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Probability, Intuitionistic Logic and Strong Negation

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Abstract

In this paper I present a sequent calculus for intuitionistic first-order logic with strong negation. The semantic is probabilistic, more precisely, it is based on partial conditional probability functions. Soundness and completeness are proved.

Keywords: Probabilistic interpretation, Intuitionistic logic, Strong negation, Conditional probability

Introduction

The notation $\Pr(A, \Gamma)$ will be used for the probability of A conditional on the set of sentences Γ. The main feature of this semantics is that, $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$ i.e. the conditional probability of the intuitionistic conditional is the probability of the consequent conditionalized on the antecedent, when $\Pr(B, \Gamma \cup \{A\})$ is defined.

It is a well-known fact that intuitionistic logic is closed to 3-value logic even if it is not a n-value logic. In fact, intuitionistic logic is not verifunctional. For example, $A \supset B$ is true iff, when we find a proof of A, we find a proof of B, i.e., if we discover that A is true, then we discover that B is true.

In the probabilistic context, the situation is slightly more complex. This is so, because intuitionistic negation is not the adequate negation for probabilistic interpretation. For example, $\Pr(\neg(A \supset B), \Gamma)$ cannot be, in general, be $(1 - \Pr((A \supset B), \Gamma))$.

1 A Sequent Calculus for Intuitionistic Predicate Logic with Strong Negation (SCIPLSN)

In what follows, $\sim$ is the strong negation, F is falsity and $\neg$ is the intuitionistic negation and is not a primitive: $\neg A =_{def} A \supset F$.

Definition 1.1. *Let $Con = \{c_1, \ldots, c_n, \ldots\}$ be the set of constants, $Var = \{x_1, \ldots, x_n, \ldots\}$ the set of variables, $Fon = \{f_0^0, \ldots, f_n^m, \ldots\}$ the set of function letters and $Pre = \{A_0^0, \ldots, A_n^m, \ldots\}$ the set of predicate letters (the superscript indicates the number of arguments).*

The set Ter of terms is defined as:
(i) $Con \cup Var \subseteq Ter$
(ii) If $t_{i_1}, \ldots, t_{i_m} \in Ter$, then $f_n^m(t_{i_1}, \ldots, t_{i_m}) \in Ter$
(iii) Nothing else is in Ter.

The set $WFFs$ of well-formed formulas is defined as:
(i) $\{F, A_n^m(t_{i_1}, \ldots, t_{i_m})\} \subseteq WFF$ for any $i, n, m \in \mathbb{N}$;
(ii) if $A, B \in WFF$, then $\sim A, \forall x_i A, \exists x_i A, (A \wedge B), (A \vee B), (A \supset B) \in WFF$;
(iii) Nothing else is in WFF.

For a wff $\forall x_i A$ (resp. $\exists x_i A$), A is call the *scope* of $\forall x_i$ (resp. $\exists x_i$).

An occurrence of a variable x_i in A which is not in the scope of $\forall x_i$ (resp. $\exists x_i$) nor immediately preceded by $\forall$ (resp. $\exists$) is said to be $free$.

An occurrence of a variable x_i in A which is in the scope of $\forall x_i$ (resp. $\exists x_i$) or immediately preceded by $\forall$ (resp. $\exists$) is said to be *bound*.

Definition 1.2. *Let A be a wff and t a term. $A[t|x_i]$ is the wff obtained by the substitution of all the free occurrences of x_i in A by t.*
t is said to be free for x_i in A iff no variable occurring in t is bound in $A[t|x_i]$.

Axioms

$A, \Gamma \Rightarrow A$ A1

$F, \Gamma \Rightarrow C$ A2

$\Gamma \Rightarrow \ \sim F$ A3

Logical Rules

$$\frac{\Gamma \Rightarrow A}{\sim A, \Gamma \Rightarrow C}\ L{\sim}$$

$$\frac{A, \Gamma \Rightarrow C}{\sim\sim A, \Gamma \Rightarrow C}\ L{\sim\sim} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\sim A}\ R{\sim\sim}$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B,}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{\sim A, \Gamma \Rightarrow C \qquad \sim B, \Gamma \Rightarrow C}{\sim(A \wedge B), \Gamma \Rightarrow C}\ L{\sim}\wedge \qquad \frac{\Gamma \Rightarrow\ \sim A}{\Gamma \Rightarrow\ \sim(A \wedge B)}\ R{\sim}\wedge_1$$

$$\frac{\Gamma \Rightarrow\ \sim B}{\Gamma \Rightarrow\ \sim(A \wedge B)}\ R{\sim}\wedge_2$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow C}{\sim(A \vee B), \Gamma \Rightarrow C}\ L{\sim}\vee \qquad \frac{\Gamma \Rightarrow \sim A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \vee B)}\ R{\sim}\vee$$

$$\frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R\supset$$

$$\frac{A, \sim B, \Gamma \Rightarrow C}{\sim(A \supset B), \Gamma \Rightarrow C}\ L{\sim}\supset \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \supset B)}\ R{\sim}\supset$$

$$\frac{A[t|x], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ L\,\forall \qquad \frac{\Gamma \Rightarrow A[y|x]}{\Gamma \Rightarrow \forall x A}\ R\,\forall$$

$$\frac{A[y|x], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\,\exists \qquad \frac{\Gamma \Rightarrow A[t|x]}{\Gamma \Rightarrow \exists x A}\ R\,\exists$$

Here we suppose that t is free for x in A and $A[t|x]$ is the wff obtained from A by replacing all the free occurrences of x by t. For $R\ \forall$, we also need the following restriction: y must not be free in Γ, $\forall xA$. For $L\ \exists$, we also need the following restriction: y must not be free in Γ, $\forall xA$ and C.

$$\frac{\sim A[t|x], \Gamma \Rightarrow C}{\sim\exists xA, \Gamma \Rightarrow C}\ L{\sim}\exists \qquad \frac{\Gamma \Rightarrow\ \sim A[y|x]}{\Gamma \Rightarrow\ \sim\exists xA}\ R{\sim}\exists$$

$$\frac{\sim A[y|x], \Gamma \Rightarrow C}{\sim\forall xA, \Gamma \Rightarrow C}\ L{\sim}\forall \qquad \frac{\Gamma \Rightarrow\ \sim A[t|x]}{\Gamma \Rightarrow\ \sim\forall xA}\ R{\sim}\forall$$

For $R\ {\sim}\exists$, we also need the following restriction: y must not be free in Γ, ${\sim}\exists xA$. For $L\ {\sim}\forall$, we also need the following restriction: y must not be free in Γ, ${\sim}\forall xA$ and C.

From a proof-theoretic point of view, $\Gamma \Rightarrow\ \sim A$ can be interpreted as "from Γ we have a constructive proof of the falsity of A". The introduction of strong negation gives us a conservative extension of intuitionistic logic: every derivable sequent of intuitionistic logic is a derivable sequent of intuitionistic logic with strong negation. Moreover, in intuitionistic logic we have $\Gamma \Rightarrow A$ or $\Gamma \not\Rightarrow A$. In intuitionistic logic with strong negation we have $\Gamma \Rightarrow A$ or $\Gamma \Rightarrow\ \sim A$ or ($\Gamma \not\Rightarrow A$ and $\Gamma \not\Rightarrow\ \sim A$.

We will also use the *Cut Rule*

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Delta, \Gamma \Rightarrow C}{}_{Cut}$$

without proving its admissibility. See Negri and von Plato [19].

2 Partial Conditional Probability Functions

We now characterize the notion of *partial conditional probability function*.

Definition 2.1. *A partial conditional probability function is any partial function*

$$\Pr : WFF \times 2^{WFF} \to [0,1]$$

which satisfies some postulates that will be specified below.

A background Γ is said to be Pr-abnormal iff, for any A, $\Pr(A, \Gamma) = 1$. Otherwise, it is Pr-normal.

Two partial conditional probability functions that give the same value for the same argument when defined are identical, i.e. they *are* the same function. When $\Pr(A, \Gamma)$ is not defined, we will say that the probability of A is unknown (the interpretation of probability is clearly subjective) giving the background Γ. A first general constraint on partial conditional probability functions is the condition:

Probabilistic Equivalence (PE). Let A and B be two wffs. We will say that A and B are probabilistically equivalent iff, for any Pr and any Γ
(i) $\Pr(A, \Gamma)$ and $\Pr(B, \Gamma)$ are both defined or both undefined;
(ii) $\Pr(A, \Gamma) = \Pr(B, \Gamma)$ when defined.

Probabilistic equivalence is the strongest semantic equivalence relation. See Theorem 2.5 below.

The following definition will be useful.

Definition 2.2. *A n-permutation is a bijection* $per_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

We will write $\bigwedge_{i=1}^{n} A_i$ for $(A_1 \wedge (\ldots \wedge A_n) \ldots)$ and $\bigvee_{i=1}^{n} A_i$ for $(A_1 \vee (\ldots \vee A_n) \ldots)$.

We restrict the set of partial probability functions to those which satisfy the following postulates:

DF. 1. *If* $\Pr(A_j, \Gamma) = 0$ *for some* $1 \leq j \leq n$, *then* $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$*;*

DF. 2. *If* $\Pr(A_j, \Gamma) = 1$ *for some* $1 \leq j \leq n$, *then* $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = 1$*;*

Postulates DF.1 and DF.2 are the rules governing "unknown".

The following postulates are also satisfied. When the probabilities are known:

POS. 3. $0 \leq \Pr(A, \Gamma) \leq 1$*;*

POS. 4. *If* $A \in \Gamma$*, then* $\Pr(A, \Gamma) = 1$*;*

POS. 5. $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$;

POS. 6. $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\})$;

POS. 7. $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$

POS. 8. $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$;

POS. 9. *If* Γ *is Pr-normal, then* $\Pr(\sim A, \Gamma) =$
(1) $1 - \Pr(A, \Gamma)$ *if* A *is an atom or* F *or* $(B \wedge C)$ *or* $(B \vee C)$ *or* $\forall x A$ *or* $\exists x A$;
(2) $\Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\})$ *if* A *is* $(B \supset C)$;
(3) $\Pr(B, \Gamma)$ *if* A *is* $\sim B$;

POS. 10. $\Pr(C, \Gamma \cup \{\bigwedge_{i=1}^{n} A_i\}) = \Pr(C, \Gamma \cup \{A_1, \ldots, A_n\})$;

POS. 11. *If* Γ *is Pr-normal, then* $\Pr(F, \Gamma) = 0$;

POS. 12. *If, for any* Δ, $\Pr(A, \Gamma \cup \Delta) = 1$,
then for any B *and* C, $\Pr(C, \Gamma \cup \Delta \cup \{B\}) = \Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$

POS. 13. *If* $\Pr(C, \Gamma \cup \{A_i\}) = 1$ *for any* i *such that* $1 \leq i \leq n$, *then* $\Pr(C, \Gamma \cup \{\bigvee_{i=1}^{n} A_i\}) = 1$;

POS. 14. *If* $\Pr(C, \Gamma \cup \{\sim A_1, \ldots, \sim A_n\}) = 1$, *then* $\Pr(C, \Gamma \cup \{\sim(\bigvee_{i=1}^{n} A_i\})) = 1$;

POS. 15. *If* $\Pr(C, \Gamma \cup \{\sim A_i\}) = 1$ *for any* i *such that* $1 \leq i \leq n$, *then* $\Pr(C, \Gamma \cup \{\sim(\bigwedge_{i=1}^{n} A_i\})) = 1$;

POS. 16. $\Pr(\forall x A, \Gamma) = \lim_{n \to \infty} \Pr(\bigwedge_{i=1}^{n} A[t_i|x], \Gamma)$ *where* $t_1, \ldots, t_n, \ldots$ *is an enumeration of all the terms that are free for* x *in* A ;

POS. 17. $\Pr(\exists x A, \Gamma) = \lim_{n \to \infty} \Pr(\bigvee_{i=1}^{n} A[y_i|x], \Gamma)$ *where* $y_1, \ldots, y_n, \ldots$ *is an enumeration of all the variables which are not free in* A *and* Γ;

POS. 18. *If* $\Pr(C, \Gamma \cup \{A[t|x]\}) = 1$, *then* $\Pr(C, \Gamma \cup \{\forall x A\}) = 1$ *where* t *is free for* x *in* A;

POS. 19. *If* $\Pr(C, \Gamma \cup \{A[y|x]\}) = 1$, *then* $\Pr(C, \Gamma \cup \{\exists x A\}) = 1$ *where* y *is not free in* A, Γ *and* C;

POS. 20. *If* $\Pr(A[y|x], \Gamma) = 1$ *with* y *not free in* Γ *nor in* A *(or* $y = x$*), then* $\Pr(A[t|x], \Gamma) = 1$ *where* t *is free for* x *in* A.

Remarks 2.3.

- *DF.1-DF.2 are quite intuitive. For example, if* $\Pr(A, \Gamma) = 0$*, then* $\Pr(B \wedge A, \Gamma) = 0$*,* $\Pr(B, \Gamma)$ *being defined or not.*
- *Unknown is not a value but a lack of value. So, arithmetical operations cannot be applied to expressions with unknown values, even equality. Even if both* $\Pr(A, \Gamma)$ *and* $\Pr(B, \Delta)$ *are both unknown, this doesn't mean that they are equal.The only legitimate uses of expressions that have unknown values are those explicitly given by DF.1-DF.2. So, if* $\Pr(p, \Gamma)$ *is unknown, POS.6 does not hold, i.e.* $\Pr(p \wedge \sim p, \Gamma)$ *is undefined if* Γ *is Pr-normal.*

One can easily prove that

Theorem 2.4. $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(\bigvee_{i=1}^{n} A_{per_n(i)}, \Gamma)$.

Proof. We just give a sketch of the proof. We proceed by induction using POS.5, POS.6 and POS.7. It is clear and that any permutation of the disjuncts preserves the value or lack of value. □

Furthermore, using POS.10 together with POS.5 and POS.6, we can easily prove that, when defined, for any $n \in \mathbb{N}$, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) \leq \Pr(\bigwedge_{i=1}^{n-1} A_i, \Gamma)$ and $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) \geq \Pr(\bigvee_{i=1}^{n-1} A_i, \Gamma)$. So the sequences $\Pr(A_1, \Gamma), \ldots, \Pr(\bigwedge_{i=1}^{n} A_i, \Gamma), \ldots,$ and $\Pr(A_1, \Gamma), \ldots, \Pr(\bigvee_{i=1}^{n} A_i, \Gamma), \ldots,$ are respectively decreasing and increasing. As these sequences are bounded (by POS.3), it follows, by an elementary result of real numbers analysis, that their limits exist and are in $[0, 1]$. This insures that POS.16 and POS.17 are not only very intuitive constraints but are adequate.

POS.8, $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$ calls for some comments. It is simply the expression of the very intuitive interpretation of the probability of the conditional as the conditional probability. David Lewis showed that this cannot apply to material conditional, i.e. to $\sim A \vee B$. But it can be applied to $A \supset B$ when " $\supset$ " is the probabilistic conditional: The probability of $A \supset B$ given the background Γ is just the probability of B when A is hypothetically add to Γ. Lewis' proof does not hold in intuitionistic logic nor in intuitionistic logic with strong negation. See [9].

The following theorem will be useful.

Theorem 2.5. *(Substituability of Probabilistic Equivalents in Background) Let A and B be two wffs and Γ a set of wffs. If for any Δ, $\Pr(A, \Gamma \cup \Delta)$ is known iff $\Pr(B, \Gamma \cup \Delta)$ is known and $\Pr(A, \Gamma \cup \Delta) = \Pr(B, \Gamma \cup \Delta)$ when both are known, then, for any C, $\Pr(C, \Gamma \cup \Delta \cup \{A\}) = \Pr(C, \Gamma \cup \Delta \cup \{B\})$ when both are known.*

Proof.

$\Pr(C \wedge A, \Gamma \cup \Delta) =$	
$\quad \Pr(C, \Gamma \cup \Delta) \times \Pr(A, \Gamma \cup \Delta \cup \{C\})$	POS.13
$\Pr(C \wedge B, \Gamma \cup \Delta) =$	
$\quad \Pr(C, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{C\})$	POS.13
$\Pr(C \wedge A, \Gamma \cup \Delta) = \Pr(C \wedge B, \Gamma \cup \Delta)$	Assumption + algebra
$\Pr(A \wedge C, \Gamma \cup \Delta) = \Pr(B \wedge C, \Gamma \cup \Delta)$	POS.15
$\Pr(A, \Gamma \cup \Delta) \times \Pr(C, \Gamma \cup \Delta \cup \{A\}) =$	
$\Pr(B, \Gamma \cup \Delta) \times \Pr(C, \Gamma \cup \Delta \cup \{B\})$	POS.13
$\Pr(C, \Gamma \cup \Delta \cup \{A\}) = \Pr(C, \Gamma \cup \Delta \cup \{B\})$	Assumption + algebra

□

Theorem 2.6. *$\sim(A \supset B)$ and $(A \wedge \sim B)$ are substitutable in backgrounds.*

Proof. Trivial by POS.9 (2).

□

The following definition will be useful.

Definition 2.7. *Let $PB \subseteq WFF$ be the set of wffs that are not of the form $A \supset B$ (PB for pseudo boolean).*

Theorem 2.8. *For any wff $A \in PB$, if Γ is Pr-normal, then $\Pr(\sim A, \Gamma) = 1 - \Pr(A, \Gamma)$ when $\Pr(A, \Gamma)$ is defined.*

Proof. This is a trivial consequence of POS.9 (1) and when A is $\sim B$, the conclusion follows from induction using POS.9 (3). □

This is the "classical" case. Consider the case where $A \notin PB$. We have $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\})$, when defined. The easy case is when $C \in PB$: $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times (1 - \Pr(C, \Gamma \cup \{B\})) = \Pr(B, \Gamma) \times (1 - \Pr(B \supset C), \Gamma)$. We have three sub-cases:

(1) If $\Pr(B \supset C, \Gamma) = 0$, then $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma)$.

(2) If $0 < \Pr((B \supset C), \Gamma) < 1$, then $0 < \Pr({\sim}(B \supset C), \Gamma) < \Pr(B, \Gamma)$

(3) If $\Pr(B \supset C), \Gamma) = 1$, then $\Pr({\sim}(B \supset C), \Gamma) = 0$.

(1) If $\Pr((B \supset C), \Gamma) = 0$, then $\Pr(C, \Gamma \cup \{B\}) = 0$. By hypothesis, $C \in PB$ and thus $\Pr({\sim}C, \Gamma \cup \{B\})) = (1 - \Pr(C, \Gamma \cup \{B\})) = 1$ As $\Pr({\sim}(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\}))$ we have $\Pr({\sim}(B \supset C), \Gamma) = Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\}) = Pr(B, \Gamma) \times (1 - 0) = Pr(B, \Gamma)$.

(2) We have $0 < \Pr((B \supset C), \Gamma) < 1$ and thus $0 < \Pr(C, \Gamma \cup \{B\}) < 1$
$\Pr({\sim}(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\})$.
This implies that $0 < \Pr({\sim}(B \supset C), \Gamma) < \Pr(B, \Gamma)$ because $\Pr({\sim}C, \Gamma \cup \{B\}) = 1 - \Pr(C, \Gamma \cup \{B\})$ and $0 < 1 - \Pr(C, \Gamma \cup \{B\}) < 1$

(3) We have $\Pr((B \supset C), \Gamma) = 1 = \Pr(C, \Gamma \cup \{B\})$
But $\Pr({\sim}(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\}) = \Pr(B, \Gamma) \times (1 - \Pr(C, \Gamma \cup \{B\})) = \Pr(B, \Gamma) \times (1 - 1) = 0$.

We now have to take a closer look to the general case. The problem is with C:
$\Pr((B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\})$. If $C \notin PB$ i.e. C is $D \supset E$, we are back to square one. We clearly need a proof based on the number of "$\supset$".

Theorem 2.9. *Let A be $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))$ with $C_n \in PB$ (any A has this form, for some $n \geq 0$). Then, when $\Pr({\sim}A, \Gamma)$ is defined*

(1) If $\Pr(A, \Gamma) = 0$ then $\Pr({\sim}A, \Gamma) = \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr({\sim}C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$;

(2) $0 < \Pr(A, \Gamma) < 1$, then
$0 < \Pr({\sim}A, \Gamma) < \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr({\sim}C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$;

(3) If $\Pr(A, \Gamma) = 1$ then $\Pr({\sim}A, \Gamma) = 0$

Proof. First of all, by applying POS.8 n times, we have $\Pr(C_0 \supset (C_1 \supset (C_2 \supset$

$(\ldots \supset (C_{n-1} \supset C_n)\ldots))), \Gamma) = \Pr(C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$

By applying POS.9 (2) n times, we have
$\Pr({\sim}(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots)))), \Gamma) = \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr({\sim}C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$

(1) We need to make sure that: If $\Pr(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))), \Gamma) = 0$, then $\Pr({\sim}(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))), \Gamma) = \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr(C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$
i.e. if $\Pr(C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\}) = 0$, then $\Pr({\sim}C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})) = 1$.
This is trivial because $C_n \notin PB$ and thus, $\Pr({\sim}C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\}) = (1 - \Pr(C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})) = (1 - 0) = 1$.

(2) and (3) are also quite trivial along the same lines. □

Corollary 2.10. *Let A be $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))$ with $C_n \in PB$. Then, when defined,* $\Pr({\sim}A, \Gamma) \leq 1 - \Pr(A, \Gamma)$.

Proof. Trivial. □

The following theorems will be useful.

Theorem 2.11. *If* $\Pr(A, \Gamma) = 1$ *and* $\Pr(B, \Gamma)$ *is defined, then* $\Pr(B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$.

Proof.

$\Pr(A \vee B), \Gamma) = 1$	DF.2
$= \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A \wedge B), \Gamma)$	POS.5
$= 1 + \Pr(B, \Gamma) - \Pr(A \wedge B), \Gamma)$	$\Pr(A, \Gamma) = 1$
$\Pr(B, \Gamma) = \Pr(A \wedge B, \Gamma)$	Algebra
$\Pr(B, \Gamma) = \Pr(A, \Gamma) \times \Pr(B, \Gamma \cup \{A\})$	POS.6
$\Pr(B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$	$\Pr(A, \Gamma) = 1$

□

Theorem 2.12. *If* $\Pr({\sim}A, \Gamma) = 1$*, then* $\Pr(A, \Gamma) = 0$

Proof. If $A \in PB$, it is a trivial consequence of POS.9 (1)-(3).
If not, A is $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))$
with $C_n \in PB$.

$\Pr({\sim}(C_0 \supset (C_1 \supset (C_2 \supset (\ldots$ $\supset (C_{n-1} \supset C_n)\ldots)))), \Gamma) = 1$	Hypothesis
$= \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{\, C_0\}) \times \ldots \times$ $\Pr({\sim}(C_n, \Gamma \cup \{C_0, \ldots, C_{n-1}\})$	POS.9 (2) n times
$\Pr(C_0, \Gamma) = \Pr(C_1, \Gamma \cup \{C_0\}) = \ldots =$ $\Pr(C_{n-1}, \Gamma \cup \{C_0, \ldots, C_{n-2}\}) = 1$	Algebra
$\Pr(C_0, \Gamma) = \Pr(C_1, \Gamma) = \ldots =$ $\Pr(C_{n-1}, \Gamma) = 1$	Thm 2.11 $(n-1)$ times
$\Pr({\sim}C_n, \Gamma \cup \{C_0, \ldots, C_{n-1}\}) = 1$	Algebra
$\Pr(C_n, \Gamma \cup \{C_0, \ldots, C_{n-1}\}) = 0$	$C_n \in PB$
$\Pr(C_n, \Gamma) = 0$	$\Pr(C_i, \Gamma) = 1$ for $0 \leq i \leq (n-1)$ and Thm 2.11 $(n-1)$ times

$\Pr(C_0 \supset (C_1 \supset (C_2 \supset (\ldots$ $\supset (C_{n-1} \supset C_n)\ldots)))), \Gamma) =$ $\Pr(C_n, \Gamma \cup \{C_0, \ldots, C_{n-1}\}) =$ $\Pr(C_n, \Gamma) = 0$	$\Pr(C_i, \Gamma) = 1$ for $0 \leq i \leq (n-1)$ and Thm 2.11 $(n-1)$ times

□

The above theorems show that the probabilistic semantic value of formulas containing strong negation of "horseshoe" is far from being trivial.

Theorem 2.13. *If $A_1, A_2 \in PB$, then ${\sim}(A_1 \wedge A_2)$ and $({\sim}A_1 \vee {\sim}A_2)$ are probabilisticaly equivalent.*

Proof. We have to prove that, for any Γ, $\Pr({\sim}(A_1 \wedge A_2), \Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$ or both are unknown. Let us suppose that $\Pr({\sim}(A_1 \wedge A_2), \Gamma)$ is known. $\Pr({\sim}(A_1 \wedge A_2), \Gamma) = 1 - \Pr((A_1 \wedge A_2), \Gamma)$ by POS.9 (1).

As the general proof uses Bayes' theorem, we need to consider a special case. Let us suppose that $\Pr({\sim}A_1, \Gamma) = 0$ (a similar proof holds for A_2).

(α)

$\Pr({\sim}(A_1 \wedge A_2), \Gamma) = 1 - \Pr((A_1 \wedge A_2), \Gamma)$	POS.9 (1)
$\Pr({\sim}A_1, \Gamma) = 0$	Hypothesis
$\Pr(A_1, \Gamma) = 1$	$A_1 \in PB$
$\Pr((A_1 \wedge A_2), \Gamma) = \Pr(A_2, \Gamma)$	DF.1
$1 - \Pr((A_1 \wedge A_2), \Gamma) = 1 - \Pr(A_2, \Gamma)$	Algebra
$\Pr({\sim}(A_1 \wedge A_2), \Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$	DF.2

(β) Let us suppose that $\Pr({\sim}A_1,\Gamma) \neq 0$ and $\Pr({\sim}A_2,\Gamma) \neq 0$.

In that case, $\Pr({\sim}(A_1 \wedge A_2),\Gamma)$ and $\Pr(({\sim}A_1 \vee {\sim}A_2),\Gamma)$ are both undefined if and only if one of them is undefined.

We prove: $\Pr({\sim}(A_1 \wedge A_2),\Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2),\Gamma)$.

$$
\begin{array}{ll}
\Pr({\sim}(A_1 \wedge A_2),\Gamma) = 1 - \Pr((A_1 \wedge A_2),\Gamma) & \text{POS.9 (1)} \\
= 1 - \Pr(A_1, \Gamma \cup \{A_2\})) \times \Pr(A_2,\Gamma) & \text{POS.6} \\
= \Pr({\sim}A_2,\Gamma) + \Pr(A_2,\Gamma) - \Pr(A_1,\Gamma \cup \{A_2\})) & \\
\quad \times \Pr(A_2,\Gamma) & A_2 \in PB \\
= \Pr({\sim}A_2,\Gamma) + \Pr(A_2,\Gamma) \times (\Pr({\sim}A_1,\Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1,\Gamma) - \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) & \\
\quad + \Pr(A_2,\Gamma) \times (\Pr({\sim}A_1,\Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) - \Pr({\sim}A_1,\Gamma) & \\
\quad + \Pr(A_2,\Gamma) \times (\Pr({\sim}A_1,\Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) - \Pr({\sim}A_1,\Gamma)\times & \\
\quad \left(1 - \dfrac{\Pr(A_2,\Gamma) \times (\Pr({\sim}A_1,\Gamma \cup \{A_2\}))}{\Pr({\sim}A_1,\Gamma)}\right) & \text{Algebra} \\
= \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) - & \\
\quad \Pr({\sim}A_1,\Gamma) \times (1 - \Pr A_2,\Gamma \cup \{{\sim}A_1\}) & \text{Bayes} \\
= \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) - & \\
\quad \Pr({\sim}A_1,\Gamma) \times (\Pr({\sim}A_2),\Gamma \cup \{{\sim}A_1\}) & A_2 \in PB \\
= \Pr({\sim}A_1,\Gamma) + \Pr({\sim}A_2,\Gamma) - \Pr(({\sim}A_1 \wedge {\sim}A_1),\Gamma) & \text{POS.6} \\
= \Pr(({\sim}A_1 \vee {\sim}A_2),\Gamma) & \text{POS.5}
\end{array}
$$

□

Similarly, we have:

$\Pr({\sim}\exists x_i A,\Gamma) = \Pr(\forall x_i {\sim}A,\Gamma)$

$$
\begin{array}{ll}
\Pr({\sim}\exists x_i A,\Gamma) = 1 - \Pr(\exists x_i A,\Gamma) & \text{POS.9 (1)} \\
= 1 - \lim\limits_{n\to\infty} \Pr(\bigvee\limits_{i=1}^{n} A[t_{i_n}|x],\Gamma) & \text{POS.17} \\
= \lim\limits_{n\to\infty} (1 - \Pr(\bigvee\limits_{i=1}^{n} A[t_{i_n}|x],\Gamma)) & \text{Elementary calculus} \\
= \lim\limits_{n\to\infty} \Pr({\sim}\bigvee\limits_{i=1}^{n} A[t_{i_n}|x],\Gamma) & \text{POS.9 (1)}
\end{array}
$$

In what follows, we will use "unknown", "unknown value" and "undefined" for the same purpose depending on the context.

$\Gamma \Rightarrow A$ is a valid sequent according to partial probabilistic interpretations iff, for any Pr satisfying all the DF and all the POS,

$$\Pr(A, \Gamma \cup \Delta) = 1$$

for all Δ. This intuition is very robust. A is a valid consequence of Γ iff, the probability that A is 1 and remains invariant, regardless of what is added to the background Γ.

Theorem 2.14. *If* Γ *is Pr-abnormal, then* $\Gamma \cup \{A\}$ *is Pr-abnormal.*

Proof.

$\Pr(A, \Gamma) = \Pr(B, \Gamma) = 1$	Γ is Pr-abnormal
$\Pr(B, \Gamma) = \Pr(B, \Gamma \cup \{A\}) = 1$	Theorem 2.11

□

Theorem 2.15. *If* $\Pr(A \wedge \sim A, \Gamma) = 1$*, then* Γ *is Pr-abnormal.*

Proof. Let us suppose that Γ is Pr-normal.

$\Pr(A \wedge \sim A, \Gamma) = 1$	Hypothesis
$\Pr(A, \Gamma) \times \Pr(\sim A, \Gamma \cup \{A\}) = 1$	POS.6
$\Pr(A, \Gamma) = \Pr(\sim A, \Gamma \cup \{A\}) = 1$	Algebra
$\Pr(\sim A, \Gamma) = 1$	Theorem 2.11
$\Pr(\sim\sim A, \Gamma) = 0$	Theorem 2.12
$\Pr(A, \Gamma) = 0$	POS. 9 (3)
$1 = 0$	

Thus Γ is Pr-abnormal. □

Theorem 2.16. *If* Γ *is Pr-normal but* $\Gamma \cup \{A\}$ *is Pr-abnormal and* $\Pr(A, \Gamma)$ *is defined, then* $\Pr(A, \Gamma) = 0$.

Proof.

$\Pr(F, \Gamma \cup \{A\}) = 1$	$\Gamma \cup \{A\}$ is Pr-abnormal
$\Pr(A, \Gamma) \neq 0$	Hypothesis
$\Pr(F \wedge A, \Gamma) = \Pr(A \wedge F, \Gamma)$	POS.7
$\Pr(A \wedge F, \Gamma) = \Pr(A, \Gamma) \times \Pr(F, \Gamma \cup \{A\}$	POS.6
$\Pr(A \wedge F, \Gamma) = \Pr(A, \Gamma)$	$\Gamma \cup \{A\}$ is Pr-abnormal

But this is impossible because, by POS.11 and DF.1 $\Pr(F \wedge A, \Gamma) = 0$. Thus $\Pr(A, \Gamma) = 0$.

□

Theorem 2.17. *If* $\Pr(A, \Gamma) = 1$ *and* $\Pr(A \supset B, \Gamma) = 0$*, then* $\Pr(B, \Gamma) = 0$.

Proof.

$0 = \Pr(A \supset B, \Gamma)$	Hypothesis
$= \Pr(B, \Gamma \cup \{A\})$	POS.8
$= \Pr(B, \Gamma)$	Theorem 2.11

□

Theorem 2.18. *If* $\Pr(A, \Gamma) \neq 0$ *or* $(\Pr(B, \Gamma) \neq 0)$*, then* $\Pr(A \vee B, \Gamma) \neq 0$.

Proof. We proceed by contraposition.

$0 = \Pr(A \vee B, \Gamma)$	Hypothesis
$= \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A \wedge B, \Gamma)$	POS.5
$= \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A, \Gamma) \times \Pr(B, \Gamma \cup \{A\})$	POS.6
$= \Pr(A, \Gamma) \times (1 - \Pr(B, \Gamma \cup \{A\})) + \Pr(B, \Gamma)$	Algebra
$\Pr(A, \Gamma) \times (1 - \Pr(B, \Gamma \cup \{A\})) = 0$ and $\Pr(B, \Gamma) = 0$	POS.3
$\Pr(A, \Gamma) = 0$	Algebra

□

3 Soundness

Let us recall the definition of validity: $\Gamma \Rightarrow A$ is a valid sequent according to partial probabilistic interpretations iff, for any Pr satisfying DF.1-DF.2 and POS.3-POS.20,

$$\Pr(A, \Gamma \cup \Delta) = 1$$

for all Δ. We write $\Gamma \mid\mid\!- A$.

We have two types of rules:

$$\frac{\Gamma \Rightarrow A}{\Delta \Rightarrow C} \text{ and } \frac{\Gamma \Rightarrow A \qquad \Lambda \Rightarrow B}{\Delta \Rightarrow C}.$$

The former is sound iff $\Gamma \mid\mid\!- A$ implies $\Delta \mid\mid\!- C$.

The latter is sound iff $\Gamma \mid\mid\!- A$ and $\Lambda \mid\mid\!- B$ implies $\Delta \mid\mid\!- C$.

Theorem 3.1. *The sequent calculus SCILSN is sound according to partial probabilistic interpretations.*

We need to verify the validity of the axioms and the soundness of the rules.

Axioms

A1 $\quad A, \Gamma \Rightarrow A$ is valid.

Proof. By POS.4, for any A, Γ and Pr, $\Pr(A, \Gamma \cup \{A\} \cup \Delta) = 1$

□

A2 $F, \Gamma \Rightarrow A$ is valid.

Proof. We show that $\Gamma \cup \{F\} \cup \Delta$ is Pr-abnormal for any Γ, Pr and Δ.

$\Pr(F, \Gamma \cup \{F\} \cup \Delta) = 1$	POS.4
$\Gamma \cup \{F\} \cup \Delta$ is Pr-abnormal	POS.11
$\Pr(A, \Gamma \cup \{F\} \cup \Delta = 1$	Theorem 2.14
$F, \Gamma \Rightarrow A$ is valid	Definition of validity.

□

A3 $\Gamma \Rightarrow {\sim}F$ is valid.

Proof. If Γ is Pr-abnormal, il is trivial. If not

$\Pr({\sim}F, \Gamma \cup \Delta) = (1 - \Pr(F, \Gamma \cup \Delta))$	POS.9
$\Pr(F, \Gamma \cup \Delta) = 0$	POS.11
$\Pr({\sim}F, \Gamma \cup \Delta) = 1$	Algebra
$\Gamma \Rightarrow {\sim}F$ is valid	Definition of validity.

□

Logical rules

$$\frac{\Gamma \Rightarrow A}{{\sim}A, \Gamma \Rightarrow C}\ L{\sim} \text{ is sound.}$$

Proof. If Γ is Pr-abnormal, then by the Theorem 2.14 $\Gamma \cup \{{\sim}A\}$ is Pr-abnormal and we are done. Else, let us suppose that Γ is Pr-normal. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta \cup \{{\sim}A\}) = 1$	with $\Delta' = \Delta \cup \{{\sim}A\}$
$\Pr({\sim}A, \Gamma \cup \Delta \cup \{{\sim}A\}) = 1$	POS.4
$\Pr(({\sim}A \wedge A, \Gamma \cup \Delta \cup \{{\sim}A\}) =$ $\Pr((A \wedge {\sim}A, \Gamma \cup \Delta \cup \{{\sim}A\})$	POS.7

$\Pr((\sim A \wedge A, \Gamma \cup \Delta \cup \{\sim A\}) =$	
$\quad \Pr((\sim A, \Gamma \cup \Delta \cup \{\sim A\})$	
$\quad \times \Pr((A, \Gamma \cup \Delta \cup \{\sim A\} \cup \{\sim A\})$	POS.6
$\Pr((A \wedge \sim A, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	Algebra
$\Gamma \cup \Delta \cup \{\sim A\}$ is Pr-abnormal	Theorem 2.15
$\Gamma \cup \{\sim A\} \cup \{C\}$ is Pr-abnormal	Theorem 2.14
$\Pr(C, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	Definition of abnormality
$\sim A, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{A, \Gamma \Rightarrow C}{\sim\sim A, \Gamma \Rightarrow C}\ L{\sim\sim}$$

Proof. For any Δ,

$A, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim\sim A\} \cup \Delta) = 1$	POS.9 (3) and Theorem 2.5
$\sim\sim A, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow\ \sim\sim A}\ R{\sim\sim}$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim\sim A, \Gamma \cup \Delta) = 1$	POS.9 (3)
$\Gamma \Rightarrow\ \sim\sim A$	Definition of validity

□

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge$$

Proof. For any Δ,

$A, B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A, B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{A \wedge B\} \cup \Delta) = 1$	POS.10
$A \wedge B, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} R\wedge$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Gamma \Rightarrow B$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(B, \Gamma \cup \Delta \cup \{A\}) = 1$	Theorem 2.11
$\Pr(A \wedge B, \Gamma \cup \Delta) =$ $\Pr(A, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{A\})$	POS.6
$\Pr(A \wedge B, \Gamma \cup \Delta) = 1 \times 1 = 1$	Algebra
$\Gamma \Rightarrow A \wedge B$	Definition of validity

□

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} L\vee$$

Proof. For any Δ,

$A, \Gamma \Rightarrow C$	Hypothesis
$B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{A \vee B\} \cup \Delta) = 1$	POS.13
$A \vee B, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} R\vee_1$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A \vee B, \Gamma \cup \Delta) = 1$	DF.2
$\Gamma \Rightarrow A \vee B$	Definition of validity

□

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} R\vee_2$$

Proof. Trivial by the soundness of $R\vee_1$ and Theorem 2.4

□

$$\frac{\sim A, \Gamma \Rightarrow C \quad \sim B, \Gamma \Rightarrow C}{\sim (A \wedge B), \Gamma \Rightarrow C} \; L{\sim}\wedge$$

Proof. For any Δ,

$\sim A, \Gamma \Rightarrow C$	Hypothesis
$\sim B, \Gamma \Rightarrow C$	
	Hypothesis
$\Pr(C, \Gamma \cup \{\sim A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim (A \wedge B)\} \cup \Delta) = 1$	POS.15
$\sim (A \wedge B), \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow \; \sim A}{\Gamma \Rightarrow \sim (A \wedge B)} \; R{\sim}\wedge_1$$

Proof. For any Δ,

$\Gamma \Rightarrow \; \sim A$	Hypothesis
$\Pr(\sim A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(\sim (A \wedge B), \Gamma \cup \Delta) =$	
$1 - \Pr((A \wedge B), \Gamma \cup \Delta)$	POS.9 (1)
$\Pr((A \wedge B), \Gamma \cup \Delta) =$	
$\Pr(A, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{A\})$	POS.6
$\Pr((A \wedge B), \Gamma \cup \Delta) = 0$	Algebra
$\Pr(\sim (A \wedge B), \Gamma \cup \Delta) = 1 - 0 = 1$	Algebra
$\Gamma \Rightarrow \sim (A \wedge B)$	Definition of validity

□

$$\frac{\Gamma \Rightarrow \; \sim A}{\Gamma \Rightarrow \sim (A \wedge B)} \; R{\sim}\wedge_2$$

Proof. Trivial from the soundness of $R{\sim}\wedge_1$ and POS.7.

□

$$\frac{\sim A, \sim B, \Gamma \Rightarrow C}{\sim (A \vee B), \Gamma \Rightarrow C} \; L{\sim}\vee$$

Proof. For any Δ

$\sim A, \sim B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \Delta \cup \{\sim A, \sim B\}) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{\sim(A \vee B)\}) = 1$	POS.14
$\sim(A \vee B), \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow\ \sim A \quad \Gamma \Rightarrow\ \sim B}{\Gamma \Rightarrow\ \sim(A \vee B)}\ R{\sim}\vee$$

Proof. For any Δ,

$\Gamma \Rightarrow\ \sim A$	Hypothesis
$\Gamma \Rightarrow\ \sim B$	Hypothesis
$\Pr(\sim A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(B, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(A \vee B, \Gamma \cup \Delta) = 0$	POS.5, POS.6 and algebra
$\Pr(\sim(A \vee B), \Gamma \cup \Delta) = 1$	POS.9 (1)
$\Gamma \Rightarrow\ \sim(A \vee B)$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A \quad B, \Gamma \Rightarrow\ C}{A \supset B, \Gamma \Rightarrow\ C}\ L \supset$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$B, \Gamma \Rightarrow\ C$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{B\}) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{B\}) =$ $\Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$	POS.12
$\Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\}) = 1$	Algebra
$A \supset B, \Gamma \Rightarrow\ C$	Definition of validity

□

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R \supset$$

Proof.

$A, \Gamma \Rightarrow B$	Hypothesis
$\Pr(B, \Gamma \cup \Delta \cup \{A\}) = 1$	Definition of validity
$\Pr(A \supset B, \Gamma \cup \Delta) = 1$	POS.8
$\Gamma \Rightarrow A \supset B$	Definition of validity

□

$$\frac{A, \sim B, \Gamma \Rightarrow C}{\sim(A \supset B), \Gamma \Rightarrow C} \; L{\sim}{\supset}$$

Proof. For any Δ,

$A, \sim B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \Delta \cup \{A, \sim B\} = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{A \wedge \sim B\} = 1$	POS.10
$\Pr(C, \Gamma \cup \Delta \cup \{\sim(A \supset B)\} = 1$	Theorem 2.6
$\sim(A \supset B), \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \supset B)} \; R{\sim}{\supset}$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Gamma \Rightarrow \sim B$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim(A \supset B), \Gamma \cup \Delta) =$	
$\quad \Pr(A, \Gamma \cup \Delta) \times \Pr(\sim B, \Gamma \cup \Delta \cup \{A\})$	POS.9 (2)
$\Pr(\sim B, \Gamma \cup \Delta) = \Pr(\sim B, \Gamma \cup \Delta \cup \{A\}) = 1$	Theorem 2.11
$\Pr(\sim(A \supset B), \Gamma \cup \Delta) = 1$	Algebra
$\Gamma \Rightarrow \; \sim(A \supset B)$	Definition of validity

□

$$\frac{A[t|x], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C} \; L\,\forall$$

Proof. We have to show that for any $\Pr, A, C, t, \Gamma$, if $\Pr(C, \Gamma \cup \{A[t|x]\} \cup \Delta) = 1$ for all Δ, then $\Pr(C, \Gamma \cup \{\forall x A\} \cup \Delta) = 1$ for all Δ.
This corresponds exactly to what POS.18 says.

□

$$\frac{\Gamma \Rightarrow A[y|x]}{\Gamma \Rightarrow \forall x A}\ R\,\forall$$ where y is not free in Γ and y is x or y is not free in A.

Proof. Let $t_1, \ldots, t_n, \ldots$ be an enumeration of all the terms that are free for x in A.

$\Gamma \Rightarrow A[y\|x]$	Hypothesis
$\Pr(A[y\|x], \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A[t_1\|x], \Gamma \cup \Delta) = 1$	POS.20
.	.
.	.
.	.
$\Pr(A[t_n\|x], \Gamma \cup \Delta) = 1$	POS.20
$\Pr(A[t_1\|x] \wedge \ldots \wedge, A[t_1\|x], \Gamma \cup \Delta) = 1$	DF.10 n times, Thm 2.11
$\lim_{n\to\infty} \Pr((A[t_{i_1}\|x] \wedge \ldots \wedge A[t_{i_n}\|x]), \Gamma \cup \Delta) = 1$	Elementary calculus
$\Pr(\forall x A, \Gamma \cup \Delta)) = 1$	POS.16
$\Gamma \Rightarrow \forall x A$	Definition of validity

□

$$\frac{A[y|x], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\,\exists$$ where y is not free in Γ and C and y is x or y is not free in$A$.

Proof. This corresponds exactly to what POS.19 says.

□

$$\frac{\Gamma \Rightarrow A[t|x]}{\Gamma \Rightarrow \exists x A}\ R\,\exists$$

Proof. The proof is quite similar to that of $R\,\forall$ and left to the reader.

□

The proofs for the rules with strong negation of quantifiers are dual of the preceding ones and are left to the reader.

This completes the proof of soundness.

□

4 Completeness

The strategy we use to prove completeness is the following. Following Kripke's idea [6, 1, 5] for designing models for intuitionistic logic, we define a 3-valued model$\{1, 0, u\}$ where the three values are standing respectively for true, false and unknown. Counter to what we have said at the beginning of this paper, in this very particular model structure, u can be considered as a value. We then show that any consistent set of sentences of SCILSN admits a 3-valued model. Using this model, we define partial conditional probability functions taking only the three values and we finally show that these functions satisfy DF.1-DF.2 and POS.3-POS.20. Calling these partial probability functions $partial\, opinated\, functions$, we show that every consistent set defines a partial opinated function, which is stronger than to show that every consistent set defines a partial probability function. Moreover, this model is canonical : if A and Γ are such that $\Gamma \not\Rightarrow A$, then there is a Δ such that $\Pr(A, \Gamma \cup \Delta) \neq 1$.

Definition 4.1. *A Deductively Closed Saturated Set (DCSS) is a set of wffs* Δ *such that*
(i) If $\Delta \Rightarrow A$*, then* $A \in \Delta$ *(closure);*
(ii) If $A \vee B \in \Delta$*, then* $A \in \Delta$ *or* $B \in \Delta$ *(saturation);*
(iii) It is not the case that $\Delta \Rightarrow F$ *(consistency).*

Definition 4.2. *If* Γ *is consistent,* $U(\Gamma) = \{\Delta : \Delta\, is\, a\, DCSS\, and\, \Delta \subseteq \Gamma\}$.

Theorem 4.3. *If* $\Gamma \not\Rightarrow A$*, there is a DCSS* Δ *such that* $\Gamma \subseteq \Delta$ *and* $\Delta \not\Rightarrow A$.

Proof. (This proof is not constructive.)
A is called the $test\ formula$. Let $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all wffs where each wff appears denumerably many times. We define the following sequence of sets:

$\Gamma_0 = \Gamma$;

.

.

.

$\Gamma_{k+1} = \Gamma_k$ if $\Gamma_k \cup \{E_k\} \Rightarrow A$;

$\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ if $\Gamma_k \Rightarrow E_k$, and E_k is not $(B \vee C)$;

if E_k is $(B \vee C)$,

$\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ if $\Gamma_k \cup \{E_k\} \cup \{B\} \not\Rightarrow A$
else $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$.

We define

$$\Delta = \bigcup_{k=0}^{\infty} \Gamma_k$$

Claim

(1) $\Delta \not\Rightarrow A$

We first show that, for any k, $\Gamma_k \not\Rightarrow A$.

For $k = 0$, it is trivial. Let us suppose that $\Gamma_k \not\Rightarrow A$, we show that $\Gamma_{k+1} \not\Rightarrow A$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ because $\Gamma_k \Rightarrow E_k$, and E_k is not $(B \vee C)$, we get the result by the Cut rule.

Let us suppose that E_k is $(B \vee C)$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ because $\Gamma_{k+1} \not\Rightarrow A$, it is trivial;

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$ because $\Gamma_k \cup \{E_k\} \cup \{B\} \Rightarrow A$, we have to show that $\Gamma_k \cup \{E_k\} \cup \{C\} \not\Rightarrow A$.

Let us suppose that $\Gamma_k \cup \{E_k\} \cup \{C\} \Rightarrow A$.

From $L\vee$ we have:

$\Gamma_k \cup \{E_k\} \cup \{(B \vee C)\} \Rightarrow A$. But E_k is $(B \vee C)$, so $\Gamma_k \cup \{E_k\} \Rightarrow A$ which contradicts the hypothesis.

(2) If $\Delta \Rightarrow B$, then $B \in \Delta$ because B is one of the E_k.

(3) Δ is saturated, i.e., if $A \vee B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$. It is a trivial consequence of the definition of Δ.

(4) Δ is consistent. This follows from the fact that $\Delta \not\Rightarrow A$.

□

The most interesting consequence of Theorem 4.3 is that if A is a classical tautology and Γ is such that $\Gamma \not\Rightarrow A$, then there is a DCSS Δ such that $\Gamma \subseteq \Delta$ and $\Delta \not\Rightarrow A$.

Corollary 4.4. *If Γ is consistent, there is a DCSS Δ such that $\Gamma \subseteq \Delta$.*

Proof. As Γ is consistent, there is a A such that $\Gamma \not\Rightarrow A$. We define a *DCSS* Δ starting from Γ using A as the test formula. □

Corollary 4.5. *Let Γ be a consistent set. If there is no DCSS Δ such that $\Gamma \subseteq \Delta$ and $A \in \Delta$, then $\Gamma \cup \{A\}$ is inconsistent.*

Proof. This is a trivial consequence of Corollary 4.4. □

Corollary 4.6. *If $A \in \Delta$ for any DCSS Δ such that $\Gamma \subseteq \Delta$, then $\Gamma \Rightarrow A$.*

Proof. The above is the contraposition of Theorem 4.3. □

Theorem 4.7. *Let W be the set of all DCSS and $\Delta \in W$. If $A \supset B \in \Delta$ and $\Delta' \in W$ with $\Delta \subseteq \Delta'$ and $A \in \Delta'$, then $B \in \Delta'$.*

Proof. Let $\Delta, \Delta' \in W$, $A \supset B \in \Delta$, $\Delta \subseteq \Delta'$ and $A \in \Delta'$. We have $A \supset B \in \Delta'$ and by

$$\frac{\Delta' \Rightarrow A \qquad B, \Delta' \Rightarrow B}{A \supset B, \Delta' \Rightarrow B} L\supset$$

and by closure $B \in \Delta'$. □

Theorem 4.8. *Let Γ be a consistent set of wffs such that $\Gamma \not\Rightarrow A \supset B$. Then there is a DCSS Δ such that $\Gamma \subseteq \Delta$, $A \in \Delta$ and $B \notin \Delta$.*

Proof. We have $A, \Gamma \not\Rightarrow B$, Otherwise, by $R\supset$, $\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}$ which contradicts the hypothesis. We then start again the Theorem 4.3 using $\Gamma_0 = \Gamma \cup \{A\}$ and B as test formula. □

Definition 4.9. *Let* Γ *be a consistent set of wffs.* $U(\Gamma) = \{\Delta : \Gamma \subseteq \Delta$ *and* Δ *is a* $DCSS$.

Theorem 4.10. *Let* $\Delta \in U(\Gamma)$. *If* $B \in \Delta$, *then* $\Delta \in U(\Gamma \cup \{B\})$.

Proof. (This proof is not constructive)
If $B \in \Gamma$, the proof is trivial. Let us suppose it is not the case that $B \in \Gamma$. Let $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all the wffs of Δ where every $A \in \Delta$ appears denumerably many times. Let us consider the following two sequences:

$$\begin{array}{ll} \Delta_0 = \Gamma & \Delta'_0 = \Gamma \cup \{B\} \\ \vdots & \vdots \\ \Delta_{i+1} = \Delta_i \cup \{E_i\} & \Delta_{i+1} = \Delta'_i \cup \{E_i\} \\ \vdots & \vdots \end{array}$$

It is clear that $\Delta = \bigcup_{k=0}^{\infty} \Delta_k$.

It is also clear that, for any i, $\Delta_i \subseteq \Delta'_i$.
Let k be the smallest integer such that E_k is B. We then have $\Delta_{k+1} = \Delta'_{k+1}$ and for any $k' \geq k+1$, $\Delta_{k'} = \Delta'_{k'}$.

$$\Delta' = \bigcup_{k=0}^{\infty} \Delta'_k = \Delta. \text{ But } \Delta' \in U(\Gamma \cup \{B\}).$$

□

Corollary 4.11. *If* $A \in \Delta$ *for any* $\Delta \in U(\Gamma)$, *then* $U(\Gamma) = U(\Gamma \cup \{A\})$.

Proof. It is a trivial consequence of Theorem 4.10. □

Theorem 4.12. *Let* Δ *be a DCSS such that* A, $A \supset B \in \Delta$. *Then* $B \in \Delta$.

Proof. The above is a trivial consequence of Theorem 4.7. □

Theorem 4.13. *For any* Γ *and any* $A_1, \ldots, A_n$, $(A_1 \wedge \ldots \wedge A_n), \Gamma \Rightarrow C$ *iff* $A_1, \ldots, A_n, \Gamma \Rightarrow C$

Proof. We just give a sketch of the proof which is quite trivial.
$\rightarrow$

We have:

$$\frac{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \qquad A_1, \ldots, A_n, \Gamma \Rightarrow A_2}{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \wedge A_2,} R\wedge$$

$$\frac{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \wedge A_2 \qquad A_1, \ldots, A_n, \Gamma \Rightarrow A_3}{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \wedge A_2 \wedge A_3} R\wedge$$

After $(n-1)$ steps, we get:

$$\frac{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1} \qquad A_1, \ldots, A_n, \Gamma \Rightarrow A_n}{A_1, \ldots, A_n, \Gamma \Rightarrow A_1 \wedge \ldots \wedge A_n} R\wedge$$

By the Cut rule, if $(A_1 \wedge \ldots \wedge A_n), \Gamma \Rightarrow C$, then $A_1, \ldots, A_n, \Gamma \Rightarrow C$.

$\leftarrow$

Let us suppose that $A_1, \ldots, A_n, \Gamma \Rightarrow C$. We have

$$\frac{A_1, \ldots, A_{n-2}, A_{n-1}, A_n, \Gamma \Rightarrow C}{A_1, \ldots, A_{n-2}, (A_{n-1} \wedge A_n), \Gamma \Rightarrow C} L\wedge$$

Applying $L\wedge$ $(n-1)$ times, we get

$$\frac{A_1, \ldots, A_n, \Gamma \Rightarrow C}{(A_1 \wedge \ldots \wedge A_n), \Gamma \Rightarrow C} L\wedge$$

$\square$

Theorem 4.14. *For any Γ and any $A_1, \ldots, A_n$, $(A_1 \vee \ldots \vee A_n), \Gamma \Rightarrow C$ iff for any i, $1 \leq i \leq n$, $A_i, \Gamma \Rightarrow C$*

The proof is quite elementary and is left to the reader.

Theorem 4.15. $\dfrac{(\sim A_1 \wedge \ldots \wedge \sim A_n), \Gamma \Rightarrow C}{(\sim(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow C}$ *and* $\dfrac{(\sim(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow C}{(\sim A_1 \wedge \ldots \wedge \sim A_n), \Gamma \Rightarrow C}$

We merely give a sketch of the proof. By Theorem 4.13 we have, if $A_1, \ldots, A_n, \Gamma \Rightarrow C$ then $(A_1 \wedge \ldots \wedge A_n), \Gamma \Rightarrow C$.
We show that $\sim A_1, \ldots, \sim A_n, \Gamma \Rightarrow (\sim(A_1 \vee \ldots \vee A_n))$ and the result follows by

the Cut rule.
By $R{\sim}\vee$, we have ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow ({\sim}(A_1 \vee A_2))$. Applying $R{\sim}\vee$ $(n-1)$ times, we get the result we are looking for.

For the converse, we have to show that $({\sim}(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n)$. In order to do this, we proceed in two steps. We first show that if ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow C$ then $({\sim}(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow C$ using $L{\sim}\vee$ $(n-1)$ times. Then we use Theorem 4.13 which implies that ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n)$.

□

Theorem 4.16. $\dfrac{({\sim}A_1 \vee \ldots \vee {\sim}A_n), \Gamma \Rightarrow C}{({\sim}(A_1 \wedge \ldots \wedge A_n)), \Gamma \Rightarrow C}$ *and* $\dfrac{({\sim}(A_1 \wedge \ldots \wedge A_n)), \Gamma \Rightarrow C}{({\sim}A_1 \vee \ldots \vee {\sim}A_n), \Gamma \Rightarrow C}$

The proof is left to the reader.

Theorem 4.17. *If* $U(\Gamma) = U(\Gamma')$, *then* $U(\Gamma \cup \{A\}) = U(\Gamma' \cup \{A\})$.

Proof. Let $\Delta \in U(\Gamma$ and $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all the wffs where every wff appears denumerably many times.
We define the following sequence $\Lambda_0,\ \Lambda_1,\ \Lambda_2,\ \ldots\ \Lambda_n,\ \ldots$:

$$\Lambda_0 = \Delta \cup \{A\};$$
$$\vdots$$
$$\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \text{ if } \Lambda_n \Rightarrow E_n \text{ and } E_n \text{ is not } (B \vee C);$$
$$\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \cup \{B\} \text{ if } \Lambda_n \Rightarrow E_n \text{ and } E_n \text{ is } (B \vee C) \text{ and } \Lambda_n \cup \{E_n\} \cup \{B\} \text{ is consistent;}$$
$$\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \cup \{C\} \text{ otherwise;}$$
$$\vdots$$

Let $\Lambda_\Delta = \bigcup_{n=0}^{\infty} \Lambda_n$

Claim: $\Lambda \in U(\Gamma \cup \{A\})$.

It is clear that Λ_Δ is a DCSS and that $\Gamma \cup \{A\} \in \Lambda_\Delta$. As $U(\Gamma) = U(\Gamma')$, a similar argument leads us to conclude that $\Gamma' \cup \{A\} \in \Lambda_\Delta$.

In order to conclude that $U(\Gamma \cup \{A\}) = U(\Gamma' \cup \{A\})$, we have to show that, for any $\Lambda \in U(\Gamma \cup \{A\})$, there is a $\Delta' \in U(\Gamma)$ such that $\Lambda = \Lambda_{\Delta'}$.
Let $\Lambda \in U(\Gamma \cup \{A\})$ and $E' =< E'_0, E'_1, E'_2, \ldots >$ be an enumeration of the wffs of Λ where every wff appears denumerably many times. We define the following sequence of sets:

$$\Lambda'_0 = \emptyset \text{ if } E'_0 \Rightarrow A,\ \Lambda'_0 = \{E'_0\} \text{ otherwise;}$$

$$\vdots$$

$$\Lambda'_{n+1} = \Lambda'_n \text{ if } \Lambda'_n \cup \{E'_n\} \Rightarrow A,\ \Lambda'_n \cup \{E'_n\} \text{ otherwise;}$$

$$\vdots$$

Let $\Lambda' = \bigcup_{n=0}^{\infty} \Lambda'_n$

Claims:
(1) $\Lambda' \in U(\Gamma)$
(2) If $\Lambda' \Rightarrow B$ then $B \in \Lambda'$
(3) If $(B \vee C) \in \Lambda'$, then $B \in \Lambda'$ or $C \in \Lambda'$

(1) $\Lambda' \in U(\Gamma)$	Trivial
(2) If $\Lambda' \Rightarrow B$ then $B \in \Lambda'$	
$\Lambda' \Rightarrow B$	Assumption
There is a Λ'_n such that $\Lambda'_n \Rightarrow B$	A proof in Λ' is finite.
There is a m such that B is E'_{n+m}	Definition of E'
$B \in \Lambda'_{n+m+1}$	Definition of Λ'_{n+m+1}
$B \in \Lambda'$	Definition of Λ'
(3) $(B \vee C) \in \Lambda'$	Assumption
$B \notin \Lambda'$ and $C \notin \Lambda'$	Assumption
$\Lambda' \cup \{B\} \Rightarrow A$ and $\Lambda' \cup \{C\} \Rightarrow A$	Definition of Λ'
$\Lambda' \cup \{(B \vee C)\} \Rightarrow A$	$L \vee$
$A \in \Lambda'$	Contradiction

□

Definition 4.18. *The pair $< W, \subseteq >$ is called the Kripkean canonical frame.*

Definition 4.19. *Let $A*$ be B or $\sim B$ according to whether A is $\sim B$ or B.*

Definition 4.20. *The canonical partial probabilistic model is the 3-uple $< W, \subseteq, \mathrm{Pr}_{<W,\subseteq>} >$ where $\mathrm{Pr}_{<W,\subseteq>} : L \times 2^L \to \{0, 1, u\}$ is such that, for any A, Γ*

(i) If A is a literal,

$$\mathrm{Pr}_{<W,\subseteq>}(A, \Gamma) = \begin{cases} 1 \text{ if } A \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ 0 \text{ if } A* \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ u \text{ otherwise} \end{cases}$$

(ii) If $\sim\sim B$,

$$\mathrm{Pr}_{<W,\subseteq>}(A, \Gamma) = \begin{cases} 1 \text{ if } B \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ 0 \text{ if } \sim B \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ u \text{ otherwise} \end{cases}$$

(iii) If A is $B \wedge C$,

$$\mathrm{Pr}_{<W,\subseteq>}(A, \Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ and } C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ or } \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(iv) If A is $\sim(B \wedge C)$,

$$\mathrm{Pr}_{<W,\subseteq>}(A, \Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in \mathrm{U}(\Gamma), \sim B \in \Delta \text{ or } \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in \mathrm{U}(\Gamma), B \in \Delta \text{ and } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(v) If A is $B \vee C$,

$$\mathrm{Pr}_{<W,\subseteq>}(A, \Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ or } C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ and } \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(vi) If A is $\sim(B \vee C)$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ and} \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ or } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(vii) If A is $B \supset C$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma) \text{ such that } B \in \Delta, C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma), B \in \Delta \text{ and} \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(viii) If A is $\sim(B \supset C)$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ and} \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ or } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(ix) If A is $\forall xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B[y|x] \in \Delta \text{ for any } y \text{ not free in } \forall xB \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ u \text{ otherwise} \end{cases}$$

(x) If A is $\sim\forall xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ \sim B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B[y|x] \in \Delta \text{ for any } t \text{ free in } \forall xB \\ u \text{ otherwise} \end{cases}$$

(xi) If A is $\exists xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ {\sim}B[y|x] \in \Delta \text{ for any } y \text{ not free in } \exists xB \\ u \text{ otherwise} \end{cases}$$

(xii) If A is ${\sim}\exists xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ {\sim}B[y|x] \in \Delta \text{ for any } y \text{ free } x \text{ in } {\sim}\exists xB \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ u \text{ otherwise} \end{cases}$$

Theorem 4.21. *(We drop the index.) For any* Pr*,* A *and* Γ*, if, for any* $\Delta \in U(\Gamma), A \in \Delta$*, then* $\Pr(A,\Gamma) = 1$

Proof.
(i) A is a literal. It is trivial by definition 4.20(i).

(ii) A is ${\sim}{\sim}B$. By definition 4.20(ii), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ for any $\Delta \in U(\Gamma)$. But by $R{\sim}{\sim}$, ${\sim}{\sim}B \in \Delta$.

(iii) A is $B \wedge C$. By definition 4.20(iii), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ and $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$.

(iv) A is ${\sim}(B \wedge C)$. By definition 4.20(iv), $\Pr(A,\Gamma) = 1$ if ${\sim}B \in \Delta$ or ${\sim}C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\wedge_1$ or $R{\sim}\wedge_2$, ${\sim}(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(v) A is $B \vee C$. By definition 4.20(v), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ or $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\vee_1$ or $R\vee_2$, $B \vee C \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vi) A is ${\sim}(B \vee C)$. By definition 4.20(vi), $\Pr(A,\Gamma) = 1$ if ${\sim}B \in \Delta$ and ${\sim}C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\vee$, ${\sim}(B \vee C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vii) A is $B \supset C$. By definition 4.20(vii), $\Pr(A,\Gamma) = 1$ if for any $\Delta \in U(\Gamma)$

such that if $B \in \Delta$, then $C \in \Delta$. In that case, by $R \supset$, for any Δ such that $B \in \Delta$, $B \supset C \in \Delta$.

(viii) A is $\sim(B \supset C)$. By definition 4.20(viii), $\Pr(A, \Gamma) = 1$ if $B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}{\supset}$, $\sim(B \supset C) \in \Delta$ for any $\Delta \in U(\Gamma)$. □

Theorem 4.22. *For any* Pr*,* A *and* Γ*, if, for any* $\Delta \in U(\Gamma), \sim A \in \Delta$*, then* $\Pr(A, \Gamma) = 0$*.*

Proof.
(i) A is a literal. If A is p, then A^* is $\sim p \in \Delta$. If A is $\sim p$, A^* is p and by $R{\sim}{\sim}$, $\sim\sim p \in \Delta$ i.e. $\sim A \in \Delta$.

(ii) A is $\sim\sim B$. By definition 4.20(ii), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ for any $\Delta \in U(\Gamma)$. By $R{\sim}{\sim}$, $\sim\sim\sim B \in \Delta$ for any $\Delta \in U(\Gamma)$ i.e. $\sim A \in \Delta$.

(iii) A is $B \wedge C$. By definition 4.20(iii), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ or $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. By $R{\sim}{\wedge}_1$ or $R{\sim}{\wedge}_2$, $\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(iv) A is $\sim(B \wedge C)$. By definition 4.20(iv), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ and $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$ and by $R{\sim}{\sim}$, $\sim\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(v) A is $B \vee C$. By definition 4.20(v), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}{\vee}$, $\sim(B \vee C)$.

(vi) A is $\sim(B \vee C)$. By definition 4.20(vi), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ or $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\vee_1$ or $R\vee_2$, $B \vee C \in \Delta$ for any $\Delta \in U(\Gamma)$ and by $R{\sim}{\sim}$, $\sim\sim(B \vee C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vii) A is $B \supset C$. By definition 4.20(vii), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}{\supset}$, $\sim(B \supset C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

□

Theorem 4.23. *For any* Pr*,*A *and* Γ*,* $\Pr(A, \Gamma) = 1$ *iff for any* $\Delta \in U(\Gamma), A \in \Delta$ *and* $\Pr(A, \Gamma) = 0$ *iff or any* $\Delta \in U(\Gamma), \sim A \in \Delta$*.*

Proof. The ifs come from Theorem 4.21 and Theorem 4.22. The only ifs come from the u otherwise clause of definition 4.20. □

Theorem 4.24. *In the canonical model, for any* Γ *and any* Pr, Γ *is consistent iff* Γ *is Pr-normal.*

Proof.
(1) If Γ is consistent, then $\Gamma \not\Rightarrow F$. Furthermore, $F \notin \Delta$ for any $\Delta \in U(\Gamma)$ and by Theorem 4.23, $\Gamma \Rightarrow \sim F$ and thus $\Pr(F, \Gamma) = 0$. So, Γ is Pr-normal.

(2) If Γ is inconsistent then $\Gamma \Rightarrow F$. By axiom 2, for any C, $F, \Gamma \Rightarrow C$ and by the Cut rule, $\Gamma \Rightarrow C$. By Theorem 4.21, $\Pr(C, \Gamma) = 1$ and Γ is Pr-abnormal. □

So, with respect to the canonical model, the two expressions are equivalent.

Theorem 4.25. *The canonical model gives to the connectives* $\wedge$ *and* $\vee$ *the value of the Kleene strong connectives in the following sense:*

$$\Pr_{<W,\subseteq>}(A \wedge B, \Gamma) = \begin{cases} 1 \text{ iff } \Pr_{<W,\subseteq>}(A,\Gamma) = \Pr_{<W,\subseteq>}(B,\Gamma) = 1 \\ 0 \text{ iff } \Pr_{<W,\subseteq>}(A,\Gamma) = 0 \text{ or } \Pr_{<W,\subseteq>}(B,\Gamma) = 0 \\ u \text{ otherwise} \end{cases}$$

and

$$\Pr_{<W,\subseteq>}(A \vee B, \Gamma) = \begin{cases} 1 \text{ iff } \Pr_{<W,\subseteq>}(A,\Gamma) = 1 \text{ or } \Pr_{<W,\subseteq>}(B,\Gamma) = 1 \\ 0 \text{ iff } \Pr_{<W,\subseteq>}(A,\Gamma) = \Pr_{<W,\subseteq>}(,\Gamma) = 0 \\ u \text{ otherwise} \end{cases}$$

Proof. The proof is straightforward using Theorem 4.21 and Theorem 4.22 and is left to the reader. □

Theorem 4.26. *If* $\Pr(A, \Gamma) = u$ *and* $\Pr(A, \Gamma \cup \{B\}) = 0$, *then* $\Gamma \cup \{A, B\})$ *is inconsistent.*

Proof. Let us suppose that $\Gamma \cup \{A, B\})$ is consistent. In that case $U(\Gamma \cup \{A, B\})$ is not empty. So there is a $\Delta \in U(\Gamma)$ which contains A and B and thus $\Pr(A, \Gamma \cup \{B\}) \neq 0$. □

We need to make sure of one last thing: Does $\Pr_{<W,\subseteq>}$ define a partial conditional probability function?

Theorem 4.27. $\Pr_{<W,\subseteq>}$ *satisfies DF.1-DF.2 and POS.3-POS.20.*

Proof. (We drop the index.)

Let us begin with
POS. 7 $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$

We proceed by induction on the number of $\wedge$.

$n = 1$
We have to prove that, when both are defined, $\Pr(A_1 \wedge A_2, \Gamma) = \Pr(A_2 \wedge A_1, \Gamma)$.
We have two cases:
(1) $\Pr(A_1 \wedge A_2, \Gamma) = 1$ and
(2) $\Pr(A_1 \wedge A_2, \Gamma) = 0$

(1)

$\Pr(A_1 \wedge A_2, \Gamma) = 1$	Assumption
$(A_1 \wedge A_2) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A_1, A_2 \in \Delta$ for any $\Delta \in U(\Gamma)$	Definition 4.20(iii)
$(A_2 \wedge A_1) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R\wedge$ and closure
$\Pr(A_2 \wedge A_1, \Gamma) = 1$	Theorem 4.23

(2)

$\Pr(A_1 \wedge A_2, \Gamma) = 0$	Assumption
$({\sim}(A_1 \wedge A_2)) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A_1 \in \Delta or A_2 \in \Delta$ for any $\Delta \in U(\Gamma)$	Definition 4.20(iv)
$(A_2 \wedge A_1) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R{\sim}\wedge_1$ or $R{\sim}\wedge_2$ and closure
$\Pr(A_2 \wedge A_1, \Gamma) = 0$	Theorem 4.23

Let us suppose it is the case for $n - 1$ conjuncts. We have

$\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1 \wedge (\bigwedge_{i=2}^{n} A_i, \Gamma)$	Definition of $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma)$
$= \Pr(A_{per_n(1)} \wedge (\bigwedge_{i=2}^{n} A_{per_n(i)}, \Gamma)$	Induction hypothesis
$= \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$	Algebra

DF. 1 If $\Pr(A_j, \Gamma) = 0$ for some $1 \leq j \leq n$, then $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$;

Let per_n be a permutation such that $per_n(1) = j$

$\Pr(A_j, \Gamma) = 0$	Assumption
$\sim A_j \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\sim(A_j \wedge (\bigwedge_{i=2}^{n} A_i, \Gamma)) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R{\sim}_1$ + closure
$\sim(A_{per_n(1)} \wedge (\bigwedge_{i=2}^{n} A_{per_n(i)})) \in \Delta$ for any $\Delta \in U(\Gamma)$	$A_{per_n(1)} = A_j$
$\Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}) = 0$	Theorem 4.23
$\Pr(\bigwedge_{i=1}^{n} A_i) = 0$	Pr satisfies DF. 7

DF. 2 If $\Pr(A_j, \Gamma) = 1$ for some $1 \leq j \leq n$, then $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = 1$.

The proof is quite similar to the preceding one.

POS.3 $0 \leq \Pr(A, \Gamma) \leq 1$.
Trivial.

POS.4 If $A \in \Gamma$, then $\Pr(A, \Gamma) = 1$.
Trivial.

POS.6 $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\})$.

We have two cases.

(1) At least one of the A_j is such that $\Pr(A_j, \Gamma) = 0$. By the adequation of DF. 1, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$.

In that case, either $j = 1$ or $j \neq 1$. In both cases, $\Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = 0$ because either $\Pr(A_1, \Gamma) = 0$ or, by the adequation of DF. 1 again, $\Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = 0$.

(2) All of the A_j's are such that $\Pr(A_j, \Gamma) = 1$.

In that subcase, by the definition of 4.20 (iii) for all j, $A_j \in \Delta$ for all $\Delta \in U(\Gamma)$ and applying $R\wedge$ $n-1$ times and by the closure, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 1$,

$\Pr(A_j, \Gamma) = 1$ and $\Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma) = 1$ (because $\Gamma = \Gamma \cup \{A_1\}$). We get $1 = 1 \times 1$ and we are done.

POS.5 $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$.

We merely have to verify all the possibilities when all the probabilities are defined. When $\Pr(A_1, \Gamma)$ and $\Pr(\bigvee_{i=2}^{n} A_i, \Gamma)$ are defined, $\Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$ and $Pr(\bigvee_{i=1}^{n} A_i, \Gamma)$ are also defined and by definition 4.20 (iii) and (v), we have the following table:

$\Pr(A_1, \Gamma)$	$\Pr(\bigvee_{i=2}^{n} A_i, \Gamma)$	$\Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$	$\Pr(\bigvee_{i=1}^{n} A_i, \Gamma)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

One can easily see that $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$.

POS.9 If Γ is Pr-normal, then $\Pr({\sim}A, \Gamma) =$
(1) $1 - \Pr(A, \Gamma)$ if A is an atom or F or $(B \wedge C)$ or $(B \vee C)$ or $\forall x B$ or $\exists x B$;
(2) $\Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\})$ if A is $(B \supset C)$;
(3) $\Pr(B, \Gamma)$ if A is ${\sim}B$.

(1) If A is an atom or F or $(B \wedge C)$ or $(B \vee C)$ or $\forall x B$ or $\exists x B$.

(i) $\Pr({\sim}p, \Gamma) = 1$

$\Pr({\sim}p, \Gamma) = 1$	Assumption
iff ${\sim}p \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
iff $\Pr(p, \Gamma) = 0$	Theorem 4.20
iff $\Pr({\sim}p, \Gamma) = 1 - \Pr(p, \Gamma)$	Algebra

(ii) $\Pr(\sim p, \Gamma) = 0$

$\Pr(\sim p, \Gamma) = 0$	Assumption
iff $\sim\sim p \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
iff $p \in \Delta$ for any $\Delta \in U(\Gamma)$	$L\sim\sim$ and $R\sim\sim$
iff $\Pr(p, \Gamma) = 1$	Theorem 4.20
iff $\Pr(\sim p, \Gamma) = 1 - \Pr(p, \Gamma)$	Algebra

(iii) If A is F
$\Pr(\sim F, \Gamma) = 1$ iff $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$ which is the case by **A3.**

But $\Pr(F, \Gamma) = 0$ iff $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$ which is the case by **A3.**
$\Pr(\sim F, \Gamma) = 1 - \Pr(F, \Gamma) = 1$ by algebra.

(iv)$\Pr(\sim F, \Gamma) = 1$ is not the case if Γ is consistent.

(v) We show that

(α) $\Pr(\sim(B \wedge C), \Gamma) = 1 - \Pr((B \wedge C), \Gamma)$

$\Pr(\sim(B \wedge C), \Gamma) = 1$	Assumption
$\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\Pr((B \wedge C), \Gamma) = 0$	Theorem 4.23
$1 = 1 - 0$	Algebra

$\Pr(\sim(B \wedge C), \Gamma) = 0$	Assumption
$(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\Pr((B \wedge C), \Gamma) = 1$	Theorem 4.23
$0 = 1 - 1$	Algebra

(β) $\Pr(\sim(B \wedge C), \Gamma) = 1 - \Pr((B \wedge C), \Gamma)$

This case is as trivial as (α) and is left to the reader.

Cases (2) and (3) are also trivial.

POS.10 $\Pr(A, \Gamma \cup \{\bigwedge_{i=1}^{n} A_i\}) = \Pr(A, \Gamma \cup \{A_1, \ldots, A_n\})$

It is a straightforward consequence of Theorem 4.13.

POS.8 $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$

We have to show that

(1) $\Pr(A \supset B, \Gamma) = 1$ iff $\Pr(B, \Gamma \cup \{A\}) = 1$

and

(2) $\Pr(A \supset B, \Gamma) = 0$ iff $\Pr(B, \Gamma \cup \{A\}) = 0$.

(1) We have to prove that, if $A \supset B \in \Delta$ for any $\Delta \in U(\Gamma)$ then $B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$

$\Pr(A \supset B, \Gamma) = 1$	Assumption
$A \supset B \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A \supset B, A \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$U(\Gamma \cup \{A\}) \subseteq U(\Gamma)$, $A \in \Gamma \cup \{A\}$
$B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.12
$\Pr(B, \Gamma \cup \{A\}) = 1$	Theorem 4.23

We also have to prove the converse, i.e., if $\Pr(B, \Gamma \cup \{A\}) = 1$, then $\Pr(A \supset B, \Gamma) = 1$.

$\Pr(B, \Gamma \cup \{A\}) = 1$	Assumption
$B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.23
$\Delta \cup \{A\} \Rightarrow B$ for any $\Delta \in U(\Gamma)$	Corollary 4.6
$\Delta \Rightarrow (A \supset B)$ for any $\Delta \in U(\Gamma)$	$R \supset$
$(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma)$	Δ is a $DCSS$
$\Pr(A \supset B, \Gamma) = 1$	Theorem 4.23

(2) We have to prove that
If $\Pr(A \supset B, \Gamma) = 0$, then $\Pr(B, \Gamma \cup \{A\}) = 0$.

$\Pr(A \supset B, \Gamma) = 0$	Assumption
$\sim(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A, \sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	PR.5 (2)
$\Pr(A, \Gamma) = 1$ and $\Pr(B, \Gamma) = 0$	Theorem 4.23
$\Pr(B, \Gamma \cup \{A\}) = 0$	Corollary 4.11

We also have to prove the converse.

$\Pr(B, \Gamma \cup \{A\}) = 0$	Assumption
$\sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.23
$A \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$\Gamma \cup \{A\} \subseteq \Delta$
$A \wedge \sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$R\wedge$
$\sim(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	PR.5 (2)
$\Pr(A \supset B, \Gamma) = 0$	Theorem 4.23

POS.12 If, for any Δ, $\Pr(A, \Gamma \cup \Delta) = 1$,
then for any B and C, $\Pr(C, \Gamma \cup \Delta \cup \{B\}) = \Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$

It is a trivial consequence of $L\supset$.

POS.11 If Γ is Pr-normal, then $\Pr(F, \Gamma) = 0$.

It follows from Ax. 3 that $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$.

POS.13 $\Pr(C, \Gamma \cup \{A_i\}) = 1$ for any i such that $1 \leq i \leq n$, then $\Pr(C, \Gamma \cup \{\bigvee_{i=1}^{n} A_i\}) = 1$.

It is a straightforward consequence of $L\vee$ applies $(n-1)$ times.

POS.15 If $\Pr(C, \Gamma \cup \{\sim A_i\}) = 1$ for any i such that $1 \leq i \leq n$, then $\Pr(C, \Gamma \cup \{\sim(\bigwedge_{i=1}^{n} A_i\})) = 1$.

It is a straightforward consequence of $L{\sim}\wedge$ applies $(n-1)$ times.

POS.14 If $\Pr(C, \Gamma \cup \{\sim A_1, \ldots, \sim A_n\}) = 1$, then $\Pr(C, \Gamma \cup \{\sim(\bigvee_{i=1}^{n} A_i\})) = 1$

It is a straightforward consequence of $L{\sim}\vee$ applies $(n-1)$ times.

POS.16

$\Pr(\forall x A, \Gamma) = \lim_{n \to \infty} \Pr(\bigwedge_{i=1}^{n} A[t_i|x], \Gamma)$ where $t_1, \ldots, t_n, \ldots$ is an enumeration of all the terms free for x in A.

There are two cases.

(1)

$\Pr(\forall x A, \Gamma) = 1$	Assumption
For all $\Delta \in U(\Gamma)$, $\forall x A \in \Delta$	Theorem 4.23

$A[t_i\|x], \Delta \Rightarrow A[t_i\|x]$	axiom A1
$A[t_i\|x] \in \Delta$ for all t_i free for x in A	$L\,\forall$ and closure of Δ
$\Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) \in \Delta$	$R \wedge$ (n-1) times
$\Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 1$	Theorem 4.23
$\lim_{n\to\infty} \Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 1$	calculus

(2)

$\Pr(\forall x A, \Gamma) = 0$	Assumption
For all $\Delta \in U(\Gamma)$, $\sim\forall x A \in \Delta$	Theorem 4.23
$\sim A[y_i\|x], \Delta \Rightarrow \ \sim A[y_i\|x]$	axiom A1
$\sim A[y_i\|x] \in \Delta$ for y_i not free in A, $\sim\forall x A$ and Δ	$R\sim\forall$ and the closure of Δ
$\Pr(A[y_i\|x], \Gamma) = 0$	Theorem 4.23
$\Pr(A[t_j\|x], \Gamma) = 0$	for $t_j = y_i$
$\Pr(\bigwedge_{i=1}^{j} A[t_i\|x], \Gamma) = 0$	validity of DF.1
$\lim_{n\to\infty} \Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 0$	calculus + validity of DF.1

POS.17

$\Pr(\exists x A, \Gamma) = \lim_{n\to\infty} \Pr(\bigvee_{i=1}^{n} A[y_i|x], \Gamma)$ where $y_1, \ldots, y_n, \ldots$ is an enumeration of all the variables that are not free in A and Γ.

The proof is quite similar to that of POS.16 and is left to the reader.

POS.18 If $\Pr(C, \Gamma \cup \{A[t|x]\}) = 1$, then $\Pr(C, \Gamma \cup \{\forall x A\}) = 1$ where t is free for x in A.

We show that $U(\Gamma \cup \{\forall x A\}) \subseteq U(\Gamma \cup \{A[t|x]\})$

$\Delta \in U(\Gamma \cup \{\forall x A\})$	Assumption
$A[t\|x], \Delta \Rightarrow A[t\|x]$	axiom A1
$\forall x A, \Delta \Rightarrow A[t\|x]$	$L\forall$
$A[t\|x] \in \Delta$	closure of Δ
$U(\Gamma \cup \{\forall x A\}) \subseteq U(\Gamma \cup \{A[t\|x]\} \cup \{\forall x A\})$	set theory
$\Pr(C, \Gamma \cup \{\forall x A\}) = \Pr(C, \Gamma \cup \{\forall x A\} \cup \{A[t\|x]\}) = 1$	Theorem 4.10 $+\Pr(C, \Gamma \cup \{A[t\|x]\}) = 1$

POS.19

If $\Pr(C, \Gamma \cup \{A[y|x]\}) = 1$, then $\Pr(C, \Gamma \cup \{\exists x A\}) = 1$ where y is not free in A, Γ and C.

The proof is similar that of POS.18 and is left to the reader.

POS.20 If $\Pr(A[y|x], \Gamma) = 1$ with y not free in Γ nor in A (or $y = x$), then $\Pr(A[t|x], \Gamma) = 1$ where t is free for x in A.

$\Pr(A[y\|x], \Gamma) = 1$	Assumption
$A[y\|x] \in \Delta$ for all $\Delta \in U(\Gamma)$	Theorem 4.23
$\forall x A \in \Delta$ for all $\Delta \in U(\Gamma)$	$R\,\forall$
$A[t\|x], \Gamma \Rightarrow A[t\|x]$	A1
$\forall x A, \Gamma \Rightarrow A[t\|x]$	$L\,\forall$
$A[t\|x] \in \Delta$ for all $\Delta \in U(\Gamma)$	Closure of Δ
$\Pr(A[t\|x], \Gamma) = 1$	Theorem 4.23

□

Theorem 4.28. *SCILSN is complete according to the partial probabilistic interpretation.*

Proof. Let us suppose that $\Gamma \not\Rightarrow A$. There is a $\Delta \in U(\Gamma)$ such that $A \notin \Delta$. Thus $\Pr(A, \Gamma) \neq 1$.

□

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