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More on Categorial Forms of the Axiom of Choice

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Abstract

In this work, we will be interested on the investigation of **categorial forms** of the Axiom of Choice (**AC**). The results are intended to be further results with respect to [3]. We introduce some more new categorial forms of **AC**, and we discuss a number of categorial versions of the Zorn's Lemma.

Keywords: Axiom of Choice, Category Theory.

Introduction

In the paper [3], the notions of **set forms** and **class forms** of the Axiom of Choice (**AC**) in Category Theory were introduced. For instance, the following statements are set-forms of **AC**:

- *"Every epic arrow of the category has a right inverse"* (that is, *"Every epimorphism has a* **section***"*).
- "*Every product of a non-empty family of non-initial objects is a non-initial object*".
- "*Every discrete, non-empty diagram of non-initial objects has more than one cone*".
- "*Every non-empty discrete diagram of non-initial objects has a non-skeletal cone*".

Those set forms are taken to be statements in the language of category theory which presuppose some equivalence with **AC** in the category **Set** (in a precise way, which will be explained presently).

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Among the statements above mentioned, the one which is more usually related to **AC** within category theory is the first statement, say, ζ:

$$\zeta \equiv \text{"Every epic arrow has a section"}.$$

Indeed, in Category Theory, it is said that "a category $\mathcal{C}$ satisfies **AC**" if the **relativization** $\zeta_{\mathcal{C}}$ of ζ holds in $\mathcal{C}$ – that is, every epic arrow **in** $\mathcal{C}$ has a section **in** $\mathcal{C}$.

Moreover, notice that, clearly, the relativization $\zeta_{\mathbf{Set}}$ may be regarded as the Axiom of Choice itself.

The **class forms** are considered in the same vein, i.e., those are the statements in category theory which presuppose some equivalence of the Axiom of Choice **for Classes** when one assumes the validity of of all of its relativizations for any category; e.g. let ξ be the statement

$$\xi \equiv \text{"The category has a skeleton"}\ ^2.$$

It was proved in [8], that "Every category has a skeleton" – that is, for all $\mathcal{C}$, $\xi_{\mathcal{C}}$ – is an equivalent of the Axiom of Choice for Classes.

The Axiom of Choice for Classes states that every **conglomerate** whose elements are non-empty classes has a choice class-function. Now, let φ be any statement from the language of Category Theory. We have the following definition from [3]:

Definition 0.1 (Categorial forms of the Axiom of Choice, [3]) *1. A statement φ from Category Theory is a* **Categorial Set–Form of the Axiom of Choice** *if the Axiom of Choice is equivalent to the statement $\varphi_{\mathbf{Set}}$.*

2. A statement φ from Category Theory is a **Categorial Class–Form of the Axiom of Choice** *if the validity of $\varphi_{\mathcal{C}}$ for* **all** *categories $\mathcal{C}$ is equivalent to the Axiom of Choice for* **Classes**.

This paper begins with a brief discussion about **CFL**. which is how we refer to the **Categorial Foundation Language**; this is the language were we assume the statements φ as above are actually written. The next section is dedicated to the statement from **CFL** given by *"Epimorphisms are surjective"* (**ES**). This is not a set form of the Axiom of Choice because it is relativization to **Set** is a theorem of **ZF**. However, even in the context of set forms there is

[2]Recall that a skeleton of a category $\mathcal{C}$ is a full, isomorphism-dense subcategory $\mathcal{C}$ in which no two distinct objects are isomorphic; roughly speaking, it is a "minimal" subcategory capturing the categorical properties of $\mathcal{C}$.

some subtleties regarding **ES**, namely certain forms which the category of sets is unable to differentiate from **ES**. By the end of this section we introduce a new set form of AC, namely the statement *"Projections have a right inverse"*

Before starting to explore several categorial versions of the Zorn's Lemma, we dedicate Section 3 to present some considerations on final objects in Category Theory. It is, indeed, some suitable version of the notion of final object will allow us to provide a certain categorial version of "maximality " in order to define categorial versions of Zorn's Lemma. So, we introduce in this section some notions related to final objects, and some of them are obtained by dualizing the variants of initial objects reported in [3] – and others are new. In the Section 4 we establish the categorial "Small" Zorn's Lemma and we proved in 4.3 that this categorial statement of Zorn's Lemma is in fact a set form of AC. Corollary 4.6 provides us that the categorial Small Zorn's Lemma restrict to any category, the categorial Small Zorn's Lemma restrict to poset and the categorial Poset Zorn's Lemma restrict to a Poset, are all equivalent to the usual **AC**.

In the final section, we rise questions and notes about well-ordering of the Universe, the Beth Definability Property and algebraic class forms of **AC**.

1 The notions of CFL and CFL*

Let us introduce, for this continuation of the research of the paper [3], some new terminologies and notations.

Notation 1.1 *We will denote as* **CFL** *the* **Categorial Foundations Language**, *i.e., the language of category theory which is considered to be done starting from some set-theoretical foundation of Category Theory which allows one to deal with sets, classes and conglomerates. After the setting of such structure – and there are several options to do it[3] –, we assume the Axiom of Global Choice (and its equivalences) as a part of the foundation of categories.*

So, if **M** is such a set-theoretic structure on which one can properly deal with the notions of sets, classes and conglomerates (see also the discussion at page 411 of [3]), let us say that

CFL = set theoretical structure **M** + all of the usual categorial constructions + Global Choice

Let us explain the notions of $\mathbf{AC}_{\text{Sets}}, \mathbf{AC}_{\text{Categories}}, \mathbf{AC}_{\text{Global}}$.

First notice that **CFL**, besides of being the environment where we have founded Category Theory, it is also the environment where we actually **work** when we do research with categories. In particular, we are allowed to use Global

[3]We will explain this (in more detail) in what follows.

Choice (together with all of its equivalences and, as one of its consequences, choice for sets) in any categorial argument.

Let $\mathbf{AC}_{\text{Sets}}$ denote the Axiom of Choice for sets. Its presence, as it is well-known, is crucial in Category Theory.

$\mathbf{AC}_{\text{Categories}}$ will denote the statement of the **CFL** language given by "Every epic arrow has a section". Notice that, as **any of our set-forms of the Axiom of Choice**, $\mathbf{AC}_{\text{Categories}}$ **is not** a property of all categories: given a category $\mathcal{C}$, $(\mathbf{AC}_{\text{Categories}})_{\mathcal{C}}$ may hold or not; that is to say, "some categories satisfy **AC**, while others don't". For example, in category of rings with unity, we have that the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is epic and in the presence of a section, this map would be isomorphism.

By definition, categorial set-forms of choice are equivalent in **Set** – but all the interest (and, in fact, all the fun) comes from the fact that they are **not necessarily equivalent** for all categories.

Obviously, $\mathbf{AC}_{\text{Global}}$ will denote the Axiom of Global Choice.

When one proves, in Set Theory, that a certain statement is an equivalence of the Axiom of Choice, the proof has to be performed – of course – in **ZF**, which is choiceless Set Theory. In the same way, we have to take off the choice of **CFL** when we have to proof that some statement of Category Theory is a set-form or a class-form of the Axiom of Choice. So, let

$\mathbf{CFL}^* = \mathbf{CFL}$ - $\mathbf{AC}_{\text{Global}} = \mathbf{M} +$ All usual categorial constructions

Throughout this paper, we are working always **within** $\mathbf{CFL}^*$. Also, we will assume that our set-theoretical structure **M** is given by "**ZF** + two strongly inaccessible cardinals" – but we could also work with some suitable **Grothendieck universe**, which requests only one strongly inaccessible cardinal. Also notice that, when we take off $\mathbf{AC}_{\text{Global}}$, we are taking off all instances of choice of our language – including $\mathbf{AC}_{\text{Sets}}$.

However, the authors are aware that, at some point of the future, intermediate situations could emerge, on which could be somehow more natural to assume only $\mathbf{AC}_{\text{Sets}}$ and take off only the choices strictly above the set level.

Indeed, to show equivalence of set-forms of the axiom of choice, we could say that we would only need to assume choice **up to the level of the first inaccessible cardinal**. This is explained in the next remark.

Remark 1.2 *In order to prove that a certain statement φ is a set-form of choice, we have to prove that*

$$\varphi_{\mathbf{Set}} \iff \mathbf{AC}_{Sets},$$

and of course the proof of such equivalence could be done in a very small fragment of $\mathbf{CFL}^*$ *– more precisely, the argument could be done at the κ-th level of* **M**, *where κ is the* **first** *of the two inaccessible cardinals whose existence we* *have assumed for –, for the argument only needs to deal with* **sets**, *and the κ-th level is precisely where the sets live.* ■

In fact, we could (in an even easier way) identify $\varphi_{\mathbf{Set}}$ with its translation to **ZF** and proceed in **ZF** – as we were just doing some routinary proof of some usual equivalence of the Axiom of Choice.

To facilitate and unify, we will assume that the proofs of equivalences of set-forms are also done in **CFL***.

The next example shows that a category may satisfy one set form and does not satisfy another.

Example 1.3 *We make use of the notation from [3]. Let* **PNI** *denote the set form*

"Every product of a non-empty family of non-initial objects is a non-initial object."

and **CEM** *denote*

"Every non-empty discrete diagram whose objects are all non-initial has a cone where all constituent morphisms are epic arrows".

Then, the locally small category **Top**, *the category of all topological spaces with continuous functions as morphisms, satisfies* **PNI** *(due to* $\mathbf{AC}_{Sets}$ *and Tychonoff product) and* **CEM** *(since the projections are continuous and surjective in* **ZFC**) *– but it* **does not** *satisfy* $\mathbf{AC}_{Categories}$. *To see this, one only needs to pick a continuous bijection between two non-homeomorphic spaces to get an epimorphism which does not have a section. For details, see [3]. Another (and even simplest) example to see this, is the following: consider a space X with two topologies τ_1 and τ_2, such that $\tau_2 \subsetneq \tau_1$ (that is, τ_1 is strictly finer than τ_2), and consider the identity map $id_X : (X, \tau_1) \to (X, \tau_2)$. This map is epic, but cannot have a section.* ■

2 ES – Epimorphisms are surjective

Consider the form **ES**, which is the statement from **CFL** given by *"Epimorphisms are surjective"*.

As surjectiveness is a concept of functions between sets, of course the form **ES** can only be considered within **concrete categories** – which are those categories equipped with a faithful functor to the category **Set**[4].

At a first glance, one could conjecture that **ES** is a set form of the Axiom of Choice. As a first fact about **ES**, we remark that this is not the case. It is easy to show that, in **ZF**, a function f is an epic arrow (that is, it is **right**

[4]In many cases, such faithful functor is given by the well-known "forgetful functor", but not *always*.

cancellable, meaning that $g = h$ whenever $gf = hf$) if, and only if, it is surjective.

Well, if such statement is not even a set form of choice, why are we interested on **ES** ? First of all, **ES** recently showed up playing a remarkable role in the context of algebraic logic. In [2], the authors show equivalence between **ES** and the Beth definibility, in some deductive systems which are Blok-Pigozzi-algebraizable.

However, even in the context of set forms we have some subtleties regarding **ES** – more precisely, regarding certain forms which the category of sets is **unable** to differentiate from **ES**.

Remark 2.1 **ZF**-*Equivalence of* $\mathbf{AC}_{Sets}$

ZF *proves that* $\mathbf{AC}_{Sets}$ *is equivalent to the statement* **"Given a product of non-empty sets, all projections are surjective"** *.* ∎

This gives rise to a new set form of choice, which will denote by **PE**, which states as: *"Given a product of non-initial objects, all projections are epic arrows"*.

Of course, in categories with all small products **PE** coincide with **CEM**.

We have also the following **ZF**-equivalence of $\mathbf{AC}_{\mathrm{Sets}}$, stated in the next remark.

Remark 2.2 **ZF**-*Equivalence of* $\mathbf{AC}_{Sets}$

ZF *proves that* $\mathbf{AC}_{Sets}$ *is equivalent to the statement* **"Given a product of non-empty sets, all projections have sections"** *.* ∎

We have therefore the following

Corollary 2.3 *The existence of sections only for those epimorphisms which are projections is sufficient to ensure the validity of the Axiom of Choice for Sets.* ∎

Of course, we will take this opportunity to define a new set form of the Axiom of Choice, which we will denote as **PRI**. So, **PRI** is the statement **"Projections have a right inverse"** .

During the rest of this section, *"projections"* will be always a short for *"projections of a product of non-initial objects"*.

Let us discuss a little bit the new axiom **PRI** in relation with **PE**. These two set forms of choice, which we have introduced inspired by **ES**, are related in the purely **CFL** language (meaning, there is no need to restrict ourselves to concrete categories, for instance):

Remark 2.4 *(a) We have the following implication:*

$$\mathbf{PRI} \Rightarrow \mathbf{PE},$$

which says that, for any category $\mathcal{C}$, $\mathbf{PRI}_{\mathcal{C}} \Rightarrow \mathbf{PE}_{\mathcal{C}}$.
(b) The reverse implication does not hold. For instance in the category of rings with unity, **PE** *holds, but* **PRI** *is false: Take the product* $\mathbb{Z} \times \mathbb{Z}_2$ *whose projections are epic arrows, but we do not have right inverse in the case of the projection to* $\mathbb{Z}_2$. ■

In fact, the existence of a right inverse for a given, arbitrary morphism easily implies that such morphism is right cancellable (and so, epic). In particular, there is no need to introduce some form as "Every epic projection has a section" , since if we say that every projection has a right inverse then we are automatically saying that projections are epic arrows.

By a similar reasoning we show that **ES** is a consequence of $\mathbf{AC}_{\text{Categories}}$ in the next remark.

Remark 2.5 *It holds that*

$$\mathbf{AC}_{Categories} \Rightarrow \mathbf{ES},$$

in the sense that, for any concrete category $\mathcal{C}$, $(\mathbf{AC}_{Categories})_{\mathcal{C}} \Rightarrow \mathbf{ES}_{\mathcal{C}}$. ■

Indeed: as having a section, in a concrete category, is enough to ensure that a given morphism is surjective, and if we say that all epimorphisms have sections we are automatically saying that all epimorphisms are surjective.

In particular, via a contrapositive arguing we may use **ES** as a kind of test for the validity of $\mathbf{AC}_{\text{Categories}}$; if **ES** is not satisfied by a given category, then $\mathbf{AC}_{\text{Categories}}$ will also fail in such category. This is applied in the next example.

Example 2.6 *Denote by* **DLat** *the category of distributive lattices with lattice homomorphisms. We show that this concrete category does not satisfy* **ES** *– and so, by the above remark, does not satisfy* $\mathbf{AC}_{Categories}$.

Let M_2 *be the diamond-like lattice with four distinct elements given by* $\{\bot, a, b, \top\}$, *where* a *and* b *are incomparable under the lattice order, and let* L *be the sublattice* $\{\bot, a, \top\}$. *Then, the inclusion morphism* $i : L \to M_2$ *is a* **non-surjective** *epimorphism. One has only to notice that, if* $g : M_2 \to L'$ *is a lattice homomorphism from* M_2 *to any distributive lattice* L' *then the value of* $g(b)$ *has to be the (unique) complement of* $g(a)$ *in the interval* $[g(\bot), g(\top)]$. *We omit the details.*

In particular, the contrapositive arguing we have just mentioned give us that the category **DLat** *does not satisfy* $\mathbf{AC}_{Categories}$. *On the other hand, the concrete category of Boolean Algebras satisfies* **ES**, *cf. [7].* ■

In the next example, we show that the category **Top** does satisfy **PRI**.

Example 2.7 *Denote by* **Top** *the category of topological spaces introduced in 1.3. We know that* **Top** *does not satisfy* $\mathbf{AC}_{Categories}$*, cf. 1.3. We will show that it does satisfy* **PRI**. *It suffices to show that at least those epimorphisms which* **are** *projections* **do** *have a section.*
Indeed: let $I \neq \emptyset$ *and* $\{X_i : i \in I\}$ *be a family of non-empty topological spaces. Fix* $j \in I$*. We claim that the projection*

$$\pi_j : \prod_{i \in I} X_i \to X_j$$

has a right inverse.

As $\mathbf{AC}_{Sets}$ *is available in* **CFL**, *fix* $z \in \prod_{i\in I} X_i$*. Define* $g : X_j \to \prod_{i\in I} X_i$ *in the following way: for any* $x \in X_j$*,* $g(x) = c = (c_i)_{i\in I}$*, where* $c_i = z_i$ *if* $i \neq j$ *and* $c_j = x$*. It is easy to check that* g *is a continuous, right inverse of* π_j *(and, moreover,* g *is a homeomorphism between* X_j *and the subspace of* $\prod_{i\in I} X_i$ *given by* $\prod_{i\in I\setminus\{j\}} \{z_i\} \times X_j$*.* ■

3 Some considerations on final objects

In the paper [3], some considerations were made on different versions of initial objects in the Chapter 3. As remarked in 2.25, page 420 op.cit. , the Axiom of Choice could be arguably regarded as a statement which is "*mainly talking about the empty set, and not about the products*", and with this in mind some different variations of the notion of *initial object* were introduced and discussed in the paper referred to. Clearly, all notions introduced there can be extended (by duality) to corresponding notions of final objects. In this section, we will talk about some new versions of *final objects*, in order to be able to introduce some categorial variations of the Zorn's Lemma[5].

We recall the notion of final object, denoted by **1** in a given category $\mathcal{C}$ as an object, that permits for every other object a in category $\mathcal{C}$, an unique morphism $a \to \mathbf{1}$. Let us introduce more versions of kinds of final elements. In the next definition, 3.1, the first four versions are the obvious duals of the (versions of) initial objects reported in Section 3 of [3]. The fifth version is mentioned in [10], and the last two are, to the best of the authors knowledge, unprecedented in the literature and are being introduced in this very work.

Definition 3.1 *Let* $\mathcal{C}$ *be a category with the final object* **1** *and* a *an object in* $\mathcal{C}$*. Then we have*

[5]In some further research, we intend to proceed analogously with respect to the Hausdorff's Maximal Principle – which states as *"Every parcial order includes a maximal chain"* –, cf. [4]

(a) The object a is **nearly final** *iff there exists a monomorphism $a \to \mathbf{1}$.*
(b) The final object **1** *is* **strict final** *iff for every object $a \in ob(\mathcal{C})$ and every $f : \mathbf{1} \to a$, f is an isomorphism.*
(c) The object a is **quasi final** *iff for every object b in $\mathcal{C}$ there is at most one morphism form b to a.*
(d) The object a is **weakly final** *iff for every object b in $\mathcal{C}$ there is a morphism from b to a.*
(e) The object a is **almost terminal**[6] *iff for every $b \in ob(\mathcal{C})$ and every $u : a \to b$, there is $v : b \to a$ such that $v \circ u = 1_a$ – that is, u has an left inverse.*
(f) The object a is **almost maximal** *iff for every object b in $\mathcal{C}$, if there is a morphism from a to b, then there is a morphism from b to a.*
(g) The object a is **maximal** *iff for every object b in $\mathcal{C}$ a morphism from a to b is an isomorphism.*

Clearly, one could dualize all of these notions and obtain the correspondent variants of initial objects; recall that, for the first four of these last ones, the corresponding notions of initiality were already investigated in [3] – and these versions of initial objects were very important, regarding the categorial forms of **AC** in the previous work referred to. In a forthcoming work, we have more versions of categorial Zorn's Lemma, cf. [4], connecting these with Hausdorff's Maximal Principle – which is also an equivalent of the classical Axiom of Choice. In the following section we will talk about a *small* Zorn's Lemma. There, we will need some little bit *finer* notions of final objects in a category, and this justifies Definition 3.1. Before getting to the next section, let us prove some simple facts about our new objects introduced above.

Proposition 3.2 *1. Given a category associated to a poset, then holds:*
(a) Every object is quasi final.
(b) An object is final iff is nearly final iff is strict final iff weakly final.
(c) An object is maximal iff it is almost maximal iff it is almost terminal.

2. Let $\mathcal{C}$ be a category. **1** *is final object $\Rightarrow$ is almost terminal $\Rightarrow$* **1** *is almost maximal.*

3. An object is strictly final iff it is final and maximal.

4. **1** *is final object iff* **1** *is quasi final and weakly final.*

5. A weakly final object always is almost maximal.

6. Let $\mathcal{C}$ be a category with the zero (i.e., initial and final) object **0**. *Then* **0** *is almost terminal.*

[6] This notion is due to Kashiwara and Shapira, [10]

Proof: *1.* is straightforward. *2.* Let $\mathbf{1}$ be final object and consider an object b in $\mathcal{C}$ with a morphism $u : \mathbf{1} \to b$. But always we have an unique morphism $! : b \to \mathbf{1}$, and so the composition $! \circ u$ is a morphism between $\mathbf{1}$. The fact that $\mathbf{1}$ is final object gives that $! \circ u$ is the identity on $\mathbf{1}$. The second implication is immediate from the definitions.

3., *4.* and *5.* follow from definitions.

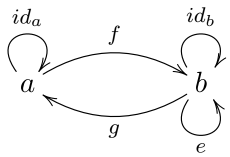
6. Suppose the morphism $u : \mathbf{0} \to a$ for an object a in category $\mathcal{C}$. The fact that $\mathbf{0}$ is initial implies that u is unique. But $\mathbf{0}$ is also final and so there is $v : a \to \mathbf{0}$ unique. Therefore, $v \circ u = 1_{\mathbf{0}}$ and u has a left inverse. ■

Example 3.3 *1. We know that the category of groups with morphisms the group morphisms has a zero object given by* $\mathbf{0} := \{e\}$. *By the above remark,* $\{e\}$ *is almost terminal.*

2. Considering $\mathcal{C}$ *the category of Boolean algebras without the trivial algebra.*

- 2 *is a almost terminal object.*
- *Every object in* $\mathcal{C}$ *is weakly final.*
- *There is no final object* ($hom(B, 2) \cong Stone(B)$).

3. Considering the category $\mathcal{C}'$ *represented by the diagram below such that* $f \circ g = e$ *and* $e \circ e = e$.



$$\mathcal{C} = \dot{\coprod_{\alpha<\xi}} \mathcal{C}'_\alpha, \ \mathcal{C}'_\alpha = \mathcal{C}', \ \xi \geq 2.$$

Then in $\mathcal{C}$ *we have the following*
(i) there is no weakly final object,
(ii) every object is almost maximal
(iii) an object is quasi final (respectively almost terminal) iff it is of the following form (a, α) *with* $\alpha \in \xi$. ■

4 The "Small" Zorn's Lemma

In [3] is introduced (and is dealt with) a number of categorial set forms of choice (as **PNI**, **CEM**, and many others). However, a categorial form of Zorn's Lemma didn't show up – and, as far as our knowledge goes, there is no "canonical" categorial version of Zorn's Lemma in Category Theory.

As mathematicians, the authors believe that **there should be** some categorial version of Zorn's Lemma. Well, we have some proposals. And, in order to present them, we first introduce some notions.

The notions of **chain**, **maximal objects** and **almost maximal objects** are used. The last two have been introduced in Section 3, and the other will be clarified in the following.

Recall that a **diagram** in a category $\mathcal{C}$ is a functor $\mathcal{D} : I \longrightarrow \mathcal{C}$, where I is an **index category** (or **scheme**) of the diagram; intuitively, you may think that I is simply an oriented multigraph.

Note that a functor is monic[7] iff the functor is injective in the objects and faithful. Accordingly, we will say that a diagram $\mathcal{D}$ is a **monic diagram** if $\mathcal{D}$ is a monic functor.

Definition 4.1 *(a) A* **chain** *in a category* $\mathcal{C}$ *is a monic functor* $\mathcal{F} : \mathbb{T} \longrightarrow \mathcal{C}$ *, where* $\mathbb{T}$ *is a totally ordered set viewed as a category.*

(b) A **chain in a diagram** $\mathcal{D}$*, will be given by the composition of a chain in the index category* $\mathcal{F} : \mathbb{T} \longrightarrow I$ *, with the diagram functor* $\mathcal{D} : I \longrightarrow \mathcal{C}$ *, i.e.,* $\mathcal{D} \circ \mathcal{F}$*.*

In the preceding definition, we asked the funtors to be monic since this requirement captures the notion of "subcategory" – so, we will be able to formally regard chains in categories as "totally ordered subcategories", as will be seen presently.

In what follows, we introduce a form which is a a categorial version of Zorn's Lemma, denoted by $\mathbf{ZL}_{\text{small}}$ as follows:

For a given category $\mathcal{C}$, $(\mathbf{ZL}_{\text{small}})_{\mathcal{C}}$ is the following statement:

"For every **small** *index category* I *not empty, and for every monic diagram* $\mathcal{D} : I \longrightarrow \mathcal{C}$ *, then the image of the diagram* $\mathcal{D}(I)$ *has an almost maximal object whenever every chain in* $\mathcal{D}(I)$ *has a cocone in* $\mathcal{D}(I)$*".*

Of course, the "small" refers to the above restriction we have made in the index category.

[7] One can see a monic functor as a monic morphism in the category of categories, denoted by CAT, where the objects are categories and the morphisms are functors.

For a given category $\mathcal{C}$, $(\mathbf{ZL}_{\text{Poset}})_{\mathcal{C}}$ is the following statement:

For every category P not empty, where P is the category associated to a poset and for every monic diagram $\mathcal{D}: P \longrightarrow \mathcal{C}$, then the image of the diagram $\mathcal{D}(P)$ has an almost maximal object whenever every chain in $\mathcal{D}(P)$ has a cocone in $\mathcal{D}(P)$.

Now, we are able to present our main results.

Theorem 4.2 *Zorn's Lemma in the category* **Set** *is equivalent to the statement that for all non empty poset $(Q; \leq)$, it holds $(\mathbf{ZL}_{Poset})_{\mathcal{Q}}$.*

Proof: For $\Rightarrow$, take a non empty poset $(Q; \leq)$ and the associated category $\mathcal{Q}$. Consider the identity map $id : Q \to Q$ (obviously it is a monic functor). Suppose that every chain in Q has a cocone, that is an upper bound, then by Zorn's Lemma in **Set**, there is a maximal element in Q. By 3.2, this is almost maximal in $\mathcal{Q}$.

For the other implication, let $(Q; \leq)$ be a poset such that every chain has an upper bound. Consider the monic diagram $id : Q \to Q$. Remark that the upper bounds are cocones in the associated category $\mathcal{Q}$, and so by $(\mathbf{ZL}_{\text{Poset}})_{\mathcal{Q}}$, $\mathcal{Q}$ has an almost maximal element, which is maximal in Q. ∎

The proof of next theorem is similar.

Theorem 4.3 *Zorn's Lemma in the category* **Set** *is equivalent to the statement that for all not empty poset $(Q; \leq)$ seen as category $\mathcal{Q}$, it holds $(\mathbf{ZL}_{small})_{\mathcal{Q}}$.*

Proof: For the right to left direction, take simply $I := Q$ and $\mathcal{D} := id$, then the proof of $\Leftarrow$ of the previous theorem gives our result. For the other direction, if given a non empty small I and $\mathcal{D} : I \to \mathcal{Q}$ a monic diagram, such that every chain in $\mathcal{D}(I)$ has a cocone, i.e., every chain in $\mathcal{D}(I) \subseteq \mathcal{Q}$ has an upper bound, the use of Zorn's Lemma gives an almost maximal element in $\mathcal{D}(I)$. ∎

Theorem 4.4 $(\mathbf{ZL}_{Poset})_{\mathbf{Set}}$ *implies Zorn's Lemma in* **Set**.

Proof: Let $(Q; \leq)$ be a partial order satisfying the usual requirements of Zorn's Lemma (non-empty, every chain has an upper bound). Denote by $\mathcal{Q}$ the usual associated category to $(Q; \leq)$ - with morphisms given by the order relation. Remark that $\mathcal{Q}$ is a small category.

Let $\mathcal{D}: \mathcal{Q} \longrightarrow \mathbf{Set}$ be the diagram functor which maps every $a \in Q$ to the **downset** of a, given by $\downarrow a = \{x \in Q : x \leqslant a\}$, and maps every arrow $\leqslant$ to the inclusion function[8].

[8] Notice that this is not a full, but is a monic functor

As every chain in $\mathcal{Q}$ has an upper bound, in the diagram $\mathcal{D}(\mathcal{Q})$ we have that every chain has a cocone – and so there is an almost maximal object in $\mathcal{D}(\mathcal{Q})$.

If $\downarrow b$ is such almost maximal object, it is easy to check that b is a maximal element of $\mathcal{Q}$ and we are done. ∎

Our last theorem connects the Axiom of Choice in category *Set*, $\mathbf{AC}_{\text{Sets}}$, with our version $\mathbf{ZL}_{\text{small}}$.

Theorem 4.5 *Zorn's Lemma in* **Set** *implies for all category* $\mathcal{C}$, $(\mathbf{ZL}_{small})_{\mathcal{C}}$.

Proof: Let $\mathcal{C}$ be a category and I a small not empty category with $\mathcal{D} : I \to \mathcal{C}$ a monic functor. Then we define the following poset

$$\mathbb{P} := \{\mathbb{T} \subseteq \mathcal{D}(I) | \quad \mathbb{T} \text{ is a chain in } \mathcal{D}(I) \ \}.$$ [9]

with the following order $\leq$:

$$\mathbb{T}_1 \leq \mathbb{T}_2 \qquad \text{iff} \qquad \mathbb{T}_1 \subseteq \mathbb{T}_2,$$

where the inclusion $\subseteq$ on the right side means inclusion of subcategories. Remark that $\mathbb{P}$ is not empty, because from $I \neq \emptyset$, there exists $a \in ob(\mathcal{D}(I))$ and $\mathbb{T} := \{\{a\}, id_a : a \to a\}$ is a chain in $\mathbb{P}$.

Let now $\{\mathbb{T}_x\}_{x\in X}$ with $(X; \leq_X)$ linearly ordered by $x \leq_X y$ iff $\mathbb{T}_x \leq \mathbb{T}_y$, be a chain in $\mathbb{P}$. Then we show that $\bigcup_{x\in X} \mathbb{T}_x$ is an upper bound of the family $\{\mathbb{T}_x\}_{x\in X}$. First, $\bigcup_{x\in X} \mathbb{T}_x$ is a subcategory of $\mathcal{D}(I)$ and a chain. For this, take $a, b \in \bigcup_{x\in X} \mathbb{T}_x$. Then exists $y \in X$ such that $a, b \in \mathbb{T}_y$. If $a \neq b$[10], we have either $f : a \to b$ or $g : b \to a$. Remark that these morphisms are unique. And so either $f \in \mathbb{T}_y$ or $g \in \mathbb{T}_y$. Clearly, $\bigcup_{x\in X} \mathbb{T}_x$ is an upper bound in $\mathbb{P}$.

Applying Zorn's Lemma, we have $\mathbb{T}'$ maximal in $\mathbb{P}$. We show that $\mathbb{T}'$ has a maximum. If not, let $K' := (k', \{f_i\}_{i\in I}) \cup \{id_{k'}\}$ where $(k', \{f_i\}_{i\in I})$ is a cocone in $\mathcal{D}(I)$ about $\mathbb{T}'$, we extend properly $\mathbb{T}'$ by the chain $\mathbb{T}' \cup K'$. We can show that $\mathbb{T}' \cup K'$ is a chain and so a real extension of $\mathbb{T}'$, contradicting the maximality of $\mathbb{T}'$.

It remains to show that $a := max(\mathbb{T}')$ is almost maximal. For this, remark that for all $b \in ob(\mathcal{D}(I))$ with $f : a \to b$, we have that $b \in \mathbb{T}'$. Suppose not, it is $b \notin \mathbb{T}'$. Then we can extend the chain $\mathbb{T}'$ by joining $f : a \to b$ and id_a, but this contradicts maximality of $\mathbb{T}'$. So $b \in \mathbb{T}'$, and consequently, there is a morphism $g : b \to a$ in $\mathbb{T}'$, finishing our proof. ∎

We have the immediate

Corollary 4.6 *With the above notation are equivalent:*

(a) Zorn's Lemma in **Set**.

(b) For all poset $\mathcal{Q}$, $\quad (\mathbf{ZL}_{Poset})_{\mathcal{Q}}$.

[9] or equivalently one can see $\mathbb{T}$ as a subcategory of $\mathcal{D}(I)$ limearly ordered.

[10] If $a = b$, then clearly $id_a : a \to a$ is an element of $\mathbb{T}_y$.

(c) For all poset $\mathcal{Q}$, $(\mathbf{ZL}_{small})_{\mathcal{Q}}$.
(d) For all category $\mathcal{C}$, $(\mathbf{ZL}_{small})_{\mathcal{C}}$. ■

When taken together, the three theorems have some remarkable consequences.

Remark 4.7 *(a) As the equivalences were all proved in the same language* $\mathbf{CFL}^*$*, after combining Theorems 4.2, 4.3 and 4.4, it should be clear we have proved that* **a certain property of all categories is equivalent to the Axiom of Choice for sets**. *This suggests that the statement of* $\mathbf{ZL}_{small}$ *allows us to work in every level of generality.*

(b) In **CFL***, the Axiom of Choice is available. So, when it comes to Category Theory itself,* $\mathbf{ZL}_{small}$ *is, indeed, a valid property of all categories.*

(c) At least when we restrict ourselves to the categories, $\mathbf{ZL}_{small}$ *behaves differently of all previously introduced set forms of the Axiom of Choice (including* $\mathbf{AC}_{Categories}$ *itself), because there will not be the case that, considering a given category, "*$\mathbf{ZL}_{small}$ *may hold or not" – it* **will do hold**. ■

Considering the above remark, it seems like $\mathbf{ZL}_{\text{small}}$ is a set form of choice which is even more intrinsecal than $\mathbf{AC}_{\text{Categories}}$ itself.

A remarkable manifestation of such phenomenon (namely, the above described "separation" between the roles of $\mathbf{AC}_{\text{Categories}}$ and $\mathbf{ZL}_{\text{small}}$ within Category Theory) appears in the context of Boolean-valued and Heyting-valued models of Set Theory. According to [6], given a complete Heyting algebra H, one obtains the H-valued universe V^H carrying out the well-known definition of the Boolean-valued universe V^B with H in place of B. In V^H all axioms of intuitionistic first-order logic are true. Since **AC** implies the *law of the excluded middle* (which holds in V^H if and only if H is a Boolean algebra), one concludes that **AC** does not hold in any V^H for which H is not a Boolean algebra. But, Zorn's lemma is always valid in V^H. This shows that, in IZF, Zorn's lemma does not imply **AC**.

5 Notes and Questions

There are lots of questions which could raise from this ongoing work. For instance:

Question 5.1 *Considering any of the set forms of the Axiom of Choice we have mentioned in this presentation; are there functors which preserve such (or some of these) forms ?*

In what follows, we discuss some more specific issues, in three subsections:

5.1 Well-ordering the Universe

Here we discuss strong forms of the Axiom of Choice: those with ensure that there is a well-ordering of the universe. We understood a *well-ordering of the universe* as a class-relation which well-orders the universe **V** of sets. As discussed in [3] (pages 412 and 413), the existence of a well-ordering of the universe is equivalent to a number of statements, including the *Axiom of Global Choice* – which states that there is a choice-class function defined on the class of all non-empty sets. More precisely:

Remark 5.2 (see [3], Proposition 2.3) *The following statements are equivalent:*
(a) The existence of a well ordering of the universe;
(b) The Axiom of Global Choice;
(c) The Axiom of Choice for Classes; and
(d) The Axiom of Choice for Conglomerates. ■

We also recall that, as introduced in [3] (and already noticed in the Introduction of this paper), a *categorial class-form the Axiom of Choice* is a statement φ from Category Theory such that the validity of $\varphi_{\mathcal{C}}$ for *all* categories $\mathcal{C}$ is equivalent to the Axiom of Choice for Classes. Our archetypical example of a categorial class-form of choice – which is the statement ζ which says "There is a skeleton in the category" – has, indeed, a nice property: the validity of $\zeta_{\mathcal{C}}$ for all *locally small* category $\mathcal{C}$ is equivalent to the Axiom of Choice for Classes (see, again, [3]).

The preceding observation suggests the following questions:

Question 5.3 *(a) Is there some statement of a form* $\mathbf{ZL}_{\textit{locally small}}$ *(allowing* I *to be locally small in the diagram* $\mathcal{D} : I \longrightarrow \mathcal{C}$ *, and changing the definition of chain in a category in order to consider the presence of totally ordered classes) which could imply the Axiom of Global Choice, in any of its equivalent forms?*

(b) How far would it be reasonable to go if we want/need some kind of $\mathbf{ZL}_{\textit{full}}$*?*

Of course, $\mathbf{ZL}_{\text{full}}$ would correspond to the strongest (however, still reasonable) categorial form of Zorn's Lemma we could come out with. As one could expect, if we accept that categorial versions of Zorn's Lemma could range over highly complex structures such as *illegitimate conglomerates*[11], then one could prove, indeed, the well-ordering the universe.

[11] A conglomerate $\mathcal{X}$ is said to be a *illegitimate conglomerate* if it cannot be indexed by a class (see [1], page 16).

For instance, let

$\mathcal{W} = \{\langle \mathbf{Y}, <_{\mathbf{Y}} \rangle : \mathbf{Y}$ is a class and $<_{\mathbf{Y}}$ is a class relation which well-orders $\mathbf{Y}\}$.

Consider, in the illegitimate conglomerate $\mathcal{W}$, the conglomerate-relation $\prec_{\mathcal{W}}$ given by the well-known *end-extension* order[12]. Exactly as in a proof of **ZL** $\Rightarrow$ **WO** (see, e.g., [11], page 107), one can easily check that any $\prec$-conglomerate-chain has a upper bound in $\mathcal{W}$, obtained by the union of the members of the conglomerate-chain (ordered by the union of the orders on such members). So, let us consider the statement $(\mathbf{ZL}_{\text{Poconglomerates}})_{\mathcal{C}}$ (where $\mathcal{C}$ range over generalized categories where we allow objects to be classes) given by

For every non-empty category P, where P is a category associated to a poconglomerate, i.e., a conglomerate partially ordered by a conglomerate-relation, and for every monic diagram $\mathcal{D} : P \to \mathcal{C}$, then the image of the diagram $\mathcal{D}(P)$ has an almost maximal object whenever every conglomerate-chain in $\mathcal{D}(P)$ has a (commutative) cocone in $\mathcal{D}(P)$.

If we take, in our context, $\mathcal{C} = \mathcal{W} = P$ and $\mathcal{D}$ the identity functor, the almost maximal object of $\mathcal{W}$ (which is given by $(\mathbf{ZL}_{\text{Poconglomerates}})_{\mathcal{W}}$, since we have already noticed the existence of upper bounds in $\mathcal{W}$ for every conglomerate $\prec$-chain) has to be a well-ordering of the universe $\mathbf{V}$ (otherwise we would be able to properly extend it to a $\prec$-larger well-ordering on a class). This shows that $\mathbf{ZL}_{\text{Poconglomerates}}$ (that is, the validity of $(\mathbf{ZL}_{\text{Poconglomerates}})_{\mathcal{C}}$ for all generalized category $\mathcal{C}$) would give us the Axiom of Global Choice. Of course, the interest of the question raised above is to look for some weaker version of $\mathbf{ZL}_{\text{Poconglomerates}}$ which could perform the very same job; $\mathbf{ZL}_{\text{Poconglomerates}}$ is, clearly, too much to ask. It would be very nice if some "locally small Zorn's Lemma" could well-order the universe of sets[13].

[12] If $\langle A, <_A \rangle$, $\langle B, <_B \rangle$ are well-ordered structures, we say that $\langle A, <_A \rangle$ precedes $\langle B, <_B \rangle$ in the end-extensior order $\prec$ if $(i) A \subseteq B$; $(ii) \forall x, y \in A[x <_A y \iff x <_B y]$, i.e., $<_A \subseteq <_B$; and (iii) $\forall x \in A \forall y \in B \setminus A[x <_B y]$. In words: $\langle A, <_A \rangle \prec \langle B, <_B \rangle$ if $\langle A, <_A \rangle$ is an initial segment of $\langle B, <_B \rangle$.

[13] Let us describe another construction of a well-ordering of the universe – which, unfortunately, still relies on conglomerates. Let $\mathbf{W} = \{\langle Y, <_Y \rangle : Y$ is a set and $<_Y$ is a set relation which well-orders $Y\}$ - that is, $\mathbf{W}$ is the class of all well-ordered sets, and considered $\mathbf{W}$ ordered by the end-extension relation. Let $\mathcal{E}$ be the conglomerate of all well-ordered class-chains in $\mathbf{W}$, and consider $\mathcal{E}$ ordered by the end-extension relation over the well-ordered class-chains. A variation of the above arguments with $\mathbf{ZL}_{\text{Poconglomerates}}$ would provide a maximal well-ordered conglomerate-chain in $\mathcal{E}$; the union of such maximal conglomerate-chain clearly provides a well-ordering of the universe. Constructions of this kind – aiming to exhibit maximal chains in ordered structures, after applying suitable categorial forms of choice – will be explored in further research ([4]).

5.2 Hausdorff Maximal Principle and Teichmüller-Tuckey principle

Besides pursuing new categorial forms of **AC**, we intend to explore categorial forms of statements on partially ordered sets which are equivalent to the Axiom of Choice, namely Hausdorff's Maximal Principle and Teichmüller-Tuckey's Maximal principle – and their relationships with those here established categorial versions of Zorn's Lemma.

These categorial forms on partially ordered sets are slightly different from the categorial set-forms of Axiom of Choice viewed until here. As we have seen, a statement φ is a *categorial Set-Form of the Axiom of Choice* if the Axiom of Choice for Sets is equivalent to the statement $\varphi_{\mathbf{Set}}$. The *categorial poset-form of the AC* is a categorial statement φ such that $\varphi_{(P,\leq)}$ ($(P,\leq)$ is viewed as a category), for any poset $(P,\leq)$, is equivalent to Axiom of Choice. This approach will probably unify some of the results provided in 4.2.

5.3 ES and the Beth Definability Property

The form **ES** – viewed as a purely algebraic property – was shown (by Blok and Hoogland, [2]) to be related to a meta-logical property in Abstract Algebraic Logic (more specifically, the Beth Definability Property; and, in fact, it was shown by Blok and Hoogland that a perfect correspondence between Beth Property and **ES** holds for a large class of equivalential logics).

The authors believe that forms like the ones mentioned in this work could give raise to a number of **bridge theorems**, meaning, theorems which connect meta-logical properties with algebraic, categorial properties (in logics as algebraizable logics, equivalential logics, etc.).

The authors believe that some suitable reformulation of the Beth Definability Property – which asserts that, under certain assumptions, whenever a certain set of variables is definable **implicitly** in terms of other set of variables, then there is some **explicit** witness of such definability – will be able to constitute a set form of the Axiom of Choice.

5.4 Algebraic class forms of the Axiom of Choice

An even more abstract approach to class and set forms of choice will correspond to **algebraic class forms** of the Axiom of Choice, which we intent to introduce in the continuation of this research.

Those forms will be given in the setting of Algebraic Set Theory (Joyal, Moerdijk, [9]). This categorial approach to Set Theory is developed in a framework given by a "category of classes", a pre-topos of Heyting, together with a notion of "small maps". The notion of "set" itself in this particular setting is, indeed, introduced **a posteriori**, through the notion of **ZF**-algebra: the free **ZF**-algebra satisfies an intuitionistic form of the **ZF** axioms.

We intend to investigate the possible relations between our "meta-external" versions of **AC** in the underlying Heyting pre-topos – the category of classes – and our "internal" forms of **AC** in some versions of intuitionistic **NBG** associated to the free **ZF**-algebra

6 Summary of definitions

Here we present some basic categorial definitions for benefit or convenience to the reader [14].The definitions 1 to 17 were taken from [1] and [10]. The last definitions are a summary of categorial forms and related forms of AC.

1. **Category**: Is a quadruple $\mathbf{A} = (O, hom_{\mathbf{A}}, id, \circ)$ consisting of
   (a) A class O whose members are called $\mathbf{A} - Object$
   (b) For each pair (A, B) of $\mathbf{A}objects$, a class $hom_{\mathbf{A}}(A, B)$ whose members are called $\mathbf{A}morphism$ from A to B.
   (c) For each $\mathbf{A} - object$ A, a morphism $id_A : A \to A$ called **A**-identity on A
   (d) A **composition law** associating with each **A**-morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ an **A**-morphism $A \xrightarrow{g \circ f} C$ called the composite of f and g, subject to the following conditions:
      i. composition is associative; i.e., for morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$, the equation $h \circ (g \circ f) = (h \circ g) \circ f$
      ii. **A**-identities act as identities with respect to composition; i.e., for **A**- morphism $A \xrightarrow{f} B$, we have $f \circ if_A = f = id_B \circ f$.
2. **Locally Small and Small Categories:** A category is said to be **locally small** if $hom(A, B)$ is a set instead of a proper class.
   A category is **small** if it is locally small and if O is also a set.
3. **Subcategory:** A category **A** is said to be a subcategory of a category **B** provided that the following conditions are satisfied:
   (a) $Ob(\mathbf{A}) \subseteq Ob(\mathbf{B})$
   (b) For each $A, A' \in Ob(\mathbf{A})$, $hom_{\mathbf{A}}(A, A') \subseteq hom_{\mathbf{B}}(A, A')$
   (c) For each $A \in Ob(\mathbf{A})$, **B**-identity on A is the **A**-identity on A

[14] Thanks to the Referee for suggest it.

   (d) The composition law in $\mathbf{A}$ is the restriction of the composition law in $\mathbf{B}$ to the morphisms of $\mathbf{A}$

4. **Duality:** For any category $\mathbf{A} = (O, hom_{\mathbf{A}}, id, \circ)$ the **dual** (or **opposite**) category of $\mathbf{A}$ is the category $\mathbf{A} = (O, hom_{\mathbf{A^{op}}}, id, \circ^{op})$, where $hom_{\mathbf{A^{op}}}(A, B) = hom_{\mathbf{A}}(B, A)$ and $f \circ^{op} g = g \circ f$ (Thus $\mathbf{A}$ and $\mathbf{A^{op}}$ have the same objects and, except for their direction, the same morphisms.)

5. **Monomorphism (Epimorphism):** A morphism $f : A \to B$ is called **monomorphism** (or monic arrow) if for every pair $g, h : C \to A$, then

$$(f \circ g = f \circ h) \implies (g = h)$$

   The dual of a monomorphism is called **epimorphism** (or epic arrow), i.e., a epimorphism in a category $\mathbf{A}$ is a monomorphism in $\mathbf{A^{op}}$.

6. **Isomorphism:** A morphism $f : A \to B$ is called **isomorphism** provided that there exists a morphism $g : B \to A$ with $g \circ f = id_A$ and $f \circ g = id_B$. Such a morphism g is called **inverse** of f.

7. **Section of a morphism:** A **section** of a morphism $f : A \to B$ is a right inverse morphism $g : B \to A$, i.e., $f \circ g = id_B$

8. **Functor:** If $\mathbf{A}$ and $\mathbf{B}$ are categories, a functor F from $\mathbf{A}$ to $\mathbf{B}$ is a function that assigns to each $\mathbf{A}$-object A a $\mathbf{B}$-object $F(A)$, and to each $\mathbf{A}$-morphism $f : A \to A'$ a $\mathbf{B}$-morphism $F(f) : F(A) \to F(A')$, in such way that

   (a) F preserves composition; i.e., $F(g \circ f) = F(g) \circ F(f)$.

   (b) F preserves identity; i.e., $F(id_A) = id_{F(A)}$ for each $\mathbf{A}$-object A.

9. **Full and Faithful Functors:** A functor F is called **Full** provided that the restriction function $F : hom_{\mathbf{A}}(A, A') \to hom_{\mathbf{B}}(F(A), F(A'))$ is surjective for each $A, A' \in Ob(\mathbf{A})$.
   F is **Faithful** if the restriction above is injective.

10. **Concrete Category:** Let $\mathbf{X}$ be a category. A **concrete category** over $\mathbf{X}$ is a pair $(\mathbf{A}, U)$, where $\mathbf{A}$ is a category and $U : \mathbf{A} \to \mathbf{X}$ is a faithful functor. U is called the **forgetful** (or **underlying**) functor of the concrete category and $\mathbf{X}$ is called the **base category** for ($\mathbf{A}$,U).

11. **Isomorphism-dense Subcategory:** A full subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is called **Isomorphism-dense** provided that every $\mathbf{B}$-object is isomorphic to some $\mathbf{A}$-object.

12. **Skeleton of a category:** is a full, isomorphism-dense subcategory in which no two distinct objects are isomorphic.

13. **Diagram:** A **diagram** in a category $\mathbf{A}$ is a functor $D : \mathbf{I} \to \mathbf{A}$. The domain $\mathbf{I}$ is called **scheme** for the diagram. A diagram with small (finite) scheme is said to be **small** (**finite**).

14. **Cone (Cocone):** Let $D : \mathbf{I} \to \mathbf{A}$ be a diagram. A **Cone** $\mathbf{c} = (c, \{\phi_i\}_{i \in Ob(\mathbf{I})})$ over D is a $\mathbf{A}$-object c together with a family of morphisms $\phi_i : c \to F(i)$ for each object $i \in Ob(\mathbf{I})$ such that for any morphism $h : i \to j$, we have that $F(h) \circ \phi_i = \phi_j$. A **Cocone** is the dual of a cone.

15. **Limits (Colimits):** A limit over a diagram $D : \mathbf{I} \to \mathbf{A}$ is a cone $\mathbf{L} = (l, \{\phi_i\}_{i \in Ob(\mathbf{I})})$ with the universal property with respect to cones, i.e., for every cone $\mathbf{c} = (c, \{\phi'_i\}_{i \in Ob(\mathbf{I})})$ over D, there exists a unique morphism $k : c \to l$ such that $\phi_i \circ k = \phi'_i$ for each $i \in Ob(\mathbf{I})$.**Colimit** is the dual of a limit.

16. **Product (Coproduct):** Let $F = \{a_i\}_{i \in I}$ be a family of $\mathbf{A}$-object. The **product** of F is the limit of the diagram $D : I \to \mathbf{A}$, considering the index set I as the scheme of the diagram such that $hom_I(i, j) = \emptyset$ for each $i, j \in I$ (discrete diagram). A **coproduct** is the dual of a product.

17. **Equalizer (Coequalizer):** An **equalizer** over a parallel pair $f, h : a \to b$ is the limit of the finite diagram of the shape

$$a \underset{f}{\overset{h}{\rightrightarrows}} b$$

A **coequalizer** is the dual of a equalizer.

From now on we summarize the categorial forms and related forms of AC.

18. **PNI:** "*Every product of a non-empty family of non-initial objects is a non-initial object*"

19. **CEM:** "*Every non-empty discrete diagram whose objects are all non-initial has a cone where all constituent morphisms are epic arrows*"

20. **ES:** "*Epimorphisms are surjective*"

21. **PE:** "*Given a product of non-initial objects, all projections are epic arrows*"

22. **PRI:** "*Projections have a right inverse*".

23. $(\mathbf{ZL})_{\mathbf{Small}}$: "*For every* **small** *index category I not empty, and for every monic diagram $\mathcal{D} : I \longrightarrow \mathcal{C}$, then the image of the diagram $\mathcal{D}(I)$ has an almost maximal object whenever every chain in $\mathcal{D}(I)$ has a cocone in $\mathcal{D}(I)$*"

24. $(\mathbf{ZL})_{\mathbf{Poset}}$: "*For every category P not empty, where P is the category associated to a poset and for every monic diagram $\mathcal{D} : P \longrightarrow \mathcal{C}$, then the image of the diagram $\mathcal{D}(P)$ has an almost maximal object whenever every chain in $\mathcal{D}(P)$ has a cocone in $\mathcal{D}(P)$*".

25. $(\mathbf{ZL})_{\mathbf{Poconglomerates}}$: "*For every non-empty category P, where P is a category associated to a poconglomerate, i.e., a conglomerate partially ordered by a conglomerate-relation, and for every monic diagram $\mathcal{D} : P \to \mathcal{C}$, then the image of the diagram $\mathcal{D}(P)$ has an almost maximal object whenever every conglomerate-chain in $\mathcal{D}(P)$ has a (commutative) cocone in $\mathcal{D}(P)$*".

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