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Congruences on Bounded Hilbert Algebras with Moisil Possibility Operators

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Abstract

In this paper, we will introduce the variety of bounded Hilbert algebras with Moisil possibility operators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, called MI_n^0-algebras. First, we give a characterization of MI_n^0-congruences in terms of a particular class of deductive systems, namely modal deductive systems. Furthermore, from the above results on MI_n^0-congruences, the principal ones are described. In addition, we proved that the variety of MI_n^0-algebras is semisimple.

Keywords: Hilbert algebras; bounded Hilbert algebras; Moisil possibility operators.

Introduction

In 1923, David Hilbert proposed to study implicative fragment of intuitionistic propositional calculus. This fragment is well-known as *positive implicative calculus* and its study was begun in 1935 by D. Hilbert and P. Bernays.

In 1950, L. Henkin ([13]) introduced *implicative models* as algebraic models of positive implicative calculus. Later, A. Monteiro renamed it as *Hilbert algebras* and his Ph. D. student A. Diego ([10, 11, 12]) made one of the most important contributions to this algebraic structure which we can define as follow:

A Hilbert algebra (or I-algebra) is an algebra $\langle A, \to, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in A :

(I1) $1 \to x = x$,

(I2) $x \to x = 1$,

(I3) $x \to (y \to z) = (x \to y) \to (x \to z)$, (I4) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

The variety of Hilbert algebras is denoted by $\mathcal{I}$. For each $A \in \mathcal{I}$ the following properties are verified:

(I5) $x \to 1 = 1$,

(I6) the binary relation $\leq$ defined by $x \leq y$ if and only if $x \to y = 1$ is a partial order on A with greatest element 1.

(I7) $x \to (y \to z) = y \to (x \to z)$,

(I8) $x \leq y$ implies $y \to z \leq x \to z$,

(I9) $x \to (y \to x) = 1$,

(I10) $x \leq y$ implies $z \to x \leq z \to y$.

A. Monteiro ([16]), proved that the semisimple I-algebras are those that verify the additional identity:

(I11) $(x \to y) \to x = x$.

This author called *Tarski algebras* to semisimple I-algebras and Pierce law to identity I11.

(I12) Let A be an I-algebra and let $t \in A$. We say that $t \in A$ is a tarskian element of A if t satisfies the identity:

(T) $(t \to x) \to t = t$ for all $x \in A$,

The set of all tarskian elements of an I-algebra A is denoted by $T(A)$.

Let A be a Hilbert algebra. A subset $D \subseteq A$ is a deductive system of A ([2, 14]) if $1 \in D$ and if $x, x \to y \in D$, then $y \in D$. The set of all deductive systems of a Hilbert algebra A is denoted by $\mathcal{D}(A)$.

Other interesting properties of I-algebras are the following:

(I13) The deductive system generated by a set $X \subseteq A$ is $[X) := \bigcap\{D \in \mathcal{D}(A) : X \subseteq D\}$. In particular, if $X = \{a\}$, the principal deductive system is $[a) = \{x \in A : a \leq x\}$.

(I14) If A is an I-algebra and $Con_{\mathbf{I}}(A)$ is the set of all $\mathbf{I}$-congruences of A, then $Con_{\mathbf{I}}(A) = \{R(D) : D \in \mathcal{D}(A)\}$ where $R(D) = \{(x, y) \in A^2 : x \to y \in D,\ y \to x \in D\}$. Besides, $[1]_{R(D)} = D$ and if $\Theta \in Con_{\mathbf{I}}(A)$, then $R([1]_\Theta) = \Theta$.

A bounded Hilbert algebra (see [3, 5]) is a Hilbert algebra A with an element $0 \in A$ such that $0 \to a = 1$, for every $a \in A$. The notation a^* means $a \to 0$.

The following result has been proved by Buşneag in [2, 4].

(I15) Let A be a bounded Hilbert algebra. Then, the following conditions are equivalent:

(i) A is a Boolean lattice,

(ii) for all $x \in A$, $x^{**} = x$.

1 $\mathbf{MI_n}$-algebras

Gr. C. Moisil introduced the 3-valued Łukasiewicz algebras as algebraic models of 3-valued Łukasiewicz propositional calculus. It is well known that in 3-valued Łukasiewicz algebras it is possible to define an implication operator which shows that 3-valued Łukasiewicz algebras are a special case of Hilbert algebras. This result was, in some way, the motivation of the papers [6] and [7].

L. Iturrioz introduced in [15] the notion of modal operators on symmetric Heyting algebras and defined the class of SH_n-algebras. In [7, 8] Canals Frau and Figallo consider some reducts of this class. In particular, they introduced the following definition.

A Hilbert algebra of order n, ($n \geq 2$), with the Moisil possibility operators (or MI_n-algebra) is an algebra $\langle A, \to, \sigma_1, \ldots, \sigma_{n-1}, 1\rangle$ of type (2,1,. . . ,1,0) such that the reduct $\langle A, \to, 1\rangle$ is a I-algebra and σ_1, . . ., σ_{n-1} are unary operations satisfying the following axioms:

(M1) $(\sigma_1 x \to y) \to x = x$,

(M2) $\sigma_i(x \to y) \to (\sigma_i x \to \sigma_j y) = 1$, $1 \leq i \leq j \leq n-1$,

(M3) $(\sigma_i x \to \sigma_i y) \to ((\sigma_{i+1} x \to \sigma_{i+1} y) \to \ldots ((\sigma_{n-1} x \to \sigma_{n-1} y) \to \sigma_i(x \to y))\ldots) = 1$, (M4) $\sigma_i(x \to \sigma_j y) = x \to \sigma_j y$, $1 \leq i, j \leq n-1$,

(M5) $\sigma_{n-1} x = (x \to \sigma_i x) \to \sigma_j x$, $1 \leq i \leq j \leq n-1$.

From now on, we will denote by $\mathcal{MI}_n$ the variety of MI_n-algebras.

Remark 1.1 *In [7] MI_n-algebras were called $(n+1)$-valued modal Hilbert algebras, following the terminology of Iturrioz we have called them Hilbert algebras of order n with Moisil operators.*

Now, we will summarize some useful properties of MI_n-algebras (see [7]).

(M6) $\sigma_1 x \leq x$,

(M7) $\sigma_i(\sigma_j x) = \sigma_j x$,

(M8) $\sigma_j 1 = 1$,

(M9) $\sigma_1 x \leq \sigma_2 x \leq \ldots \leq \sigma_{n-1} x$,

(M10) $x \leq \sigma_{n-1} x$,

(M11) $x \leq y$ implies $\sigma_i x \leq \sigma_i y$,

(M12) $\sigma_i(\sigma_j x \to y) = \sigma_j x \to \sigma_i y$, $i \leq j$,

(M13) $x \to \sigma_j(x \to y) = \sigma_j(x \to y)$,

(M14) $x \to \sigma_j y \leq \sigma_j(x \to y)$,

(M15) $\sigma_j(x \to y) \leq \sigma_j x \to \sigma_j y$,

(M16) $(\sigma_1 x \to \sigma_1 y) \to ((\sigma_2 x \to \sigma_2 y) \to \ldots ((\sigma_{n-1} x \to \sigma_{n-1} y) \to (x \to y))\ldots) = 1$,

(M17) $\sigma_i x = \sigma_i y$ for all $i = 1, 2, \ldots, n-1$, implies $x = y$,

(M18) $(\sigma_j x \to y) \to \sigma_j x = \sigma_j x$,

(M19) $\sigma_{n-1} x = (x \to \sigma_1 x) \to x$,

(M20) $\sigma_1(\sigma_1 y \to x) \to (\sigma_1(\sigma_1 x \to z) \to (\sigma_1 y \to z)) = 1$.

(M21) The algebra $\mathbf{C_n}^{MI} = \langle \mathcal{C}_n, \to, \sigma_1, \ldots, \sigma_{n-1}, 1 \rangle$, where $\mathcal{C}_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & x > y, \end{cases} \quad \text{and} \quad \sigma_j(\tfrac{k}{n-1}) = \begin{cases} 0 & \text{if } k+j \leq n-1, \\ 1 & \text{if } k+j > n-1 \end{cases} \quad 0 \leq k \leq n-1,$$

is a MI_n-algebra, called the standard MI_n-algebra.

In the MI_k-algebra $C_{\mathbf{k}}^{MI}$ with $2 \leq k < n-1$ we can define $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{n-1}$ being $\sigma_k = \sigma_{k+1} = \ldots = \sigma_{n-1}$. Hence, the chain $C_{\mathbf{k}}^{MI} \in \mathcal{MI}_n$.

(M22) Let $A \in \mathcal{MI}_n$. $D \in \mathcal{D}(A)$ is a modal deductive system if it satisfies the following condition: $x \in D$ implies $\sigma_1 x \in D$.

The set of all modal deductive system of a MI_n-algebra A it is denoted by $\mathcal{D}_m(A)$.

Let $A \in \mathcal{MI}_n$, $X \subseteq A$ and $a \in A \setminus X$. $D_m(X)$ denotes the modal deductive system of A generated by X and $D_m(X, a)$ denotes the modal deductive system of A generated by $X \cup \{a\}$. Moreover, if B is a subalgebra of A we will denote $B \triangleleft A$.

Next, for the purpose of describing properties of modal deductive system we use the following notation introduced by Buşneag in ([2]) and frequently used by different authors:

$$(x_1, \ldots, x_{n-1}; x_n) = \begin{cases} x_n & \text{if} \quad n = 1 \\ x_1 \to (x_2, \ldots, x_{n-1}; x_n) & \text{if} \quad n > 1 \end{cases}.$$

(M23) Let $A \in \mathcal{MI}_n$, $X \subseteq A$ and $a \in A$. Then, the following conditions are verified:

(i) $D_m(X) = \{x \in A : \exists\, h_1, \ldots, h_k \in X : (\sigma_1 h_1, \ldots, \sigma_1 h_k; x) = 1\}$,

(ii) $D_m(a) = \{x \in A : (\sigma_1 a; x) = 1\} = [\sigma_1 a)$.

(iii) $D_m(X \cup \{a\}) = \{x \in A : (\sigma_1 a; x) \in D_m(X)\}$.

On the other hand, it is easy to see that:

(M24) If $A \in \mathcal{MI}_n$, $B \triangleleft A$ and $D_B \in \mathcal{D}_m(B)$. Then, there exists $D \in \mathcal{D}_m(A)$ such that $D_B = D \cap B$.

(M25) Let $A \in \mathcal{MI}_n$ and $M \in \mathcal{D}_m(A)$. Then, the following conditions are equivalents:

(i) M is a maximal,

(ii) A/M is a simple MI_n-algebra,

(iii) $A/M \simeq S \triangleleft \mathbf{C}_{\mathbf{n}}^{\mathbf{MI}}$.

2 Bounded $\mathbf{MI_n}$-algebras

In this section we are going to introduce the variety of bounded Hilbert algebras with Moisil possibility operators.

Definition 2.1 *A bounded MI_n-algebra (or MI_n^0-algebra) is an algebra $\langle A, \rightarrow, \sigma_1, \ldots, \sigma_{n-1}, 0, 1\rangle$ of type $(2, 1, \ldots, 1, 0, 0)$ where $\langle A, \rightarrow, \sigma_1, \ldots, \sigma_{n-1}, 1\rangle$ is a MI_n-algebra and it satisfies the following additional condition:*

(A1) $0 \rightarrow x = 1$.

We will denote by $\mathcal{MI}_n^0$ the variety of MI_n^0-algebras.

Example 2.2 *The algebra $C_{\mathbf{n}}^{MI^0} = \langle \mathcal{C}_n, \rightarrow, \sigma_1, \ldots, \sigma_{n-1}, 0, 1\rangle$ considered in* (M21) *is a MI_n^0-algebra.*

We will list some basic properties valid in the MI_n^0-algebras, proving just some of them.

Proposition 2.3 *Let $A \in \mathcal{MI}_n^0$. Then, the following properties are satisfied:*

(A2) $\sigma_i 0 = 0$,

(A3) $\sigma_i x^* = x^*$,

(A4) $\sigma_j(\sigma_i x)^* = (\sigma_i x)^*$,

(A5) $(\sigma_i x)^* \rightarrow \sigma_i x = \sigma_i x$,

(A6) $\sigma_i((\sigma_i x)^* \rightarrow \sigma_i y) = (\sigma_i x)^* \rightarrow \sigma_i y$,

(A7) $(\sigma_i x)^{**} = \sigma_i x$,

(A8) $(\sigma_i x)^* \rightarrow (\sigma_i y)^* = \sigma_i y \rightarrow \sigma_i x$,

(A9) $x^* = (\sigma_{n-1} x)^*$,

(A10) $\sigma_i x^* = (\sigma_{n-1} x)^*$,

(A11) $\sigma_i(\sigma_1 x)^* = (\sigma_1 x)^*$,

(A12) $(\sigma_1 x \rightarrow (\sigma_1 y)^*)^* \rightarrow x = 1$.

Proof.

(A2): From A1 and M6, we have that $\sigma_1 0 = 0$. Then, from M19, we infer that $\sigma_{n-1} 0 = (0 \rightarrow \sigma_1 0) \rightarrow 0 = 0$. Hence, from M9, we conclude that $\sigma_i 0 = 0$.

(A6): From A4, we have that $\sigma_i((\sigma_i x)^* \to \sigma_i y) = \sigma_i(\sigma_i((\sigma_i x)^*) \to \sigma_i y)$. Hence, taking into account M12 and M7, we obtain that $\sigma_i((\sigma_i x)^* \to \sigma_i y) = (\sigma_i x)^* \to \sigma_i \sigma_i y = (\sigma_i x)^* \to \sigma_i y$.

(A7): From A1, I10 and M18, we have that $(\sigma_i x \to 0) \to 0 \leq (\sigma_i x \to 0) \to \sigma_i x = \sigma_i x$. So, $(\sigma_i x)^{**} \leq \sigma_i x$. On the other hand, from I7 and I2 we obtain that $\sigma_i x \to (\sigma_i x)^{**} = \sigma_i x \to ((\sigma_i x \to 0) \to 0) = (\sigma_i x \to 0) \to (\sigma_i x \to 0) = 1$ from which we get that $\sigma_i x \leq (\sigma_i x)^{**}$.

(A9): From M10 and I8, we have that $(\sigma_n x)^* \leq x^*$. On the other hand, since $\sigma_1 x \leq x$ and so, from I8, we obtain that $x^* \to (\sigma_1 x)^* = (x \to 0) \to (\sigma_1 x \to 0) = 1$. Besides, from I7 and M5 we have that $x^* \to (\sigma_{n-1} x)^* = (x \to 0) \to (\sigma_{n-1} x \to 0) = \sigma_{n-1} x \to ((x \to 0) \to 0) = ((x \to \sigma_1 x) \to \sigma_1 x) \to ((x \to 0) \to 0)$. Hence, from I7, I3, A1 and I5, we obtain that $x^* \to (\sigma_{n-1} x)^* = (((x \to 0) \to (x \to \sigma_1 x)) \to ((x \to 0) \to \sigma_1 x)) \to ((x \to 0) \to 0) = ((x \to (0 \to \sigma_1 x)) \to ((x \to 0) \to \sigma_1 x)) \to ((x \to 0) \to 0) = ((x \to 0) \to \sigma_1 x) \to ((x \to 0) \to 0)$. For this result I3, M6 and I8 we obtain $x^* \to (\sigma_{n-1} x)^* = (x \to 0) \to (\sigma_1 x \to 0) = 1$. Therefore, $x^* \leq (\sigma_{n-1} x)^*$.

Definition 2.4 *An element x of a MI_n^0-algebra A is invariant if $\sigma_i x = x$.*

The set of all invariant elements of a MI_n^0-algebra A is denoted by $K(A)$.

Definition 2.5 *An element x of a MI_n^0-algebra A is regular if $x^{**} = x$.*

In what follows, the set of all regular elements of A we will denote by A^{**}.

Next, we will show the relationship between the above two definitions.

Proposition 2.6 *Let $A \in \mathcal{MI}_n^0$. Then, $K(A)$ is a MI_n^0-subalgebra of A.*

Proof. Let $x, y \in K(A)$. Then, $x = \sigma_i x$ and $y = \sigma_j y$, $1 \leq i, j \leq n-1$. Hence, from M12 we have that $x \to y = \sigma_i x \to \sigma_j y = \sigma_j(\sigma_i x \to y)$ and from M7 we deduce that $\sigma_k(x \to y) = x \to y$. Therefore, $x \to y \in K(A)$. On the other hand, from M7 $\sigma_k x = \sigma_k(\sigma_i x) = x$. So, $\sigma_k x \in K(A)$. Besides, from A2 and M8, we have that $0, 1 \in K(A)$.

Proposition 2.7 *Let $A \in \mathcal{MI}_n^0$. Then, $A^{**} = K(A)$.*

Proof. Let $x \in A^{**}$. Then, from A2 and M14, we have that $x = x^{**} = (x \to 0) \to \sigma_1 0 \leq \sigma_1 x^{**} = \sigma_1 x$. The other inequality results immediately from M6. Conversely, if $x \in K(A)$ then $x = \sigma_i x$. Then, from A7 we obtain that $x^{**} = (\sigma_i x)^{**} = \sigma_i x = x$. Therefore, $x \in A^{**}$.

Proposition 2.8 *Let $A \in \mathcal{MI}_n^0$. Then, $K(A)$ is a Boolean lattice.*

Proof. From the Proposition 2.6, we have that $\langle K(A), \to, \sigma_1, \dots, \sigma_n, 0, 1\rangle$ is an MI_n^0-algebra. Hence, from A7 and I15 we obtain that $K(A)$ is a Boolean lattice.

Remark 2.9 *From Buşneag's proof of I15, it was proved that for every k_1, $k_2 \in K(A)$, the following properties hold:*

(i) $k_1 \vee k_2 = k_1^* \to k_2$,

(ii) k_1^* *is the boolean complement of* k_1.

Now, we will give another characterization of $K(A)$, using the tarskian elements of A.

Lemma 2.10 $T(A) = K(A)$.

Proof. Let $t \in T(A)$. Then, from M19, we have that $\sigma_{n-1}t = (t \to \sigma_1 t) \to t = t$. So, $t \in K(A)$. Conversely, let $k \in K(A)$ and $x \in A$. Then, we have that $(k \to x) \to k = (\sigma_i k \to x) \to \sigma_i k$ and from M18 we infer that $(k \to x) \to k = \sigma_i k = k$. Hence, $k \in T(A)$ and so, $K(A) = T(A)$.

3 Congruences

In this section we will determine the MI_n-congruences and we will establish a lattice isomorphism between $Con_{MI_n}(A)$ and $\mathcal{D}_m(A)$. Besides, we will obtain a characterization of MI_n-congruences. Furthermore, from the above results on the MI_n-congruences, the principal ones are described.

The following two theorems were stated in [7].

Theorem 3.1 *Let $A \in \mathcal{MI}_n$ and $D \in \mathcal{D}_m(A)$. Then, $Con_{MI_n}(A) = \{R(D) : D \in \mathcal{D}_m(A)\}$, where $R(D) = \{(x,y) \in A^2 : x \to y \in D,\ y \to x \in D\}$.*

Proof. Since A is a Hilbert algebra and D is a deductive system of A, by I14 we know that $R(D)$ is an I-congruence on A. Moreover, if $(x,y) \in R(D)$ since D is a modal deductive system, we have that $\sigma_1(x \to y), \sigma_1(y \to x) \in D$. Hence, from M9 we have that, $\sigma_i(x \to y)$, $\sigma_i(y \to x) \in D$, $1 \le i \le n-1$, and by M2 we infer that $\sigma_i x \to \sigma_i y$, $\sigma_i y \to \sigma_i x \in D$, $1 \le i \le n-1$. Therefore, $(\sigma_i x, \sigma_i y) \in D$, $1 \le i \le n-1$ from which we conclude that $R(D) \in Con_{MI_n}(A)$. Conversely, let $\theta \in Con_{MI_n}(A)$. Then, $\theta \in Con_I(A)$. From I14, we have that $[1]_\theta$ is a deductive system of A and $R([1]_\theta) = \theta$. Besides, from hypothesis and M8 we have that: if $(x,1) \in \theta$, then $(\sigma_1 x, 1) \in \theta$, that is, $[1]_\theta \in \mathcal{D}_m(A)$ which completes the proof.

Theorem 3.2 *Let $A \in \mathcal{MI}_n$. Then, the lattices $Con_{MI_n}(A)$ and $\mathcal{D}_m(A)$ are isomorphic.*

Proof. It is a direct consequence of I14 and Theorem 3.1 considering the applications $\theta \mapsto [1]_\theta$ and $D \mapsto R(D)$ which are inverse to one another.

Next, we will show a characterization of simples MI_n^0-algebras.

Corollary 3.3 *Let $A \in \mathcal{MI}_n^0$. Then, the following conditions are equivalent:*

(i) *A is a simple MI_n^0-algebra,*

(ii) $\sigma_1(A) = \{0, 1\}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that A is a simple MI_n^0-algebra and let $x \in A$. From (M23), we have that $[\sigma_1 x)$ is a modal deductive system of A. Hence, $[\sigma_1 x) = \{1\}$ or $[\sigma_1 x) = A$ from which it follows that $\sigma_1 x = 1$ or $\sigma_1 x = 0$.

(ii) $\Rightarrow$ (i): Suppose that $\sigma_1(A) = \{0, 1\}$. Let $D \in \mathcal{D}_m(A)$ and $x \in D$. Then, $\sigma_1 x \in D$. If $\sigma_1 x = 0$, we have that $D = A$ and if $\sigma_1 x = 1$, from M6, we have that $x = 1$. Therefore, $D = \{1\}$.

Let A be an MI_n-algebra and $a, b \in A$. By $\theta(a, b)$ we denote the principal congruence of A generated by (a, b), i.e., the smallest congruence of A that contains (a, b). In Theorem 3.4, we provide a description of the principal congruences of A.

Theorem 3.4 *Let $A \in \mathcal{MI}_n$. Then, for every $a, b \in A$ it is verified that: $\theta(a, b) = \{(x, y) \in A^2 : \sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to (\sigma_1(a \to b) \to y)\}$.*

Proof. Let $S = \{(x, y) \in A^2 : \sigma_1(a \to b) \to ((\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)\}$. Then, $(a, b) \in S$. Indeed, from M6, I5 and I3, we have that $1 = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to (a \to b)) = \big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to a)\big) \to \big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to b)\big)$. From this statement and I6 we have that $\sigma_1(b \to a) \to (\sigma_1(a \to b) \to a) \leq \sigma_1(b \to a) \to (\sigma_1(a \to b) \to b)$. In a similar way we obtain that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to b) \leq \sigma_1(a \to b) \to (\sigma_1(b \to a) \to a)$. Moreover, S is an equivalence relation on A such that:
(i): S is compatible with $\to$: Let $(x, y) \in S$ and $t \in A$. Then, we have that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(a \to b) \to (\sigma_1(b \to a) \to y)$. From this last statement, we obtain that $t \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = t \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to y)$ and from I7 we obtain that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to (t \to x)) = \sigma_1(a \to b) \to (\sigma_1(b \to a) \to (t \to y))$. So, $(t \to x, t \to y) \in S$. Moreover, from I3, we have that $\sigma_1(a \to b) \to$

$(\sigma_1(b \to a) \to (x \to t)) = (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x)) \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to t))$. From this last statement, I7 and I3 we deduce that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to (x \to t)) = (\sigma_1(b \to a) \to (\sigma_1(a \to b) \to y)) \to (\sigma_1(b \to a) \to (\sigma_1(a \to b) \to t)) = \sigma_1(b \to a) \to (\sigma_1(a \to b) \to (y \to t))$. Therefore, we conclude that $(x \to t, y \to t) \in S$.
(ii): S is compatible with σ_i : let $(x, y) \in S$. Then, $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)$ from which $\sigma_i\big(\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x)\big) = \sigma_i\big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)\big)$ and we conclude the proof by M12.

Hence, $S \in Con_{MI_n}(A)$. Finally, if $R \in Con_{MI_n}(A)$ and $(a, b) \in R$, then $S \subseteq R$. Indeed, let (1) $(x, y) \in S$. Since $(a, b) \in R$ we have that (2) $(\sigma_1(a \to b) \to x, x) \in R$ and $(\sigma_1(b \to a) \to y, y) \in R$ from which we obtain that $(\sigma_1(b \to a) \to (\sigma_1(a \to b) \to x), \sigma_1(b \to a) \to x) \in R$ and $(\sigma_1(a \to b) \to (\sigma_1(b \to a) \to y), \sigma_1(a \to b) \to y) \in R$. From (1) and I7, we conclude that $(\sigma_1(b \to a) \to x, \sigma_1(b \to a) \to y) \in R$. Hence, from (2), $(x, y) \in R$.

Corollary 3.5 *$\mathcal{MI}_n$ has equationally definable principal congruences.*

We prove that the variety $\mathcal{MI}_n$ satisfies the congruence extension property.

Lemma 3.6 *Let $A \in \mathcal{MI}_n$, $B \lhd A$ and $\theta \in Con_{MI_n}(B)$. Then, there exists $\varphi \in Con_{MI_n}(A)$ such that $\theta = \varphi \cap B^2$.*

Proof. Let $B \lhd A$ and $\theta \in Con_{MI}(B)$. Then, by Theorem 3.2, there exists $D_1 \in \mathcal{D}_m(B)$ such that $R(D_1) = \theta$. Hence, by (M24), there exists $D \in \mathcal{D}_m(A)$ such that $D \cap B = D_1$. Moreover, since $D \in \mathcal{D}_m(A)$ there exists $R(D) \in Con_{MI_n}(A)$. Let $\varphi = R(D)$ and suppose that $(x, y) \in \varphi \cap B^2$. Then, we have $x \to y$, $y \to x \in D \cap B$. So, $(x, y) \in R(D_1)$. From this last statement we have $\varphi \cap B^2 \subseteq \theta$. In a similar way we obtain that $\theta \subseteq \varphi \cap B^2$.

From Lemma 3.6 and a result of A. Day ([9]) the following property holds:

Lemma 3.7 *Let $A \in \mathcal{MI}_n$. Then, the following conditions are equivalent:*

(i) *$\mathcal{MI}_n$ satisfies the congruence extension property,*

(ii) *$\mathcal{MI}_n$ satisfies the principal congruences extension property,*

(iii) *for all A, $B \in \mathcal{MI}_n$ such that $B \lhd A$ and for all a, $b \in B$ it follows that $\theta_B(a, b) = \theta_A(a, b) \cap B^2$.*

Lemma 3.8 *$\mathcal{MI}_n$ has regular congruences.*

Proof. Let θ, $\varphi \in Con_{MI_n}(A)$ and $a \in A$ such that $[a]_\theta = [a]_\varphi$. Let us consider the quotients algebras A/θ and A/φ. Then, we have that: $[1]_\theta = [a]_\theta \to [a]_\theta = [a]_\varphi \to [a]_\varphi = [1]_\varphi$. Moreover, by Theorem 3.2, we have that $R([1]_\theta) = \theta$ and $R([1]_\varphi) = \varphi$. So, $\theta = \varphi$.

Lemma 3.9 *$\mathcal{MI}_n$ has distributive congruences.*

Proof. It is a direct consequence of [1] and taking into account that this variety has the EDPC property.

Let $A \in \mathcal{MI}_n^0$. We denote by $\mathcal{D}_m^P(A)$ the lattice of all principal deductive systems of a MI_n^0-algebra A and by $Con^P_{MI^0}(A)$ the lattice of all principal congruences of a MI_n^0-algebra A.

Lemma 3.10 *Let $A \in \mathcal{MI}_n^0$ and let a, $b \in A$. Then, $\big[w_{a,b}\big) \in \mathcal{D}_m^P(A)$, where $w_{a,b} := (\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*$.*

Proof. From A4, we have that $\sigma_j w_{a,b} = w_{a,b}$, from which we conclude that $w_{a,b} \in K(A)$. From this last statement and from (ii) of (M23) we have that $[\,w_{a,b}) \in \mathcal{D}_m^P(A)$.

Proposition 3.11 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $K(A)$ and $\mathcal{D}_m^P(A)$ are anti-isomorphic.*

Proof. It follows from considering the application $\alpha : K(A) \longrightarrow \mathcal{D}_m^P(A)$ define by $\alpha(k) = [\,k)$ for all $k \in K(A)$.

In the following theorem we will obtain a good characterization of principal congruences in MI_n^0-algebras.

Theorem 3.12 *Let $A \in \mathcal{MI}_n^0$ and let a, $b \in A$. Then, $\theta(a,b) = \theta\big(w_{a,b}, 1\big)$.*

Proof. It is sufficient to show that:

(i) $\big(w_{a,b}, 1\big) \in \theta(a,b)$,

(ii) $(a,b) \in \theta\big(w_{a,b}, 1\big)$.

(i): From $(a,b) \in \theta(a,b)$ we infer that $(\sigma_1(b \to a), 1) \in \theta(a,b)$. Hence, we have that $(\sigma_1(a \to b) \to (\sigma_1(b \to a))^*, \sigma_1(a \to b) \to 0) \in \theta(a,b)$. From this last statement we have that $((\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*, (\sigma_1(a \to b))^{**}) \in \theta(a,b)$ and by A7 we conclude that $((\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*, \sigma_1(a \to b)) \in \theta(a,b)$. Besides, since $(\sigma_1(a \to b), 1) \in \theta(a,b)$ we have that $(w_{a,b}, 1) \in \theta(a,b)$.
(ii): (1) $(a \to b, b \to a) \in \theta(w_{a,b}, 1)$. Indeed, by Theorem 3.4, we must prove that $\sigma_1(w_{a,b} \to 1) \to (\sigma_1(1 \to w_{a,b}) \to (a \to b)) = \sigma_1(1 \to w_{a,b}) \to$

$(\sigma_1(w_{a,b} \to 1) \to (b \to a))$, which is equivalent to prove that $w_{a,b} \to (a \to b) = w_{a,b} \to (b \to a)$ and from A12 is equivalent to $1 = w_{a,b} \to (b \to a)$, which follows immediately from I7 and A12. Hence, from (1), I3, I2 and I1 we have that $\big((a \to b) \to ((b \to a) \to a), (b \to a) \to a\big) \in \theta(w_{a,b}, 1)$ and $\big((b \to a) \to ((a \to b) \to b), (a \to b) \to b\big) \in \theta(w_{a,b}, 1)$. From this last statement and I4 we deduce (2) $\big((a \to b) \to b, (b \to a) \to a\big) \in \theta(w_{a,b}, 1)$. On the other hand, by (1), we have that $(a \to b, 1) \in \theta(w_{a,b}, 1)$ and $(b \to a, 1) \in \theta(w_{a,b}, 1)$. So, we obtain $\big((a \to b) \to b, b\big) \in \theta(w_{a,b}, 1)$ and $\big((b \to a) \to a, b\big) \in \theta(w_{a,b}, 1)$. From these two above statements and (2) we conclude that $(a, b) \in \theta(w_{a,b}, 1)$.

Proposition 3.13 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $Con^P_{MI_n^0}(A)$ and $\mathcal{D}^P_m(A)$ are isomorphic.*

Proof. It follows from Theorem 3.12 and by considering the application $\Psi : Con^P_{MI_n^0}(A) \longrightarrow \mathcal{D}^P_m(A)$ defined by the prescription $\Psi\big(\theta(a, b)\big) = \big[w_{a,b}\big)$ for all $\theta(a, b) \in Con^P_{MI_n^0}(A)$.

Corollary 3.14 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $K(A)$ and $Con^P_{MI_n^0}(A)$ are anti-isomorphic.*

Proof. It is a direct consequence of Proposition 3.11 and 3.13.

Corollary 3.15 *Let A be a finite MI_n^0-algebra. Then, $|Con^P_{MI^0}(A)| = 2^m$ where m is the number of atoms of $K(A)$.*

Proof. It is a direct consequence of Proposition 3.14 and Proposition 2.8.

Next, we prove that the variety $\mathcal{MI}_n^0$ is semisimple.

Proposition 3.16 *Let $A \in \mathcal{MI}_n^0$. Then, the following conditions are equivalent:*

(i) *A is a subdirectly irreducible MI_n^0-algebra,*

(ii) *$K(A)$ has a unique dual atom,*

(iii) *A is a simple MI_n^0-algebra.*

Proof. (i) $\Rightarrow$ (ii): Let Θ_0 be a unique nontrivial minimal congruence. Then, there exists $(a, b) \in \Theta_0$ where $a \neq b$. Hence, $\theta(a, b) \subseteq \Theta_0$ and $\theta(a, b) \neq \Delta$ from which we get that $\Theta_0 = \theta(a, b)$. So, by Theorem 3.12 and Lemma 3.13, there exists a unique minimal principal modal deductive system $D_0 = [w_{a,b})$ where

$w_{a,b} \in K(A)$. Besides, taking into account that the lattices $K(A)$ and $\mathcal{D}_m^P(A)$ are anti-isomorphic we obtain that $w_{a,b}$ is a dual atom of $K(A)$.

(ii) $\Rightarrow$ (iii): Since A has a unique dual atom k, from Remark 2.9 we have that k^* is the unique atom of $K(A)$. Therefore, $K(A) = \{0, 1\}$ and from Theorem 3.3 we conclude that A is simple.

(iii) $\Rightarrow$ (i): It is easy to check.

Corollary 3.17 *$\mathcal{MI}_n^0$ is semisimple.*

Proof. It is a direct consequence of Proposition 3.16.

□

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