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Monadic Algebras for the Notion of Many

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Abstract

The 'logic of many' is characterized as an extension of classical first-order logic by the inclusion of a new generalized quantifier for formalizing the notion of 'many'. We present in this paper a monadic algebra for a monadic constraint of logic of many.

Keywords: Logic of many, Monadic algebra, Monadic algebra of many.

Introduction

The logic of many was introduced by Grácio [4]. This logic formalizes in first-order the central aspects of any notion for the quantifier 'many'.

As the logic of many extends the first-order logic, then its models naturally extend the first-order structures. They take a first-order structure and include especial mathematical aspects to describe the notion of 'many' considered fundamental by Grácio.

Just like a first-order logical system in general, the logic of many is undecidable.

On the other hand, Paul Halmos ([5] and [6]) developed a particular way to investigate first-order logic in a genuine algebraic approach.

Considering the interest in algebraic models, decidability and Halmos' approach, we introduce, in this paper, a monadic algebra of many and the monadic logic of many.

In this way, we propose to develop the Halmos' strategy for the monadic logic of many, which permits the formalization of essential characteristics of 'many' in a monadic logic that can be shown decidable.

We begin the paper by presenting the logic of many as a modulated logic. Then, we review some results of the Halmos' approach.

As original contribution, we introduce the monadic algebra of many, the monadic logic of many, and the proofs of soundness and completeness between these two systems using the Halmos' strategy.

1 The logic of many

The *logic of many* is a particular case of *modulated logic*, motivated by a formalization of the notion of 'many'. This vague notion is associated with the concept of a large evidence set, but not necessarily tied to the notion of majority, in terms of cardinality.

1.1 Proper family of upper closed sets

Carnielli and Grácio [2] consider that the following properties must be contemplated if we propose to formalize some notion of 'many', which expresses "a great amount of evidence":

(i) if φ is a true sentence for all members of an universe, then φ is true for many individuals of that universe;

(ii) if φ is a true sentence for many individuals, then there is some element in the universe that satisfies φ;

(iii) if the set of elements that satisfy φ is included in the set of elements that satisfy ψ and there are many members that satisfy φ, then also many members satisfy ψ.

This notion of 'many' can be represented by the mathematical concept of proper family of upper closed sets.

Definition 1.1 *A proper family of upper closed sets over an universe E is a set $\mathcal{M} \subseteq \mathcal{P}(E)$ such that:*

(i) $E \in \mathcal{M}$*,*

(ii) $\emptyset \notin \mathcal{M}$*,*

(iii) *if $B \in \mathcal{M}$ and $B \subseteq C$, then $C \in \mathcal{M}$.*

1.2 An axiomatic system for 'many'

Let **L** be the classical first-order logic with equality. The *logic of many*, denoted by $\mathbf{L}(M)$, is obtained from **L** in the following way.

The axioms of $\mathbf{L}(M)$ are those of **L** plus the following additional axioms for the new quantifier M, called *quantifier of many* and meaning "for many $x, \varphi(x)$", as follows:

(Ax_1) $\forall x\ (\varphi(x) \leftrightarrow \psi(x)) \to (Mx\ \varphi(x) \leftrightarrow Mx\ \psi(x))$

(Ax_2) $Mx\ \varphi(x) \to My\ \varphi(y)$, when y is free for x in $\varphi(x)$

(Ax_3) $\forall x\ \varphi(x) \to Mx\ \varphi(x)$

(Ax_4) $Mx\ \varphi(x) \to \exists x\ \varphi(x)$

(Ax_5) $\forall x(\varphi(x) \to \psi(x)) \to (Mx\ \varphi(x) \to Mx\ \psi(x))$.

The deduction rules of $\mathbf{L}(M)$ are the same defined for $\mathbf{L}$, specifically:

(MP) *Modus Ponens*: $\varphi, \varphi \to \psi\ /\ \psi$
(Gen) Generalization: $\varphi\ /\ \forall x\ \varphi(x)$.

The original intuition of 'many' is formalized in the axioms $(Ax_3) - (Ax_5)$.
The two first axioms have the task of making possible the adequacy with the proposed model.

The usual syntactical notions for $\mathbf{L}(M)$, such as sentence, proof, theorem, logical consequence, consistency and others are defined in an analogue way as in the classical first-order logic $\mathbf{L}$.

1.3 First-order model for the logic of many

Let $\mathcal{A}$ be a classical first-order structure with universe A.

Definition 1.2 *A structure of proper family of upper closed sets, denoted by $\mathcal{A}^{\mathcal{M}}$, for $\boldsymbol{L}(M)$, is obtained from the structure $\mathcal{A}$ plus a proper family of upper closed sets $\mathcal{M}$ over the universe A.*

The interpretation of relations, functions and constant symbols is as the interpretation of $\mathbf{L}$ with respect to $\mathcal{A}$.

Definition 1.3 *The satisfaction of a formula φ of $\boldsymbol{L}(M)$ in a structure $\mathcal{A}^{\mathcal{M}}$, is inductively defined, as in usual way plus the following clause:*

• *if φ is a formula whose set of free variables is contained in $\{x\} \cup \{y_1, ..., y_n\}$ and $\overline{a} = (a_1, ..., a_n)$ is a sequence of elements of A, then:*

$$\mathcal{A}^{\mathcal{M}} \vDash Mx\ \varphi[x, \overline{a}] \Leftrightarrow \{b \in A : \mathcal{A}^{\mathcal{M}} \vDash [b, \overline{a}]\} \in \mathcal{M}.$$

For a sentence $Mx\ \varphi(x)$, we have:

$$\mathcal{A}^{\mathcal{M}} \vDash Mx\ \varphi(x) \Leftrightarrow \{a \in A : \mathcal{A}^{\mathcal{M}} \vDash \varphi(a)\} \in \mathcal{M}.$$

Other usual semantic notions as model, validity, semantic consequence, among others, for $\mathbf{L}(M)$, are appropriately adapted from classical logic.

Carnielli and Grácio [2] proved that structures of proper family of upper closed sets $\mathcal{A}^{\mathcal{M}}$ are sound and complete models for $\mathbf{L}(M)$.

2 Monadic algebras

We present the monadic algebras of Halmos [5] and from them we introduce the monadic algebras of many.

So we propose an algebraic model for a monadic part of the *logic of many.*

2.1 Functional monadic algebra

We begin with the Boolean monadic algebra of Halmos [5].

According to Halmos (cf. [5]), propositions, in general, tend to band together and form a Boolean algebra. Hence, the expression “propositional function” is a function whose values are in a Boolean algebra.

In this way, we consider a set X, as a domain, such that $X \neq \emptyset$ and a Boolean algebra $\mathcal{B}$, the *value-algebra.*

Besides, we consider $\mathcal{B}^X$, the set of all functions from X to $\mathcal{B}$. If $p \in \mathcal{B}^X$ and $q \in \mathcal{B}^X$, then the supremum of p and q, denoted $p \vee q$, and the complement of p, denoted $\sim p$, are defined by:

$(p \vee q)(x) = p(x) \vee q(x)$ and $\sim p(x) = \sim (p(x))$, for each $x \in X$.

The zero and the unit of $\mathcal{B}^X$ are the constant functions 0 and 1, respectively.

The main interest in $\mathcal{B}^X$ is the possibility of associating with each element p a subset $R(p)$ of $\mathcal{B}$, where:

$R(p) = \{p(x) : x \in X\}$, is the range of the function $p(x)$.

With respect to the set $R(p)$ we can associate an element of $\mathcal{B}$ as the supremum or the infimum from that set.

Thereby, we should consider a Boolean subalgebra $\mathcal{A}$ of $\mathcal{B}^X$ such that:

For every $p \in \mathcal{A}$ there exist the supremum $Sup\ R(p)$ and the infimum $Inf\ R(p)$ in $\mathcal{A}$.

The functions $\exists p$ and $\forall p$, are defined by:

$$\exists p(x) = Sup\ R(p) \quad \text{and} \quad \forall p(x) = Inf\ R(p).$$

Definition 2.1 *Every such subalgebra $\mathcal{A}$ is a functional monadic algebra.*

Given that $\mathcal{A}$ is subalgebra of $\mathcal{B}$, hence it is Boolean and since the Sup and the Inf of all subsets of $\mathcal{A}$ belong to $\mathcal{A}$, then $\mathcal{A}$ is a complete algebra.

Now, let $\mathcal{B}$ be a complete Boolean algebra of all subsets of a set Y, such that $y \in Y$.

Then a value $p(x)$ of a function $p \in \mathcal{A} \subseteq \mathcal{B}^X$ corresponds to the proposition ‘$y \in p(x)$’.

Provided that the supremum in $\mathcal{B}$ is a set-theoretic union, then each value of $\exists p$ corresponds to 'there is an x such that y belongs to $p(x)$'. Dually, each value of $\forall p$ corresponds to 'for all x, $y \in p(x)$'.

Definition 2.2 *The operator $\exists$ is the functional existential quantifier, and the operator $\forall$ is the functional universal quantifier.*

We can define one of the quantifiers from the other one, as follows.

$$\forall p = \sim (\exists \sim p) \quad \text{and} \quad \exists p = \sim (\forall \sim p).$$

In reason of this duality between $\exists$ and $\forall$, we focus the development on the quantifier $\exists$ and we just comment on the $\forall$.

2.2 Quantifiers

To deal with the concept of quantification, we forget the domain X and the value-algebra $\mathcal{B}$.

Definition 2.3 *An existential quantifier is a unary operator $\exists$ of a complete Boolean algebra $\mathcal{A}$ into itself, that is, $\exists : \mathcal{A} \to \mathcal{A}$ such that for every $p, q \in \mathcal{A}$:*
(i) $\exists 0 = 0$,
(ii) $p \leq \exists p$,
(iii) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.

Definition 2.4 *A universal quantifier is an operator $\forall : \mathcal{A} \to \mathcal{A}$ such that for every $p, q \in \mathcal{A}$:*
(i) $\forall 1 = 1$,
(ii) $\forall p \leq p$,
(iii) $\forall(p \vee \forall q) = \forall p \vee \forall q$.

Definition 2.5 *The relative complement of p and q is defined by:*
$p - q = p \wedge (\sim q)$.

Definition 2.6 *The symmetric difference of p and q is defined by:*
$p + q = (p - q) \vee \sim (q - p)$.

The proofs of next proposition can be seen in [6].

Proposition 2.7 *Let $\exists : \mathcal{A} \to \mathcal{A}$ be an existential quantifier and $p, q \in \mathcal{A}$. Then:*
(i) $\exists 1 = 1$,
(ii) $\exists\exists q = \exists q$,

(iii) $p \in \exists(\mathcal{A}) \Leftrightarrow \exists p = p$,
(iv) $p \leq \exists q \Rightarrow \exists p \leq \exists q$,
(v) $p \leq q \Rightarrow \exists p \leq \exists q$,
(vi) $\exists(\sim \exists p) = \sim \exists p$,
(vii) $\exists(\mathcal{A})$ *is a subalgebra of* $\mathcal{A}$,
(viii) $\exists(p \vee q) = \exists p \vee \exists q$,
(ix) $\exists p - \exists q \leq \exists(p - q)$,
(x) $\exists p + \exists q \leq \exists(p + q)$.

2.3 Boolean monadic algebra

Now we can define the algebra of monadic first-order logic as [5].

Definition 2.8 *A Boolean monadic algebra is a pair* $(\mathcal{A}, \exists)$*, in which* $\mathcal{A}$ *is a Boolean algebra and* $\exists$ *is an existential quantifier on* $\mathcal{A}$.

Of course, with this definition, $\mathcal{A}$ is a complete Boolean algebra.

The elementary algebraic theory of monadic algebras is similar to that of every other algebraic system, with respect to the concepts of subalgebras, homomorphisms and ideals.

Definition 2.9 *A subset* $\mathcal{B}$ *is a monadic subalgebra of* $\mathcal{A}$ *if* $\mathcal{B}$ *is a Boolean subalgebra of* $\mathcal{A}$ *and it is a monadic algebra w.r.t. the quantifier of* $\mathcal{A}$.

Thereby, a Boolean subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a monadic subalgebra of $\mathcal{A}$ if, and only if, $\exists p \in \mathcal{B}$ whenever $p \in \mathcal{B}$.

Definition 2.10 *A monadic homomorphism is a mapping* h *from one monadic algebra into another monadic algebra, such that* h *is a Boolean homomorphism and* $h(\exists p) = \exists h(p)$*, for all* p.

Definition 2.11 *The kernel of homomorphism* h *is the set* $K(h) = \{p : h(p) = 0\}$.

Definition 2.12 *A Boolean monadic ideal* $\boldsymbol{I}$ *in* $\mathcal{A}$ *is a Boolean ideal such that if* $p \in \boldsymbol{I}$*, then* $\exists p \in \boldsymbol{I}$.

Proposition 2.13 *The kernel of a monadic homomorphism is a monadic ideal.*

Let $h : \mathcal{A} \to \mathcal{B}$ be a surjective Boolean homomorphism and $\mathcal{A}$ is a monadic algebra.

There is a unique way of converting $\mathcal{B}$ into a monadic algebra such that h becomes a monadic homomorphism. Given $q \in \mathcal{B}$, there exists $p \in \mathcal{A}$ so that $h(p) = q$, and we define the operator $\exists$ on q by $\exists q = h(\exists p)$. Therefore, $h(\exists p) = h(\exists q)$.

Definition 2.14 *A monadic algebra is simple if* $\{0\}$ *is the only proper ideal in it.*

The only simple Boolean algebra is the two-element algebra, designated by $\mathbf{2} = \{0, 1\}$.

The Boolean algebra $\mathbf{2}$ besides being a subalgebra it is a relatively complete subalgebra of every Boolean algebra.

Definition 2.15 *A monadic ideal is maximal if it is proper and it is not a proper subset of any other monadic proper ideal.*

Proposition 2.16 *The kernel of a homomorphism is a maximal ideal if, and only if, its range is a simple algebra.*

Definition 2.17 *A quantifier* $\exists$ *in a monadic algebra* $\mathcal{A}$ *is simple if:*
$\exists 0 = 0$ *and for every* $p \neq 0$, $\exists p = 1$.

Analogously, a quantifier $\forall$ in a monadic algebra $\mathcal{A}$ is simple if:
$\forall 1 = 1$ and, for every $p \neq 1$, $\forall p = 0$.
The next two important results are proved in [5].

Proposition 2.18 *A monadic algebra is simple if, and only if, its quantifier is simple.*

Theorem 2.19 *A monadic algebra is simple if, and only if, it is isomorphic to a* $\mathbf{2}$*-valued functional monadic algebra.*

3 Monadic algebras of many

In this section we introduce the monadic algebras of many.

Definition 3.1 *A quantifier of many is an operator* $\flat$ *from a monadic algebra into itself, that satisfies the following conditions:*
(i) $\forall p \leq \flat p$,
(ii) $\flat p \leq \exists p$,
(iii) $\flat(p \wedge q) \leq \flat p$.

Definition 3.2 *A monadic algebra of many is a triple* $(\mathcal{A}, \exists, \flat)$ *where* $(\mathcal{A}, \exists)$ *is a monadic algebra and* $\flat$ *is a quantifier of many on* $(\mathcal{A}, \exists)$.

Proposition 3.3 *If* $(\mathcal{A}, \exists, \flat)$ *is a monadic algebra of many, then:*
(i) $\flat 1 = 1$,
(ii) $\flat 0 = 0$,
(iii) $p \leq q \Rightarrow \flat p \leq \flat q$.
Proof. From Definition 3.1:
(i) Since $1 = \forall 1 \leq \flat 1$, then $\flat 1 = 1$.
(ii) Since $\flat 0 \leq \exists 0 = 0$, then $\flat 0 = 0$.
(iii) $p \leq q \Rightarrow p = q \wedge p \Leftrightarrow \flat p = \flat(q \wedge p) \leq \flat q$. ■

Proposition 3.4 *If* $(\mathcal{A}, \exists, \flat)$ *is a monadic algebra of many, then:*
(i) $\flat p \leq \flat(p \vee q)$,
(ii) $\flat p \vee \flat q \leq \flat(p \vee q)$,
(iii) $\flat(p \wedge q) \leq \flat p \wedge \flat q$,
(iv) $\flat(p \wedge q) \leq \flat(p \vee q)$,
(v) $\sim \flat 0 = 1 = \flat \sim 0$,
(vi) $\sim \flat 1 = 0 = \flat \sim 1$.
Proof. Immediate. ■

The elementary algebraic theory of monadic algebras of many is similar to that of every other algebraic system, with respect to the concepts of subalgebras, homomorphisms and ideals.

Definition 3.5 *A monadic subalgebra of many of* $(\mathcal{A}, \exists, \flat)$ *is a monadic subalgebra of* $(\mathcal{A}, \exists)$ *which is closed under the operator* $\flat$.

Definition 3.6 *A monadic homomorphism of many is a mapping* h *from a monadic algebra of many into another, such that* h *is a monadic homomorphism and* $h(\flat p) = \flat h(p)$, *for every* p.

The definition of kernel remains the same.

Definition 3.7 *A monadic ideal of many is a monadic ideal* $\boldsymbol{I}$ *in* $(\mathcal{A}, \exists, \flat)$, *such that if* $p \in \boldsymbol{I}$, *then* $\flat p \in \boldsymbol{I}$.

The kernel of a monadic homomorphism of many is a monadic ideal of many.

3.1 Logic as algebra

Before we deal with the algebraic adequacy of the logic of many, we will formalize the concept of Monadic Logic of Many.

In the logical context, certain elements are called 'provable' or 'demonstrable'. The corresponding algebraic structure of all provable elements is very important in algebraic study. Sometimes, when examining an algebraic structure, we may also consider not the provability, but the refutability, for which there is an immediate relation between the two concepts, namely: p is refutable if, and only if, $\sim p$ is provable.

Definition 3.8 *A monadic filter is a Boolean filter* $\boldsymbol{F}$ *in* $(\mathcal{A}, \exists, \flat)$*, such that if* $p \in \boldsymbol{F}$*, then* $\forall p \in \boldsymbol{F}$*.*

Definition 3.9 *A monadic filter of many is a Boolean filter* $\boldsymbol{F}$ *in* $(\mathcal{A}, \exists, \flat)$*, such that if* $p \in \boldsymbol{F}$*, then* $\flat p \in \boldsymbol{F}$*.*

Proposition 3.10 *Each monadic filter is a monadic filter of many in* $(\mathcal{A}, \exists, \flat)$*.*
Proof. If $\mathbf{F}$ is monadic, then always that $p \in \mathbf{F}$, it follows that $\forall p \in \mathbf{F}$. Since in $(\mathcal{A}, \exists, \flat)$, we know that $\forall p \leq \flat p$ holds, then $\flat p \in \mathbf{F}$. ■

If $(\mathcal{A}, \exists, \flat)$ is a monadic algebra of many, whose elements are propositional functions and $\mathbf{F}$ is a subset of $\mathcal{A}$ of provable elements, then to $\mathbf{F}$ should hold the following conditions:

(i) if p and q are provable, then $p \wedge q$ must be provable;

(ii) if p is provable, then $p \vee q$ must also be provable, regardless of q.

Hence, $\mathbf{F}$ should be, at least, a Boolean filter in $(\mathcal{A}, \exists, \flat)$.

However, it is not enough to $(\mathcal{A}, \exists, \flat)$, because $\mathbf{F}$ should also hold the following properties to quantification:

(iii) if p is provable, then $\flat p$ must also be provable;

(iv) if p is provable, then $\forall p$ must also be provable.

From these requirements, $\mathbf{F}$ should be a monadic filter.

Definition 3.11 *A monadic logic of many is a pair* $(\mathcal{M}, \boldsymbol{F})$*, where* $\mathcal{M} = (\mathcal{A}, \exists, \flat)$ *is a monadic algebra of many and* $\boldsymbol{F}$ *is a monadic filter in* $\mathcal{M}$*.*

On the basis of the above considerations, we have that the elements $p \in \mathbf{F}$ are the provable elements of the monadic logic of many $(\mathcal{M}, F)$, and if $\sim p \in \mathbf{F}$, then p is called refutable.

Moreover, we will use the filter concept and, as a consequence, we will define the kernel from the notion of filter.

Definition 3.12 *The kernel of a monadic homomorphism of many h is defined by $N(f) = \{p : h(p) = 1\}$.*

Obviously, the kernel of a monadic homomorphism of many is a monadic filter.

The definition of the quotient monadic algebra of many follows as usual.

Let us now consider that $\mathcal{M} = (\mathcal{A}, \exists, \flat)$ is a monadic algebra of many, $\mathbf{U}$ is a ultrafilter in $\mathcal{M}$, and let us assume the quotient monadic algebra $\mathcal{B} = \mathcal{M}/\mathbf{U}$.

Let h be a monadic homomorphism of many, such that $h : \mathcal{M} \to \mathcal{B}$ and h maps each element p into its equivalence class $[p]$ modulo $\mathbf{U}$.

There exists a natural way to make $\mathcal{M}$ be a monadic algebra of many in order to h be a surjective monadic homomorphism of many with kernel $\mathbf{U}$.

Definition 3.13 *For $\flat p \in \mathcal{M}$ and $\flat[p] \in \mathcal{B}$, we define $\flat[p] = [\flat p]$.*

Proposition 3.14 *The quantifier of many, as above, is well defined in $\mathcal{B}$.*

Proof. If $p_1, p_2 \in \mathcal{M}$ and $[p_1] = [p_2]$, then we have two cases: (i) $p_1, p_2 \in \mathbf{U}$ and (ii) $p_1, p_2 \in \mathbf{U}^C$.

For (i), as $[p_1] = [p_2]$ and $p_1, p_2 \in \mathbf{U}$, then $p_1 \wedge p_2 \in \mathbf{U}$, moreover $\flat(p_1 \wedge p_2) \in \mathbf{U}$, since $\mathbf{U}$ is an ultrafilter of many. By Definition 3.1, $\flat(p_1 \wedge p_2) \leq \flat(p_1), \flat(p_2)$, hence $\flat(p_1), \flat(p_2) \in \mathbf{U}$. We conclude that $[\flat(p_1)] = [\flat(p_2)]$.

The case (ii) is similar. ■

Definition 3.15 *A monadic algebra of many is simple if $\{1\}$ is its only proper monadic filter.*

Definition 3.16 *A quantifier $\flat$ in a monadic algebra of many is simple if:*
$\flat 1 = 1$ *and, for every $p \neq 1$, $\flat p = 0$.*

If $\flat$ is simple, then $\forall$ also is simple, for $\forall 1 = 1$ and, for every $p \neq 1$, $\forall p \leq \flat p = 0$.

Proposition 3.17 *A monadic algebra of many $\mathcal{M}$ is simple if, and only if, its quantifier $\flat$ is simple.*

Proof. ($\Rightarrow$) If $\mathcal{M}$ is simple, then $\{1\}$ is its only proper monadic filter, and therefore, there is an element $p \in \mathcal{M}$, such that $p \neq 1$.

If $p \in \mathcal{M}$ is such that $p \neq 1$, we define the monadic filter $\mathbf{F}$ generated by p as follows $\{q : \flat p \leq q\}$. Since $\mathbf{F}$ is a monadic filter distinct of $\{1\}$, then $\mathbf{F} = \mathcal{M}$, and hence, in particular, $0 \in \mathbf{F}$. This means that if $p \neq 1$, then $\flat p = 0$. Therefore, the quantifier of many is simple.

($\Leftarrow$) Since $\mathcal{M}$ has at least two elements, if $p \neq 1$, then $\flat p = 0$.

Certainly $\{1\}$ is a proper monadic filter on $\mathcal{M}$. Now, if $\mathbf{F}$ is a monadic filter on $\mathcal{M}$ distinct of $\{1\}$, then it must contain an element p, such that $p \neq 1$. Hence, $\flat p \in \mathbf{F}$, because $\mathbf{F}$ is monadic. So, provided that $\flat$ is simple, then $0 \in \mathbf{F}$ and, consequently, $\mathbf{F} = \mathcal{M}$.

Therefore, the only proper monadic filter on $\mathcal{M}$ is $\{1\}$ and $\mathcal{M}$ is simple. ■

Definition 3.18 *A monadic filter of many is maximal if it is a proper filter of many that is not a proper subset of any other proper filter of many.*

Proposition 3.19 *The kernel of a homomorphism is a maximal filter of many if, and only if, its range is a simple algebra.*

Proof. ($\Rightarrow$) The range of homomorphism h, $Im(h)$, is a monadic algebra of many.

By the hypothesis, $N(h) = \{p : h(p) = 1\}$ is a maximal filter of many. By Theorem of Homomorphism, it follows that $\mathcal{M}/N(h) \cong Im(h)$. Hence, $\{1\}$ is the only proper filter of many on $\mathcal{M}$, therefore, $\mathcal{M}$ is simple.

($\Leftarrow$) If $Im(h)$, the range of homomorphism h, is a simple algebra, then $\{1\}$ is the only proper monadic filter in it, i.e., the kernel $N(h) = \{p : f(p) = 1\} = \{1\}$ is a maximal monadic filter, because it is proper and since it is the only proper filter of many in the range, then there is no other proper monadic filter such that $N(h)$ is its proper subset. ■

Corollary 3.20 *Every subalgebra of a simple monadic algebra of many is also simple.*

Proof. Immediate from previous proposition. ■

Corollary 3.21 *The only simple Boolean algebra is the two-element algebra $\mathit{2} = \{0, 1\}$.*

Proof. Since a monadic algebra of many is simple if and only if its quantifier is simple, moreover, the quantifier of functional monadic algebra $\mathbf{2}^X$ is simple, then $\mathbf{2}^X$ is a simple monadic algebra, whenever $X \neq \emptyset$. The Corollary 3.20 implies that every subalgebra of $\mathbf{2}^X$, i.e., every $\mathbf{2}$-valued functional (monadic) algebra with a nonempty domain is also simple. ■

Proposition 3.22 *A monadic algebra of many is simple if and only if it is isomorphic to a 2-valued functional monadic algebra with a nonempty domain.*

Proof. [$\Leftarrow$] It suffices to observe that every $\mathbf{2}$-valued functional algebra with a nonempty domain is simple.

[⇒] Conversely, it is necessary to use a relevant fact, namely Stone's Theorem on the representation of Boolean algebra (cf. [6]). If $\mathcal{M}$ is a simple monadic algebra of many, then it is, particularly, a simple monadic algebra, hence, a nontrivial Boolean algebra, to which Stone's Theorem is applicable. Thus there exist: (i) a nonempty set X; (ii) a Boolean subalgebra $\mathcal{B}$ of $\mathbf{2}^X$, and (iii) a Boolean isomorphism f from $\mathcal{M}$ onto $\mathcal{B}$. If $\mathcal{B}$ is considered a monadic subalgebra of many of $\mathbf{2}^X$, then it is a $\mathbf{2}$-valued functional lagebra, whose quantifier is simple. Consequently, f preserves quantification. Indeed, f preserves 0 and 1 (it is a Boolean isomorphism) and, the quantifier is simple for both $\mathcal{M}$ and $\mathcal{B}$, hence it assumes the value 1 for every elements different that zero. Therefore, f is a monadic isomorphism of many between the monadic algebras $\mathcal{M}$ and $\mathcal{B}$. ■

Proposition 3.23 *For every Boolean filter $\boldsymbol{F}$ on $\mathcal{M}$, if $\boldsymbol{F}^*$ is the set of all $p \in \mathcal{M}$ in which $\forall p \in \boldsymbol{F}$, then $\boldsymbol{F}^*$ is a monadic filter.*

Proof. Indeed, if p and q are in $\mathbf{F}^*$, then $\forall p$ and $\forall q$ are in $\mathbf{F}$. Since $\mathbf{F}$ is a filter, then $\forall p \wedge \forall q \in \mathbf{F}$. From distributive law of operator $\forall$, follows that $\forall(p \wedge q) \in \mathbf{F}$, and so $p \wedge q \in \mathbf{F}^*$. If $p \in \mathcal{M}$ and $q \in \mathbf{F}^*$, then $\forall q \in \mathbf{F}$, and consequently $p \vee \forall q \in \mathbf{F}$. From modular law, it follows that $\forall(p \vee \forall q) \in \mathbf{F}$, so $\forall(p \vee q) \in \mathbf{F}$. Thereby, $p \vee q \in \mathbf{F}^*$. The idempotent law implies that $\forall\forall p \in \mathbf{F}$, so $\forall p \in \mathbf{F}^*$. Thus $\mathbf{F}^*$ is a monadic filter ■

Proposition 3.24 *If $\boldsymbol{F}$ is a maximal Boolean filter on $\mathcal{M}$, then $\boldsymbol{F}^*$ is a maximal monadic filter on $\mathcal{M}$.*

Proof. Assume that $\mathbf{J}$ is a monadic filter that extends properly $\mathbf{F}^*$. Then, there exists an element p in $\mathbf{J}$ that does not belong to $\mathbf{F}^*$. Since $\mathbf{J}$ is a monadic filter, it contains the element $\forall p$, but this element is not in $\mathbf{F}$, otherwise p would be in $\mathbf{F}^*$. Moreover, since $\mathbf{F}$ is a maximal filter, then the element $\sim \forall p$ has to belong to $\mathbf{F}$. This means that $\forall \sim \forall p$ is in $\mathbf{F}$, so $\sim \forall p$ is in $\mathbf{F}^*$ and consequently, it is also in $\mathbf{J}$. Since $\mathbf{J}$ has both an element and its complement, then $\mathbf{J}$ is not proper. ■

In this way, the correspondence from each Boolean filter $\mathbf{F}$ to $\mathbf{F}^*$ maps the set of Boolean filter on $\mathcal{M}$ into the set of monadic filters on $\mathcal{M}$, preserving the inclusion, and maps maximal filter into maximal filter. The filters left invariant, by the correspondence, are precisely the monadic filters.

Theorem 3.25 *Every proper monadic filter on $\mathcal{M}$ is included in a maximal monadic filter on $\mathcal{M}$.*

Proof. Let $\mathbf{F}_0$ be a proper monadic filter of a monadic algebra $\mathcal{M}$. Hence $\mathbf{F}_0$ is a proper Boolean filter and, consequently, it can be extended to a maximal Boolean filter $\mathbf{F}$. From the previous observations, the set $\mathbf{F}^*$ is a maximal monadic filter such that $\mathbf{F}_0 \subseteq \mathbf{F}^*$. Therefore, $\mathbf{F}^*$ is a maximal monadic filter that extends $\mathbf{F}_0$. ■

Definition 3.26 *The chain of monadic filters of many is a sequence* $(\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, ...)$ *of monadic filters of many, such that* $\boldsymbol{F}_1 \subseteq \boldsymbol{F}_2 \subseteq \boldsymbol{F}_3 \subseteq ....$

Proposition 3.27 *Let* $\mathcal{M}$ *be a monadic algebra of many and* $(\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, ...)$ *a chain of proper monadic filters of many on* $\mathcal{M}$. *The union* $\cup \boldsymbol{F}_n$ *is a proper monadic filter of many.*

Proof. Let $\mathbf{F} = \cup \mathbf{F}_n$, $p \in \mathbf{F}$ and $q \in \mathcal{M}$, with $p \leq q$. As $\mathbf{F} = \cup \mathbf{F}_n$, then for each $n \in \mathbb{N}$, we have that $p \in \mathbf{F}_n$ and since $\mathbf{F}_n$ is filter, then $q \in \mathbf{F}_n$. Hence, $q \in \cup \mathbf{F}_n = \mathbf{F}$.

Now, if $p, q \in \mathbf{F} = \cup \mathbf{F}_n$. Considering that $\mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \mathbf{F}_3 \subseteq ...$, for each $n \in \mathbb{N}$, then $p \wedge q \in \mathbf{F}_n \subseteq \mathbf{F}$ and as each $\mathbf{F}_n$ is a monadic filter of many, then $\forall p$, $\forall q$, $\flat p$, $\flat q \in \mathbf{F}_n$. Thus, $\forall p$, $\forall q$, $\flat p$, $\flat q \in \cup \mathbf{F}_n = \mathbf{F}$.

Consequently, $\mathbf{F}$ is a monadic filter of many.

Besides, for each $n \in \mathbb{N}$, $0 \notin \mathbf{F}_n$ and so $0 \notin \mathbf{F}$.

Hence, $\mathbf{F}$ is a proper monadic filter of many. ■

Proposition 3.28 *Every proper monadic filter of many on* $\mathcal{M}$ *is included in a maximal monadic filter of many.*

Proof. From previous proposition and Lemma of Zorn. ■

Theorem 3.29 *Let* $\mathcal{M}$ *be a monadic algebra of many and* $p \in \mathcal{M}$, *such that* $p \neq 1$. *Then, there exists a maximal monadic filter of many* $\boldsymbol{F}$ *such that* $p \notin \boldsymbol{F}$.

Proof. Let $\mathbf{F}_0$ the monadic filter of many defined by $\mathbf{F}_0 = \{q : \sim \forall p \leq q\}$. And let $\mathbf{F}$ be a maximal monadic filter of many that include $\mathbf{F}_0$.

If $p \in \mathbf{F}$, then the element $p \wedge \sim \forall p \in \mathbf{F}$ and, hence, $\forall(p \wedge \sim \forall p) \in \mathbf{F}$. But this element is $\forall p \wedge \sim \forall p$. Thus, $0 \in \mathbf{F}$ and, therefore, $\mathbf{F}$ is not proper. So, $p \notin \mathbf{F}$. ■

Under what conditions on a monadic algebra of many $\mathcal{M}$ is it true that whenever an element $p \in \mathcal{M}$ is mapped on 1 by every homomorphism from $\mathcal{M}$ into a simple monadic algebra of many, then $p = 1$?

Due to correspondence between homomorphisms with simple ranges and maximal filters of many, the question could be raised this way: under what conditions on a monadic algebra of many $\mathcal{M}$ is it true that whenever an element p of $\mathcal{M}$ belongs to all maximal filters of many, then $p = 1$? By means of the analogy with other parts of algebra, we put the definition as follows.

Definition 3.30 *A monadic algebra of many is semisimple if the intersection of all maximal filters of many in* $\mathcal{M}$ *is* $\{1\}$.

Theorem 3.31 *Every monadic algebra of many is semi-simple.*
Proof. We must to show that if $\mathcal{M}$ is a monadic algebra of many and $r \neq 1$ is an element of $\mathcal{M}$, then there exists a maximal monadic filter of many $\mathbf{F} \subseteq \mathcal{M}$ such that $r \notin \mathbf{F}$. From the Boolean version Theorem of Ultrafilter, there exists a maximal (ultrafilter) Boolean filter $\mathbf{F}_0 \subseteq \mathcal{M}$ such that $r \notin \mathbf{F}_0$.

Now, let $\mathbf{F}$ the set of all elements $p \in \mathcal{M}$ such that $\forall p \in \mathbf{F}_0$. Then $\flat p \in \mathbf{F}_0$ too and it is easy to verify that $\mathbf{F}$ is monadic filter and so a monadic filter of many for whose $r \notin \mathbf{F}$.

For the semi-simplicity we need to show that $\mathbf{F}$ is maximal.

Suppose that $\mathbf{J}$ is a monadic filter that properly includes $\mathbf{F}$. From Proposition 3.10, $\mathbf{J}$ is a monadic filter of many.

Thus, there is $p \in \mathbf{J}$ but $\forall p \notin \mathbf{F}_0$. Like $\mathbf{J}$ is a monadic filter, then $\forall p \in \mathbf{J}$. As $\forall p \notin \mathbf{F}_0$ and $\mathbf{F}_0$ is a maximal Boolean filter, then $\sim \forall p \in \mathbf{F}_0$. But, then $\forall p \wedge \sim \forall p = 0 \in \mathbf{J}$, and $\mathbf{J}$ is not proper. ■

4 The adequacy

In this section we provide the adequacy between the monadic algebra of many and the monadic part of $\mathbf{L}(M)$ using only algebraic tools.

Definition 4.1 *A model is a monadic logic of many* $(\mathcal{M}, \boldsymbol{F})$*, such that* $\mathcal{M}$ *is a monadic functional algebra of many* 2*-valued with nonempty domain and* $\boldsymbol{F}$ *is the filter of many* $\{1\}$.

It is natural to regard **2**-valued functional monadic algebras of many because, as we have already seen, they can be presented as monadic algebras enriched with the simple monadic quantifier. Note that since an **2**-valued functional monadic algebra with a nonempty domain is simple, then $\mathbf{F}$ only can be $\{1\}$ or $\mathcal{M}$, and the second option should be discarded.

Definition 4.2 *An interpretation of a monadic logic of many* $(\mathcal{M}, \boldsymbol{F})$ *in a model* $(\mathcal{B}, \{1\})$ *is a monadic homomorphism of many* $h : \mathcal{M} \to \mathcal{B}$ *such that, if* $p \in \boldsymbol{F}$*, then* $h(p) = 1$.

Thus, an interpretation of $(\mathcal{M}, \mathbf{F})$ in $(\mathcal{B}, \{1\})$ is a monadic homomorphism of many from $\mathcal{M}/\mathbf{F}$ into $\mathcal{B}$.

Definition 4.3 *A monadic logic of many* $(\mathcal{M}, \boldsymbol{F})$ *is semantically sound if it has an interpretation.*

Theorem 4.4 *The logic* $(\mathcal{M}, \boldsymbol{F})$ *is semantically sound if, and only if, the filter* $\boldsymbol{F}$ *is proper.*

Proof. $(\Rightarrow)$ If $\mathbf{F}$ is not proper, then $0 \in \mathbf{F} = \mathcal{M}$ and there is no isomorphism into $(\mathcal{B}, \{1\})$.

$(\Leftarrow)$ If $\mathbf{F}$ is proper, there exits a maximal filter of many $\mathbf{H}$ such that $\mathbf{F} \subseteq \mathbf{H}$. So we take the canonical homomorphism $h : \mathcal{M}/\mathbf{H} \to \mathcal{B}$. Besides, as $\mathbf{H}$ is maximal, then $\mathcal{M}/\mathbf{H}$ is simple and so $(\mathcal{M}/\mathbf{H}, \{1\})$ is isomorphic to $(\mathcal{B}, \{1\})$. Hence, h is an interpretation. ∎

Definition 4.5 *An element* $p \in \mathcal{M}$ *is valid if for any* $h : \mathcal{M} \to \mathcal{B}$, $h(p) = 1$.

Theorem 4.6 *(Soundness) If* p *is provable in* $(\mathcal{M}, \boldsymbol{F})$ *and* $(\mathcal{M}, \boldsymbol{F})$ *has an interpretation* $(\mathcal{B}, \{1\})$*, then* p *is valid in* $(\mathcal{B}, \{1\})$*.*

Proof. If p is provable in $(\mathcal{M}, \mathbf{F})$, then $p \in \mathbf{F}$ and so $h(p) = 1$. ∎

The soundness means that every provable element is valid in $(\mathcal{B}, \{1\})$. However, there may be some unprovable element that would be valid.

Definition 4.7 *A logic is semantically complete if all its valid elements are provable.*

Semantic completeness can be described by saying that every truth is provable.

This definition can also be expressed by its contrapositive: a logic is semantically complete if every refutable element is not valid.

Given a model $(\mathcal{B}, \{1\})$, we say that an element p is not valid if $h(p) = 0$, for some interpretation, that is, if p is false in some interpretation.

Algebraic elements help us in the completeness as well.

Let us suppose that the filter of many $\mathbf{F}$ on $(\mathcal{M}, \mathbf{F})$ is relatively large. If, in particular, $\mathbf{F}$ is very large, i.e. $\mathcal{M} = \mathbf{F}$, then the logic is semantically complete, since every element of $\mathcal{M}$ is provable. This case is not relevant.

Now, if $\mathcal{M} \neq \mathbf{F}$, then the quotient algebra $\mathcal{M}/\mathbf{F}$ must be considered. We evaluate if the logic $(\mathcal{M}, \mathbf{F})$ is semantically complete by the analysis of algebra $\mathcal{M}/\mathbf{F}$.

As each interpretation of $h : (\mathcal{M}, \mathbf{F}) \to (\mathcal{B}, \{1\})$ naturally induces a homomorphism of $\mathcal{M}/\mathbf{F}$ in $\mathcal{B}$ with the only constraint that $\mathcal{B}$ is a simple algebra, then the question is to indicate the conditions on the quotient algebra $\mathcal{M}/\mathbf{F}$ in order that if p is refutable in $\mathcal{M}/\mathbf{F}$, then $h(p) = 0$ on $\mathcal{B}$, for some homomorphism h.

It is enough to take $\mathcal{M}/\mathbf{F}$ semisimple.

Theorem 4.8 *Each monadic logic of many is semantically complete.*
Proof. From Theorem 3.31, every monadic algebra of many is semisimple. Then every monadic logic of many is semantically complete. ■

Corollary 4.9 *(Completeness) If p is valid in $(\mathcal{B}, \{1\})$, then p is provable in $(\mathcal{M}, \boldsymbol{F})$.*
Proof. From previous Theorem, if p is refutable in $(\mathcal{M}, \mathbf{F})$, then p is not valid in $(\mathcal{B}, \{1\})$. ■

5 Final remarks

In this paper, we have introduced the monotonic logic of many via algebra, inspired by the development of monadic logics according to Paul Halmos. The logic of many was introduced by Grácio, as a way to formalize some central aspects of generalized quantifier ‘many’.

Thereby, it is a first-order logic with added formal elements to characterize a notion of ‘many’.

Halmos initiated a tradition on algebraic logic, particularly, interested in first-order logic and its algebraic models. Besides, all the systems developed by Halmos are algebraic and using just algebraic tools. He has produced the adequacy of first-order logic relative to his algebraic models. In a first moment for monadic logic and subsequently for the usual approach.

The monadic logics are interesting, in general, because they are decidable, while the non-monadic are not. For a next step, we need to show that the monadic logic, introduced here, is decidable and another improvement will be the adequacy of the logic of many following the Halmos constructions.

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