*South American Journal of Logic*
Vol. 4, n. 2, pp. 407–421, 2018
ISSN: 2446-6719

# Analyzing Categories of Signatures

Caio A. Mendes and Hugo L. Mariano

**Abstract**

Motivated by the phenomena of combinations of logics, we analyze two categories of finitary signatures underlying to (propositional) logics: the categories $\mathcal{S}_s$ ([4]) and $\mathcal{S}_f$ ([10]). $\mathcal{S}_s$ has a very **s**imple notion of morphism, but it is too **s**trict: $\mathcal{S}_s$ has good categorial properties (is a complete/cocomplete $\omega$-accessible category), but does not allows a good treatment of the identity problem for logics ([5]). $\mathcal{S}_f$ has a more **f**lexible notion of morphism: it allows a better treatment of the identity problem for logics but, on the other hand, $\mathcal{S}_f$ has serious categorial defects. We define a pair of (faithful) functors $\mathcal{S}_s \underset{(-)}{\overset{(+)}{\rightleftarrows}} \mathcal{S}_f$, such that $(+)$ is left adjoint to $(-)$. We consider the (endo)functor in $\mathcal{S}_s$, $T := (-) \circ (+)$ and we prove that $T$ preserves filtered colimits and reflects isos/epis/monos. We consider the monad (or triple) canonically associated to this adjunction, $\mathcal{T} = (T, \mu, \eta)$, and we prove that $\mathcal{S}_f = Kleisli(\mathcal{T})$: this result entails that the category of logics $\mathcal{L}_f$ build over $\mathcal{S}_f$ has: unconstrained fibrings, i.e. coproducts, and "constrained" fibrings, i.e. colimits with base diagram "in" $\mathcal{S}_s$ (i.e., obtained via $(+) : \mathcal{S}_s \longrightarrow \mathcal{S}_f$).

**Keywords:** Universal Logic, Propositional Signatures, Category theory.

# Introduction

In this work we consider (finitary) signatures for (propositional) logics through the original use of Category Theory: the study of the "sociology of mathematical objects", aligning us with a recent, and growing, trend of study logics through its relations with other logics, in particular by the processes of combinations of logics.

The phenomenon of combinations of logics ([13]), emerged in the mid-1980s, was the main motivation for considering categories of logics. There are two aspects of combination of logics: (i) splitting of logics: a analytical process; (ii) splicing of logics: a synthesis. The "Possible-Translations Semantics", introduced in [12], is an instance of the splitting process: a given logic system is decomposed into other (simpler) systems, providing, for instance a conservative translation of the logic in analysis into a "product" (or weak product) of simpler or better known logics. The "Fibring" of logics, introduced originally in the context of modal logics ([16]), is "the least logic which extends simultaneously the given logics"; after, this was recognized as a coproduct construction ([19]): this provides an example of synthesis of logics.

In the field of categories of logics there are, of course, two choices that must be done: (i) the choice of objects (how represent a logical system?); (ii) the choice of arrows (what are the relevant notions of morphisms between logics?).

The main flow of research on categories of logics, represented by the groups of CLE-Unicamp (Brazil) and IST-Lisboa (Portugal) focus on the determination of the conditions for preservation of metalogical properties under the process of combination of logics. On the other hand, the "global aspects" of categories of logics, that ensure for example the abundance or scarcity of constructions, seem to have not been adequately studied.

The present work provides the first part of a project of considering categories of logical systems satisfying *simultaneously* four natural requirements:
**(i)** If they represent the major part of the usual logical systems;
**(ii)** If they have good categorial properties (e.g., if they are a complete and/or cocomplete category, if they are accessible categories ([1]));
**(iii)** If they allow a natural notion of *algebraizable* logical system (as in the concept of Blok-Pigozzi algebraizable logic ([6]) or Czelakowski's proto-algebraizability ([14]));
**(iv)** If they provide a satisfactory treatment of the *identity problem* of logical systems (when logics can be considered "the same"? ([5], [11])). * in the series of articles [2], [3], [4], was considered a **s**imple (but too **s**trict) notion of morphism of signatures, where are founded some categories of logics that satisfy simultaneously three requirements ((i), (ii) and (iii)); here we will denote by $\mathcal{S}_s$ the category of signatures therein;
* in the series of papers [8], [9], [10], [11],[1] [15] is developed a more **f**lexible notion of morphism of signatures based on **f**ormulas as connectives (our notation for the associated category will be $\mathcal{S}_f$), it encompass other three requirements ((i), (ii) and (iv)).

**Overview of the paper:** In section 1 we recall the basic properties of the categories of signatures $\mathcal{S}_s$ and $\mathcal{S}_f$ and we add some new information. In section 2 we compare these two categories of signatures by means of functors $\mathcal{S}_s \overset{(+)}{\underset{(-)}{\leftrightarrows}} \mathcal{S}_f$ and we prove that they provide an adjoint pair of functors. In section

[1]We want to thank professor Marcelo Coniglio for suggesting that reference.

3 we identify the monad (or triple) associated to the described adjunction, we identify some properties of the monad, we prove that $\mathcal{S}_f$ is, precisely, the *Kleisli category* of that monad and we extract some consequences. We finish this work presenting some future directions.

All the basic notions on category-theory, freely used in this work, can be found in [17].

*In the sequel,* $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ *will denote a fixed enumerable set (written in a fixed order).*[2]

# 1 Categories of signatures

We recall here the basic notions and results on (finitary) signatures.

- A signature $\Sigma$ is a sequence of sets $\Sigma = (\Sigma_n)_{n\in\omega}$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i < j < \omega$ .
- We write $|\Sigma| = \bigcup_{n\in\omega} \Sigma_n$ for the *support of* $\Sigma$.
- We denote by $F(\Sigma)$, the *formula algebra of* $\Sigma$, i.e. the set of all (propositional) formulas built with signature $\Sigma$ over the variables in $X$. More precisely $F(\Sigma) = \bigcup_{i\in\mathbb{N}} F_i$, where the $(F_i)_{i\in\mathbb{N}}$ are recursively defined by:
  * $F_0 = X$;
  * $F_{i+1} = F_i \cup \{\langle c_n, \psi_0, \cdots, \psi_{n-1}\rangle$ : where $c_n \in \Sigma_n$ for some $n \in \mathbb{N}$ and $\{\psi_0, \cdots, \psi_{n-1}\} \subseteq F_i\}$.
- As usual, we will also denote:
  * $c_n(\psi_0, \cdots, \psi_{n-1}) := \langle c_n, \psi_0, \cdots, \psi_{n-1}\rangle$;
  * $\phi(x_{i_0}, \cdots, x_{i_{n-1}})$ means just that the formula $\phi$ has its variables *contained* in the set $\{x_{i_0}, \cdots, x_{i_{n-1}}\}$.
- For all $n \in \mathbb{N}$ let $F(\Sigma)[n] = \{\varphi \in F(\Sigma) : var(\varphi) = \{x_0, x_1, \ldots, x_{n-1}\}\}$, where $var(\varphi)$ is the set of all variables that occur in the $\Sigma$-formula $\varphi$.
- The notion of complexity $compl(\varphi)$ of the formula $\varphi$ is, as usual, the number of occurrences of connectives in $\varphi$.
- **On composition/substitution:**
  If $\{\phi, \psi_0, \cdots \psi_{n-1}\} \subseteq F(\Sigma)$, $var(\phi) \subseteq \{x_{i_0}, \ldots, x_{i_{n-1}}\}$, $\vec{x} = (x_{i_0}, \ldots, x_{i_{n-1}})$ and $\vec{\psi} = (\psi_0, \cdots, \psi_{n-1})$, then $(\phi(\vec{x})[\vec{x} \mid \vec{\psi}])$ will denote the corresponding composition (or substitution), recursively defined on $compl(\phi)$ by:

[2]We thank the reviewer for his/her thorough review and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the publication.

* $compl(\phi) = 0$: $\phi = x_n$, for some $n \in \mathbb{N}$
$x_n[\vec{x}|\vec{\psi}] := \psi_k$, if $x_n = x_{i_k}$;
* $compl(\phi) > 0$: $\phi = c_n(\varphi_0, \cdots, \varphi_{n-1})$
$c_n(\varphi_0, \cdots, \varphi_{n-1})[\vec{x}|\vec{\psi}] := c_n(\varphi_0[\vec{x}|\vec{\psi}], \cdots, \varphi_{n-1}[\vec{x}|\vec{\psi}])$.

### 1.1 The category $\mathcal{S}_s$

We will write $\mathcal{S}_s$ for the category of signatures and *strict* morphisms of signatures presented in [2], [3], [4], and described below.

The objects of $\mathcal{S}_s$ are signatures.

If $\Sigma, \Sigma'$ are signatures then a *strict* morphism $f : \Sigma \longrightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n\in\omega}$, where $f_n : \Sigma_n \longrightarrow \Sigma'_n$. Composition and identities in $\mathcal{S}_s$ are componentwise.

For each morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_s$ there is a unique function $\hat{f} : F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of* $f$, such that:
(i) $\hat{f}(x) = x$, if $x \in X$;
(ii) $\hat{f}(c_n(\psi_0, \ldots, \psi_{n-1}) = f_n(c_n)(\hat{f}(\psi_0), \ldots, \hat{f}(\psi_{n-1}))$, if $c_n \in \Sigma_n$.

Then, by induction on the complexity of formulas:
**(0)** $compl(\hat{f}(\theta)) = compl(\theta))$, for all $\theta \in F(\Sigma)$.
**(1)** If $var(\theta) \subseteq \{x_{i_0}, \ldots, x_{i_{n-1}}\}$, then $\hat{f}(\theta(\vec{x})[\vec{x} \mid \vec{\psi}]) = (\hat{f}(\theta(\vec{x}))[\vec{x} \mid \hat{f}(\vec{\psi})]$. Moreover $var(\hat{f}(\theta)) = var(\theta)$ and then $\hat{f}$ restricts to maps $\hat{f}\restriction_n: F(\Sigma)[n] \longrightarrow F(\Sigma')[n]$, $n \in \mathbb{N}$.
**(2)** The extension to the formula algebra of a composition is the extension's composition. The extension of an identity is the identity function on the formula algebra.

Observe that $\mathcal{S}_s$ is equivalent to the functor category $\mathbf{Set}^{\mathbb{N}}$, where $\mathbb{N}$ is the discrete category with object class $\mathbb{N}$, then $\mathcal{S}_s$ has all small limits and colimits and they are componentwise. Moreover, the category $\mathcal{S}_s$ is a finitely locally presentable category, i.e., $\mathcal{S}_s$ is a finitely accessible category that is cocomplete and/or complete ([1]). The finitely presentable signatures are precisely the signatures of finite support.

**(Sub)** For any *substitution* function $\sigma : X \longrightarrow F(\Sigma)$, there is unique *extension* $\widetilde{\sigma} : F(\Sigma) \longrightarrow F(\Sigma)$ such that $\widetilde{\sigma}$ is an "homomorphism": $\widetilde{\sigma}(x) = \sigma(x)$, for all $x \in X$ and $\widetilde{\sigma}(c_n(\psi_0, \ldots, \psi_{n-1}) = c_n(\widetilde{\sigma}(\psi_0)), \ldots, \widetilde{\sigma}(\psi_{n-1}))$, for all $c_n \in \Sigma_n$, $n \in \omega$; it follows that for any $\theta(x_0, \ldots, x_{n-1}) \in F(\Sigma)$ $\widetilde{\sigma}(\theta(x_0, \ldots, x_{n-1})) = \theta(\sigma(x_0), \ldots, \sigma(x_{n-1}))$. The *identity substitution* induces the identity homomorphism on the formula algebra; the *composition substitution* of the substitutions $\sigma', \sigma : X \longrightarrow F(\Sigma)$ is the substitution $\sigma'' : X \longrightarrow F(\Sigma)$ , $\sigma'' = \sigma' \star \sigma := \widetilde{\sigma'} \circ \sigma$ and $\widetilde{\sigma''} = \widetilde{\sigma' \star \sigma} = \widetilde{\sigma'} \circ \widetilde{\sigma}$.

**(3)** Let $f : \Sigma \longrightarrow \Sigma'$ be a $\mathcal{S}_s$-morphism. Then for each substitution $\sigma : X \longrightarrow F(\Sigma)$ there is a substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma'} \circ \hat{f} = \hat{f} \circ \widetilde{\sigma}$.

## 1.2 The category $\mathcal{S}_f$

We will write $\mathcal{S}_f$ for the category of signatures and *flexible* morphisms of signatures presented in the series of papers [8], [9], [10], [11], [15] and described below.

We introduce the following notations:
• If $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ is a signature, then write $T(\Sigma) := (F(\Sigma)[n])_{n\in\mathbb{N}}$; clearly $T(\Sigma)$ satisfies the "disjunction condition", then it is a signature too.
• For each signature $\Sigma$ and $n \in \mathbb{N}$, let the function:
$(j_\Sigma)_n : \Sigma_n \longrightarrow F(\Sigma)[n] \quad : \quad c_n \mapsto c_n(x_0, \dots, x_{n-1})$.
We have the inverse bijections (just notations):

$$h \in \mathcal{S}_f(\Sigma, \Sigma') \leftrightsquigarrow h^\sharp \in \mathcal{S}_s(\Sigma, T(\Sigma'));$$
$$f \in \mathcal{S}_s(\Sigma, T(\Sigma')) \leftrightsquigarrow f^\flat \in \mathcal{S}_f(\Sigma, \Sigma').$$

Thus: $\mathcal{S}_f(\Sigma, \Sigma') = \mathcal{S}_s(\Sigma, T(\Sigma'))$; moreover $j_\Sigma = id_\Sigma^{(f)} \in \mathcal{S}_f(\Sigma, \Sigma)$.

For each morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_f$ there is a unique function $\check{f} : F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of* $f$, such that:
(i) $\check{f}(x) = x$, if $x \in X$;
(ii) $\check{f}(c_n(\psi_0, \dots, \psi_{n-1}) = (f_n(c_n)(x_0, \dots, x_{n-1}))[x_0 \mid \check{f}(\psi_0), \dots, x_{n-1} \mid \check{f}(\psi_{n-1}))$, if $c_n \in \Sigma_n$.
The notion of extension of $\mathcal{S}_f$-morphism to formula algebras shares many properties with notion of extension of $\mathcal{S}_s$-morphism to formula algebras: e.g., the properties **(1)**, **(2)**, **(3)**.
The composition in $\mathcal{S}_f$ is given by $(f' \bullet f)^\sharp := (\check{f'}\restriction_n \circ (f^\sharp)_n)_{n\in\mathbb{N}}$. The identity $id_\Sigma^{(f)}$ in $\mathcal{S}_f$ is given by $id_\Sigma^{(f)\sharp} = ((j_\Sigma)_n)_{n\in\mathbb{N}}$.

Remark that the "information encoded" by the of extension of $\mathcal{S}_f$-morphism is enough to determine that morphism. More precisely, given $g, f \in \mathcal{S}_f(\Sigma, \Sigma')$, note that:
$*$ $(f^\sharp)_n = \check{f}\restriction_n \circ (j_\Sigma)_n$, $n \in \mathbb{N}$;
$*$ $\check{f} = \check{g} \Rightarrow f^\sharp = (\check{f}\restriction_n)_{n\in\mathbb{N}} \circ j_\Sigma = (\check{g}\restriction_n)_{n\in\mathbb{N}} \circ j_\Sigma = g^\sharp \Rightarrow f = g$.
For the reader's convenience we add here the proof that $\mathcal{S}_f$ is a category:
$*$ identity: $f \bullet id_\Sigma = f = id_{\Sigma'} \bullet f$:
$(f \bullet id_\Sigma)_n^\sharp = \check{f}\restriction_n \circ (id_\Sigma)_n^\sharp = \check{f}\restriction_n \circ (j_\Sigma)_n = (f^\sharp)_n$;
$(id_{\Sigma'} \bullet f)_n^\sharp = \check{id}_{\Sigma'}\restriction_n \circ (f^\sharp)_n = id_{F(\Sigma')[n]} \circ (f^\sharp)_n = (f^\sharp)_n$.
$*$ associativity: $(f'' \bullet f') \bullet f = f'' \bullet (f' \bullet f)$:

$((f'' \bullet f') \bullet f)^{\sharp}_{n} = (f'' \check{\bullet} f')\lceil_n \circ (f^{\sharp})_n =$
$(\check{f''} \circ \check{f'})\lceil_n \circ (f^{\sharp})_n = (\check{f''}\lceil_n \circ \check{f'}\lceil_n) \circ (\check{f}\lceil_n \circ (j_\Sigma)_n) =$
$\check{f''}\lceil_n \circ (\check{f'}\lceil_n \circ \check{f}\lceil_n) \circ (j_\Sigma)_n = \check{f''}\lceil_n \circ (\check{f'} \circ \check{f})\lceil_n) \circ (j_\Sigma)_n = \check{f''}\lceil_n \circ (f' \check{\bullet} f)\lceil_n$
$\circ (j_\Sigma)_n = \check{f''}\lceil_n \circ ((f' \bullet f)^{\sharp})_n = (f'' \bullet (f' \bullet f))^{\sharp}_{n}.$

**Definition 1.1** *A $\mathcal{S}_f$-morphism $f : \Sigma \longrightarrow \Sigma'$ is* regular *if $compl(\check{f}(\theta)) \geq compl(\theta)$, any $\theta \in F(\Sigma)$.*

**Proposition 1.2** *(a) The mapping $f \in \mathcal{S}_s(\Sigma, \Sigma') \mapsto (j_{\Sigma'} \circ f)^{\flat} \in \mathcal{S}_f(\Sigma, \Sigma')$ is a* (natural) bijection $\mathcal{S}_s(\Sigma, \Sigma') \overset{\cong}{\longrightarrow} \{h \in \mathcal{S}_f(\Sigma, \Sigma') : compl(\check{h}(\theta)) = compl(\theta),$ *for any $\theta \in F(\Sigma)\}$.*
*(b) If $f \in \mathcal{S}_f(\Sigma, \Sigma')$, then: $f$ is regular iff $f(c_1) \neq x_0$ , all $c_1 \in \Sigma_1$.*
*(c) The "empty" signature is the unique initial object of $\mathcal{S}_f$ (as in $\mathcal{S}_s$).*
*(d) If a (non full) subcategory of $\mathcal{S}_f$ with the same objects and "copies" of $\mathcal{S}_s$ arrows have only regular morphisms, then it has a* strict [3] *initial object.*

**Proof.**

(a) The proof follows by induction in the complexity of formulas.

(b) If $f(c_1) = x_0$ for some $c_1 \in \Sigma_1$, so $compl(\check{f}(c_1(x_0))) = compl(x_0) = 0 < 1 = compl(c_1(x_0))$. Therefore, $f$ isn't regular.

If $f(c_1) \neq x_0$, so exists a formula $\varphi \in F(\Sigma)[0]$ such that $f(c_1) = \varphi \neq x_0$. So, for definition of $\Sigma_0$ as free algebra, exists a $c'_n \in \Sigma_n$ and $\psi_0, \cdots, \psi_{n-1}$ such that $\varphi = c'_n(\psi_0, \cdots, \psi_{n-1})$. In this way, $compl(f(c_n))f(c'_n(\psi_0, \cdots, \psi_{n-1})) = 1 + compl(\psi_0) + \cdots + compl(\psi_{n-1})) > 1 = compl(c_n(x_0, \cdots, x_{n-1}))$.

So, by induction on complexity of formulas, if for all $c_1 \in \Sigma_1$, $f(c_1) \neq x_0$, $f$ is regular.

(c) The initial signature is empty in every coordinate.

(d) If a subcategory $\mathcal{S}$ of $\mathcal{S}_f$ contains $\mathcal{S}_s$, then it has the empty signature as unique initial object. On the other hand, if $\Sigma_i$ is the initial object and $\Sigma \xrightarrow{f} \Sigma_i$ is a $\mathcal{S}$ morphism, so the image of each connective can be only a variable, that has arity 1. Therefore, $\Sigma$ can only have connectives of arity 1. But if $f(c_1) = x_0$, $f$ is not regular. So, $\Sigma$ needs to be the initial object too, thus $\Sigma_i$ is *strict* initial object. ■

**Proposition 1.3** *(a) $\mathcal{S}_f$ has weak terminal objects.*[4] *More precisely, a signature $\Sigma'$ is an weak terminal object iff $F(\Sigma')[n] \neq \emptyset, \forall n \in \mathbb{N}$ iff $\Sigma'_0 \neq \emptyset$ and*

[3] A initial object 0 is *strict* if, and only if, for all morphism $f : x \to 0$, $f$ is a isomorphism.

[4] A object 1 is weak terminal if, and only if, for all object $x$, exists a (not necessarily unique) morphism $f : x \to 1$.

*exists* $k \geqslant 2$ *such that* $\Sigma'_k \neq \emptyset$.
*(b)* $\mathcal{S}_f$ *does not have terminal object. (Example: let* $\Sigma'$ *be a weak terminal object and take a signature* $\Sigma$ *with only one conective and it is binary: as* $F(\Sigma')[2]$ *is* infinity, *there are many* $\mathcal{S}_f$*-morphisms from* $\Sigma$ *into* $\Sigma'$*.)*

**Remark 1.4** *It is easy to see that* $\mathcal{S}_f$ *has weak products: a weak product of a (small) family of signatures can be given by taking the product signature in the* strict category $\mathcal{S}_s$ *and the corresponding* $\mathcal{S}_s$*-projections, transformed into* $\mathcal{S}_f$*-morphisms (see the next section). As* $\mathcal{S}_f$ *has initial object, any family of parallel arrows has an weak equalizer.*

## 2 The fundamental adjunction

In this section we define functors between the categories $\mathcal{S}_s$ and $\mathcal{S}_f$ and prove that they establishes an adjunction, thus allowing pass some information from one category to another.

**Proposition 2.1** Connecting categories of signatures:
*(a) We have the (faithful) functors:*

$$(+) : \mathcal{S}_s \longrightarrow \mathcal{S}_f \quad : \quad (\Sigma \xrightarrow{f} \Sigma') \mapsto (\Sigma \xrightarrow{(j_{\Sigma'} \circ f)^\flat} \Sigma');$$

$$(-) : \mathcal{S}_f \longrightarrow \mathcal{S}_s \quad : \quad (\Sigma \xrightarrow{h} \Sigma') \mapsto ((F(\Sigma)[n])_{n\in\mathbb{N}} \xrightarrow{(\check{h}\restriction_n)_{n\in\mathbb{N}}} (F(\Sigma')[n])_{n\in\mathbb{N}}).$$

$$T : \mathcal{S}_s \longrightarrow \mathcal{S}_s \quad (\Sigma \xrightarrow{f} \Sigma') \overset{T}{\mapsto} ((F(\Sigma)[n])_{n\in\mathbb{N}} \xrightarrow{(\hat{f}\restriction_n)_{n\in\mathbb{N}}} (F(\Sigma')[n])_{n\in\mathbb{N}}).$$

*(b) For each* $f \in \mathcal{S}_s(\Sigma, \Sigma')$*, we have* $(\check{f^+}) = \hat{f} \in Set(F(\Sigma), F(\Sigma'))$*.*
*(c) We have the* natural transformations*:*

$$\eta : Id_{\mathcal{S}_s} \longrightarrow (-) \circ (+) \ : \ (\eta_\Sigma)_n := (j_\Sigma)_n$$
$$\varepsilon : (+) \circ (-) \longrightarrow Id_{\mathcal{S}_f} \ : \ (\varepsilon_\Sigma)^\sharp_n := id_{F(\Sigma)[n]}$$

*and we write*

$$\mu = (-)\varepsilon(+). \ (\mu_\Sigma)_n : T \circ T(\Sigma)_n \to T(\Sigma)_n$$

Note that the endofunctor $T$ maps each formula $\varphi$ to a connective $\llcorner\varphi\lrcorner$. The natural transformation $\varepsilon_\Sigma : T(\Sigma) \to \Sigma$ maps each connective $\llcorner\varphi\lrcorner$ to a formula $\varphi(x_0, \cdots, x_{n-1})$

The following technical results are fundamental to establish the main results of this work.

**Lemma 2.2** *The explicit definition of* $\mu$ *is:*
$\mu_{\Sigma n} = \check{\varepsilon}_\Sigma \restriction_n : F(T\Sigma)[n] \to F(\Sigma)[n]$
$\mu_{\Sigma n}(\llcorner\varphi\lrcorner(\psi_0, \cdots, \psi_{n-1})) = \check{\varepsilon}_\Sigma \restriction_n (\llcorner\varphi\lrcorner(\psi_0, \cdots, \psi_{n-1})) =$
$\varphi(x_0, \cdots, x_{n-1})[x_0|(\check{\varepsilon}_\Sigma \restriction_n (\psi_0), \cdots, x_{n-1}|\check{\varepsilon}_\Sigma \restriction_n (\psi_{n-1}))]$

**Proof.** The proof follows directly for definitions and notions in the Proposition 2.1:

$$\mu_\Sigma = \varepsilon^-_{+\Sigma} = \varepsilon^-_{\Sigma} = (\check{\varepsilon}_\Sigma \restriction_n)_{n\in\mathbb{N}} \qquad \blacksquare$$

Note that the natural transformation $\mu$ modifies the complexities (since it transforms a derived connective $\llcorner\varphi\lrcorner$ into a formula $\varphi$) but does not loses "information".

**Lemma 2.3** *Let $g \in \mathcal{S}_s(\Sigma, \Sigma')$, $f \in \mathcal{S}_s(\Sigma', \Sigma'')$, $k \in \mathcal{S}_f(\Sigma, \Sigma')$ and $h \in \mathcal{S}_f(\Sigma', \Sigma'')$. Then:*

*(a)* $T(id_\Sigma) = id_{T\Sigma}$

*(b)* $T(f \circ g) = Tf \circ Tg$

*(c)* $Tf \circ j_\Sigma = j_{\Sigma'} \circ f$

*(d)* $\mu_\Sigma \circ Tj_\Sigma = id_{T\Sigma}$

*(e)* $f^{+\sharp} = j_{\Sigma'} \circ f$

*(f)* $h^- = \mu_{\Sigma''} \circ Th^\sharp$

*(g)* $(h \bullet k)^\sharp = \mu_{\Sigma''} \circ Th^\sharp \circ k^\sharp$

*(h)* $(f \circ g)^+ = (Tf \circ j_{\Sigma'} \circ f)^\flat$

*(i)* $\mu_{\Sigma''} \circ Th^- = h^- \circ \mu_{\Sigma'}$

**Proof.**

(a) $T(id_\Sigma) = (\check{id} \restriction_n)_\mathbb{N} = id_{T\Sigma}$

(b) $T(f \circ g) = ((f \check{\circ} g) \restriction_n)_\mathbb{N} = (\check{f} \restriction_n \circ \check{g} \restriction_n)_\mathbb{N} = T(f) \circ T(g)$

(c) $Tf \circ j_\Sigma(c_n) = Tf(c_n(x_0, \cdots, x_{n-1})) = \check{f} \restriction_n (c_n(x_0, \cdots, x_{n-1})) = f(c_n)(x_0, \cdots, x_{n-1}) = j_{\Sigma'}(f(c_n)) = j_{\Sigma'} \circ f(c_n)$

(d) $\mu_\Sigma \circ Tj_\Sigma(\llcorner\varphi\lrcorner) = \mu_\Sigma(\llcorner\llcorner\varphi\lrcorner(x_0, \cdots, x_{n-1}\lrcorner)) =$
$\llcorner\varphi\lrcorner(x_0, \cdots, x_{n-1})[x_0|x_0, \cdots, x_{n-1}|x_{n-1}] = \llcorner\varphi\lrcorner = id_{T\Sigma}(\llcorner\varphi\lrcorner)$

(e) $f^+ = (j_{\Sigma'} \circ f)^\flat \Leftrightarrow f^{+\sharp} = j_{\Sigma'} \circ f$

(f) $h^-(\llcorner\varphi(x_0, \cdots, x_{n-1})\lrcorner) = \check{h}(\varphi(x_0, \cdots, x_{n-1})) = \mu_{\Sigma''}(\llcorner\varphi(x_0, \cdots, x_{n-1})\lrcorner)$

(g) $(h \bullet k)^\sharp = (\check{h} \restriction_n \circ k^\sharp_n)_{n\in\mathbb{N}} = \mu_{\Sigma''} \circ Th^\sharp \circ k^\sharp$

(h) $(f \circ g)^+ = (j_\Sigma \circ f \circ g)^\flat = (Tf \circ j_{\Sigma'} \circ f)^\flat$

(i) $Th^- \circ j_{\Sigma'} = j_{\Sigma''} \circ h^- \Rightarrow \mu_{\Sigma''} \circ Th^- \circ j_{\Sigma'} = \mu_{\Sigma''} \circ j_{\Sigma''} \circ h^- \Rightarrow \mu_{\Sigma''} \circ Th^- \circ j_{\Sigma'} = h^- \Rightarrow \mu_{\Sigma''} \circ Th^- = h^- \circ \mu_{\Sigma'}$ ■

**Proposition 2.4** $(+)$ *and* $(-)$ *are functors.*

**Proof.**
$*$ $(+)$ is a functor:

$(id^{(s)}{}_\Sigma)^+ \underset{def}{=} (j_\Sigma)^\flat = id^{(f)}{}_\Sigma$

$f^+ \bullet g^+ \underset{2.3g}{=} (\mu_{\Sigma''} \circ Tf^{+\sharp} \circ g^{+\sharp})^\flat \underset{2.3e}{=} \mu_{\Sigma''} \circ T(j_{\Sigma'} \circ f) \circ j_{\Sigma'} \circ g)^\flat \underset{2.3b}{=} \mu_{\Sigma''} \circ T(j_{\Sigma'}) \circ Tf \circ j_{\Sigma'} \circ g)^\flat \underset{2.3d}{=} Tf \circ j_{\Sigma'} \circ g \underset{2.3h}{=} (f \circ g)^+$

$*$ $(-)$ is a functor:

$(id_\Sigma)^- = ((\check{id_\Sigma})_{\restriction n})_{n\in\mathbb{N}} = (id_{F(\Sigma)[n]})_{n\in\mathbb{N}}$

$(h \bullet k)^- \underset{2.3f}{=} \mu_{\Sigma''} \circ T(h \bullet k)^\sharp \underset{2.3g}{=} \mu_{\Sigma''} \circ T(\mu_{\Sigma''} \circ Th^\sharp \circ k^\sharp) \underset{2.3b}{=} \mu_{\Sigma'} \circ T\mu_{\Sigma''} \circ T^2h^\sharp \circ Tk^\sharp \underset{2.3f}{=} \mu_{\Sigma''} \circ Th^- \circ Tk^\sharp \underset{2.3i}{=} h^- \circ \mu_{\Sigma'} \circ Tk^\sharp \underset{2.3f}{=} h^- \circ k^-$ ■

**Theorem 2.5** *The (faithful) functor* $(+)$ *is a* left adjoint *of the (faithful) functor* $(-)$*:* $\eta$ *and* $\varepsilon$ *are, respectively, the unit and the counit of the adjunction.*

**Proof.** (*Sketch*)
$*$ $(+)$ is the left adjoint of $(-)$. Is suficient to show the triangular identities:

$(\varepsilon_{+\Sigma} \bullet \eta^+{}_\Sigma)^\sharp \underset{2.3g}{=} \mu_\Sigma \circ T\varepsilon^\sharp_\Sigma \circ \eta^\sharp_\Sigma \underset{2.1c}{=} \mu_\Sigma \circ T(id_\Sigma) \circ \eta^\sharp_\Sigma \underset{2.3a}{=} \mu_\Sigma \circ Tj_\Sigma \underset{2.3d}{=} id_{T\Sigma} \Rightarrow \varepsilon_{+\Sigma} \bullet \eta^+{}_\Sigma = id_\Sigma$

$\varepsilon^-_\Sigma \circ \eta_{-\Sigma} \underset{2.3f}{=} \mu_\Sigma \circ T\varepsilon^\sharp_\Sigma \circ \eta_{T\Sigma} \underset{2.1c}{=} \mu_\Sigma \circ T(id_\Sigma) \circ \eta_{T\Sigma} \underset{2.1c}{=} \mu_\Sigma \circ T(id_\Sigma) \circ Tj_\Sigma \underset{2.3a}{=} \mu_\Sigma \circ Tj_\Sigma \underset{2.3d}{=} id_{T\Sigma}$

$*$ $(+)$ is faithful. Lets $f$ and $g \in \mathcal{S}_s(\Sigma, \Sigma')$:

If for all $c_n \in \Sigma$, $f^+(c_n) = g^+(c_n) = c'{}_n(x_0, \cdots, x_{n-1})$, then, for all $c_n \in \Sigma$, $f(c_n) = g(c_n)$.

$*$ $(-)$ is faithful. Let $f$ and $g \in \mathcal{S}_f(\Sigma, \Sigma')$:

If for all $c_n \in \Sigma$, $f^-(c_n) = g^-(c_n) = \varphi(x_0, \cdots, x_{n-1})$, then, for all $c_n \in \Sigma$, $f(\llcorner c_n \lrcorner) = g(\llcorner c_n \lrcorner) = \llcorner \varphi \lrcorner$. ∎

The above Theorem immediately yields the:

**Corollary 2.6** *(a) The functor* $(+)$ *preserves colimits and the functor* $(-)$ *preserves limits.*
*(b) Both the functors* $(+)$ *and* $(-)$ *reflect epimorphisms and monomorphisms.*

We finish this section stating the following result.

**Proposition 2.7** *Let* $h \in \mathcal{S}_f(\Sigma, \Sigma')$*:*
*(a) If* $h^-$ *is a* $\mathcal{S}_s$*-epimorphism, then* $h$ *is* $\mathcal{S}_f$*-epimorphism.*
*(b)* $h$ *is a* $\mathcal{S}_f$*-monomorphism if and only if* $h^-$ *is a* $\mathcal{S}_s$*-monomorphism.*
*(c) If* $h$ *is an* $\mathcal{S}_f$*-isomorphism, then "*$h \in \mathcal{S}_s$*", i.e. there is a (unique)* $\mathcal{S}_s$*-(iso)morphism* $f$ *such that* $h = f^+$*; in particular,* $h$ *is regular.*
*(d) If* $h$ *is a* $\mathcal{S}_f$*-section, then* $h$ *is regular and if* $g \bullet h = id$ *for some* $\mathcal{S}_f$*-morphism* $g$ *that is regular over the "image signature of* $h$*" (i.e. the signature whose connectives effectively occur in the image of some* $h_n$*,* $n \in \mathbb{N}$*), then "*$h \in \mathcal{S}_s$*".*

**Proof.**
(a) $h^- \in \mathcal{S}_s(\Sigma, \Sigma')$ is a $\mathcal{S}_s$-epimorphism iff for all $\psi \in F(\Sigma')$, exists $\varphi \in F(\Sigma)$, such that $h^-(\llcorner \varphi \lrcorner) = \llcorner \psi \lrcorner$. Therefore, for definition (2.1), $h(\varphi) = \psi$.

(b) The prove follows by induction on complexity. Note that $h(c_n) = h(c'{}_n) \Leftrightarrow h^-(\llcorner c_n \lrcorner) = h^-(\llcorner c'{}_n \lrcorner)$. So, $h$ is $\mathcal{S}_s - monomorphism$ iff $h(c_n) = h(c'{}_n) \Rightarrow c_n = c'{}_n$ iff $h^-(\llcorner c_n \lrcorner) = h^-(\llcorner c'{}_n \lrcorner) \Rightarrow c_n = c'{}_n$. Suppose that if $compl(\varphi) < k$ and $compl(\varphi') < k$, $h^-(\llcorner \varphi \lrcorner) = h^-(\llcorner \varphi' \lrcorner) \Leftrightarrow h(c_n) = h(c'{}_n)$, for all $c_n$ that occurs in $\varphi$ and all $c'{}_n$ that occurs in $\varphi'$. Then, if $\psi = c_n(\varphi_0, \cdots, \varphi_{n-1})$ and $\psi' = c'_n(\varphi'{}_0, \cdots, \varphi'{}_{n-1})$, for formulas with complexity less than $k$, $h^-(\llcorner \psi \lrcorner) = h^-(\llcorner \psi' \lrcorner)$ iff $h^-(\llcorner c_n \lrcorner)(h^-(\llcorner \varphi_0, \cdots, \varphi_{n-1} \lrcorner)) =$
$h^-(\llcorner c'_n \lrcorner)(h^-(\llcorner \varphi'_0, \cdots, \varphi'_{n-1} \lrcorner))$ iff $h(c_n)(h^-(\llcorner \varphi_0, \cdots, \varphi_{n-1} \lrcorner)) =$
$h^-(c'_n)(h^-(\llcorner \varphi'_0, \cdots, \varphi'_{n-1} \lrcorner))$ iff $h(c_n) = h^-(c'_n)$ and for all $0 \leq i < n$, $h^-(\varphi_i) = h^-(\varphi'_i)$.

(c) If $h \in \mathcal{S}_f(\Sigma, \Sigma')$ is a isomorphism, then exists $k \in \mathcal{S}_f(\Sigma', \Sigma)$ such that $k \bullet h = id_{\Sigma}^{(f)}$ and $h \bullet k = id_{\Sigma'}^{(f)}$. So, the complexity of the formulas in $F(\Sigma)$ or $F(\Sigma')$ need to be preserved by $k \bullet h$ and $h \bullet k$ respectively. If $h$ or $k$ are not regular, then the composite of then are regular too, and exists a connective $c_1 \in \Sigma_1$ such that $h(c_1) = x_0$ or exists a connective $k \in \Sigma'_1$ such that $k(c_1) = x_0$. So, for this fixed $c_1$, $k \bullet h(c_1) = x_0$ or $h \bullet k(c_1) = x_0$. Thus, for absurd, $h$ and $k$ are regular morphisms. In other hand, if $h$ or $k$ increases the complexity, $h \bullet k$ and $h \bullet k$ will increases the complexity too. Thus, $h$ is not a isomorphism. Finally, since isomorphism $h$ preserves complexity, for all $c_n \in \Sigma_n$, exists $c'_n \in \Sigma'_n$ such hat $h(c) = c'(x_0, \cdots, x_{n-1})$. Fix $f \in \mathcal{S}_s(\Sigma, \Sigma')$ defined by $f(c_n) = c'_n$, then $h = f^+$.

(d) Analogously, if $h \in \mathcal{S}_f(\Sigma, \Sigma')$ is a section, exists $k \in \mathcal{S}_f(\Sigma', \Sigma)$ such that $k \bullet h = id_{\Sigma}^{(f)}$ and the complexity also need to be conserved as above. If $g \bullet h = id^{(s)}$, as above in item (c), $g(h(c_1))$ needs to be equal to $c_1$, for all $c_1$ in the "image" of $h$. ■

## 3 The monad and its properties

Clearly, $T = (-) \circ (+)$ and it is a faithful functor. Let $\mathcal{T} = (T, \eta, \mu)$ be the monad (or triple) associated to the adjunction $(\eta, \varepsilon) : \mathcal{S}_s \underset{(-)}{\overset{(+)}{\rightleftarrows}} \mathcal{S}_f$.

**Proposition 3.1** *The functor $T$ reflects isomorphisms (respectively: monomorphisms, epimorphisms).*

**Proof.** First remark that, for each signature $\Sigma$ and $n \in \mathbb{N}$, $(\eta_\Sigma)_n : \Sigma_n \rightarrowtail F(\Sigma)[n]$ establish a bijection between $\Sigma_n$ and $\{\theta \in F(\Sigma)_n : compl(\theta) = 1\}$. Now let $f : \Sigma \longrightarrow \Sigma'$ a $\mathcal{S}_s$-morphism such that $T(f)$ is a $\mathcal{S}_s$-isomorphism (respectively: a $\mathcal{S}_s$-monomorphism, a $\mathcal{S}_s$-epimorphism). Then, for each $n \in \mathbb{N}$, $\hat{f}\restriction_n: F(\Sigma)[n] \longrightarrow F(\Sigma')[n]$ is a bijection (respectively: a injection, a surjection) and, as $compl(\hat{f}(\theta)) = compl(\theta)$ for each $\theta \in F(\Sigma)$, $\hat{f}\restriction_n$ restricts to a bijection (respectively: a injection, a surjection) between $\{\theta \in F(\Sigma)[n] : compl(\theta) = 1\}$ and $\{\theta' \in F(\Sigma')[n] : compl(\theta') = 1\}$. Finally, as $\eta_{\Sigma'} \circ f = T(f) \circ \eta_\Sigma$, we conclude that $f_n : \Sigma_n \longrightarrow \Sigma'_n$ is a bijection (respectively: a injection, a surjection), for each $n \in N$, as we need. ■

**Proposition 3.2** *The functor $T$ preserves directed colimits (i.e., colimits of diagrams over upward directed posets). More explicitly, let $(I, \leqslant)$ be an upward directed poset and $D : (I, \leqslant) \longrightarrow \mathcal{S}_s : i \mapsto \Sigma_i$ be a diagram in $\mathcal{S}_s$;*

$(\Sigma, (\Sigma_i \overset{\alpha_i}{\to} \Sigma)_{i \in I})$ *denotes the colimit of* $D$ *in* $\mathcal{S}_s$; $(\Sigma', (T(\Sigma_i) \overset{\alpha'_i}{\to} \Sigma')_{i \in I})$ *be the colimit of* $T \circ D$ *in* $\mathcal{S}_s$; $(S, (F(\Sigma_i) \overset{\beta_i}{\to} S)_{i \in I})$ *denotes the colimit of* $(\hat{\ }) \circ D$ *in the category Set, then:*

*(a) The canonical function* $S \to F(\Sigma)$, *denoted* $k : colim_{i \in I} F(\Sigma_i) \longrightarrow F(colim_{i \in I} \Sigma_i)$, *i.e. the unique function such that* $k \circ \beta_i = \hat{\alpha}_i$, $i \in I$, *is a* bijection.

*(b) The canonical* $\mathcal{S}_s$*-morphism* $can : colim_{i \in I} T(\Sigma_i) \longrightarrow T(colim_{i \in I} \Sigma_i)$, *i.e. the unique* $\mathcal{S}_s$*-morphism such that* $can \circ \alpha'_i = T(\alpha_i)$, $i \in I$, *is a* $\underline{\mathcal{S}_s\textit{-isomorphism}}$. *It is given by sequence of* bijections

$$can_n : colim_{i \in I}(F(\Sigma_i)[n]) \longrightarrow F(colim_{i \in I} \Sigma_i)[n], n \in \mathbb{N}$$

*obtained from the "restrictions" of the canonical bijection* $k$ *just above.*

**Proof.** (*Sketch*) For a proof of item (a) we apply a "global reasoning": we consider formula algebras and apply induction on complexity of formulas. For (b): we extract "local" information from the result is (a), i.e., we consider convenient "restrictions" to the subsets $F(\Sigma)[n]$, $n \in \mathbb{N}$. ■

The same technique of proof in the Proposition above gives us the Theorem below:

**Theorem 3.3** *Let* $\mathcal{T} = (T, \eta, \mu)$ *be the monad associated to the adjunction* $(\eta, \varepsilon) : \mathcal{S}_s \underset{(-)}{\overset{(+)}{\rightleftarrows}} \mathcal{S}_f$ *(i.e.,* $\mu = (+)\varepsilon(-)$*) is such that* $Kleisli(\mathcal{T}) = \mathcal{S}_f$. *Moreover, the functors* $(+)$ *and* $(-)$ *are precisely the canonical functors associated to the adjunction of the Kleisli category of a monad. More explicitly: given* $(\Sigma \xrightarrow{f} \Sigma' \xrightarrow{f'} \Sigma'')$ *in* $\mathcal{S}_f$, *then* $f' \bullet f = (\mu_{\Sigma''} \circ T(f'^{\sharp}) \circ f^{\sharp})^{\flat}$ *,i.e., we have in* $\mathcal{S}_s$:

$$(\Sigma \xrightarrow{f^{\sharp}} T(\Sigma') \xrightarrow{(\check{f}'\restriction_n)_{n \in \mathbb{N}}} T(\Sigma'')) \quad =$$
$$(\Sigma \xrightarrow{f^{\sharp}} T(\Sigma') \xrightarrow{T(f'^{\sharp})} T \circ T(\Sigma'') \xrightarrow{\mu_{\Sigma''}} T(\Sigma'')).$$

**Proof.** This follows directly from item (g) in Lemma 2.3 ■

**Corollary 3.4** *The category* $\mathcal{S}_f$ *has colimits for any (small) diagram "in* $\mathcal{S}_s$*", i.e., given* $\mathcal{I}$ *a small category and a diagram* $D : \mathcal{I} \longrightarrow \mathcal{S}_s$, *the category* $\mathcal{S}_f$ *has a colimit for the diagram* $(+) \circ D : \mathcal{I} \longrightarrow \mathcal{S}_f$. *In particular,* $\mathcal{S}_f$ *has all (small) coproducts.*

**Remark 3.5** *Let $\mathcal{L}_f$ be the category of (Tarskian) logics and* flexible *signature morphisms that induces consequence relations preserving functions on the formula algebras. Then there is an obvious forgetful functor $U_f : \mathcal{L}_f \longrightarrow \mathcal{S}_f$, and it has left and right adjoints, then $U_f$ preserves limits and colimits. As $U_f$ also "lift" limits and colimits, then given a small category $\mathcal{I}$, $\mathcal{L}_f$ is $\mathcal{I}$-complete (respectively, $\mathcal{I}$-cocomplete)* if and only if *$\mathcal{S}_f$ is $\mathcal{I}$-complete (respectively, $\mathcal{I}$-cocomplete). Thus the Corollary above entails that $\mathcal{L}_f$ has colimits for any (small) diagram "in $\mathcal{L}_s$", in particular, it has "unrestricted fibrings" (= coproducts).*

# 4 Final remarks and future works

In the present work we have provided a detailed account on two categories of signatures considered by logicians that works with propositional logics.

In the sequence of works, [2], [3], [4] is proven that the set of (Tarskian) logics defined over a given signature has a natural structure of algebraic lattice; the category of (Tarskian) logics and *strict* signature morphisms that induces consequence relations preserving functions on the formula algebras is a $\omega$-locally presentable category ([1]); the category of Blok-Pigozzi algebraizable logics ([6]) and *strict* signature morphisms that induces consequence relations and algebraizing pairs preserving functions on the formula algebras is a relatively complete $\omega$-accessible category ([1]).

In [18], a sequel of the present work, we will focus on categories of propositional logics: (i) analyzing categories previously defined in other papers; (ii) presenting new (and more suitable) categories of logics. In there, we consider (finitary, propositional) logics through the original use of Category Theory: the study of the " sociology of mathematical objects", aligning us with a recent, and growing, trend of study logics through its relations with other logics (e.g. process of combinations of logics as fibring [16] and possible translation semantics [12]). So will be objects of study the classes *of* logics, i.e. categories whose objects are logical systems (i.e., a signature with a Tarskian consequence relation) and the morphisms are related to (some concept of) translations between these systems. This provides the first steps of a project of considering categories of logical systems satisfying *simultaneously* certain natural requirements (it seems that in the literature ([2], [3], [4], [8], [9], [10], [11], [15]) this is achieved only partially): **(i)**, **(ii)**, **(iii)** and **(iv)**.

## References


[1] J. Adámek, J. Rosický, **Locally Presentable and Accessible Categories**, Lecture Notes Series of the LMS **189**, Cambridge University Press, Cambridge, Great Britain, 1994.

[2] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *Fibring and Sheaves*, Proceedings of IICAI-05, Special Session at the 2nd Indian International Conference on Artificial Intelligence, Pune, India, 2005.

[3] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *On the category of algebraizable logics*, *CLE e-Prints* **vol.6 n.1**, (2006), `http://www.cle.unicamp.br/e-prints`.

[4] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *A global glance on categories in Logic*, Logica Universalis **1(1)** (2007), 3–39.

[5] J.-Y. Béziau, *From Consequence Operator to Universal Logic: A Survey of General Abstract Logic*, in **Logica Universalis: Towards a General Theory of Logic** (J.-Y. Béziau, ed.), Birkhäuser, 2007, 3–17.

[6] W. J. Blok, D. Pigozzi, **Algebraizable logics**, Memoirs of the AMS **396**, American Mathematical Society, Providence, USA, 1989.

[7] F. Borceux, **Handbook of Categorical Algebra**, Encyclopedia of Mathematics and its Applications **50,51,52**, Cambridge University Press, Cambridge, Great Britain, 1994.

[8] J. Bueno-Soler, W.A. Carnielli, *Possible-translations algebraization for paraconsistent logics*, *Bulletin of the Section of Logic*, University of Lodz, Poland, vol. 34, n. 2, 2005, pp. 77–92. *CLE e-Prints* **vol.5 n.6**, (2005), `http://www.cle.unicamp.br/e-prints`.

[9] J. Bueno, M.E. Coniglio, W.A. Carnielli, *Finite algebraizability via possible-translations semantics*, *Proceedings of CombLog'04 — Workshop on Combination of Logics: Theory and Applications*, Departamento de Matemática, Instituto Superior Técnico), Lisboa, Portugal, 2004.

[10] J. Bueno-Soler, M.E. Coniglio, W.A. Carnielli, *Possible-Translations Algebraizability*, in: **Handbook of Paraconsistency** (J.-Y. Béziau, W. Carnielli and D.M. Gabbay, eds.), Studies in Logic (Logic & Cognitive Systems), vol. 9, pp. 321–340, College Publications, 2007.

[11] C. Caleiro, R. Gonçalves, *Equipollent logical systems*, **Logica Universalis: Towards a General Theory of Logic** (J.-Y. Béziau, ed.), Birkhäuser, 2007, 97–110.

[12] W. Carnielli, *Many-valued logics and plausible reasoning*, in: Proceedings of the XX International Congress on Many-Valued Logics, IEEE Computer Society, University of Charlotte, USA , (1990), 328-335.

[13] W. Carnielli, M. E. Coniglio, *Combining Logics*, Stanford Encyclopedia of Philosophy, `http://plato.stanford.edu/entries/logic-combining/`, 2011.

[14] J. Czelakowski, **Protoalgebraic logic**, Trends in Logic, Studia Logica Library, Kluwer Academic Publishers, 2001.

[15] V. L. Fernández, M. E. Coniglio, *Fibring in the Leibniz Hierarchy*, Logic Journal of the IGPL **15(5-6)** (2007), 475–501.

[16] D. Gabbay, *Fibred semantics and the weaving of logics: Part 1*, Journal of Symbolic Logic **61(4)** (1996), 1057–1120.

[17] S. MacLane, **Categories for the Working Mathematician**, Graduate Texts in Mathematics **5**, $2^{nd}$ ed., Springer-Verlag, 1998.

[18] H. L. Mariano, C. A. Mendes, *Towards a good notion of categories of logics*, in preparation, preliminary version (2016) available at `https://arxiv.org/pdf/1404.3780.pdf`.

[19] A. Sernadas, C. Sernadas, C. Caleiro, *Fibring of logics as a categorial construction*, Journal of Logic and Computation **9(2)** (1999), 149–179.

[20] J. Szigeti, *On limits and colimits in the Kleisli category*, Cahiers de topologie et géométrie différentielle catégorique **24(4)** (1983), 381–391.

Caio de Andrade Mendes
Department of Mathematics
University of São Paulo (USP)
Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil
*E-mail:* `caio.mendes@usp.br`

Hugo Luiz Mariano
Department of Mathematics
University of São Paulo (USP)
Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil
*E-mail:* `hugomar@ime.usp.br`