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On Rings of Fractions of Reduced f-Rings by Non Zero-Divisors

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Abstract

We present a new and elementary proof of a result of Knebusch and Zhang, namely that the ring of fractions of an f-ring, A, by a multiplicative set of non zero-divisors in A is an *f-ring extension* of A. (Corollary 10.13, p. 84 in M. Knebusch, D. Zhang, *Convexity, Valuations and Prüfer Extensions in Real Algebra*, Documenta Math. **10** (2005), 1-109.)

The purpose of these notes is to present an elementary proof of Corollary 10.13, p. 84 in [8] for *reduced* f-rings, namely that the ring of fractions of such ring, A, by a multiplicative set of non zero-divisors in A is an *f-ring extension* of A (cf. Theorem 3.6); in [8] this obtained as a consequence of the more general Theorem 10.9 (p. 80) for the *complete* ring of quotients of A, as define by J. Lambek in [9]. For applications of these results to quadratic form theory, see, among others, Theorem 8.21 and Corollary 8.23 (pp. 90 ff) in [6]. Moreover, we also give elementary proofs that rings of quotients of weakly real closed and real closed rings by multiplicative sets of non zero-divisors are again weakly real closed and real closed, respectively.

Section 1 contains, for the convenience of the reader, some standard and basic facts on Real Algebra and on preorders of rings, while in section 2 we introduce and discuss the basic properties of a certain class of preorders having a cancellation property with respect to non zero-divisors, that will be useful in section 3.

All notational conventions in [6], indicated by FQR, remain in force. In particular, all rings are commutative and unitary, wherein 2 is a unit. If A is a ring and $B \subseteq A$, write $B^{\times}$ for the set of units in B; in particular, $A^{\times}$ is the group of units in A.

1 Preliminaries

We recall some basic facts of Real Algebra that can be found in 6.1 and 7.7 of **FQR**.

1.1 Let A be a ring.

a) A **preorder** on A is a subset T of A, containing the squares (A^2) and closed under addition and multiplication. A preorder T on A is **proper** if $T \neq A$; since $2 \in A^{\times}$, T is proper iff $-1 \notin T$. Clearly, ΣA^2 (sums of squares) is the least preorder on A; A is said to be **semi-real** if $-1 \notin \Sigma A^2$. A **p-ring** is a pair $\langle A, T \rangle$, with T a preorder on A; henceforth, all p-rings shall be assumed to be proper; in particular, they are all semi-real, that is, $-1 \notin \Sigma A^2$.

b) If T is a preorder on A, the set $\mathrm{supp}(T) = T \cap -T$ is called the **support** of T. T is a **partial order** if $\mathrm{supp}\, T = \{0\}$; in this case, $\langle A, T \rangle$ is a **partially ordered ring (po-ring)** and T is a called a **ring-po**.

c) If $\langle A, T \rangle$ is a p-ring, it is sometimes convenient to write $x \geq_T y$ iff $x - y \in T$. Note that for all x, y, $z \in A$

1. $x \geq_T y \;\Rightarrow\; x + z \geq_T y + z$; 2. $x \geq_T y$ and $z \geq_T 0 \;\Rightarrow\; xz \geq_T yz$.

Indeed, we have $(x + z) - (y + z) = x - y \in T$, entailing (1); for (2), the closure of T under multiplication obtains $z(x - y) = xz - zy \in T$, as needed.

The relations (1) and (2) shall be used below, frequently without explicit mention.

d) If $\langle A, T \rangle$, $\langle A', T' \rangle$ are p-rings, a **p-ring morphism** is a ring morphism, $f : A \longrightarrow A'$, such that $f[T] \subseteq T'$; f is an **embedding** if it is injective and for all $a \in A$, $a \in T \;\Leftrightarrow\; f(a) \in T'$ (in this case $\langle A', T' \rangle$ is a *p-ring extension* of $\langle A, T \rangle$). ∎

1.2 An ideal I in a p-ring $\langle A, T \rangle$ is

∗ **T-convex** if for all s, $t \in T$, $s + t \in I \;\Rightarrow\; s, t \in I$;

∗ **T-radical** if for all $a \in A$ and $t \in T$, $a^2 + t \in I \;\Rightarrow\; a \in I$.

A ΣA^2-radical ideal is called **real**.

By Proposition 4.2.5 in [3] an ideal of A is T-radical iff it is T-convex and radical. In particular, a prime ideal is T-radical iff it is T-convex. ∎

Proposition 1.3 a) ([3], Prop. 4.2.7, p. 87) *A preorder T on a ring A is proper iff A has a proper T-convex ideal.*

b) *If T is a preorder of A, any ideal of A, maximal for the property of being T-convex, is prime.* ∎

Proposition 1.4 ([3], Prop. 4.2.6, p. 87) *Given a preorder T of A, every ideal I of A is contained in a smallest T-radical ideal (possibly improper), namely:*

$$\sqrt[T]{I} \quad = \quad \{a \in A : \exists\, m \in \mathbb{N} \text{ and } t \in T \text{ such that } a^{2m} + t \in I\},$$

called the ***T*-radical of *I***, *the intersection of all T-convex prime ideals containing I.* ■

Remark 1.5 With notation as in 1.4:

a) If $a \in A$, write $\sqrt[T]{a}$ for the T-radical of the principal ideal (a). In particular, $\sqrt[T]{0}$ is the T-radical of the zero ideal. By 1.4, an ideal I is T-radical iff $\sqrt[T]{I}$ $= I$.

b) If $T = \Sigma A^2$ and I is an ideal in A, we write $\sqrt[\mathrm{re}]{I}$ for $\sqrt[T]{I}$, the **real radical** of I, equal to the intersection of all real primes of A containing I.

c) A ring A is **reduced** if it has no non-zero nilpotent element The next definition describes the analog of the notion of reduced in the case of preordered rings. ■

Definition 1.6 *A p-ring $\langle A, T \rangle$ is* ***T*-reduced** *if $\sqrt[T]{0} = (0)$. In case $T = \Sigma A^2$, i.e., $\sqrt[re]{0} = (0)$, we say that A is a* **real ring**. *Clearly, a T-reduced ring is reduced and semi-real.*[1]

We recall the following

Lemma 1.7 *If $\langle A, T \rangle$ is a proper p-ring, then:*

a) $\mathrm{supp}(T)$ *is a proper ideal in A.*

b) $\sqrt[T]{\mathrm{supp}(T)} = \sqrt[T]{0}$. *In particular,* $\mathrm{supp}(T)$ *is contained in all T-convex prime ideals of A.*

c) The following conditions are equivalent:

(1) *T is a partial order of A;*

(2) *The zero ideal is T-convex* (1.1).

d) $\langle A, T \rangle$ is T-reduced $\Leftrightarrow$ A is reduced and T is partial order of A. In particular,

(1) *A is real $\Leftrightarrow$ it is reduced and ΣA^2 is a partial order of A.*

(2) *The following are equivalent:*

[1] Since $\sqrt[T]{0} = (0)$ is the intersection of all T-convex prime ideals, the intersection of all primes in A is (0) (i.e., A is reduced); further, A has a proper real prime ideal and so ΣA^2 is a proper preorder of A (by 1.3.(a)). Moreover, our definition of real ring coincides with the usual one, i.e. (0) is a real ideal (cf. 1.5.(a)).

(i) A is Pythagorean (i.e., $A^2 = \Sigma A^2$) and real;

(ii) A is reduced and A^2 is a partial order of A.

e) Let S be a proper multiplicative subset of A and let

$$T_S = \left\{\frac{t}{s^2} \in AS^{-1} : t \in T \text{ and } s \in S\right\}.$$

Then, T_S is a proper preorder of AS^{-1} iff $S \cap \text{supp}(T) = \emptyset$.

In particular, this holds if T is a partial order, i.e., $\text{supp}(T) = \{0\}$.

Proof. a) Clearly, $\text{supp}(T)$ is closed under addition and $1 \in \text{supp}(T)$ iff $-1 \in T$. If $t \in \text{supp}(T)$, i.e., $t, -t \in T$ and $x \in A$, recalling that $2 \in T^{\times}$, we have

$$-xt = \frac{-t(1+x)^2 + t(1-x)^2}{4} \in T;$$

similarly one shows that $xt \in T$, and $\text{supp}(T)$ is indeed an ideal in A.

b) The second assertion in (b) follows from the first assertion and Proposition 1.4. For the former, since taking the T-radical (defined in 1.4) is an increasing operation on ideals, and $(0) \subseteq \text{supp}(T)$, it suffices to verify $\sqrt[T]{\text{supp}(T)} \subseteq \sqrt[T]{0}$. Fix $a \in A$ and suppose there is a positive integer k and $t \in T$ such that $a^{2k} + t \in \text{supp}(T)$; then, for some $s \in T$, we have $a^{2k} + (t + s) = 0$; since $s + t \in T$, we conclude $a \in \sqrt[T]{0}$, as needed.

c) $\underline{(1) \Rightarrow (2)}$: Assume $s + t = 0$, with $s, t \in T$; since T is the positive cone of a ring-po in A, we have $0 \leq s \leq s + t = 0$, which implies $s = t = 0$ and (0) is T-convex.

$\underline{(2) \Rightarrow (1)}$: Since T is a preorder of A, it will be a ring-po if $T \cap -T = \{0\}$. If $x \in T \cap -T$, there is $t \in T$ such that $-x = t$ and so $x + t = 0$. Since (0) is T-convex, we conclude $x = 0$, as needed.

d) If A is T-reduced, then A is reduced (cf. 1.6 and its footnote). Moreover, since $(0) = \sqrt[T]{0}$, it follows immediately from (b) that $\text{supp}(T) = \{0\}$ and T is a partial order of A. Conversely, by the equivalence in (c), (0) is T-convex and reducibility implies it to be T-radical, because $a^2 = 0$ entails $a = 0$. But then Proposition 1.4 yields $\sqrt[T]{0} = (0)$, as needed. The remaining assertions in (d) are now clear. Item (e) is straightforward, recalling that since $2 \in A^{\times}$, a preorder P is proper iff $-1 \notin P$ (e.g., if $-1 = \dfrac{t}{s^2}$, then there is $s' \in S$ so that $s'(-s^2 - t) = 0$, whence $(ss')^2 = -s'^2 t \in \text{supp}\, T \cap S$.) ■

1.8 a) Let A be a unitary ring and let $\mathcal{N}$ be the proper multiplicative subset of **non zero-divisors** in A, i.e.,

$$\mathcal{N} = \{a \in A : \forall\, b \in A, \;\; ab = 0 \;\Rightarrow\; b = 0\}.$$

b) The set $\mathcal{N}$ is saturated (i.e., $xy \in \mathcal{N} \;\Leftrightarrow\; x, y \in \mathcal{N}$); moreover, by Exercise

9, page 44 in [1], $\mathcal{N}$ is the largest multiplicative subset of A for which the canonical ring morphism, $\iota_{\mathcal{N}} : A \longrightarrow Q(A) =: A\mathcal{N}^{-1}$, given by $a \longmapsto \frac{a}{1}$, is injective. Note that for $a, b \in A$ and $x, y \in \mathcal{N}$,

$$ax = 0 \;\;\Leftrightarrow\;\; a = 0 \quad \text{and} \quad \frac{a}{x} = \frac{b}{y} \;\;\Leftrightarrow\;\; ay = bx.$$

The ring $Q(A)$ is the **total ring of fractions of A**.

c) If T is a preorder of A, define

$$Q(T) \;=\; \left\{\frac{t}{x^2} : t \in T \text{ and } x \in \mathcal{N}\right\}.$$

By 1.7.(e), $Q(T)$ is a preorder of $Q(A)$, which is proper iff $\operatorname{supp} T \cap \mathcal{N} = \emptyset$. ■

Remark 1.9 Let A be a ring and let S be a proper multiplicative subset of A ($0 \notin S$ and $1 \in S$). It is well-known that the set

$$\mathcal{P}(S) \;=\; \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a proper prime ideal and } \mathfrak{p} \cap S = \emptyset\}$$

is non-empty.[2] Let $\widehat{S} = \bigcap \{A \setminus \mathfrak{p} : \mathfrak{p} \in \mathcal{P}(S)\}$. Then, $\widehat{S}$ is the smallest saturated (i.e., $x, y \in \widehat{S} \Leftrightarrow xy \in \widehat{S}$) proper multiplicative set containing S. Moreover, for all multiplicative sets M satisfying $S \subseteq M \subseteq \widehat{S}$, the rings of fractions AS^{-1} and AM^{-1} are naturally isomorphic over A, i.e., there is a unique ring isomorphism, f, making the following diagram commutative, where $\iota_\bullet$ are the canonical maps from A to the respective ring of fractions:

$$\begin{tikzcd} A \arrow[r, "\iota_S"] \arrow[dr, "\iota_M"'] & AS^{-1} \arrow[d, "f"] \\ & AM^{-1} \end{tikzcd}$$

Hence, we may assume, whenever necessary, that the multiplicative set under consideration is saturated. ■

2 The Weak Cancellation Property. The Weak Cancellation Closure of a Preorder

A crucial property in what follows is contained in the following

[2] Apply Zorn's Lemma to the partially ordered set $\mathcal{V}$ of ideals that do not meet S; note that $(0) \in \mathcal{V}$. Any maximal element in $\mathcal{V}$ is prime and disjoint from S.

Definition 2.1 *A p-ring* $\langle A, T \rangle$ *has the* **weak cancellation property (wcp)** (*with respect to non zero-divisors*) *if for all* a, $b \in A$,

[canc] $$ab \in T \quad and \quad b \in \mathcal{N} \cap T \quad \Rightarrow \quad a \in T.$$

Remark 2.2 In general, there is no hope of extending the cancellation property beyond non zero-divisors. If $\langle A, T \rangle$ is a *partially* ordered ring and d is a zero-divisor in $\mathcal{N} \cap T$, there is $a \in A \setminus \{0\}$ such that $da = 0 = d(-a)$; hence, the cancellation of d would imply $a \in \mathrm{supp}(T)$, an impossibility. ■

Lemma 2.3 *a) The class of p-rings with the weak cancellation property is closed under arbitrary non-trivial reduced products. In particular, it is closed under arbitrary non-trivial products.*

b) If A is a ring, the class of preorders of A with the wcp is closed under arbitrary intersections and upward directed unions.

Proof. a) Let $\{\langle A_i, T_i \rangle : i \in I\}$ be a non-empty family of p-rings with the wcp and let D be a proper filter in I. For each $i \in I$, write $\mathcal{N}_i$ for the multiplicative set of non zero-divisors in A_i and

$$\langle A, T \rangle = \langle \textstyle\prod_{i \in I} A_i, \prod_{i \in I} T_i \rangle \quad \text{and} \quad \langle A_D, T_D \rangle = \langle \prod_D A_i, \prod_D T_i \rangle$$

for the product and the reduced product mod D of the $\langle A_i, T_i \rangle$, respectively. Let $\mathcal{N}_D$ be the set of non zero-divisors in A_D. We start with the following

Fact. *For all $z \in \prod_{i \in I} A_i$, we have* $z/D \in \mathcal{N}_D \quad \Leftrightarrow \quad \{i \in I : z(i) \in \mathcal{N}_i\} \in D$.

Proof. $\underline{\Rightarrow:}$ Suppose $z/D \in \mathcal{N}_D$ and set $K = \{i \in I : z(i) \in \mathcal{N}_i\}$. For each $i \in K^c = I \setminus K$, $z(i) \notin \mathcal{N}_i$ and so there is $a_i \in A_i \setminus \{0\}$ such that $z(i)a_i = 0$. Define, for $i \in I$,

$$a(i) = \begin{cases} a_i & \text{if } i \in K^c \\ 0 & \text{if } i \in K. \end{cases}$$

Then, $az = 0$ and so, $a/Dz/D = (az)/D = 0/D$. Since $z/D \in \mathcal{N}_D$, we get $a/D = 0/D$ whence, $\{i \in I : a(i) = 0\} = K \in D$, as needed.

$\underline{\Leftarrow:}$ Assume $a \in A$ satisfies $a/D \cdot z/D = (az)/D = 0$, the definition of reduced product entails that $J = \{i \in I : a(i)z(i) = 0\} \in D$; set $K = \{i \in I : z(i) \in \mathcal{N}_i\} \in D$. Hence, $G = K \cap J \in D$ and for all $i \in G$, $a(i) = 0$, whence $a/D = 0$ and $z/D \in \mathcal{N}_D$. □

Let $a/D \in A_D$ and assume $c/D \in T_D \cap \mathcal{N}_D$ is so that $(a/D)(c/D) = (ac)/D \in T_D$. Then, $U = \{i \in I : a(i)c(i) \in T_i\} \in D$. By the Fact, $V = \{i \in I : c(i) \in T_i \cap \mathcal{N}_i\} \in D$. Hence, $W = U \cap V \in D$ and for all $i \in W$, $a(i) \in T_i$, whence $a/D \in T_D$, as desired.

b) Let $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$ be a family of preorders of A with the wcp. Clearly, $\bigcap_{\lambda\in\Lambda} T_\lambda$ is a preorder of A with the wcp. Assume $\mathcal{T}$ is upward directed[3] and set $T = \bigcup_{\lambda\in\Lambda} T_\lambda$. If $a \in A$ and $c \in \mathcal{N} \cap T$ satisfy $cx \in T$, the upward directedness of $\mathcal{T}$ yields $\lambda \in \Lambda$ such that $c \in \mathcal{N} \cap T_\lambda$ and $ca \in T_\lambda$, whence, $a \in T_\lambda \subseteq T$, as desired. ∎

Remarks 2.4 a) Any ring -po which is a chain (or linear; a lo- ring) has the wcp and so, by 2.3, so does every reduced product of lo-rings. To establish the first assertion, let $\langle R, \leq \rangle$ be an lo-ring and assume $yx > 0$, with $y > 0$ being a non zero-divisor in R; then, we must have $x \geq 0$, otherwise $-yx = y(-x) > 0$, a contradiction.

b) If A is a ring, then $A^\times \subseteq \mathcal{N}$. Whenever, $A^\times = \mathcal{N}$, then A is its own total ring of quotients and *every* preorder of A has the wcp. This is the case of fields and, by 2.3, the case of any reduced product of rings in which units and non zero-divisors coincide. For instance, if A is a von Neumann regular ring (i.e., all principal ideals are generated by an idempotent), then $A^\times = \mathcal{N}$. To see this, assume $0 \neq d \notin A^\times$, that is $(d) \neq A$; then, there is an idempotent $e \in A$ such that $(d) = (e)$, with $e \neq 1$. But then we have $1 - e \neq 0$ and $d(1 - e) = 0$, showing that d is a zero-divisor in A.

c) It follows from Lemma 2.3 and Theorem 6.2.5 (p. 412 in [4]; without need of the continuum hypothesis by the observations in the penultimate paragraph of page 414 in [4]), the first-order theory of p-rings with the wcp has a Horn axiomatization; it would be of interest to exhibit an explicit axiomatization of this form.

d) Propositions 2.5 and 3.3, below, describe yet other classes of rings with the wcp. ∎

If X is a topological space, $\mathbb{C}(X)$ is the f-ring of continuous real-valued functions defined on X. For $f \in \mathbb{C}(X)$,

$$[\![f = 0]\!] \quad = \quad \{x \in X : f(x) = 0\}$$

is the zero-set of f, a closed set in X. Similarly, one defines $[\![f \geq 0]\!]$, $[\![f > 0]\!]$, $[\![f \neq 0]\!]$, etc. (see 8.22, p. 108 in **FQR**).

If K is a closed set in X, the set

$$P_K \quad = \quad \{f \in \mathbb{C}(X) : K \subseteq [\![f \geq 0]\!]\}$$

is a proper preorder of $\mathbb{C}(X)$, which is of bounded inversion iff $K = X$ (cf. Lemma 8.28, p. 113 in **FQR**). Recall that a closed set K in X is

• **regular** if it is the closure of its interior; • **rare** (in X) if it has empty interior.

[3] I.e., $\Lambda \neq \emptyset$ and for all $\lambda_1, \lambda_2 \in \Lambda$, there is $\lambda \in \Lambda$ so that $T_{\lambda_1}, T_{\lambda_2} \subseteq T_\lambda$.

Proposition 2.5 *Let X be a completely regular Hausdorff space and let $\mathcal{N}$ be the multiplicative set of non-zero-divisors in $\mathbb{C}(X)$.*

a) For $f \in \mathbb{C}(X)$, the following are equivalent:

(1) $f \in \mathcal{N}$; (2) $[\![f = 0]\!]$ *is rare in* X.

b) If K is a regular closed set in X, then the preorder P_K defined above has the weak cancellation property (cf. 2.1). In particular, the natural partial order of $\mathbb{C}(X)$, namely $\mathbb{C}(X)^2$, has the wcp.

Proof. a) (1) $\Rightarrow$ (2) : If $[\![f = 0]\!]$ has non-empty interior, select $p \in U \subseteq [\![f = 0]\!]$, with U open in X. By complete regularity, there is $g \in \mathbb{C}(X)$, such that $g(p) = 1$ and $g \restriction (X \setminus U) = 0$. Clearly, $fg = 0$, with $g \neq 0$, contradicting (1).

(2) $\Rightarrow$ (1) : By (2), $V = [\![f \neq 0]\!]$ is a dense open set in X. If $gf = 0$, then $V \subseteq [\![g = 0]\!]$ and so, since the zero-set of g is closed, we obtain $g = 0$ everywhere on X, as needed.

b) We first note

Fact 2.6 *If F is a rare closed set in X, then $K \setminus (K \cap F)$ is dense in K.*

Proof. Let $V = X \setminus F$ (a dense open in X) and let U be the interior of K. Since K is regular, it suffices to check that $K \setminus (K \cap F) = K \cap V$ is dense in U. If not, there are $p \in W \subseteq U$, with W open in X, such that $W \cap (K \cap V) = W \cap V = \emptyset$, which is impossible, since V is dense in X. □

Let $f \in \mathcal{N} \cap P_K$ and assume that $fg \geq 0$ on all of K. Set $V = X \setminus [\![f = 0]\!]$. By Fact 2.6, $K \cap V$ is dense in K and contained in $[\![f > 0]\!]$ (because $f \geq 0$ on K). Since $fg \geq 0$ on K, we must have $K \cap V \subseteq [\![g \geq 0]\!]$, whence, since $[\![g \geq 0]\!]$ is closed, we conclude $K \subseteq [\![g \geq 0]\!]$, as desired. ■

The following example shows that the hypothesis of regularity in 2.5.(b) is necessary for P_K to have the wcp, yielding, in particular, an example of a preorder of $\mathbb{C}(X)$ without the weak cancellation property. This same example will also be useful in section 2.

Example 2.7 With notation as above, let $A = \mathbb{C}(\mathbb{R})$ and let $K = \{0\} \cup [1, 2]$. Clearly, K is a closed set, which is not regular. It is plain that the function $|x|$ is a non zero-divisor in A, belonging to P_K. Let g be any continuous function on $\mathbb{R}$, such that $g(0) = -1$ and is non-negative on $[1, 2]$. Then, $|x|g \in P_K$, but $g \notin P_K$ because its value at $0 \in K$ is strictly negative. Hence, P_K does not have the wcp. Note that

$$\text{supp}(P_K) \quad = \quad \{f \in A : f \restriction K = 0\},$$

and so, 2.5.(a) entails $\mathrm{supp}(P_K) \cap \mathcal{N} = \emptyset$. Moreover, by Lemma 8.28, p. 113 in **FQR**, P_K is a unit-reflecting preorder of A; hence, in general, unit-reflecting preorders may not have the weak cancellation property. If X is a *metric space*, then P_K has the wcp *iff* K is a regular closed set in X (in fact, perfect normality suffices, i.e., a normal space in which closed sets are G_δ; cf. paragraphs before Theorem 1.5.19 in [7]). ■

We now introduce the notion of weak cancellation closure:

Definition 2.8 *If $\langle A, T \rangle$ is a p-ring and $\mathcal{N}$ be the multiplicative set of non zero-divisors in A. Define*

$$\mathfrak{c}T \quad = \quad \{x \in A : \exists\, c \in T \cap \mathcal{N} \textit{ such that } cx \in T\},$$

called the **weak cancellation closure** *of T in A.*

Lemma 2.9 *Let A be a ring and let T, T' be preorders of A. With notation as in* 2.8*,*

a) $\mathfrak{c}T$ is a preorder of A and $T \subseteq \mathfrak{c}T$. Moreover,

(1) *$\mathfrak{c}T$ is proper iff $\mathcal{N} \cap \mathrm{supp}(T) = \emptyset$.*

(2) $T \subseteq T' \quad \Rightarrow \quad \mathfrak{c}T \subseteq \mathfrak{c}T'$.

b) If T is a proper partial order of A, the same is true of $\mathfrak{c}T$.

c) $\mathfrak{c}(\mathfrak{c}T) \;=\; \mathfrak{c}T$.

d) $\mathfrak{c}T$ is the least preorder with the wcp (in the inclusion partial order) containing T.

Proof. a) Clearly, $T \subseteq \mathfrak{c}T$ ($1 \in \mathcal{N}$); in particular, $A^2 \subseteq \mathfrak{c}T$. Let a, $b \in \mathfrak{c}T$ and let c, $d \in \mathcal{N} \cap T$ be such that ca, $db \in T$. Then, $cdab \in T$ and cda, $cdb \in T$, whence $cd(a + b) \in T$, with $cd \in T \cap \mathcal{N}$. Hence, $\mathfrak{c}T$ is closed under sums and products, and so a preorder of A. Since

$$-1 \in \mathfrak{c}T \quad \Leftrightarrow \quad \exists\, c \in T \cap \mathcal{N} \text{ such that } -c \in T,$$

we conclude that $\mathfrak{c}T$ is proper iff $\mathcal{N} \cap \mathrm{supp}(T) = \emptyset$, establishing (1); item (2) is clear.

b) In view of (a), it suffices to check that $\mathrm{supp}(T) = \{0\}$ implies $\mathrm{supp}(\mathfrak{c}T) = \{0\}$. If $a \in \mathrm{supp}(\mathfrak{c}T)$, there are c, $d \in \mathcal{N} \cap T$ such that ca, $-da \in T$. Hence, cda, $-cda \in T$, and so $cda = 0$, yielding $a = 0$, as needed.

c) By (a), it is enough to check that $\mathfrak{c}(\mathfrak{c}T) \subseteq \mathfrak{c}T$. If $a \in \mathfrak{c}(\mathfrak{c}T)$, there is $c \in \underline{\mathcal{N} \cap \mathfrak{c}T}$ such that $ca \in \mathfrak{c}T$. Thus, there are u, $v \in \mathcal{N} \cap T$ such that $uc \in T$ and $vca \in T$. Hence, $(uc)va \in T$, with $(uc)v \in \mathcal{N} \cap T$, and so $a \in \mathfrak{c}T$, as desired.

d) The second assertion is straightforward; the first follows from (c): if $a \in A$ and $c \in \mathcal{N} \cap \mathfrak{c}T$ verify $ca \in \mathfrak{c}T$, then $a \in \mathfrak{c}(\mathfrak{c}T) = \mathfrak{c}T$. ■

3 Rings of Fractions of f-rings by non Zero-Divisors

A standard reference for f-rings is [2], particularly chapters 8 and 9. For the convenience of the reader, we register some basic facts concerning this class of rings (see also section 1 of Chapter 8 (p. 86) in **FQR**).

A. Lattice-Ordered Rings. A partially ordered ring (po-ring) $\langle A, \leq \rangle$ is **lattice-ordered (ℓ-ring)** if for all $a, b \in A$,

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}$$

exist in A, where join (or sup) and meet (or inf) are considered with respect to the partial order $\leq$. Let A be a ℓ-ring and let $a \in A$. Define

(av) $$a^+ = a \vee 0, \quad a^- = -a \vee 0 \quad \text{and} \quad |a| = a^+ \vee a^-,$$

called the **positive part, negative part** and **absolute value of a in A**. It is clear that a^+, a^-, $|a| \geq 0$. We note the following

Lemma 3.1 ([2], 8.1.4, p. 151) *If A is an ℓ-group*[4] *and a, b, $x \in A$, then*

a) $x + (a \wedge b) = (x + a) \wedge (x + b)$ *and*
$x + (a \vee b) = (x + a) \vee (x + b)$.

b) $-(a \wedge b) = -a \vee -b$ *and* $-(a \vee b) = -a \wedge -b$.

c) $a + b = (a \wedge b) + (a \vee b)$.

d) $|a| = a \vee -a = a^+ + a^-$.

e) $|a + b| \leq |a| + |b|$.

f) ([2], 1.3.2, 1.3.3, p. 22) $a = a^+ - a^-$ *and* $a^+ \wedge a^- = 0$.

g) ([2], Proposition 1.3.4, p. 22) *For x, y, $z \in A$, the following are equivalent:*
(1) $x = y - z$ *and* $y \wedge z = 0$;
(2) $y = x^+$ *and* $z = x^-$. ■

B. f-rings. A lattice-ordered ring is called an **f-ring** if it is isomorphic to a subdirect product of linearly ordered rings ([2], Definition 9.1.1, p. 172). We have

Lemma 3.2 *Let A be a ring.*

[4] For the definition and basic properties of ℓ-groups see Chapter 1 of [2]. The laws that follow will be used only for the (commutative) additive group of a ℓ-ring.

a) ([2], Proposition 9.1.10, p. 175) *If A is an f-ring and a, b, $x \in A$, then*

(1) $x \geq 0 \quad \Rightarrow \quad x(a \vee b) \; = \; xa \vee xb$ *and* $x(a \wedge b) \; = \; xa \wedge xb.$

(2) $|ab| \; = \; |a| \cdot |b|.$

(3) $a \wedge b = 0 \quad \Rightarrow \quad ab = 0.$ (4) $a^2 \geq 0.$

b) ([2], Corollary 9.1.14, p. 176) *In a ℓ-ring A, properties* (*a*.1) *and* (*a*.2) *are equivalent. Moreover, any unitary ring verifying* (*a*.1) *is an f-ring.*

c) ([2], Theorem 9.3.1, pp. 178-179) *If A is a ℓ-ring, the following are equivalent:*

(1) *A is a reduced f-ring;*

(2) *A is a subdirect product of linearly ordered integral domains.*

(3) *For all a, $b \in A$, $\;|a| \wedge |b| = 0$ iff $ab = 0$.* ■

If A is a f-ring, we write $T_\sharp$ for its natural *partial order*, which, whenever convenient, will be denoted by $\leq$. Regarding the weak cancellation property, we have the following

Proposition 3.3 *Any reduced f-ring satisfies the weak cancellation property.*

Proof. If $\langle A, T_\sharp \rangle$ is a f-ring, by items (a) and (b) of 2.9, $\mathfrak{c}T_\sharp$ is a proper partial order, containing $T_\sharp$. But then Proposition 1.11, p. 33, in [12] entails $T_\sharp = \mathfrak{c}T_\sharp$ (see also Proposition 8.15.(a), p. 82 in **FQR**); by 2.9.(e), $T_\sharp$ has the wcp. ■

Henceforth, let $\langle A, T_\sharp \rangle$ be a reduced f-ring. Notation is as in 1.8. Let S be a multiplicative subset of $\mathcal{N}$ (containing 1). Write A_S for the ring of fractions AS^{-1} and

$$T_{\sharp S} \;=\; \{\frac{t}{s^2} \in A_S : t \in T \text{ and } s \in S\}$$

for the preorder defined in Lemma 1.7.(e). Recall that $Q(A)$ is the total ring of fractions of A and we write $Q(T_\sharp)$ for the extension of $T_\sharp$ to $Q(A)$ (cf. 1.8.(c)).[5]

Lemma 3.4 *Notation as above, let S be a* saturated *multiplicative subset of $\mathcal{N}$.*[6]

a) For all $z \in A$, $\quad z \in S \;\Leftrightarrow\; |z| \in S \cap T_\sharp$.

b) A_S is reduced and $T_{\sharp S}$ is a proper ring partial order on A_S, to be written $\leq_S$.

c) For f, $g \in A$ and x, $y \in S$,

[5] In [8], $Q(A)$ denotes the *complete* ring of quotients of A. Hence, our notation deviates from that in [8].

[6] By 1.9, there is no loss of generality in assuming S is saturated, as is the case of $\mathcal{N}$.

(1) $\dfrac{f}{x} \geq_S 0 \quad \Leftrightarrow \quad fx \geq 0$.[7]

(2) $\dfrac{f}{x^2} \geq_S 0 \;\Leftrightarrow\; f \geq 0$. *In particular,* $\dfrac{f}{1} \geq_S 0 \;\Leftrightarrow\; f \geq 0$.

(3) (i) $\dfrac{f}{x} \geq_S \dfrac{g}{x} \;\Leftrightarrow\; fx \geq gx$; $\qquad (ii)$ $\dfrac{f}{x^2} \geq_S \dfrac{g}{x^2} \;\Leftrightarrow\; f \geq g$.

(4) $\dfrac{f}{x} \geq_S \dfrac{g}{y} \;\Leftrightarrow\; fxy^2 \geq gyx^2$.

d) The canonical ring morphism, $\iota_S : \langle A, T_\sharp \rangle \longrightarrow \langle A_S, T_{\sharp S} \rangle$, *is a p-ring embedding.*

Proof. a) Since A is an f-ring, it is clear that $|z| \in T_\sharp$. Moreover, items (2) and (4) in 3.2.(a) yield $z^2 = \left|z^2\right| = |z|^2$, whence, since S is saturated, we obtain $z \in S$ iff $|z| \in S \cap T_\sharp$, as needed.

b) Clearly, A_S is reduced and Lemma 1.7.(e) entails that $T_{\sharp S}$ is a proper pre-order of A_S. To see it is a partial order, assume, for $f \in A$ and $x \in S$, that $\dfrac{f}{x} \in \mathrm{supp}(T_{\sharp S})$; then there are p, $q \in T_\sharp$ and u, $v \in S$ such that

$$\text{(I)} \qquad \frac{f}{x} = \frac{fx}{x^2} = \frac{p}{u^2} \quad \text{and} \quad \frac{f}{x} = \frac{fx}{x^2} = -\frac{q}{v^2}.$$

The first equation in (I) yields $fxu^2 = px^2 \in T_\sharp$, while the second gives $fxv^2 = -qx^2 \in -T_\sharp$. Hence, multiplying the former and the latter equations by v^2 and u^2, respectively, yields $fxu^2v^2 \in \mathrm{supp}(T_\sharp) = \{0\}$. Since $xu^2v^2 \in S \subseteq \mathcal{N}$, we conclude $f = 0$, whence $\dfrac{f}{x} = 0$, as needed.

c)(1) Suppose there are $p \in T_\sharp$ and $u \in S$ such that

$$\text{(II)} \qquad \frac{f}{x} = \frac{fx}{x^2} = \frac{p}{u^2}.$$

Then, $fxu^2 = px^2 \in T_\sharp$; now, 3.3 and the wcp yield (because $u^2 \in T_\sharp \cap S \subseteq T_\sharp \cap \mathcal{N}$) $fx \in T_\sharp$, i.e., $fx \geq 0$. The converse follows immediately from the first equality in (II). Item (2) is immediate from (1) and the fact that $T_\sharp$ has the wcp (3.3).

(3.i) Item (c.1) gives

$$\frac{f}{x} \geq_S \frac{g}{x} \;\Leftrightarrow\; \frac{f}{x} - \frac{g}{x} \geq_S 0 \;\Leftrightarrow\; \frac{f-g}{x} \geq_S 0 \;\Leftrightarrow\; fx - gx \geq 0,$$

as needed. Item (ii) follows directly from (i) and (c.2).

(4) Note that $\dfrac{f}{x} = \dfrac{fx}{x^2} = \dfrac{fxy^2}{x^2y^2}$, with a similar computation yielding $\dfrac{g}{y} = \dfrac{gyx^2}{x^2y^2}$, and the desired conclusion follows from (3.(ii)).

[7] Recall that $\leq$ is the partial order $T_\sharp$.

Item (d) is an immediate consequence of (c).(2) (or (c.1)). ■

Lemma 3.5 *The p-ring* $\langle A_S, T_{\sharp S} \rangle$ *is a* ℓ*-ring and for* $f, g \in A$ *and* $x, y \in S$ *we have*

$$(1)\quad \frac{f}{x} \vee \frac{g}{y} = \frac{fxy^2 \vee gyx^2}{x^2y^2}; \qquad (2)\quad \frac{f}{x} \wedge \frac{g}{y} = \frac{fxy^2 \wedge gyx^2}{x^2y^2};$$

$$(3)\quad \left(\frac{f}{x}\right)^+ = \frac{(fx)^+}{x^2} \quad \text{and} \quad \left(\frac{f}{x}\right)^- = \frac{(fx)^-}{x^2};$$

$$(4)\quad \left|\frac{f}{x}\right| = \frac{|f|}{|x|}; \qquad (5)\quad \frac{f}{1} \vee \frac{g}{1} = \frac{f \vee g}{1} \quad \text{and} \quad \frac{f}{1} \wedge \frac{g}{1} = \frac{f \wedge g}{1}.$$

Proof. To see that $\langle A_S, T_{\sharp S} \rangle$ is an ℓ-ring we must check that it is a lattice with respect to the partial order $\leq_S$, i.e., it suffices to verify (1) and (2).

Proof of (1) : Clearly, in A, $fxy^2,\ gyx^2 \leq fxy^2 \vee gyx^2$; now, item 3.$(ii)$ in 3.4.(c) entails $\dfrac{f}{x} = \dfrac{fxy^2}{x^2y^2} \leq_S \dfrac{fxy^2 \vee gyx^2}{x^2y^2}$, with an analogous relation holding for $\dfrac{g}{y}$.

Now, for $a \in A$ and $b \in S$, suppose $\dfrac{f}{x}, \dfrac{g}{y} \leq_S \dfrac{a}{b}$. Then, 3.4.(c.4) yields

$$(*)\quad fxb^2 \leq abx^2 \quad \text{and} \quad (**)\quad gyb^2 \leq aby^2.$$

Multiplying (*) by y^2 and (**) by x^2 obtains

$$fxy^2b^2 \leq abx^2y^2 \quad \text{and} \quad gyx^2b^2 \leq abx^2y^2,$$

and so, since $\langle A, T_\sharp \rangle$ is an f-ring, we get, recalling 3.2.(a.1),

$$\text{(I)}\qquad (fxy^2 \vee gyx^2)b^2 = fxy^2b^2 \vee gyx^2b^2 \leq abx^2y^2.$$

But then (I) and another application of (c.4) in 3.4 entail

$$\frac{fxy^2 \vee gyx^2}{x^2y^2} = \frac{(fxy^2 \vee gyx^2)b^2}{b^2x^2y^2} \leq_S \frac{abx^2y^2}{b^2x^2y^2} = \frac{a}{b},$$

concluding the proof that $\dfrac{fxy^2 \vee gyx^2}{x^2y^2}$ is indeed the join of $\dfrac{f}{x}$ and $\dfrac{g}{y}$ in the partial order $\leq_S$. An entirely similar argument, using the second equality in 3.2.(a.1) (to factor out b^2 in the analog of (I)) will establish (2). Statement (3) follows straightforwardly from (1), since for $\xi \in A_S$, $\xi^+ = \xi \vee 0$ and $\xi^- = (-\xi) \vee 0$. For instance,

$$-\left(\frac{f}{x}\right) \vee \frac{0}{1} = \frac{-f}{x} \vee \frac{0}{1} = \frac{-(fx) \vee 0}{x^2} = \frac{(fx)^-}{x^2}.$$

Proof of (4) : By 3.4.(a), for all $x \in S$ we have $|x| \in S \cap T_\sharp$, and so the right-hand side of the equality is in A_S. Now (3) and 3.2.(a.2), together with the fact that for all elements u of a ℓ-ring, $|u| = u^+ + u^-$ (3.1.(d)) yield, recalling 3.2.(a.2),

$$\left|\frac{f}{x}\right| = \left(\frac{f}{x}\right)^+ + \left(\frac{f}{x}\right)^- = \frac{(fx)^+}{x^2} + \frac{(fx)^-}{x^2} = \frac{(fx)^+ + (fx)^-}{x^2}$$
$$= \frac{|fx|}{x^2} = \frac{|f|\,|x|}{|x|^2} = \frac{|f|}{|x|},$$

ending the proof. Item (5) is an immediate consequence of (1) and (2). ■

Theorem 3.6 (Corollary 10.13, [8]) *Let $\langle A, T_\sharp \rangle$ be a reduced f-ring. If S is a multiplicative subset of non zero-divisors in A, then $\langle A_S, T_{\sharp S} \rangle$ is a reduced f-ring extension of $\langle A, T_\sharp \rangle$. In particular, $\langle Q(A), Q(T_\sharp) \rangle$ is a reduced f-ring extension of $\langle A, T_\sharp \rangle$.*

Proof. By 1.9, we may suppose, without loss of generality, that S is saturated. Moreover, Lemmas 3.4.(d) and 3.5 imply that $\langle A_S, T_{\sharp S} \rangle$ is a reduced l-ring extension of $\langle A, T \rangle$. It remains to see that $\langle A_S, T_{\sharp S} \rangle$ is an f-ring; to this end, it suffices, by 3.2.(b), to verify that for all u, v, $w \in A_S$,

(I) $u \geq_S 0 \quad \Rightarrow \quad u(v \vee_S w) = (uv \vee_S uw)$ and $u(v \wedge_S w) = (uv \wedge_S uw)$,

where $\vee_S$ and $\wedge_S$ indicate join and meet in A_S, respectively. We shall treat the first equality; the second can be similarly proven.

Let $u = \dfrac{f}{x} \geq_S 0$, $v = \dfrac{g}{y}$ and $w = \dfrac{h}{z}$, where x, y, $z \in S \subseteq \mathcal{N}$. Note that 3.4.(c.1) entails $fx \geq 0$, and so $fx^3 \geq 0$ (in A). But then, (1) in 3.5.(1) and the fact that A is a f-ring yield,

$$\frac{f}{x}\left(\frac{g}{y} \vee_S \frac{h}{z}\right) = \frac{fx^3}{x^4}\,\frac{gyz^2 \vee hzy^2}{y^2z^2} = \frac{fx^3(gyz^2 \vee hzy^2)}{x^4y^2z^2}$$
$$= \frac{(fx)(gy)x^2z^2 \vee (fx)(hz)x^2y^2}{x^4y^2z^2} = \frac{fg}{xy} \vee_S \frac{fh}{xz},$$

as needed. A similar argument yields the second equality in the consequent of (I), and A_S is indeed a f-ring. The remaining statement is now clear. ■

We now show that rings of fractions of weakly real closed and real closed rings by multiplicative sets of non zero-divisors are still weakly real closed and real closed, respectively.

A ring A is a **weakly real closed ring (WRCR)** (introduced in Definition 8.33, p. 166 of **FQR**) if it satisfies

[WRCR 1] : A is reduced;

[WRCR 2] : A^2 is the positive cone of a partial order $\leq$ on A, with which it is a f-ring;

[WRCR 3] : For all $a, b \in A$, $0 \leq a \leq b \quad \Rightarrow \quad b$ divides a^2.

Corollary 3.7 *If A is a WRCR and S is multiplicative subset of the non zero-divisors in A, then $A_S = AS^{-1}$ is a weakly real closed ring.*

Proof. Since A is a reduced f-ring with $T_\sharp = A^2$, it is clear that $T_{\sharp S} = (A_S)^2$; moreover, by Theorem 3.6, $\langle A_S, (A_S)^2 \rangle$ is a reduced f-ring extension of $\langle A, A^2 \rangle$. It remains to check A_S satisfies [WRCR 3]. By Remark 1.9, we may assume S is saturated. Suppose

$$\text{(I)} \qquad 0 \leq_S \frac{a}{x} \leq_S \frac{b}{y},$$

with a, $b \in A$ and x, $y \in S$. By items (1) and (4) in Lemma 3.4.(c), (I) is equivalent to

$$\text{(II)} \qquad 0 \leq axy^2 \leq byx^2.$$

Hence, (II) and axiom [WRCR 3] applied in A yields $\alpha \in A$ such that

$$\text{(III)} \qquad a^2x^2y^4 = \alpha\, byx^2,$$

Since $x^4y^4 \in S \cap A^2$, (III) and $(c).(3.ii)$ in 3.4 entail,

$$\left(\frac{a}{x}\right)^2 = \frac{a^2}{x^2} = \frac{a^2x^2y^4}{x^4y^4} = \frac{\alpha}{1} \cdot \frac{byx^2}{x^4y^4} = \frac{\alpha}{x^2y^2} \cdot \frac{by}{y^2} = \frac{\alpha}{x^2y^2} \cdot \frac{b}{y},$$

as needed to verify [WRCR 3] in A_S. ∎

Proposition 3.8 *If A is a real closed ring and S is a multiplicative subset of non zero-divisors in A, then $A_S = AS^{-1}$ is real closed. In particular, $Q(A)$ is real closed.*

Proof. Recall that A is real closed if it is weakly real closed and it satisfies (cf. Remark 8.34, p. 117 in **FQR** and [11])

[RC] *For all proper prime ideals $\mathfrak{p} \subseteq A$, the field of fractions of $A/\mathfrak{p}$ is real closed and $A/\mathfrak{p}$ is integrally closed in it.*

We begin by recalling certain basic algebraic facts.

Fact 1. *Let D be a domain and k its field of fractions. If S is a proper multiplicative set in D, then $D_S = DS^{-1}$ is an integral domain and its field of fractions is (canonically isomorphic to) k.*

Proof. Clearly, D_S is an integral domain; let F be its field of fractions. Up to isomorphism, we may consider $D \subseteq D_S \subseteq F$ and $D \subseteq k$. Since F is a field containing D, we conclude that $k \subseteq F$; on the other hand, because every element of S is a unit in k, we also have $D_S \subseteq k$, wherefrom we conclude $F \subseteq k$, as needed.[8] □

Recall that if A is a ring, $\mathrm{Spec}(A)$ is the space of proper prime ideals in A, the **Zariski spectrum** of A. We register the well-known (cf. Proposition 3.11.(iv), p. 41 in [1])

Fact 2. *Let A be a ring, S be a multiplicative set in A and $\iota_S : A \longrightarrow A_S = AS^{-1}$ be the canonical morphism. The map $\mathfrak{p} \in \mathrm{Spec}(A_S) \overset{h}{\longmapsto} \iota_S^{-1}(\mathfrak{p}) \in \mathrm{Spec}(A)$ is a homeomorphism onto $\mathcal{S} = \{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \cap S = \emptyset\}$, with the topology induced by $\mathrm{Spec}(A)$; the inverse of h is given by $\mathfrak{q} \in \mathcal{S} \longmapsto \mathfrak{q}S^{-1} = \left\{\frac{a}{s} \in A_S : a \in \mathfrak{q} \text{ and } s \in S\right\}$.* ■

Fact 3. (Corollary 3.4, p. 39 in [1]) *Let A be a ring, I an ideal in A and S a multiplicative set in A with $I \cap S = \emptyset$. Let $I_S := IS^{-1} = \left\{\frac{a}{s} \in A_S : a \in I \text{ and } s \in S\right\}$. Then, I_S is a proper ideal in A_S, S/I is a proper multiplicative set in A/I and A_S/I_S is naturally isomorphic to the ring of fractions $(A/I)(S/I)^{-1}$, that is, there is a* unique *ring isomorphism, $h : A_S/I_S \longrightarrow (A/I)(S/I)^{-1}$, making the following diagram commute*

$$\begin{tikzcd} A_S \arrow[rr, "\pi"] \arrow[dr, "f"'] & & A_S/I_S \arrow[dl, "h"] \\ & (A/I)(S/I)^{-1} & \end{tikzcd}$$

where π is the canonical quotient projection. ■

Fact 4. (Proposition 5.12, p. 62 in [1]) *Let $A \subseteq B$ be rings, C the integral closure of A in B. Let S be a multiplicative subset of A. Then, CS^{-1} is the integral closure of AS^{-1} in BS^{-1}.* ■

Now, let A be a real closed ring and S be a proper multiplicative subset of the non zero-divisors in A. Since we already know (by 3.7) that A_S is weakly real closed, it remains to see that it has property [RC] stated above. Let $\mathfrak{p}$ a proper prime ideal in A_S; by Fact 2, $\mathfrak{p}$ is of the form $\mathfrak{q}_S = \mathfrak{q}S^{-1}$, for some prime $\mathfrak{q}$ in A, disjoint from S. Since $A/\mathfrak{q}$ is an integral domain, it follows from Facts 1 and 3 that the field of fractions of $A_S/\mathfrak{p} = A_S/\mathfrak{q}_S \approx (A/\mathfrak{q})(S/\mathfrak{q})^{-1}$ is isomorphic to the field of quotients of $A/\mathfrak{q}$, $k_{\mathfrak{q}}$, which, by assumption is real

[8] The universal property of rings of fractions yields a (more) formal proof: $s, t \in S$ and $a, b \in D$, with $b \neq 0$, the map $\frac{a}{s}/\frac{b}{t} \in F \longmapsto (at)(bs)^{-1} \in k$ is an isomorphism over D_S.

closed. Moreover, by Fact 4, $A_S/\mathfrak{p} \approx (A/\mathfrak{q})(S/\mathfrak{q})^{-1}$ is integrally closed in $k_\mathfrak{q}(S/\mathfrak{q})^{-1} \approx k_\mathfrak{q}$, and A_S is a real closed ring, ending the proof. ■

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