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Metamathematical Subtleties of Winning in Diophantine Games

Andrey Bovykin

Abstract

A Diophantine game on a polynomial expression with integer coefficients $p(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ is a finite-length full-information, win-lose game between two players X (first) and Y (second), who, in their turns $i = 1, 2, \ldots, n$, successively pick natural numbers x_i, then y_i. The value of the payoff-polynomial $p(x_1, y_1, \ldots, x_n, y_n)$ determines the winner. If the value is 0 then Y wins, otherwise X wins. The law of excluded middle declares that for every polynomial p, one of the two players "has" a winning strategy. But this empty declaration opens up a labyrinth of metamathematical subtleties.

Diophantine games were introduced by James Jones in [12] in 1982. The first layer of subtleties was challenged when Jones investigated existence and non-existence of computable (rather than declared to exist) winning strategies in various Diophantine games. This research has close connection to investigations of Hilbert's Tenth Problem on the integers. It was important at that time to isolate and illustrate the distinction between abstract "existence" of a winning strategy and existence of an algorithm that computes the winning moves. It was proved by Jones that for some concrete Diophantine games, neither of the two players has a computable winning strategy. But suppose one of the players is given an algorithm pertaining to provide the winning moves. How can we be sure? Perhaps, being mathematicians, we could try to prove it. For this, we introduce the notion of a T-provably computable winning strategy, where T is a strong enough axiomatic theory that can formalize our reasoning methods. But what if that algorithm is neither T-provably computable (halting), nor T-provably winning?

Here, we investigate Diophantine games by studying further layers of metamathematical intricacy, related to existence or non-existence of provably computable winning strategies. What if there "exists" an algorithm pertaining to compute the winning moves but our strongest mathematical methods are not sufficient to prove that it halts and does the job correctly? It turns out that for some rather simple polynomials (we give examples), there exists a computable winning strategy but there does not exist a T provably computable winning strategy, where T may be taken to comprise of very strong classical mathematical reasoning methods (Primitive Recursive Mathematics, Peano Arithmetic, Predicative Mathematics, etc).

In the process of proving unprovability, we notice that the common combinatorial method of proving unprovability (the method of indiscernible elements) has a clear game-theoretic interpretation: indiscernible elements become upper bounds in the (otherwise potentially infinite) searches for the winning moves.

We finish with a curious example of a Diophantine game where neither player has a Peano Arithmetic-provably winning strategy at the start, but the initial position is a deadly zugzwang for the losing player X, who can, nevertheless, inflict unboundedly high logical-resource damage on his opponent in revenge.

Keywords: Unprovability, undecidability, Diophantine games, provable strategy, logical zugzwang.

This article is multidisciplinary: it is exploring yet one more connection between logic and games, namely the issue of provability and unprovability of existence of winning strategies for a large class of seemingly simple games. This is an open-ended project, with deliberate emphasis on low generality, and on planting the seeds of further interactions and generalizations.

1 Games

Since ancient times, people played games for pleasure. Nowadays, with the creation, expansion and diversification of game theory, we can suddenly perceive and imagine games everywhere: there are situations that we may interpret and codify as games and we deliberately model various problems in mathematics, sciences and decision-making as games. There are competitive games and cooperative games. The number of players may vary. Games are studied for existence of a winning strategy, a losing strategy, the best strategy, winning and losing positions, equilibrium, etc, and for computational complexity of such, where applies. Also, for the minimal, maximal, average number of steps to win or lose or draw, and for many other properties. Some are ancient board games on finite boards, I am including their oriental spellings just in case: go (围棋, 碁, 바둑), chess, xiàngqí (象棋), shōgi (将棋)) or infinite boards (tochki, naughts-and-crosses(5)). Some are played without a board: various mathematical problems, numerous card games, games invented in combinatorial game theory, applications for negotiations, applications in sciences and decision-making). Some games involve randomness (a roll of a die, a flip of a coin, a shuffle of a deck of cards or a computer-generated "random" number), others follow only the decisions of the players. Some games allow for a draw (tie), others don't. Some games necessarily finish (hex, nim, chess, gomoku, xiàngqí (象棋)), others theoretically can continue forever (noughts-and-crosses(5), shōgi (将棋), grand shōgi (大将棋)). I will not go further describing this fascinating family of subjects and its incarnations and applications (in Economics, Finance, Negotiation, Biology, Decision-making, Psychology, etc), especially that there are many excellent books about them all.

Advances in computers and computer science recently produced some spectacular results and continue doing so. I will refrain from listing recent discoveries because the field is developing so fast that whatever I say today will shortly be out of date.

Combinatorial game theory in particular, the core of game-theoretic thought, is a clearly defined subject by now (see [6], [1]), with a rich culture and emerging body of problems and results, clearly connected to what used to be called "mathematical logic" (think of branches through trees, ordinal semi-invariants, strategy-stealing and symmetry arguments). For a brief and incomplete list of examples of the game-paradigm in various corners of former "logic" – see the end of this article.

This article is about metamathematics of Diophantine games. I deliberately chose this low level of generality, so that the concepts and solutions would emerge concrete and clear. However, the material will generalize onto a much larger class of games, where players have infinite numbers of choices at each of their finitely-many moves. Let me rephrase it, and it will be important for understanding the scope of this article. Some games can be re-codified (in many ways) to become other games, with certain mathematical properties staying invariant. We may seem to study only Diophantine games in this note, but really we simultaneously lay ground to the future study of many other classes of "arithmetically complete" games, their subclasses and their future cousins.

There is also some "reverse mathematics" of games to be built and discovered. As "first player wins in arbitrarily-large Hex" is somewhat equivalent to Brouwer's Fixed-Point Theorem [8], we should naturally associate logical complexity of provably winning in Hex with the theory WKL_0 (the strength of Brouwer's theorem). One can also extract some reverse mathematics from various results on Turing-completeness of the computation of the Nash equilibrium in the general case.

However, let me add a necessary disclaimer at this stage. This article is in pure mathematics and not computer science. The issues of feasibility, real-life algorithms and simplifications, of philosophy of computational complexity classes will be irrelevant to us in this article. It was possible, and actually quite easy, to add lots of such material, but I deliberately decided not to. Instead the article concentrates in detail on five crucial components :

1. an introductory essay on metamathematical subtleties;
2. a difference between computable strategy and a provably computable strategy (also reporting some old results by Jones);
3. first ever examples of Diophantine games without a provable strategy;
4. a new result explaining unprovability in game-theoretic terms (bounds on potentially unbounded searches);
5. a curious example of a deadly Diophantine zugzwang with revenge.

2 Basic logic

I am not assuming much knowledge of logic, but will nevertheless list some definitions, and, most importantly of all, fix the classical examples and names of several canonical classical collections of mathematical methods (encapsulated in axiomatic theories) used by mathematicians to prove theorems.

Throughout the article, we shall need arithmetical formulas, sentences, polynomials and understanding of usual mathematical reasoning formalizable as formal mechanical proofs in various axiomatic systems. I will define a small number of classical well-known axiomatic systems sufficient for our purposes. The readers may think of our axiomatic systems as various collections of methods of mathematical proof. The main results of this article are of the form: "such and such methods will never produce a proof of existence of a winning strategy in this concrete game".

Arithmetical formulas are constructed in the expected way from the symbols for variables $x, y, z, w, \ldots$, operations $+, \times, -$, relations $=, <, >$, constants $0, 1$, logical connectives $\wedge$ "and", $\vee$ "or", $\neg$ "not", $\rightarrow$ "implies", $\leftrightarrow$ "if and only if" and two quantifiers: $\forall$ "for all" and $\exists$ "there is". Parenthesis and common abbreviations like x^3 (instead of $x \times (x \times x)$) and 5 (instead of $1 + (1 + (1 + (1 + 1))))$ are routinely used. Arithmetical formulas are supposed to talk about what "natural numbers" $\{0, 1, 2, 3, \ldots\}$, that is, non-negative integers, which some decades ago seemed as a definite and non-controversial mathematical object. We could equivalently set up our whole arithmetical and Diophantine story on full integers or on the rationals (but not on the reals)[1] but let us stay with the set-up chosen by the predecessors.

[1]The fact that Hilbert's Tenth Problem on the rationals is still unsolved doesn't affect us because we allow multiple quantifiers. Real or complex numbers are not suitable because of

A formula without free variables is called a sentence. There is a standard deduction system to mechanically transform formulas into other formulas, called predicate calculus. (All deduction rules are obvious.) On the top of the predicate calculus we can add some straightforward rules of manipulations with arithmetical formulas. This weak system, which defines exponentiation, and has induction for formulas without unbounded quantifiers, is called the Exponential Function Arithmetic, and denoted EFA.

Every arithmetical formula can be (provably in EFA) transformed (up to renaming of variables and dummy variables) into an equivalent form:

$$\forall x_1 \, \exists x_2 \, \forall x_3 \, \ldots Q_n x_n \; p(x_1, x_2, x_3, \ldots, x_n) = 0,$$

where p is a polynomial expression with integer coefficients and the variables are supposed to range over "natural numbers". The quantifier Q_n is "$\exists$" when n is even and "$\forall$" when n is odd.

Let me now mention some classical collections of mathematical methods of argumentation, and their names. (I would like to apologize to my readers and ask them not to be scared of the unfortunate abbreviations used to denote axiomatic systems. They came from the era when abbreviated notation was not yet considered low taste.)

1. "Arithmetic of Basic Manipulations" comes in three forms of increasing proof-capabilities: "Arithmetic Without Induction" (all basic arithmetical rules, but no possibility of mathematical induction), "Polynomial Function Arithmetic" (denoted $I\Delta_0$ or PFA) allows for more coding, more manipulations with finite sets and mathematical induction for formulas without unbounded quantifiers. "Exponential Function Arithmetic", which we saw above, (denoted $I\Delta_0 + \exp$ or EFA) adds exponential function, and hence unrestricted manipulations with finite sets, and so, through coding, can be considered to be the Finite Set Theory. However, its coding devices go a very long way, so long that almost all existing mathematics can be conducted (at least in approximated versions) in this theory.

2. "Primitive Recursive Mathematics", in its various guises (PRA, $I\Sigma_1$, RCA_0) adds induction for one-quantifier formulas, which increases the strength of EFA methods immensely. If you further add all axioms of n-nested induction, the resulting stronger theory will be called $I\Sigma_n$, the theory of induction for n-quantifier formulas.

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decidability of the corresponding theories (it is easier to establish existence of a real root than a rational root). However, interesting questions can still be asked in an expanded language (say, by adding exponentiation), and the answer may depend on the, also as yet unproved, Schanuel Conjecture. Several deep results are known.

3. "Peano Arithmetic" (PA) is $\bigcup_{i=1}^{\infty} I\Sigma_n$, that is full mathematical induction (for all arithmetical formulas). It was barely believable 40 years ago that any examples of meaningful concrete PA-unprovable mathematical statements could ever be found. Now this is a subject in its own, with many interesting results, starting with the famous Paris-Harrington Principle (PH), a version of Ramsey's theorem, unprovable in PA (see the original in [21], and a recent exposition in [2]).

4. "Predicative Mathematics", denoted ATR_0, adds the possibility of transfinite arguments along countable well-orderings. These methods are often necessary in Infinitary Combinatorics (infinite-dimensional Ramsey theory, WQO theory), the theory of Banach Spaces, and occasionally elsewhere in functional analysis. A somewhat strengthened version of "Predicative Mathematics" is called "Borel Mathematics".

5. "The axiomatic theory of natural numbers and sets of numbers", denoted Z_2, is more commonly known as the "Axiomatic Second-Order Arithmetic". This theory has huge strength (ZFC without Powerset) and is not yet barely understood, at least in its first-order fragments (polynomial equations with quantifiers).

6. Let me also mention various set theories, like ZF and its extensions (they can all talk, among other things, about natural numbers). Also, perhaps, we shouldn't forget about methods of mathematical reasoning that have not yet been discovered. It is important that I mentioned them now because the discussion will depend on them later on.

Starting from $I\Sigma_1$ (Primitive Recursive Mathematics) onwards, all the listed axiomatic theories constitute extremely strong collections of methods, so every time their necessity to prove something or their insufficiency to prove something is discovered – it signals the presence of some deep and non-trivial mathematics[2]. This article will show, that whatever collection T of methods of proof one possesses, there will be Diophantine games without a T-provably winning strategy for either player. (This has been theoretically known to logicians around Hilbert's Tenth Problem since the 1970s and is explicitly mentioned by Jones in [12], page 69.)

The transformations of arbitrary arithmetical formulas into polynomial equations with quantifier-prefixes, and the struggle to minimize the polynomial, uses methods and tricks, some obvious and some deep, from the literature on the solution of Hilbert's Tenth Problem. There are many sources available. I will only mention [12], [11], [18], [25], [3], [4], but there are many more.

[2]However, the phenomenon does not reverse: there is a lot of deep and non-trivial mathematics unrelated to logical strength.

3 Definition of Diophantine games

Diophantine games were introduced by James Jones in [12] as a narrowing and concretising of the concept of recursion-theoretic games with a decidable winning condition pioneered by Michael O. Rabin in 1957 [22] and further developed in that era by Alistair Lachlan [17]. By now, recursion-theoretic games have become fundamental and paradigmatic in classical computability theory. Many theorems, notions and definitions can be almost uniformly presented as statements about two-player finite or infinite recursion-theoretic games. We shall not study general recursion-theoretic games here and refer the interested reader to Soare's new monograph [26], section 2.5 and chapters 14, 15, 16, as well as to Martin Kummer's article [16].

We study Diophantine games as defined by James Jones in [12] and discussed by Yuri Matiyasevich in his book [18], chapter 7 and his recent article [19]. Given a polynomial expression $p(x_1, y_1, \ldots, x_n, y_n)$ with integer coefficients in at most the variables shown (some variables are allowed to be dummy variables[3]), the Diophantine game between Player X and Player Y proceeds as follows. First, Player X picks a natural number x_1, then Player Y picks a natural number y_1, etc, until Player X chooses x_n and Player Y answers y_n. (Remember that some variables could be dummies.) The number $2n$, where n is the maximal index of a variable (x_n or y_n) that occurs explicitly in the expression p is called the length of our game. Player X is declared the winner if $p(x_1, y_1, \ldots, x_n, y_n) \neq 0$, otherwise $p(x_1, y_1, \ldots, x_n, y_n) = 0$ and Player Y is declared the winner. Hence, a Diophantine game is a two-player finite-length win-lose game with perfect information and a trivially computable winning condition. It is easy to see that the statement

$$\exists x_1 \, \forall y_1 \; \ldots \; \exists x_n \, \forall y_n \quad p(x_1, y_1, \ldots, x_n, y_n) \; \neq \; 0$$

naively means that Player X 'can always win'. The negation of this formula

$$\forall x_1 \, \exists y_1 \; \ldots \; \forall x_n \, \exists y_n \quad p(x_1, y_1, \ldots, x_n, y_n) \; = \; 0$$

naively means that Player Y 'can always win'.

So, from a naive point of view, for any polynomial expression p, one of the two players 'has' a winning strategy, depending on which of the two formulas is "true": the first formula (then Player X 'has' a winning strategy) or its negation, the second one (then Player Y 'has' a winning strategy). This is no

[3]Throughout this paper we identify a polynomial with its explicit syntactic expression, hence each polynomial p will give rise to infinitely-many Diophantine games via rewriting p into various equivalent forms, some of which may use new or old dummy variables in decorative ways. For example, we can always rewrite p into $p + x_{100} - x_{100}$.

more than the law of excluded middle from classical logic. There is no theorem or mathematical phenomenon behind this empty declaration that one of the players "has a winning strategy".

4 Arithmetical realism

But what do we mean by 'has' or 'can' or 'exists'? Non-constructive existence has always been a dodgy can of worms throughout metamathematics. The more we think about it - the less clear the meaning of 'has' or 'can' or 'exists' becomes. In the next two pages, we shall try to disambiguate and spell out some clarified and more exact versions of the inherently vague, due to metamathematical phenomena, notions of a "winning strategy", its "existence" and "being able to always win". The subtlety will go far beyond the first layer that distinguished "existence" of a winning strategy and "existence" of a computable winning strategy. (This first layer has been successfully approached by James Jones in the 1970s – 1980s, but there is still much work left to be done.)

"Arithmetical realism" (or "arithmetical platonism") is the belief that every arithmetical sentence is either "true" or "false", despite the possibility of being highly unprovable or even absolutely undecidable. Given an arithmetical sentence, isn't it or its negation true in $\mathbb{N}$? What is there to think about? This is a conservative position that tries to reconcile pre-Gödelian views with post-Gödelian metamathematical wonders such as persistent uneliminable unprovability, algorithmic undecidability and the potential of eventual discovery of first-order absolute undecidability, Arithmetical Bifurcation into equally good highly unprovable arithmetical sentences φ and its negation $\neg\varphi$, that is absolute absence of preference as to which of the two opposite alternatives is "more true". For in-depth discussion of Arithmetical Bifurcation, see [5]. The only current candidates to be absolutely undecidable, like CH (Cantor's Continuum Hypothesis), are third-order arithmetical, not yet first-order. The anti-realist camp doesn't have a single coherent neat ideology spelt out, which of course doesn't make anti-realists wrong.

In the arithmetical realism's view, the metamathematical subtleties are usually dismissed as 'epistemological difficulties' that are irrelevant to the question of 'objective' truth or falsity of arithmetical statements. It recklessly proclaims: "We may know that nobody will ever know which of the two possibilities φ or $\neg\varphi$ is true, but nevertheless one of them is true and its negation is false." For no reason at all.

Arithmetical realism claims existence of a winning strategy in every Diophantine game, but this is just an empty useless claim, an application of the law of excluded middle, with no serious mathematics behind it. In this sense, we shall treat the alleged "theorem" of the necessary existence of a winning strategy for one of the players, in its full generality (for all p), as problematic, empty of content and mathematically meaningless. However, of course there are many examples of concrete polynomials p with a genuine concrete winning strategy for one of the players in the game on p, accompanied by a mathematical proof. Just that this strategy will come from analysing p and proving a theorem, not from some dodgy philosophy or the blind application of the law of excluded middle.

5 'Computable winning strategy' is vague too

The second, still not sufficient, in full generality, as we shall see, attempt to define 'can win' comes from the notion of an algorithm and of a 'computable winning strategy'. It may seem that if for one of the players there always existed an algorithm would always, when needed, halt and compute the winning moves, then we isolated the right notion, and got away from subtleties and vagueness of the notion.

However, it was proved by Jones in [12] that for some concrete polynomial expressions p, neither of the two players has a computable winning strategy (see examples below). But this is only the beginning of our story of further metamathematical subtleties.

We know that we can't algorithmically recognize the "truth" of even one-quantifier sentences (MDRP theorem, see [18]), let alone arbitrary arithmetical sentences. We know that the halting problem for algorithms is algorithmically undecidable. Only for some particular instances, concrete algorithms can come accompanied by mathematical proofs (ideally – certified by an automatic proof-checker) of their halting and of their correct performance.

6 Provably computable winning strategy

Suppose we give one of the players an algorithm and tell him that it works (that is, computes the winning moves). How can he trust us? Perhaps he would first be interested to produce a proof that this algorithm always halts and that it indeed guarantees victory? What kind of proof-methods does he know? Maybe $I\Sigma_1$ or PA or Z_2 or ZF or even some as-yet unknown methods? Will he be able to provide the alleged computable winning strategy with a credible proof that it works and guarantees victory? This is the main topic of this note: how metamathematical intricacies related to unprovability and potential discovery of arithmetical absolute unprovability (Arithmetical Splitting) affect

Diophantine games.

However, let us not hurry and move step by step. Let us study a few examples of concrete Diophantine games and analyse “existence of a strategy”, “existence of a computable winning strategy”, “existence of a provable winning strategy” and the question of dependence or independence of existence of strategies on axiomatic theories (mathematical methods) that try to deal with them.

7 Simple introductory examples

Let me first give some simple examples of Diophantine games, where “existence” of a winning strategy for a particular player is equivalent to some classical mathematical statements (proved or unproved). I am doing this to illustrate “arithmetical completeness” of the class of all Diophantine games. Arithmetical completeness means that any statement that can be written as an arithmetical formula can also be written in the language of winning in a Diophantine game.

1) **Infinitude of primes** means that Player Y wins in the game

$$\forall x_1\ \exists y_1\ \exists y_2\ \forall x_3\ \forall x_4\ \exists y_4$$

$$(y_1 - x_1 - y_2 - 2)^2 + (x_3 \cdot x_4 - y_1 - y_4 - 1)^2 \cdot (x_3 - 1)^2 \cdot (x_4 - 1)^2 = 0.$$

Notice the sloppiness of notation, which we shall maintain as convention for interpreting the symbol of subtraction. What if in the above formula x_3 happened to be 0, so $x_3 - 1$ would be negative, while our variables range over non-negative integers? Squaring saved us, and we shall often use this convention as a shorthand instead of modifying the polynomial to its pedantically correct form.

2) **Irrationality of** $\sqrt{5}$ is existence of a winning strategy for Player X in the following game.

$$\forall y_1\ \forall y_2 \quad (y_1 + 1) \cdot (y_2 + 1) \cdot (y_1^2 - 5y_2^2)^2 \neq 0.$$

Any strategy is winning for Player X in this game because his moves don't appear in the polynomial and hence don't count.

3) **The Twin Prime Conjecture** can be written in the form that Player Y

wins in, say, the following Diophantine game.

$$\forall x_1\ \exists y_1\ \exists y_2\ \forall x_3\ \forall x_4\ \exists y_4\ \forall x_5\ \forall x_6\ \exists y_6$$
$$\big(y_1 - x_1 - y_2 - 2\big)^2 \ +\ \big(x_3 \cdot x_4 - y_1 + y_4 + 1\big)^2 \cdot \big(x_3 - 1\big)^2 \cdot \big(x_4 - 1\big)^2 +$$
$$+\ \big(x_5 \cdot x_6 - (y_1 + 2) + y_6 + 1\big)^2 \cdot \big(x_5 - 1\big)^2 \cdot \big(x_6 - 1\big)^2 = 0.$$

4) **The area of the unit circle is bigger than 3.1415** can be written as existence of winning moves for Player Y in a Diophantine game. Let us formulate it as follows: "there is a large enough n such that the grid of squares $\frac{1}{n} \times \frac{1}{n}$ in the upper-right quarter-circle sums up in area to more than 0.7856". So, we simply have to count the number of ordered pairs of natural positive numbers $\langle a, b\rangle$ such that $a^2 + b^2 \le n^2$. (Clearly, such pairs represent all upper-right corners of the relevant squares.) Here is a quick clumsy Diophantine victory saying this.

$$\exists y_1\ \exists y_2\ \exists y_3\ \exists y_4\ \forall x_5\ \exists y_5\ \forall x_6\ \exists y_6\ \forall x_7\ \exists y_7\ \exists y_8\ \exists y_9\ \exists y_{10}\ \exists y_{11}\ \exists y_{12}\ \exists y_{13}\ \exists y_{14}$$

$$\Big[y_3 - y_5 \cdot (y_4 + 1) - 1\Big]^2 + \Big[y_3 - y_6 \cdot (y_4 \cdot (x_5 + 1) + 1) - y_7\Big]^2 + \Big[y_7 + y_8 - y_4 \cdot (x_5 + 1)\Big] +$$

$$+ \Bigg[\Big(2(x_5 + 1) - (x_6 + x_7)^2 - 3x_6 - x_7\Big)^2 + \Big((x_6 + 1)^2 + (x_7 + 1)^2 + y_9 + 1 - y_1^2\Big) +$$

$$+ \Big(y_3 - y_{10} \cdot (y_4 \cdot (x_5 + 2) + 1) - y_7 - 1\Big)^2\Bigg]^2 \ \times$$

$$\times \Bigg[\Big([2(x_5 + 1) - (x_6 + x_7)^2 - 3x_6 - x_7]^2 - y_{11} - 1\Big)^2 \ \cdot\ \Big(y_1^2 + y_9 + 1 - (x_6 +$$
$$1)^2 - (x_7 + 1)^2\Big) \ +$$

$$+\ \Big(y_3 - y_{10} \cdot (y_4 \cdot (x_5 + 2) + 1) - y_7\Big)^2\ \Bigg]^2 + \Big[y_3 - y_{12} \cdot (y_4 \cdot y_1^2 + 1) - y_2\Big]^2 +$$

$$+\ \Big[y_2 + y_{13} - y_4 \cdot y_1^2\Big]\ +\ \Big[1000 \cdot y_2 - 7854 - y_{14}\Big]^2 = 0.$$

With enough practise one quickly learns to translate most kinds of mathematical assertions into the language of winning in particular Diophantine games. It is easy to express "the derivative of $\ln x$ is $\frac{1}{x}$" or "there are exactly five Platonic solids in $\mathbb{R}^3$" and many other, sometimes unexpected, sentences. These translations are not necessarily interesting or useful, especially with currently available coding techniques. However, we study winning in Diophantine games here, and I wanted to illustrate their "arithmetical completeness" and the giant expressive power that comes with it.

8 Does provability of a winning strategy make a difference?

Let $p(x_1, y_1, x_2, y_2, x_3, y_3) = (y_1 + x_2)^2 + 1 - (y_2 + 2) \cdot (y_3 + 2)$. (Here, x_1 and x_3 are dummy variables.) Clearly, Player X (who tries to make $p \neq 0$) 'can' win or 'has' a (computable) winning strategy if and only if there are infinitely-many primes of the form $m^2 + 1$ (which is the as-yet unproved and unrefuted Landau's Conjecture). Assuming Landau's conjecture (as 'true', regardless of provability considerations), here is Player X's 'truly computable' winning strategy. Set the dummy variable $x_1 = 0$ and receive the opponent's move y_1. Start the following unbounded search process: checking, successively, primality of all numbers $(y_1 + 1)^2 + 1$, $(y_1 + 2)^2 + 1$, ..., until, by Landau's statement, eventually encountering a prime number $(y_1 + n)^2 + 1$. Set $x_2 = n$ and notice that regardless of the subsequent values y_2 and y_3 chosen by Player Y, the value of p is different from 0, hence Player X wins.

Now, suppose that Landau's Conjecture has been proved in some good theory T. Will this knowledge give Player X extra information about his game? Yes. Consider the class of all T-provably computable functions. Then, instead of the potentially unbounded search for x_2, Player X can be equipped with a T-provably computable function f extracted from the proof of Landau's Conjecture in T such that his search for x_2 can be restricted to checking primality of only finitely-many, namely $\leq f(y_1)$-many, natural numbers. Hence his "theoretical computable winning strategy" can be converted into a proper T-provably winning strategy by modifying the unbounded search condition for x_2 by adding the upper bound on the search.

Now, suppose (in the 'real world') a strong negation of Landau's Conjecture, that is, that we are given the actual biggest prime number p^* of the form m^2+1. Here is the computable winning strategy for Player Y. Ignore X's first dummy move x_1 and, knowing the last prime value $p^* = m^2+1$, set $y_1 = (m+1)$. Now, regardless of X's move x_2, the number $(y_1 + x_2)^2 + 1$ will be composite and Player Y can choose its divisors (y_2+2) and (y_3+2) undisturbed. Clearly, this is a 'truly' computable winning strategy, but would there 'exist' a computable strategy without the a priori knowledge of the value p^*? It depends on your understanding of the word 'exist'. Consider the family of strategies indexed by primes of the form $p = m^2 + 1$, where the strategy with index p is defined as Player Y's strategy above, with p in place of p^*. Assuming the 'truth' (regardless of provability) of the negation of Landau's Conjecture, this is a finite set, and its final element is a concrete computable winning strategy (so - yes, it 'exists' somewhere), although we never know just on the basis of 'falsity' of Landau's Conjecture, which number p is the index of that final element (so - no, the computable winning strategy 'doesn't practically exist').

Here, we came to distinguish the 'existence' of a computable winning strategy from 'knowing' a computable winning strategy. In the above argument "there exists" a concrete algorithm, a computable winning strategy for Player Y, but he can't find it without knowing the "key" p^*, even though he knows that there are finitely many strategies to choose from. We shall encounter upper bounds for winning moves search again later.

Now suppose that the strong negation of Landau's conjecture has been proved in some good theory T, that is, for some numeral m^*, T proves "$\forall m > m^*$ ($m^2 + 1$ is composite)". Then, since this is a Π^0_1 formula, the Player Y's computable winning strategy above is T-provably computable in all theories T, even in $T = I\Delta_0$, and is thus absolute (not depending on T), contrasting with the situation with Player X's T-provably winning strategy, which is T-sensitive. This example illustrated relativity and absoluteness of 'existence' of provably winning strategies as depending on quantifier complexity of a given Diophantine game.

9 Past examples of Diophantine games by Jones

Remember that any axiomatic theory based on classical logic will blindly conclude that "one of the players has a winning strategy" by classically deducing the A or not-A without even looking at the game represented by A, a mathematically meaningless gesture.

Let us first briefly discuss the old results by Jones [12] before introducing our new material.

Fact 1.

A problem of algorithmic decidability or undecidability. Inputs are all possible polynomial expressions $p(x_1, y_1, x_2, y_2)$ defining all possible Diophantine games of length 4. There is no algorithm to decide which of the two players 'has' a winning strategy in a given Diophantine game.

I am not aware whether whether this is the strongest form of this theorem nowadays. It relies on the best-available representation of enumerable sets by polynomial equations (or non-equalities) with quantifier-prefixes.

However, one can go beyond the question of existence of a decision algorithm for many polynomials and present one single polynomial with this property.

Fact 2.

For some natural number n, neither of the two players has a computable winning strategy in the Diophantine game defined by the following polynomial expression.

$$\left\{n + x_4 + 1 - y_3\right\} \cdot \left\{\left\langle (x_4 + x_5)^2 + 3x_5 + x_4 - 2y_3 \right\rangle^2 + \right.$$
$$+\left\langle \left[(y_7 - x_5)^2 + (y_8 - x_7)^2\right] \cdot \left[(y_7 - x_4)^2 + (y_8 - x_6)^2 \cdot \left((y_3 - n)^2 + (y_8 - x_7 - \right.\right.\right.$$
$$\left.\left. x_1 - y_1)^2\right)\right] \times$$
$$\left. \times \left[(y_7 - 3y_3)^2 + (y_8 - x_6 - x_7)^2\right] \cdot \left[(y_7 - 3y_3 - 1)^2 + (y_8 - x_6 \cdot x_7)^2\right] - x_9 - 1 \right\rangle^2 \times$$
$$\left. \times \left\langle \left[y_8 + x_9 + x_9 \cdot y_7 \cdot x_3 - x_2\right]^2 + \left[y_8 + x_{10} - y_7 \cdot x_3\right]^2 \right\rangle \right\}.$$

The deficiency of this example is that we don't know the value of the number n that does the work, and we have no idea where it sits. However, there are examples without this drawback, also from Jones.

Fact 3.
In the following Diophantine game, neither player has a computable winning strategy.

$$\left\{\{x_1 + x_6 + 1 - y_4\}^2 \cdot \left\{\langle (x_6 + x_7)^2 + 3x_7 + x_6 - 2y_4\rangle^2 + \left\langle [(y_9 - x_7)^2 + (y_{10} - x_9)^2] \times \right.\right.\right.$$
$$\times \left[(y_9 - x_6)^2 + (y_{10} - x_8)^2 \cdot ((y_4 - x_1)^2 + (y_{10} - x_9 - y_1)^2)\right] \cdot \left[(y_9 - 3y_4)^2 + (y_{10} - x_8 - x_9)^2\right] \times$$
$$\left.\left. \times \left[(y_9 - 3y_4 - 1)^2 + (y_{10} - x_8 x_9)^2\right] - x_{12} - 1 \right\rangle^2 + \left\langle \left[y_{10} + x_{12} + x_{12} y_9 x_4 - x_3\right]^2 + \left[y_{10} + x_{13} - y_9 x_4\right]^2 \right\rangle \right\} -$$
$$\left. - y_{13} - 1 \right\} \cdot \left\{x_1 + y_5 + 1 - x_5\right\} \cdot \left\{\left\langle (y_5 - y_6)^2 + 3y_6 + y_5 - 2x_5 \right\rangle^2 + \left\langle [(x_{10} - y_6)^2 + (x_{11} - y_8)^2] \times \right.\right.$$
$$\times \left[(x_{10} - y_5)^2 + (x_{11} - y_7)^2 \cdot \left((x_5 - x_1)^2 + (x_{11} - y_8 - x_2)^2\right)\right] \cdot \left[(x_{10} - 3x_5)^2 + (x_{11} - y_7 - y_8)^2\right] \times$$
$$\left.\left. \times \left[(x_{10} - 3x_5 - 1)^2 + (x_{11} - y_7 y_8)^2\right] - y_{11} - 1 \right\rangle^2 + \left\langle \left[x_{11} + y_{11} + y_{11} x_{10} y_3 - y_2\right]^2 + \left[x_{11} + y_{12} - x_{10} y_3\right]^2 \right\rangle \right\}.$$

Jones in [12] also shows (using enumerability of sets of consequences of a theory) how for every theory T to construct a polynomial such that T can't prove that either of the two players in the corresponding Diophantine game can always win. However, the polynomial involves a large coefficient C_T which is hard to find. All examples by Jones use, in various forms, representations of enumerable sets by polynomials with quantifier prefixes. The battle for the size of the polynomial is related to the battle for the size of the universal Turing machine, a universal Diophantine equation or a smallest set of relations defining a group with undecidable word problem. However, since multiple quantifiers are allowed, there are trade-offs that can dramatically shrink the size of the polynomial. Probably, as our knowledge of number theory grows, eventually, new coding tricks will be discovered to compress the resulting polynomials to excitingly short lengths.

10 Diophantine games without a provable winning strategy

Every first-order arithmetical formula can be re-written into a prefixed polynomial equation which means "Player Y can always win". Equally easy is to re-write any formula into a prefixed polynomial inequality which means "Player X can always win". By doing this to known unprovable statements, we obtain Diophantine games, where existence of a winning strategy is unprovable.

Here is our first example of PH^2, the Paris-Harrington principle for pairs. Consider the statement

$$\forall\, m\, e\, \exists\, N\, \forall\, a\, b\, \exists\, c\, d\, A\, X\, \forall\, x\, y\;\; \exists\, BCF\, \exists\, hijk\ell npqrst$$

$$x \cdot \Big(y+B-x\Big) \cdot \Big(A+m+B-y\Big) \cdot \Big((A+h-d)^2+((d+1)\cdot i+A-c)^2+(B+n-dx)^2+$$
$$+((dx+1)\cdot j+B-c)^2+(C+r-dy)^2+((dy+1)\cdot k+C-c)^2+(B+s+1-C)^2+(C+t-N)^2+$$
$$+(F+p-b\cdot(B+C^2))^2+(a-\ell\cdot b\cdot(B+C^2)-F-\ell)^2+(X-F+eq)^2\Big) \;=\; 0.$$

It has been proved by the author and Michiel De Smet in [3] that this statement is EFA- equivalent to PH^2, and hence is unprovable in $I\Sigma_1$. (The equivalence is rather straightforward, using some elementary, rather coarse, coding techniques.) Hence, introducing dummy variables to satisfy the definition of a Diophantine game, Player Y 'can always win' in the following game but this is unprovable by primitive recursive methods. However, PH^2, and, hence, Player Y's ability to win, is provable in $I\Sigma_2$, thus requiring a complex nested inductive argumentation. Let $p_1(x_1, y_1, \ldots, x_{22}, y_{22})$ be the following polynomial expression.

$$x_8\cdot\Big(x_9+y_9-x_8\Big)\cdot\Big(y_6+x_1+y_9-x_9\Big)\cdot\Big((y_6+y_{12}-y_5)^2+((y_5+1)\cdot y_{13}+y_6-y_4)^2+ +(y_9+y_{17}-y_5\cdot x_8)^2+((y_5\cdot x_8+1)\cdot y_{14}+y_9-y_4)^2+(y_{10}+y_{20}-y_5\cdot x_9)^2+((y_5\cdot x_9+1)\cdot y_{15}+ +y_{10}-y_4)^2+(y_9+y_{21}+1-y_{10})^2+(y_{10}+y_{22}-y_2)^2+(y_{11}+y_{18}-x_4\cdot(y_9+y_{10}^2))^2+ +(x_3-y_{16}\cdot x_4\cdot(y_9+y_{10}^2)-y_{11}-y_{16})^2+(y_7-y_{11}+x_2\cdot y_{19})^2\Big).$$

Theorem 10.1
In the Diophantine game corresponding to the polynomial expression

$$p_1(x_1,y_1,\ldots,x_{22},y_{22}),$$

Player Y has an $I\Sigma_2$-provably winning strategy, but has no $I\Sigma_1$-provably winning strategy.

Here is another example from [3], this time equivalent to PH^3, the Paris-Harrington principle for triples (again, the equivalence is quite straightforward) hence unprovable in $I\Sigma_2$ but provable in $I\Sigma_3$.

Theorem 10.2
The following statement is equivalent to PH^3 *and therefore also to the* 1-*consistency of $I\Sigma_2$. In particular, it is not provable in $I\Sigma_2$ but is provable in $I\Sigma_3$.*

$$\forall\, m\, e\ \exists\, N\ \forall\, a\, b\ \exists\, c\, d\ A\ X\ \forall\, xyz\ \ \exists\, BCD\ F\ \exists\ hijk\ell npqrstuvw$$

$$x\cdot\Big(y+B-x\Big)\cdot\Big(z+B-y\Big)\cdot\Big(A+m+B-z\Big)\cdot\Big((A+h-d)^2+(c-A-(d+1)\cdot i)^2+(B+n-dx)^2+ +(c-B-(dx+1)\cdot j)^2+(C+r-dy)^2+(c-C-(dy+1)\cdot k)^2+(D+t-dz)^2+(c-D-(dz+1)\cdot u)^2+(B+s+1-C)^2+ +(C+v+1-D)^2+(D+w-N)^2+(F+p-b\cdot(B+C^2+D^3))^2+(a-F-\ell\cdot b\cdot(B+C^2+D^3)-\ell)^2+(F-X+qe)^2\Big)\ =\ 0.$$

Rewriting it into a polynomial with the variables suitably renamed for a Diophantine game, one obtains the following expression $p_2(x_1,y_1,\ldots,x_{27},y_{27})$:

$$x_8\cdot\Big(x_9+y_{10}-x_8\Big)\cdot\Big(x_{10}+y_{10}-x_9\Big)\cdot\Big(y_6+x_1+y_{10}-x_{10}\Big)\cdot\Big((y_6+y_{14}-y_5)^2+(y_4-y_6-(y_5+1)\cdot y_{15})^2+ +(y_{10}+y_{19}-y_5\cdot x_8)^2+(y_4-y_{10}-(y_5\cdot x_8+1)\cdot y_{16})^2+(y_{11}+y_{22}-y_5\cdot x_9)^2+(y_4-y_{11}-(y_5\cdot x_9+1)\cdot y_{17})^2+ +(y_{12}+y_{24}-y_5\cdot x_{10})^2+(y_4-y_{12}-(y_5\cdot x_{10}+1)\cdot y_{25})^2+(y_{10}+y_{23}+1-y_{11})^2+(y_{11}+y_{26}+1-y_{12})^2+ +(y_{12}+y_{27}-y_2)^2+(y_{13}+y_{20}-x_4\cdot(y_{10}+y_{11}^2+y_{12}^3))^2+(x_3-y_{13}-y_{18}\cdot x_4\cdot(y_{10}+y_{11}^2+y_{12}^3)-y_{18})^2+ +(y_{13}-y_7+y_{21}\cdot x_2)^2\Big).$$

Theorem 10.3
In the Diophantine game defined by the polynomial expression $p_2(x_1, y_1, \ldots, x_{27}, y_{27})$*, Player Y can always win. This is not provable in* $I\Sigma_2$ *but is provable in* $I\Sigma_3$*.*

In other words a 4-nested induction argument is required and unavoidable to provably win in this game. Needless to say, 4-nested induction arguments are unheard of in "real" mathematics. Remember the controversy with the 2-nestedness of the inductive argument in the original proofs of van-der Waerden's theorem on arithmetical progressions, as well as the Hales-Jewett theorem on the generalized game of naughts and crosses, which thus theoretically allowed for the possibility of $I\Sigma_1$ unprovability of these statements. (There were even indications, coming from Furstenberg's school's dynamical proofs that this must indeed be the case.) The controversy was eliminated by Saharon Shelah in [23], who circumvented the perceived difficulty and came up with a different inductive argument that doesn't require a nested induction argument at each inductive step. The readers may read the whole story in [23].

The manuscript [3] contains many other examples of polynomials with quantifier-prefixes (and hence the statements of existence of winning strategies) which are unprovable in ATR_0, fragments of Z_2 and even in ZFC plus existence of n-Mahlo cardinals for each finite order n (building on arithmetizations of deep unprovability results by Harvey Friedman [7]).

Let me mention at this stage that there is an article from as recently as the 1980s, by perfectly qualified logic experts, that openly doubted the possibility of ever finding such short and clearly "feasible" unprovable statements in the form of polynomial equations with quantifiers.

11 Winning strategies and binding sets

The source of the difficulty for both players in a Diophantine game comes from the unboundedness of quantifiers involved in the statement

$$\exists x_1\, \forall y_1\, \ldots\, \exists x_n\, \forall y_n \quad p(x_1, y_1, \ldots, x_n, y_n) \;\neq\; 0$$

('Player X can win') and in its negation

$$\forall x_1\, \exists y_1\, \ldots\, \forall x_n\, \exists y_n \quad p(x_1, y_1, \ldots, x_n, y_n) \;=\; 0$$

('Player Y can win') and from "Gödelian" arithmetical completeness of polynomials with quantifiers.

It would be already great for one of the players if he not necessarily knew (or could compute) all the winning moves himself, but at least knew how far to search in every given position, that is, existence of upper bounds on a winning answer as a function of the current position. This is the clear idea behind the next definition.

Definition 11.1 *Given a polynomial expression $p(x_1, y_1, \ldots, x_n, y_n)$, we say that an infinite set $A \subseteq \mathbb{N}$ is binding for the Diophantine game on p if for all $a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}$ and $b_1 < b_2 < \cdots < b_{2n-1} < b_{2n}$ in A,*

$$\forall x_1 < a_1 \, \exists y_1 < a_2 \, \ldots \, \forall x_n < a_{2n-1} \, \exists y_n < a_{2n} \quad p(x_1, y_1, \ldots, x_n, y_n) = 0 \iff$$

$$\iff \forall x_1 < b_1 \, \exists y_1 < b_2 \, \ldots \, \forall x_n < b_{2n-1} \, \exists y_n < b_{2n} \quad p(x_1, y_1, \ldots, x_n, y_n) = 0.$$

Knowledge of an infinite binding set for p gives enormous playing power in the Diophantine game on p.

Lemma 11.2 *Let p be a polynomial and suppose we know a binding infinite set $A \subseteq \mathbb{N}$ for the Diophantine game on p. Then we know which of the two players is supposed to win from the knowledge of the first $2n$ elements of A. More exactly,*

$$\mathbb{N} \models \forall x_1 \, \exists y_1 \, \ldots \, \forall x_n \, \exists y_n \quad p(x_1, y_1, \ldots, x_n, y_n) = 0 \iff$$

$$\iff \mathbb{N} \models \forall x_1 < a_1 \, \exists y_1 < a_2 \, \ldots \, \forall x_n < a_{2n-1} \, \exists y_n < a_{2n} \quad p(x_1, y_1, \ldots, x_n, y_n) = 0,$$

which is a bounded formula, hence algorithmically decidable.

The proof is clear.

Readers familiar with the Specker colouring (a computable colouring of size-3 subsets of natural numbers without a computable infinite monochromatic subset, namely $\langle x < y < z \rangle \to 0$ if every Turing machine below x halts within y steps if and only if it halts within z steps, and $\to 1$ otherwise) or with Paris-Kirby indiscernibles (see [2] for a modern exposition) will notice the clear analogy. Indeed, the definition of bindedness of A is a form of indiscernibility (a crucial notion in unprovability proofs), and in our context of potentially unbounded searches for winning moves, these indiscernibles serve as upper bounds for these searches, turning infinite search into finite. So, a fundamental notion from metamathematics in fact has a clear game-theoretic explanation.

Theorem 11.3 *Suppose $A \subseteq \mathbb{N}$ is an infinite binding set for the Diophantine game on p. Then the following hold.*

1. *We can compute the winning strategy for the winning player from the oracle A.*

2. *In particular, if A is a computable set then we can write the winning algorithm for the winning player explicitly, from the decision algorithm for A. Moreover, the knowledge of the first $2n$ elements of A already decides which of the players can guarantee victory.*

3. *There are polynomials p such that the Diophantine game on p has no infinite computable binding set.*

4. *Let n be a natural number. Then there exist polynomials p such that every binding infinite set A for the Diophantine game on p computes $0^{(n)}$.*

The proof is clear.

12 A deadly PA-unprovable Diophantine zugzwang with the possibility of inflicting unbounded logical damage in revenge

A zugzwang is a position in which any move by a player worsens his situation. A deadly zugzwang is a position where every move results in a losing position. In games where players have only finitely-many possible moves, a deadly zugzwang would be already a losing position. But in Diophantine games, unlike chess, xiàngqí (象棋) or shōgi (将棋)), each player has infinitely-many possibilities for a move.

It may seem paradoxical at first, but this is a common phenomenon in logic when each instance $\varphi(n)$ of a formula $\varphi(x)$ is provable using some given methods but the statement $\forall x\ \varphi(x)$ is unprovable. Just that for each instance, the proof is separate, and there doesn't exist a uniform proof that would deal with all infinitely-many instances.[4]

Now we shall describe an interesting Diophantine game. At the start, neither of the two players has a PA-provably winning strategy. However, any first move by Player X will put him in an imminent PA-provably losing position. Understanding this, Player X can, however, inflict arbitrarily large logical resource damage on his opponent, by his very first move x_1. What do we mean by "logical resource damage"? It means that in order to actually provably win, Player Y would have to use arguments of unbounded high logical strength $I\Sigma_n$ (in the hierarchy $I\Sigma_n$ of approximations of PA). And how high the needed n is comes from the first move of Player X. He knows he is doomed but at least he can exert some revenge and make Player Y struggle arbitrarily hard.

The game is based on the following theorem by the author and Michiel De Smet [3], [4]. (Again, the translation can be done quite easily, with lots of tricks, and I will not burden the reader with this long sequence of easy steps.)

Theorem 12.1

[4]Compare it with continuity and uniform continuity of functions in calculus. For continuous functions, for each separate ε, there exists its own δ, but in uniform continuity there is a δ that suits all εs.

Consider the following statement $\Phi(n)$ with one free variable n. For every $n > 1$, the statement $\Phi(n)$ is equivalent to the Paris-Harrington Principle in dimension n and hence to the 1-consistency of $I\Sigma_{n-1}$. In particular, $\forall\, n\, \Phi(n)$ is equivalent to PH *and hence to the 1-consistency of full Peano Arithmetic. Therefore $\forall\, n\, \Phi(n)$ is unprovable in Peano Arithmetic.*

$$\forall\, m\, e\, \exists\, N\, \forall\, a\, b\, u\, v\, \exists\, str\, STR\, \alpha\beta\gamma\delta\varepsilon\rho\tau\ \ \forall\, i\, j\, \exists k\ell BC\sigma\Sigma pPQU\, \Omega Mxyzw\Delta E\, FGK\, HLZ\, XW$$

$$\Big\{[s-t-r-1]^2+[S-T-R-1]^2+\Big[i\cdot(i-p-\mathbf{n}-1)\cdot\Big([s-\sigma-\Sigma(it+1)]^2+[\sigma+w-it]^2+$$
$$+[S-C-z(iT+1)]^2+[C+\Omega-iT]^2+[\sigma+B+1-C]^2\Big)\Big]\cdot\Big([a-\alpha-\beta((t+s^2)b+1)]^2+$$
$$+[\alpha+\gamma-(t+s^2)b]^2+[a-\delta-\varepsilon((T+S^2)b+1)]^2+[\delta+\rho-(T+S^2)b]^2+[\alpha+\tau+1-\delta]^2\Big]\Big\}\cdot$$
$$\cdot\Big\{i\cdot(j+B-i)\cdot(r+m+C-j)\cdot\Big[[s-r-x(t+1)]^2+[r+\ell-t]^2+[s-B-\sigma(ti+1)]^2+$$
$$+[B+w-ti]^2+[s-C-M(tj+1)]^2+[C+y-tj]^2+[B+z+1-C]^2+[C+\Omega-N]^2\Big]^2+$$
$$\Big[i\cdot(\mathbf{n}+p+1-i)\cdot k\cdot(r+m+E-k)\cdot(u+\Sigma-v)\cdot\Big[[u-F-G(iv+1)]^2+[F+K-iv]^2+$$
$$+[s-H-L(tk+1)]^2+[H+Z-tk]^2+[u-P-Q((i+1)v+1)]^2+[P+U-(i+1)v]^2+$$
$$+[P+X-F]\cdot[(H-P)^2-W-1]\Big]\cdot\Big[[a-\alpha-\beta((v+u^2)b+1)]^2+[\alpha+\gamma-(v+u^2)b]^2+[\alpha-S-\tau\cdot e]^2\Big]\Big]^2\Big\}=0.$$

Let us rewrite this statement to be equivalent to "Player Y can always win" in a certain Diophantine game by renaming the variables. It is crucial for our example that n becomes Player X's first move x_1. Let $q(x_1, y_1, \ldots, x_{46}, y_{46})$ be the following polynomial expression.

$$\Big\{[y_7-y_8-y_9-1]^2+[y_{10}-y_{11}-y_{12}-1]^2+\Big[x_{20}\cdot(x_{20}-y_{27}-x_1-1)\cdot\Big([y_7-y_{25}-y_{26}\cdot(x_{20}\cdot y_8+1)]^2+$$
$$+[y_{25}+y_{36}-x_{20}\cdot y_8]^2+[y_{10}-y_{24}-y_{35}\cdot(x_{20}\cdot y_{11}+1)]^2+[y_{24}+y_{31}-x_{20}\cdot y_{11}]^2+[y_{25}+y_{23}+1-y_{24}]^2\Big)\Big]\cdot$$
$$\cdot\Big([x_4-y_{13}-y_{14}\cdot((y_8+y_7^2)\cdot x_5+1)]^2+[y_{13}+y_{15}-(y_8+y_7^2)\cdot x_5]^2+[x_4-y_{16}-y_{17}\cdot((y_{11}+y_{10}^2)\cdot x_5+1)]^2+$$
$$+[y_{16}+y_{18}-(y_{11}+y_{10}^2)\cdot x_5]^2+[y_{13}+y_{19}+1-y_{16}]^2\Big]\Big\}\cdot\Big\{x_{20}\cdot(x_{21}+y_{23}-x_{20})\cdot(y_9+x_2+y_{24}-x_{21})\cdot$$
$$\cdot\Big[[y_7-y_9-y_{33}(y_8+1)]^2+[y_9+y_{22}-y_8]^2+[y_7-y_{23}-y_{25}\cdot(y_8\cdot x_{20}+1)]^2+[y_{23}+y_{36}-y_8\cdot x_{20}]^2+$$
$$+[y_7-y_{24}-y_{32}\cdot(y_8\cdot x_{21}+1)]^2+[y_{24}+y_{34}-y_8\cdot x_{21}]^2+[y_{23}+y_{35}+1-y_{24}]^2+[y_{24}+y_{31}-y_3]^2\Big]^2+$$
$$+\Big[x_{20}\cdot(x_1+y_{27}+1-x_{20})\cdot y_{21}\cdot(y_9+x_2+y_{38}-y_{21})\cdot(x_6+y_{26}-x_7)\cdot\Big[[x_6-y_{39}-y_{40}\cdot(x_{20}\cdot x_7+1)]^2+$$
$$+[y_{39}+y_{41}-x_{20}\cdot x_7]^2+[y_7-y_{42}-y_{43}\cdot(y_8\cdot y_{21}+1)]^2+[y_{42}+y_{44}-y_8\cdot y_{21}]^2+$$
$$+[x_6-y_{28}-y_{29}\cdot((x_{20}+1)\cdot x_7+1)]^2+[y_{28}+y_{30}-(x_{20}+1)\cdot x_7]^2+[y_{28}+y_{45}-y_{39}]\cdot[(y_{42}-y_{28})^2-y_{46}-1]\Big]\cdot$$

$$\cdot\Big[[x_4 - y_{13} - y_{14}\cdot((x_7 + x_6^2)\cdot x_5 + 1)]^2 + [y_{13} + y_{15} - (x_7 + x_6^2)\cdot x_5]^2 + [y_{13} - y_{10} - y_{19}\cdot x_3]^2\Big]^2\Big\}.$$

Here is the explanation of this situation in simpler words: PA-unprovability of winning for either player at the start, being a deadly zugzwang for Player X, with logically unbounded revenge. Suppose both players have Peano Arithmetic as their weapon. At the beginning of the game neither of the two players has a PA-provably winning strategy because the statement of Player Y's ability to always win is equivalent to PH, hence is PA-unprovable. But once Player X has chosen his first move x_1, he will make the instance of the statement "Player Y can win the game from the position x_1" equivalent to PH^{x_1}, hence provable in PA, and even in $I\Sigma_{x_1+1}$. Player X will lose but he can nevertheless impose a vengeful penalty for his loss by choosing an arbitrarily-large x_1. The larger the value of x_1 - the more logical power Player Y has to use to extract the winning moves, namely, he will have to use at least the strength of $I\Sigma_{x_1+1}$-proofs (that is, at least (x_1+2)-nested inductive arguments) to provably win.

Theorem 12.2 *(Deadly zugzwang with unlimited revenge possibility.)*
In the Diophantine game defined by the polynomial q above, neither player has a PA*-provably winning strategy. However, any first move by Player X puts Player Y in a* PA*-provably winning position. However, Player X can revenge in advance by setting x_1 arbitrarily large, hence making existence of a winning strategy for Player Y $I\Sigma_{x_1}$-unprovable.*

13 Some games in logic

The game-theoretic paradigm grew to become one of the few ingredients permeating all corners of former "mathematical logic". I will list a few famous examples, but the list is far from complete. The purpose of this list is to stimulate interest and catalyse further mutually-enriching connections between games and logic.

1. For a systematic presentation of games in model theory and parts of algebra, see Hodges [10]. For another assortment of games in model theory, one may also consult Väänänen [29].

2. The game paradigm is now central in classical computability theory. See Rabin [22], Lachlan [17], Kummer [16], Soare [26].

3. In set theory, several games are of fundamental importance: full Axiom of Determinacy (that implies existence of sharps of all reals), Projective

Determinacy and its consequences, Analytic Determinacy, Borel Determinacy. There are games for sharps (the Martin-Harrington Theorem), see Kanamori [13], there is the Solovay's game that turns the club filter on ω_1 into an ultrafilter (making ω_1 measurable) and related games. Many forcing constructions can be set up as games. There are Choquet games, Banach-Mazur games, game quantifiers, $*$-games, Wadge games, Separation games, Cantor-Bendixon games, meager-comeager games and many more. See Kechris [15] or Kanovei, Sabok, Zapletal [14].

4. Games are at the heart of metamathematically-sensitive Ramsey theory. For example, the Hales-Jewett theorem was first formulated as a generalized game of naughts and crosses. Paris-Kirby indicator games can be seen as games for largeness (of a piece of a partition). Similarly the Clote game. The now-famous Hercules versus Hydra battle is a game, as well as its cousin, unjustly less famous, but much harder to kill, the Bucholtz Hydra. There are experts, in the now-esoteric subject called "proof theory" who understand why termination of the Hydra battle, "Gentzen's cut-elimination" and well-orderedness of ordinal notations up to the ordinal ε_0 are actually one and the same statement of termination of the same game, just cast in slightly different terms.

5. Many combinatorial forcing constructions also can be viewed as games, for example in Nash-Williams theory. See Todorcevic's [27], [28] as well as the important games one can find in Halbeisen [9]. There are games of domino and tiling games (to tile the plane or the space, possibly in higher dimensions), probably with most fascinating as-yet undiscovered metamathematics.

6. There are "games for truth" and games to build a nonstandard model. Classical theorems: omitting types, arithmetized completeness, various compactness principles and recursive saturation can all be described as games: one player tries to build an infinite Henkinized branch in a tree, while the other player tries to block any emerging candidate for a branch. Which of the players can guarantee victory of course depends on the conditions of the theorem in question: consistency of a theory, non-principality of a type, etc.

7. In the analysis of predicativity, there are games of length (which player can build a longer well-ordering) and their cousins played on linear orders or on well-founded trees. See Simpson's [24] and Kechris's [15].

8. Many large cardinals affect games that happen deep below them. There are Ramsey games, van der Waerden games, Schur games. There are

games for minor-embeddings and games for indiscernibles, games for ordinal-indiscernibles and games for ordinal L-indiscernibles.

9. There is a recent series of deep results about slices of Determinacy by Montalbán and Schore [20]. Also, by now, we have achieved an understanding of the correspondence between the slices of Determinacy and slices of the infinite-dimensional Ramsey theorem $\omega \to (\omega)^\omega_2$. (The best exposition is in [24].) However, as of today (2021), the question of whether the full AD implies $\omega \to (\omega)^\omega_2$ remains open.

This list is just a fraction of what is going on. Literally, every corner of former "logic" is teeming with games. I propose that it could be very interesting and mutually rewarding to connect these games with the mathematical games that people play in Combinatorial Game Theory.

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Andrey Bovykin
Department of Mathematics
Institute of Mathematics
Federal University of Bahia (UFBA)
Campus de Ondina, Av. Adhemar de Barros, S/N,
CEP 40170-110, Salvador, BA, Brazil
E-mail: andrey.bovykin@gmail.com