as the used in [7]. For a better comparison of them, let us define "Fidel F_1-valuations" (the original definition indebted to Fidel, but with the formalism used here):

Definition 6.1 Given $\langle L, \mathfrak{f}\rangle$, a **Fidel F_1-valuation** is a map $v : Fm \to L$, verifying:

(a*) For every $\delta \in Var$:

a.1*) $v(\neg\delta) \in \mathfrak{f}(v(\delta))$, $v(\neg\neg\delta) \in \mathfrak{f}(v(\neg\delta))$, $v(\delta^\circ) \in \mathfrak{F}(v(\delta))$, $v(\delta^\circ) \in \mathfrak{f}(v(\delta\&\neg\delta))$, $v(\neg(\delta^\circ)) \in \mathfrak{f}(v(\delta^\circ))$, $v(\neg\neg(\delta^\circ)) \in \mathfrak{f}(v(\neg(\delta^\circ)))$,

a.2*) $v(\delta \& \neg\delta) = v(\delta) \wedge v(\neg\delta)$; $v(\delta) \wedge v(\neg\delta) \wedge v(\delta^\circ) = 0$; $v(\delta^\circ) \leq v(\neg\delta) \vee v(\neg\neg\delta)$; $v(\neg\neg\delta) \leq v(\delta)$; $v(\neg\delta^\circ) \leq v(\delta) \wedge v(\neg\delta)$.

(b*) For every α, $\beta \in Fm$, $v(\alpha \& \beta) = v(\alpha) \wedge v(\beta)$, $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$, $v(\alpha \supset \beta) = v(\alpha) \Rightarrow v(\beta)$;

(c*) For every α, $\beta \in Fm$, $v((\alpha \vee \beta)^\circ) \in \mathfrak{F}(v(\alpha \vee \beta))$, $v(\alpha^\circ) \wedge v(\beta^\circ) \leq v((\alpha \vee \beta)^\circ)$, $v((\alpha \supset \beta)^\circ) \in \mathfrak{F}(v(\alpha \supset \beta))$, $v(\alpha^\circ) \wedge v(\beta^\circ) \leq v((\alpha \supset \beta)^\circ)$;

(d*) For every α of the form $\beta \vee \gamma$ or $\beta \supset \gamma$, it holds $v(\alpha^\circ) \in \mathfrak{F}(v(\alpha))$, $v(\neg\alpha) \in \mathfrak{f}(v(\alpha))$, $v(\neg\neg\alpha) \in \mathfrak{f}(v(\neg\alpha))$, $v(\alpha^\circ) \in \mathfrak{f}(v(\alpha\&\neg\alpha))$, $v(\neg(\alpha^\circ)) \in \mathfrak{f}(v(\alpha^\circ))$, $v(\neg\neg(\alpha^\circ)) \in \mathfrak{f}(v(\neg(\alpha^\circ)))$, and these elements satisfy additionally:

d.i*) $v(\neg\neg\alpha) \leq v(\alpha)$;

d.ii*) $v(\alpha^\circ) \leq v(\neg\alpha) \vee v(\neg\neg\alpha)$;

d.iii)* $v(\neg\alpha^\circ) \leq v(\alpha) \wedge v(\neg\alpha)$;

d.iv)* $v(\alpha) \wedge v(\neg\alpha) \wedge v(\alpha^\circ) = 0$.

(e*) Finally, if α is of the form $\neg\beta$ or $\beta \& \gamma$, then $v(\neg\alpha) \in \mathfrak{f}(v(\alpha))$, $v(\alpha^\circ) \in \mathfrak{F}(v(\alpha))$, $v(\neg\neg\alpha) \in \mathfrak{f}(v(\neg\alpha))$, $v(\alpha^\circ) \in \mathfrak{f}(v(\alpha\&\neg\alpha))$, $v(\neg(\alpha^\circ)) \in \mathfrak{f}(v(\alpha^\circ))$, $v(\neg\neg(\alpha^\circ)) \in \mathfrak{f}(v(\neg(\alpha^\circ)))$, validating additionally conditions (i)-(iv) of **(d*)**.

It should be clear that Def. 3.1 is more operative than Def. 6.1. Moreover, it can be proved that they are not exactly the same: more specifically, both definitions *coincide in the case of saturated F_1-structures*, but they differ in the case of not saturated ones (see [10] for a detailed comparison).

Finally, we will remark that the way in which are defined the Lindenbaum F_1-structures in [7] is diferent to the given in Def. 3.20 (we are interested only in the case when $\Gamma = \emptyset$). Even when the support sets in both structures is the same (i.e. the Lindenbaum Boolean algebra defined on Fm), the main difference between both approaches lays on the definition of the function $\mathfrak{f}$. More specifically:

Definition 6.2 The **Lindenbaum-Fidel F_1-structure** (or, shorter, the LF-F_1-structure) is the F_1-structure of the form: $\langle\, Fm/\simeq_\Gamma, \mathfrak{f}^*_\simeq\rangle$, where $\mathfrak{f}^*_\simeq(\|\alpha\|) = \{\|\neg\lambda\|) : \|\alpha\| = \|\lambda\|\}$.

Of course, Fidel's original definition of Lindenbaum F_1-structures included the map $\mathfrak{F}^*_\simeq$ (see [7], page 34). Indeed, $\mathfrak{F}^*_\simeq(\|\alpha\|) = \{\|\lambda^\circ\| : \|\alpha\| = \|\lambda\|\}$. We have already seen, however, that it is not essential. Despite this latter mentioned fact, the structures $\langle\, Fm/\simeq_\Gamma, \mathfrak{f}^*_\simeq\rangle$ and $\langle\, Fm/\simeq_\Gamma, \mathfrak{f}_\simeq\rangle$ (given in Def. 3.20) are not (necessarily) the same. In a similar way to the case of F_1-valuations, a compared analysis of LF-F_1-structures and the ones given here (discussion that lies outside the scope of this paper) can be found in [10].

Summarizing, all these differences between both treatments of F_1-structures justify, under our point of view, the main results and the general approach of this paper.

7 Final Remarks

From all the definitions and results shown above, it should be clear that F_1-structures, as developed here, are not only understood as a "better formalism" than the presented originally. Moreover, they not only allow us to prove completeness and decidability in a simpler way: the definitions of F_1-valuations and Lindenbaum F_1-structures given here differ from the given in [7], and in a certain sense both notions should not to be treated as the same (even when they are equally useful to prove completeness and decidability).

In addition, if we work with F_1-structures as here, some interesting questions can be posed in a clearer way (w.r.t. [7]). For instance, the definition and study of a *category of F_1-structures* seems to be natural, here (with respect to this, see [10] and [12] for a more extensive treatment of that category and other related ones). Other topic that deserves a deep analysis is the characterization of the *minimum generator set* that determines the F_1-valuations (or, equivalently, the QM_1-valuations taking into account the results of Section 5). Finally, F_1-structures deserve a good, deep, model-theoretic treatment, such as the done in [3] for the logic **mbC**. As we said above, all these problems could be solved (or, at least, analyzed) with the help of the simple presentation given in this paper.

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