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Sheaves of Structures and the Problem of Universals

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Abstract

In this paper I demonstrate that, if classical logic is used as the logic underlying the explanantia description given to the explananda of a nominalist solution for the Problem of Universals, then the following contradiction is generated: (N) nominalist theories eliminate universals if and only if nominalist theories admit universals. The purpose of this article is, therefore, to provide nominalists with a logic capable of providing the appropriate tools so that they can try to prevent their theories from falling into (N). Such logic, as will be demonstrated herein, is the logic of a sheaf of structures over the time-line seen as a topological space. I demonstrate what a nominalist theory whose underlying logic is the logic proposed in this article would be, namely, what a nominalist theory that naturally expresses the congruence with the extended and variable feature of objects and events that extend into space-time would be. The theory obtained is a Variable Class Nominalism, where sets are conceived not as absolute entities, but as variable structures that vary over a domain of knowledge states. Finally, I demonstrate that the contradiction (N) does not occur in the Variable Class Nominalism proposed by myself herein.

Keywords: limit-idealizations, extended objects, universals, ontological commitment, nominalist theories.

Introduction

The Problem of Universals is one of those issues whose nature is as difficult or more difficult than the solution it requires. In other words, the attempt to provide a solution to the Problem of Universals leads to two more essential questions:

(E1) *What are the explananda of a solution to the Problem of Universals?*

(E2) *What types of explanantia are required for these explananda?*

Rodriguez-Pereyra ([1]: 261) significantly clarified these issues by defining the Problem of Universals as a problem that requires the description of the truthmakers of certain sentences, which led to a change in the commonly accepted perspective of the problem, i.e., a solution to the Problem of Universals must explain the Many over One, not One over Many. However, in spite of the clarification achieved, two even more essential questions compared to (E1) and (E2) seem to have not yet been detected until now:

(L1) *Is the logic underlying the description of the explanantia given to the explananda of a solution to the Problem of Universals suitable to provide this description?*

(L2) *If not, what logic is adequate to describe the explanantia given to the explananda of a solution to the Problem of Universals?*

As we shall see in Section 1, any model of reality whose underlying logic is classical logic, including realistic and nominalistic solutions to the Problem of Universals, is ontologically committed to abstract entities called 'limit-idealizations'. Such limit-idealizations are not real natural phenomena, nor do they possess any experimental or perceptible character; they are mere ideal limits of extended objects and events, and are inserted only to save the laws of classical logic; in other words, these entities are inserted only so that the mathematical models of the world whose underlying logic is the classical one can be realized.

I demonstrate that these entities are not only abstract objects, but also universals. Thus, if classical logic is used as the logic underlying the description of the explanantia given to the explananda of a realistic solution for the Problem of Universals, then this solution (model) is ontologically compromised with limit-idealizations, and since limit-idealizations are universal, the realist has to admit in his ontology not only the universally accepted universals, but the universals that are limit-idealizations, thus multiplying the entities admitted in the ontology. This, however, is not such a concern for realistic theories, except for the fact that this go against Occam's razor. However, this ontological commitment to limit-idealizations is devastating for nominalist theories. I demonstrate that if classical logic is used as the logic underlying the explanantia description given to the explananda of a nominalist solution for the Problem of Universals, then the following contradiction is generated:

(N) *Nominalist theories eliminate universals if and only if nominalist theories admit universals.*

In this way, there are only two outputs for nominalists: to become a realist; or try to prevent their theories from falling into (N). As we shall see, the only plausible way to avoid nominalist theories falling into (N) is to replace classical logic as the logic underlying nominalist theories. So, the answer to (L1) is: classical logic is not adequate if you are a realist, and should definitely be replaced if you are a nominalist.

Given that the need to insert limit-idealizations lies in the incongruity between models (theories, solutions) whose underlying logic is the classic logic and the way in which objects and events occur in space-time, a first step to avoid (N) would be to find a way to make these theories express a natural congruence with the extended and variable feature of objects. This leads directly to the answer given to question (L2): The proper logic for describing the explanantia given to the explananda of a solution (nominalistic or realistic) for the Problem of Universals is a logic that is appropriate to the objects and events that extend in the space-time, that is, a logic that revises the punctual and instantaneous properties of extended objects, namely, the properties that are ideal limits of extensive properties. The purpose of this article is, therefore, to provide nominalists with a logic capable of providing the appropriate tools so that they can try to prevent their theories from falling into (N). Such logic, as will be demonstrated in Section 2, is the logic of a sheaf of structures over the timeline seen as a topological space.

In Section 3, I show what would be a nominalist theory whose underlying logic is the logic given as a response to (L2), that is, I show how a nominalist theory would naturally express congruence with the extended and variable characteristic of objects and events that extend in space-time. For this, I review Class Nominalism, with the logic given as a response to (L2) as a framework. The idea is to present a cumulative hierarchy of variable sets that provides us with a model for designing sets not as absolute entities, but as variable structures that vary over a domain of knowledge states. Such a domain can be thought of as a partial, not necessarily linear, order where there is the possibility of alternative evolutions, for example, an evolution where the sentence 'a is F' is true and another where it is not. Finally, I demonstrate that the contradiction (N) does not occur in the Variable Class Nominalism proposed by myself herein.

1 Limit-idealizations, classical logic and universals

First, note that the notion of truth that is contained in physical reality is a contextual notion, in the following sense: given a property F, an object p and a temporal instant x, to say that p has the property F at the temporal instant x is to say p has the property F in a measurement realized in an extended interval of time in which x is contained. As we know, the mathematical models used so far to describe this contextual feature are all based on classical logic and classical set theory. However, as we also know, the notion of truth that is contained in classical logic is absolute, rather than contextual; and classical set theory, since it has as underlying logic classical logic, also has an absolute notion of truth contained within itself. But then, how is it possible that such models, whose underlying basis are classical logic and classical set theory, can properly describe a physical reality possessing a contextual notion of truth? In other words, how can these models capture the contextual characteristic of physical reality? This is possible thanks to the notion of limit. Thus, it can be said that what makes classical logic work for objects and events that are extended in the continuous space-time is the notion of limit.

The above observations present us with two issues. First, spatial points and temporal instants are idealizations-limit, designed so that the construction of mathematical models of the world can be performed. In other words, punctual and instantaneous events occurring in physical-mathematical models of the world are not in fact natural phenomena, but only ideal limits of extensive events (see Whitehead [2]). Any attempt to bring this punctual and instantaneous localization of phenomena to the ultimate consequences can lead to the loss of validity of the laws of classical logic, which are important in describing many of these phenomena. For example, imagine a paper sheet divided in half, black on the left side and blue on the right side. Now consider three points: p, q and r, where p is located on the middle of the left side of the sheet, q is located on the middle of the right side, and r lies exactly in the middle.

Now, what is the color of each point? No one would have doubts in replying that p is black, and that point q is blue, but what about point r? As Caicedo ([3]: 3) points out, it seems that, with respect to boundary points like r, the excluded middle is not valid. The solution to this kind of difficulty is given, according to the definitions of mathematical analysis, by asking whether the black extension (space) of the sheet is closed or not. However, evidently, the assumption that one of the extensions includes or not its boundary points is purely ideal, that is, without any perceptible or experimental character, so that, as mentioned previously, it is not a real natural phenomenon, nor does it display physical intuition: are inserted only to save the laws of classical logic (see Caicedo [3]: 3-4).

Now, note that we have no difficulty in saying that the point p is black and that point q is blue, because these are precisely the colors of the spatial extensions surrounding these points, so that the notion of color is in fact applicable only to extensions, not points. Similar problems occur when instant properties are assigned to objects that are extended in time (see Caicedo [3]: 4-5). Translating this into our context of interest, in the same way that it ideally only makes sense to say that the point r is black or blue, it also ideally only makes sense to say that 'a is F at temporal instant x', so that, just as we do not have any difficulty in saying that point p is black because that is the exact color of the spatial extent surrounding it, we will not have any difficulty saying that '"a is F at temporal instant x" is true' only if there is a temporal extension U surrounding the instant x such that 'a is F' is true in U. So, when we say '"a is F at temporal instant x" is true', we are actually saying that there is a temporal extension U surrounding the instant x such that 'a is F' is true in U.

Second, what is the ontological status of these limit-idealizations? As stated above, such limit-idealizations are not real natural phenomena, nor do they possess any experimental or perceptible character, they are merely ideal limits of extended objects and events, and are inserted only to save the laws of classical logic. In this way, limit-idealizations are nothing more than abstract objects, which are postulated only to make models that have as underlying logic classical logic fit the physical reality, so that they can describe the phenomena of the world of extended objects and events. With the obvious realization that limit-idealizations are abstract objects whose theoretical role is to adapt these models to physical reality, an issue immediately comes to mind: are limit-idealizations universals? Note that the theoretical role that these limit-idealizations play in the aforementioned models is exactly analogous to the theoretical role that universals have in their various uses: both are inserted with the purpose of solving problems of an ontological order. Thus, it seems very plausible to observed an enormous similarity between these two notions. Let me examine this more closely.

What are universals? A simple way to answer this question is simply to say that universals are entities postulated for the purpose of explaining qualitative identity and similarity relations between particulars. But perhaps the most striking characteristic of universals is the contrast with particulars: since universals explain qualitative identity and similarity relations between particulars, universals must possess the capacity to be in several distinct particulars (i.e, in several places) at the same time, something that is impossible for a particular. Thus, by definition, if an object exists in more than one particular at the same time, it is a universal. It is practically from this controversial characteristic that all the debates on the Problem of Universals originate. Note that by definition, universals are properties. Also note that a universal can exist in more than one universal, for example, the universal 'virtue' exists in the universal 'wisdom' and in the universal 'courage'. It is important to point out that what matters in this paragraph is only to know how to identify the notion of universal. Now that we have a definition for universal, we can know whether limit-idealizations are universal or not.

To demonstrate that limit-idealizations are universals, it is enough to see that limit-idealizations (ideal limits) of extensive properties are punctual or instantaneous properties, and, as seen above, by the definition of universal, properties are universals; therefore, limit-idealizations (ideal limits) of extensive properties are punctual or instantaneous universals.

Now, note that all models whose underlying logic is classical logic (including realist and nominalist theories) are ontologically committed to limit-idealizations, since it is the limit-idealizations that save the laws of classical logic when these models are used to describe objects and events that extend in space-time; in other words, without limit-idealizations, these models do not work. This ontological commitment is not so disturbing to the realist theories that use classical logic; one can argue only that using classical logic as the underlying logic in realist theories multiplies the number of entities, and, by Occam's razor, it is more plausible to prevent more entities from being inserted into the ontology. However, the ontological commitment to limit-idealizations is devastating to nominalist theories.

Even if a nominalist solution succeeds in eliminating universals using classical logic as the underlying logic, the nominalist would have to develop a theory to eliminate limit-idealizations as well, since limit-idealizations, as seen previously, are universals. However, as already demonstrated, any model whose underlying logic is classical logic will have to accept the existence of limit-idealizations (ontological commitment, since without limit-idealizations the model does not work). Thus, the nominalist of classical logic would find him/herself in the following impasse: eliminate universals and accept limit-idealizations; or eliminate universals and then try to develop a theory to eliminate limit-idealizations. But if he/she eliminates limit-idealizations, then the nominalist theory used to eliminate universals will not work, and so the universals return. Thus, nominalist theories eliminate universals if and only if nominalist theories admit limit-idealizations. As limit-idealizations are universals, we have:

(N) *Nominalist theories eliminate universals if and only if nominalist theories admit universals.*

Thus, there are only two outputs for nominalists: to become a realist; or try to prevent their theories from falling into (N). Assuming that becoming a realist is not a valid option for any nominalist, it remains for us to try to find a way to prevent our theories from falling into (N). That is, before proposing any solution to the Problem of Universals, nominalists must first find a way to prevent their theories from falling into (N). In turn, there are two forms of nominalist to avoid (N): the first is demonstrating that limit-idealizations are not universals, but only abstract particulars; and the second is to find a way to prevent their theories from ontologically compromising with limit-idealizations. To abstain from limit-idealizations in our mathematical models of the world is something that would make such models extremely complicated, which is not adequate. Thus, we must find a way to make the limit-idealizations allowed in our models to be abstract particulars, not universals.

The idea, then, is for a solution to the Problem of Universals to be able to eliminate (i.e, demonstrate that they are abstract particulars) not only extensive universals, but punctual (instantaneous) ones. In other words, the nominalist solution must be able to treat not only the extensive properties, but also punctual (instantaneous) properties. This is impossible to do with classical logic, since it has no formal apparatus for treating limit-idealizations as punctual (instantaneous) properties of extensive properties, so the contradiction occurs because the nominalist who uses classical logic as the underlying logic to its solution eliminates the extensive universals, but cannot even formalize the punctual (instantaneous) universals, and they remain there, for, as seen previously, the classical logic nominalist is ontologically committed to these entities.

What we need is a logic appropriate to objects that extend in space-time, that is, a logic that revises the punctual and instantaneous properties of extensive object, i.e. properties that are ideal limits of extensive properties. It is necessary that the punctual properties in this logic fulfill the following paradigm of veritative continuity:

(P) *If a property for an object holds at a point of its extension domain, then it has to hold throughout an entire neighborhood of that point* ([3]: 6).

Another version for the above paradigm is given as follows:

(P') *A proposition about an object will be true if it is true in an extension where this object lies* ([4]: 3).

To understand what this logic would be, it is necessary to understand how the Problem of Universals is connected to the fact that classical logic can not capture alone the variable and extended characteristic of objects and events that are extended in space-time, and how a model to capture this characteristic would be like.

1.1 The Many over One problem from the perspective of extended objects

I understand the Problem of Universals, according to Rodriguez-Pereyra ([1]: 261), as the problem that seeks to describe the truthmakers of the following sentences (see Oliver [5]: 49-50):

1. *a* and *b* are of the same type/have a common property;

2. a and b are both F;

3. a and b have a common property, F;

4. a has a property;

5. a is F;

6. a has the property F.

As Rodriguez-Pereyra ([1]: 266) demonstrates, this is reduced to describing the truthmakers of sentences (5) and (6). This leads to a modification of the perspective of the problem, namely the exchange of the One over Many perspective for the Many over One perspective. This means that the fact to be explained is not that of multiple individuals sharing the same property, but that of how it is possible that a particular may possess multiple properties. In this subsection I will demonstrate the Many over One problem from the perspective of extended objects.

Solutions to the Problem of Universals conceive of the objects of the problem, namely, particulars and properties, from an unreal, or at least ideal, perspective, so that any offered solution, however suitable it may be to the problem, is bound to be inadequate to reality. To understand this, note that any solution to the Problem of Universals is nothing more than a model of physical reality (although some solutions admit metaphysical objects), because, regardless of the question whether properties are abstract objects or not, in other words, whether properties lie outside of the space-time or not, it is in space-time that we perceive them in the particulars, in view of the fact that the objects and events of the world are extended in time and space.

As Caicedo ([3]: 3) points out, one can argue whether or not certain particles, such as the photon or the neutrino, have a spatial extension, but every particle considered by physics has a temporal extension, namely its 'life'. Even events that clearly lack spatial extension, such as mental events, seem to possess an obvious positive temporal extension. Thus, in order to be perceived, object properties must be present in an entire region of some point-instant of the spatiotemporal extensions of these objects. Thus, the non-adequacy to the reality cited above concerns the incongruity between the way objects and events occur in space-time and the way they are conceived in these models (solutions). Of course, a solution to the Problem of Universals should explain not only the fact that particulars possess multiple properties, but also explain sentences that seem to indicate that a property can also possess properties, such as 'wisdom is a virtue', while, of course, wisdom is not an object extending into space-time. However, 'wisdom is a virtue' indicates something that must be explained by a solution for the Problem of Universals, and not a trivial basic fact from which a solution must be given. The basic fact is: we perceive properties in objects that extend in space-time, and it is this basic fact that a logic underlying any solution must be able to seize.

Thus, the way the real world works best fits a type of frame of variation, which can be considered as spatio-temporal states, and particulars as variable objects that vary over these states. This means, as already glimpsed above, that the 'truths' and 'properties' of particulars will be conditioned by extensions within these space-time states. Let us try to make these intuitions more explicit. We have a timeline X, with each temporal instant $x \in X$, and we have a frame of variation, considered as spatio-temporal states, extending over this timeline X. However, this alone is not enough, because we wish to refer to particulars and properties in this model, so we require some kind of structure that allows us to describe, at each temporal instant $x \in X$, the instantaneous properties of the extended objects.

Thus, we will need, for each temporal instant $x \in X$, a structure $\mathfrak{A}_x$ equipped with the following components: a world E_x, whose objects constitute an instantaneous image of extended objects, at the temporal instant $x \in X$, functions $f_x, ..$ and relations $R_x, ..$ which describe the instantaneous properties of extended objects at the temporal instant $x \in X$. It is also necessary that the worlds E_x be of the same type, in the sense that the functions, relations and objects can be interpreted in an analogous way at each temporal instant $x \in X$. In addition, there must be an extended universe E, where the different E_x worlds can be attached in such a way that the attachment of objects, functions and relations is continuous, in the sense of being seen as extended objects, functions, and relations defined in that extended universe E. Thus, for each temporal instant $x \in X$, a structure $\mathfrak{A}_x = (E_x, R_x, .., f_x, ..)$ must exist, functioning in the aforementioned manner.

The previous paragraph outlined the characteristics that a model of reality must possess so that it can be properly used in a possible solution to the Problem of Universals. Now, we must find a way to indicate when it makes sense to say that an extended object has multiple different properties in the intuitive model discussed above. The appropriate way of speaking when an extended object has multiple properties is obviously saying: '"a has such and such properties at instant x" is true when there is a temporal extension U surrounding the instant x such that "a has such and such properties" is true in U'. However, note that what we must explain is how is it possible that a has multiple properties at instant x, in other words:

(M) *How is it possible that there is a temporal extension U surrounding an instant x such that 'a has such and such properties' is true in U?*

The question (M) is the Many over One problem from the perspective of extended objects, that is, from the perspective of how objects and events appear in space-time.

As stated above, nominalists require a logic that reviews the punctual and instantaneous properties of extended objects, that is, that reviews the properties that are ideal limits of extensive properties, and also require that, in this logic, punctual properties fulfill the continuity paradigm (P) above. The next section will aim to meet this need, presenting a logic that satisfies the requirements, namely, the logic of a sheaf of structures over the timeline seen as a topological space.

2 Sheaves of structures

The intuitive model discussed above can be easily translated into known mathematical notions. For example, we can think of the timeline X as a topological space, and the temporal extensions surrounding an instant $x \in X$ as neighborhoods of x in this topological space. From this, how we can translate the entire intuitive model, namely, using the notion of sheaf over a topological space immediately comes to mind. In this section, I will use the notion of sheaf of structures, whose technical results with their respective proofs have been previously presented and are contained in the paper by Caicedo [3]. Familiarity with sheaf theory, or even with topology, is not essential in order to understand this section, since I will always try to translate all the definitions and technical results used herein into our context of interest.

Definition 2.1 *Let X be a topological space. A sheaf over X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism, or, in other words, a continuous function such that for every point $e \in E$ there is an open neighborhood V such that:*

1. *$p(V)$ is open in X.*
2. *$p\restriction_V : V \to p(V)$ is an homeomorphism.*

Definition 2.2 *Given an open set U in X, a continuous function $\sigma : U \to E$ such that $p \circ \sigma = Id_U$ is named a local section; if $U = X$, σ is named a global section. For $x \in X$, the so-called fiber over x is the set $E_x = p^{-1}(x) = \{e \in E : p(e) = x\}$.*

Let me translate these definitions into our context of interest. The sheaves that will be considered herein will be thought of as sets that extend or develop over a space $X = timeline$. The 'elements' of this extended set must, therefore, be thought of as extending over open sets of X. Thus, these 'elements' will not be points in E, but the sections of the sheaf. The value $\sigma(x)$ of a section σ will be only the punctual description of σ at point x. This leads to a relation of equality between sections of a sheaf that satisfy the continuity paradigm (P):

Lemma 2.3 (Caicedo [3]: 10) *If two sections coincide at a point $x \in X$, then they coincide in a neighborhood containing the point $x \in X$.*

The definition below will present the notion of a sheaf of first-order structures, which will be the main tool used herein.

Definition 2.4 *Given a fixed vocabulary of structures $\tau = \langle R^n, .., f^m, .., c, ..\rangle$, a sheaf of τ-structures $\mathfrak{A}$ over a topological space X is given by:*

(A) A sheaf (E, p) over X.

(B) For each $x \in X$, a τ-structure $\mathfrak{A}_x = (E_x, R_x, .., f_x, .., c_x, ..)$, where $E_x = p^{-1}(x)$ (the fiber that can be empty) is the universe of the τ-structure $\mathfrak{A}_x$, and the following conditions are satisfied:

i. $R^{\mathfrak{A}} = \bigcup_x R_x$ is open in $\bigcup_x E_x^n$ seeing as subspace of E^n, where R is an n-ary relation symbol.

ii. $f^{\mathfrak{A}} = \bigcup_x f_x : \bigcup_x E_x^m \to \bigcup_x E_x$ is a continuous function, where f is an m-parameter function symbol.

iii. the function $c^{\mathfrak{A}} : X \to E$, given by $c^{\mathfrak{A}}(x) = c_x$, where c is a constant symbol, is a continuous global section.

Each fiber $\mathfrak{A}_x$ of the sheaf of structures $\mathfrak{A}$ is a structure in the classical sense, with the condition that it may be empty. Translating this into our context of interest, think of the sheaf of structures $\mathfrak{A}$ as the space-time that derives from the principle of Galilean relativity, according to which the dynamic laws are the same when they are referred to in any frame in uniform motion. Thus, we no longer have the Aristotelian conception of a fixed and absolute background space constituting a preferential frame where physical objects move. This means that it does not make sense to consider a point in space as being the same point a moment later. On the contrary, each instant corresponds to a different three-dimensional world, that is, there is no fixed Euclidean three-dimensional space where the physical world is contained, but an annexation of these different three-dimensional worlds in a continuous way, with respect to the temporal order [6, 7]. Thus, think of a sheaf of structures $\mathfrak{A}$ as an extended space over the base space X of the sheaf (E, p) as the Galilean space-time extending over the timeline.

The continuity paradigm (P) is also satisfied by the predicates and extended functions, and, in truth, is valid for any positive existential property or relation:

Lemma 2.5 (Caicedo [3]: 10) *If $\varphi(v_1, ..., v_n)$ is a first-order formula that contains only the logical operators $=, \wedge, \vee, \exists$, besides the predicates and atomic* *functions, and* $\varphi[\sigma_1(x), ..., \sigma_n(x)]$ *is true in* $\mathfrak{A}_x$*, then, there exists an open neighborhood* U *of* x *such that* $\varphi[\sigma_1(y), ..., \sigma_n(y)]$ *is true in* $\mathfrak{A}_y$*, for all* $y \in U$*.*

Putting this into the perspective of the our context of interest, what Lemma 2 is saying exactly is that if 'a is F at temporal instant x' is true, then there is a temporal extension U surrounding the instant x such that 'a is F at temporal instant y' is true, for each temporal instant $y \in U$; in other words, there is a temporal extension U surrounding the instant x such that 'a is F' is true in U.

2.1 Punctual semantics

Now that we have a logical language capable of capturing the extended and variable feature of objects and events, we need an interpretation, a semantics for our variable structures. As described above, a sheaf is considered as an extended structure over the base space (in our case, $X = timeline$), and the fibers as mere punctual or instantaneous descriptions of the structure in the course of their variation. The elements of the structure (in our case, the particulars) are the continuous sections, objects that vary over their domain in the base space. Thus, as already expected from the explained above, the interpretation of the logical language is given in such a way that the subjects of the propositions are the sections, not the geometric points of the space of the fibers. On the other hand, the properties of the sections may vary from point to point in their extension domain, or, in other words, the logical propositions will take the form:

the sections $\sigma_1, ..., \sigma_n$ *have the property* F *at point* x*.*

Thus, as already emphasized, the value $\sigma(x)$ of a section at a point x is the punctual or instantaneous description of a particular, and not a particular. A precise meaning for this idea will be given below.

Definition 2.6 *Let* L_τ *be a first order language of vocabulary* τ*. Given a sheaf of structures* $\mathfrak{A}$ *of vocabulary* τ *over* X*, for the formulas* $\varphi(v_1, ..., v_n) \in L_\tau$ *is defined, by induction, the relation '*$\mathfrak{A}$ *forces* $\varphi[\sigma_1, ..., \sigma_n]$ *in* x*, for sections* $\sigma_1, ..., \sigma_n$ *of* $\mathfrak{A}$ *defined in* $x \in X$*', in symbols,*

$$\mathfrak{A} \Vdash_x \varphi[\sigma_1, ..., \sigma_n]$$

(meaning that property φ *holds at point* x *for sections* $\sigma_1, ..., \sigma_n$ *in the sheaf* $\mathfrak{A}$*), in the following manner:*

(a) If φ *is an atomic formula and* $t_1, ..., t_k$ *are* τ*-terms:*

(i) $\mathfrak{A} \Vdash_x (t_1 = t_2)[\sigma_1, ..., \sigma_n] \Leftrightarrow t_1^{\mathfrak{A}_x}[\sigma_1(x), ..., \sigma_n(x)] = t_2^{\mathfrak{A}_x}[\sigma_1(x), ..., \sigma_n(x)]$;
(ii) $\mathfrak{A} \Vdash_x R(t_1, ..., t_k)[\sigma_1, ..., \sigma_n] \Leftrightarrow (t_1^{\mathfrak{A}_x}[\sigma_1(x), ..., \sigma_n(x)], ..., t_k^{\mathfrak{A}_x}[\sigma_1(x), ..., \sigma_n(x)]) \in R_x$.
(b) $\mathfrak{A} \Vdash_x (\varphi \wedge \psi)[\sigma_1, ..., \sigma_n] \Leftrightarrow \mathfrak{A} \Vdash_x \varphi[\sigma_1, ..., \sigma_n]$ *and* $\mathfrak{A} \Vdash_x \psi[\sigma_1, ..., \sigma_n]$;
(c) $\mathfrak{A} \Vdash_x (\varphi \vee \psi)[\sigma_1, ..., \sigma_n] \Leftrightarrow \mathfrak{A} \Vdash_x \varphi[\sigma_1, ..., \sigma_n]$ *or* $\mathfrak{A} \Vdash_x \psi[\sigma_1, ..., \sigma_n]$;
(d) $\mathfrak{A} \Vdash_x \neg\varphi[\sigma_1, ..., \sigma_n] \Leftrightarrow$ *exists* U *open neighborhood of* x *such that for all* $y \in U$, $\mathfrak{A} \nVdash_y \varphi[\sigma_1, ..., \sigma_n]$;
(e) $\mathfrak{A} \Vdash_x (\varphi \rightarrow \psi)[\sigma_1, ..., \sigma_n] \Leftrightarrow$ *exists* U *open neighborhood of* x *such that for all* $y \in U$, *if* $\mathfrak{A} \Vdash_y \varphi[\sigma_1, ..., \sigma_n]$ *then* $\mathfrak{A} \Vdash_y \psi[\sigma_1, ..., \sigma_n]$;
(f) $\mathfrak{A} \Vdash_x \exists v\varphi[v, \sigma_1, ..., \sigma_n] \Leftrightarrow$ *exists* σ *defined in* x *such that* $\mathfrak{A} \Vdash_x \varphi[\sigma, \sigma_1, ..., \sigma_n]$;
(g) $\mathfrak{A} \Vdash_x \forall v\varphi[\sigma_1, ..., \sigma_n] \Leftrightarrow$ *exists* U *open neighborhood of* x *such that for all* $y \in U$ *and all* σ *defined in* y, $\mathfrak{A} \Vdash_y \varphi[\sigma, \sigma_1, ..., \sigma_n]$.

(In (d),(e) and (g), U *must satisfy* $U \subseteq \bigcap_i dom(\sigma_i)$*).*

Now, translating the punctual semantics given above into the context that interests us, we have the following:
(a) Two particulars are equal at a certain temporal instant if and only if the equality holds for their punctual descriptions. This occurs in an analogous way for the relations.
(b) The conjunction of two properties holds for particulars at some temporal instant if and only if each property holds for these particulars at the same temporal instant.
(c) The disjunction of two properties holds for particulars in a certain temporal instant if and only if one of the properties hold for the particulars in that temporal instant.
(d) The negation of a property for particulars hold at a certain temporal instant if and only if there is a temporal extension surrounding that temporal instant such that, at each temporal instant of that temporal extension, the property does not hold for these particulars.
(e), (f) and (g) are more clearly understood using formal language (again, think of the timeline as a topological space, and the temporal extensions surrounding an temporal instant x as neighborhoods of x in this topological space).

The next theorem demonstrated that punctual forcing fulfills the principle (P) (semantic continuity):

Theorem 2.7 (Caicedo [3]: 16) $\mathfrak{A} \Vdash_x \varphi[\sigma_1, ..., \sigma_n]$ *if and only if there exists a neighborhood* U *of* x *such that* $\mathfrak{A} \Vdash_y \varphi[\sigma_1, ..., \sigma_n]$ *for every* $y \in U$.

Translating to our context, the particulars $\sigma_1, ..., \sigma_n$ have the property φ at the temporal instant x if and only if there is a temporal extension U surrounding

x such that the particulars $\sigma_1, ..., \sigma_n$ have the property φ at each temporal instant $y \in U$.

It is easy to see that the logic of sheaves of structures presented above included classical logic as a special case, and we have the classical logic exactly when space is $X = \{x\}$. This demonstrates, in a more formal manner, what I have already argued in Section 1 above, namely, that classical logic is not adequate to treat the Problem of Universals, in other words, classical logic is not adequate because it is the logic that is tied to the case in which it does not make sense to think of the sections of the sheaf as particulars that extend in the timeline. Thus, the response to (L2) provided in this section can be summarized as follows: the adequate logic to describe the explanantia given to the explananda of a solution to the Problem of Universals is the logic of a sheaf of structures over the timeline seen as a topological space.

From punctual semantics, it is also possible to define a local semantics (that is, not point forcing, but open forcing) as follows:

Definition 2.8 *Given an open $U \subseteq X$ and sections σ_i defined in U, the open forcing is defined as:*

$$\mathfrak{A} \Vdash_U \varphi[\sigma_1, ..., \sigma_n] \Leftrightarrow \forall x \in U, \mathfrak{A} \Vdash_x \varphi[\sigma_1, ..., \sigma_n].$$

The open forcing defined above trivially meets the following conditions:

$$\text{(I) } \mathfrak{A} \Vdash_U \varphi \text{ and } W \subseteq U \Rightarrow \mathfrak{A} \Vdash_W \varphi;$$

$$\text{(II) } \mathfrak{A} \Vdash_{U_i} \varphi[\sigma \restriction_{U_i}] \text{ for all } i \in I \Rightarrow \mathfrak{A} \Vdash_{\bigcup_i U_i} \varphi[\sigma].$$

The rules on connectives that characterize open forcing is a result named Kripke-Joyal Semantics (see Caicedo [3]: 19). The importance of this 'synthetic' definition is that it can be generalized to sheaves on a site, in the Grothendieck sense (see Mac Lane & Moerdijk [8]: 302).

3 Variable class nominalism and a solution to the contradiction (N)

In this section I will revise Class Nominalism using as the underlying logic the logic given in response to (L2), which was presented in Section 2. The aim of this section is to demonstrate how a nominalist theory would be able to capture the extended and variable characteristics of objects and events that extend in space-time, i.e. what would a nominalist theory capable of expressing a natural congruence with the way in which objects and events present themselves in space-time be. The theory obtained is a Variable Class Nominalism, where sets are conceived not as absolute entities, but as variable structures that vary over a domain of knowledge states.

First, note that our intuitions about classes have been axiomatized in systems like ZF. Thus, a proper way to begin a contextualization of what I have exposed in Section 1 and Section 1.1 using Class Nominalism is to present a set theory, such as ZF, with the underlying logic discussed in Section 2. Now, also note that the ZF model is called the Von Neumann Hierarchy, which is constructed in an inductive manner as follows:

$$V_0 = \emptyset;$$
$$V_{\alpha+1} = P(V_\alpha);$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \text{ if } \lambda \text{ is a limit ordinal;}$$
$$\mathbb{V} = \bigcup_{\alpha\in On} V_\alpha;$$

where On is the class of all the ordinals, and $P(V_\alpha)$ is the power set of V_α. Obviously, in the Von Neumann Hierarchy all members of sets are sets[1], and in Class Nominalism we have members of sets that are not sets. However, what matters herein is that, as stated, the Von Neumann Hierarchy, together with the classical membership relation between sets, $\in$, is the ZF model, and, thus, the model of our intuitions about operations involving sets. Therefore, what is required to revise Class Nominalism within the logic discussed in Section 2 is to present a model that, at the same time, captures our intuitions about objects extended in space-time and our intuitions about operations involving sets.

Such a model must translate the notion of 'truths' and 'properties' of particulars that are conditioned by extensions within spatial-temporal states to our intuitions involving sets. To understand how the above notion can be translated into our intuitions about sets, and at the same time understand why Class Nominalism based on classical logic is inadequate, note that using the comprehension axiom, for a given sentence $\varphi(a)$, we can construct, for any set A, the following set:

$$B = \{a \in A : \varphi(a)\}.$$

That is, set B is the set formed by each a that satisfies two conditions: a is a member of A; and the sentence $\varphi(a)$ is 'true' for a. Now, as it is easy to see, the notion of 'truth' used to define B is absolute, and as such, for the reasons given in Section 1, is inadequate when it comes to objects that extend

[1]Of course, we can gather together all those things that are not themselves sets but that we want to have as members of sets. Call such things atoms. Let A be the set of all atoms, we take $V_0 = A$ (see Enderton [9]).

in space-time. For example, if A is the set composed of cups, and $\varphi(a)$ is the sentence 'a is blue', B is the set formed by all blue cups. However, to say that the sentence 'the cup is blue' is true for the entire temporal extension of the cup is inadequate, at most we can say that 'the cup is blue' is true in the temporal extension surrounding the instant x in which we perceive it as blue. Thus, a way of translating the notion of 'truths' and 'properties' of particulars that are conditioned by extensions within spatial-temporal states to our intuitions about sets, and, thus, redefining Class Nominalism within this notion, is to make the construction of sets such as B not based on absolute 'truths', but rather on 'truths' based on a contextual paradigm of 'truth', such as the principle of veritative continuity (P).

To exemplify what a Class Nominalism capable of capturing the characteristics of objects extended in space-time would be, again consider the sentence $\varphi(a)$ as being 'a is blue'. As I write this paragraph, my location in space-time is $p :=$ 'between 2 and 3 pm on November 11th 2018, somewhere in Brazil'. Now, in the node p of the timeline, I can definitely state that the cup I am holding is a member of set B, but I cannot say that it will remain that way for the rest of its temporal extension (it may lose its color, or be painted in another color). Thus, in place of set B, it makes more sense to construct the following set:

$$\varphi(p) = \{a : \varphi(a) \text{ holds at the node } p\}.$$

Thus, the Variable Class Nominalism that I am proposing herein conceives of sets not as absolute entities, but as variable structures that vary over a domain of states of knowledge. Such a domain can be thought of as a partial, not necessarily linear, order where there is the possibility of alternative evolutions, for example, an evolution where the sentence 'my cup is blue' is true and another where it is not. The essential information here is that the model must be such that at each temporal instant x a 'cumulative hierarchy of sets' is set up, a 'universe' for each temporal instant x.

I begin by defining an analogue of Von Neumann Hierarchy that extends the notion of a set, redefining such a notion as a variable object over a sheaf, where the relation $\in$ will be provided with a new contextual meaning. The following is originally presented in [3], and has been used in [4] to obtain a new proof of the independence of the Continuum Hypothesis. It was also used in [7] to try to capture the essence of quantum logic in order to obtain a mathematical quantum universe capable of capturing quantum reality. Herein, I will only present what is necessary to the Variable Class Nominalism that I am proposing. A more detailed presentation can be found in the aforementioned texts.

3.1 The cumulative hierarchy of variable sets

Definition 3.1 *Let X be a topological space and $U \in \mathcal{O}(X)$, where $\mathcal{O}(X)$ is the set of open sets of X. The hierarchy of variable sets*[2] *is defined as follows:*

$\mathbb{V}_0(U) = \emptyset$;

$\mathbb{V}_{\alpha+1}(U) = \{f : \mathcal{O}(U) \to \bigcup_{W \subseteq U} P(\mathbb{V}_\alpha(W)) : 1. W \subseteq U \Rightarrow f(W) \subseteq \mathbb{V}_\alpha(W);$
$2. V \subseteq W \subseteq U \Rightarrow \forall g \in f(W), g \restriction_{\mathcal{O}(V)} \in f(V);$
3. *Given* $\{U_i\}_i$ *an open cover of* U *and* $g_i \in f(U_i)$ *such that* $g_i \restriction_{\mathcal{O}(U_i \cap U_j)} = g_j \restriction_{\mathcal{O}(U_i \cap U_j)}$, *for any* i, j, *there exists* $g \in f(U)$ *such that* $g \restriction_{\mathcal{O}(U_i)} = g_i$, *for all* $i.\}$;

$\mathbb{V}_\lambda(U) = \bigcup_{\alpha < \lambda} \mathbb{V}_\alpha(U)$ *if* λ *is a limit ordinal;*

$\mathbb{V}(U) = \bigcup_{\alpha \in On} \mathbb{V}_\alpha(U)$.

For each ordinal $\alpha \in On$, and open $U \in \mathcal{O}(X)$, the set $\mathbb{V}(U)$ is a set of functions defined over $\mathcal{O}(U)$, whose values for $W \in \mathcal{O}(U)$ are sets of functions defined over $\mathcal{O}(W)$, whose values for $V \in \mathcal{O}(W)$, are, in turn, sets of functions defined over $\mathcal{O}(V)$, and thus, hereditarily. The membership relation between these sets is defined as follows:

Definition 3.2 $\Vdash_U f \in g \Leftrightarrow f \in g(U)$.

Where the $\in$ on the right in the above definition is just the relation of classical membership between sets.

The cumulative hierarchy of variable sets[3] defined above is given for an arbitrary topological space. However, it is possible to take as topological space a partial order whose topology is given by its hereditary subsets. If we interpret this partial order as the structure of time, we will have a true hierarchy of variable sets, capable of adequately capturing the behavior of the sets supported in the Variable Class Nominalism proposed by myself. In addition, the hierarchy of variable sets over a partial order is only a Kripke model, which in turn can be interpreted as a sheaf of structures (cf. Caicedo [3]: 22). Now, remember that, as stated in Section 2, the individuals that extend over open sets of X (in this case on the open ones of the timeline seen as a partial order) are the

[2] Analogously, we can gather together all those things that are not themselves sets but that we want to have as members of sets. Let A be the set of all atoms, we take $\mathbb{V}_0(U) = A$.

[3] Actually, the valuation $\mathbb{V}$ over the open sets constitute an exact presheaf of structures, which sheaf of germs $\mathcal{G}\mathbb{V}$ constitute the cumulative hierarchy of variable sets (see Caicedo [3]: 29-31).

sections of the sheaf in question, whose value $\sigma(x)$ will be only the punctual (instantaneous) description of σ at point x. With this in mind, the membership relation in the Variable Class Nominalism proposed by myself has the following intuitive description based on the cumulative hierarchy of variable sets seen as a sheaf of structures:

(V) *A particular p belongs to a variable set φ at temporal instant x if p belongs to the punctual (instantaneous) description $\varphi(x)$ of φ at the temporal instant x as in the classical membership relation between sets.*

Now, we will verify how to answer the Many over One problem from the perspective of the extended objects, i.e. how to answer the question (M). The answer can now be given using the Variable Class Nominalism as follows:

(T) *A particular p has the properties $F_1, F_2, F_3, ..., F_m$ in an temporal instant x, in other words, there exists a temporal extension U surrounding the instant x such that 'p has the properties $F_1, F_2, F_3, ..., F_m$' is true in U, when p belongs to multiple punctual (instantaneous) descriptions $\varphi_1(x), \varphi_2(x), \varphi_3(x), ..., \varphi_m(x)$ of multiple variable sets $\varphi_1, \varphi_2, \varphi_3, ..., \varphi_m$ of particulars which are $F_1, F_2, F_3, ..., F_m$, respectively.*

The solution to the problem pointed out by myself herein now becomes obvious. With Variable Class Nominalism we can treat not only the extended universals, but also puntual universals. Variable Class Nominalism accounts for the punctual (instantaneous) properties in the same way as the Class Nominalism whose underlying logic is classical logic accounts for extensive properties; thus, the problem disappears instantaneously, since instantaneous properties, say F_x of F, will be abstract particulars, namely, the set of all the particulars that are F at temporal instant x. The same occurs for punctual (instantaneous) universals that are relations.

4 Conclusion

In this paper I have demonstrated that nominalist theories that use classical logic as the underlying logic generate the following contradiction: (N) nominalist theories eliminate universals if and only if nominalist theories admit universals. In order to attempt to overcome this impasse, I proposed that nominalist theories should use a logic appropriate to objects that extend in space-time as the underlying logic, that is, a logic that revises the punctual and instantaneous properties of extended objects, i.e. the properties that are ideal limits of extensive properties, and that the punctual properties fulfill the paradigm of veritative continuity (P).

I suggested the logic of a sheaf of structures over the timeline seen as a topological space to be the logic underlying nominalist theories in Section 2, and, in Section 3, I demonstrated how a nominalist theory with the logic presented in Section 2 as the underlying logic would be, resulting in a Variable Class Nominalism, where sets are conceived not as absolute entities, but as variable structures that vary over a domain of states of knowledge. As observed herein, in Variable Class Nominalism, the contradiction (N) pointed out by myself herein does not occur.

Further work also includes research into the potentialities of Variable Class Nominalism proposed by myself herein. For example, it can be demonstrated that there exists a homeomorphism between the space of the fibers of a sheaf and the space of the fibers of the so-called sheaf of germs of the presheaf of sections associated with that sheaf (see Caicedo [3]: 12). These fibered spaces can be considered, respectively, as an intensional space and an extensional space. Leading this homeomorphism into the context of Variable Class Nominalism, such a homeomorphism could be used to demonstrate that properties can be reduced to classes, thereby dissolving the Wolterstorff problem [10]. Moreover, the implications of this homeomorphism are not limited to the dissolution of the Wolterstorff problem: since this homeomorphism can be used to demonstrate that extension and intension are topologically the same thing, it can be used in all matters involving the obscurity of intentional notions. For example, it can be used in theories of truth that seek to clarify the concepts present in the theory of meaning (intension) in terms of the concepts present in the theory of reference (extension).

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