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Finitary Filter Pairs and Propositional Logics

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Abstract

We present the notion of filter pair as a tool for creating and analyzing logics. We show that every Tarskian logic arises from a filter pair and that translations of logics arise from morphisms of filter pairs, establishing a categorial connection.

Keywords: Abstract Algebraic Logic, Category Theory.

Introduction

In this work we present the notion of filter pair, introduced in [28], as a tool for creating and analyzing logics. We sketch the basic idea of this notion:

Throughout the article the word *logic* will mean a pair $(\Sigma, \vdash)$ where Σ is a signature, i.e. a collection of connectives with finite arities, and $\vdash$ is a Tarskian consequence relation, i.e. an idempotent, increasing, monotonic, finitary and structural relation between subsets and elements of the set of formulas $Fm_\Sigma(X)$ built from Σ and a set X of variables.

It is well-known that every Tarskian logic gives rise to an algebraic lattice contained in the powerset $\wp(Fm_\Sigma(X))$, namely the lattice of theories. This lattice is closed under arbitrary intersections and directed unions.

Conversely an algebraic lattice $L \subseteq \wp(Fm_\Sigma(X))$ that is closed under arbitrary intersections and directed unions gives rise to a finitary closure operator (assigning to a subset $A \subseteq Fm_\Sigma(X)$ the intersection of all members of L containing A). This closure operator need not be structural — this is an extra requirement.

We observe that the structurality of the logic just defined is equivalent to the *naturality* (in the sense of category theory) of the inclusion of the algebraic

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lattice of theories into the power set of formulas with respect to endomorphisms of the formula algebra: Structurality means that the preimage under a substitution of a theory is a theory again or, equivalently, that the following diagram commutes:

$$
\begin{array}{ccccc}
Fm_\Sigma(X) & \quad & L & \overset{i}{\hookrightarrow} & \wp(Fm_\Sigma(X)) \\
\sigma \big\downarrow & & \sigma^{-1}_{|_L} \big\uparrow & & \big\uparrow \sigma^{-1} \\
Fm_\Sigma(X) & & L & \overset{i}{\hookrightarrow} & \wp(Fm_\Sigma(X))
\end{array}
$$

More generally, for any Σ-structure we have the inclusion of the algebraic lattice of filters for the logic into the power set of the underlying set of the structure. The fact that the preimage of a filter under a homomorphism is a filter again can again be expressed as the commutativity of a square as above.

We thus arrive at the definition of *filter pair*, Definition 2.1: A filter pair for the signature Σ is a contravariant functor G from Σ-algebras to algebraic lattices together with a natural transformation $i\colon G \to \wp(-)$ from G to the functor taking an algebra to the power set of its underlying set, which preserves arbitrary infima and directed suprema

The logic associated to a filter pair (G, i) is simply the logic associated (in the above fashion) to the algebraic lattice given by the image $i(G(Fm_\Sigma(X))) \subseteq \wp(Fm_\Sigma(X))$.

In particular, it is clear that different filter pairs can give rise to the same logic, indeed this will happen precisely if the images of i for the formula algebra are the same. A filter pair can thus be seen as a *presentation* of a logic, and there can of course be different presentations of the same logic. We could have removed a bit of this ambiguity by requiring that the natural transformation i be an inclusion, but it is one of the insights of our work with filter pairs that it is beneficial not to do this. We give an example of this point in Section 4.2.

Overview of the article: In Section 2 we fix notation and recall some definitions we need.

In Section 3 we introduce the notion of filter pair sketched above and show how, for a fixed set of variables, to a filter pair one can associate a logic with this set of variables. More generally, the map i gives us a lattice of "theories" not just for formula algebras but for any Σ-structure, see Proposition 2.4, thus yielding a "generalized logic", whose set of formulas need not be an absolutely free algebra. We show in Proposition 2.9 that if one fixes one generalized logic coming from a filter pair, then the subsets of any other Σ-structure that lie in the image of i are filters for this generalized logic – these subsets are called the i-filters. We investigate the relation between the different generalized logics, on different Σ-structures, arising from a fixed filter pair:

In Proposition 2.12 we establish that homomorphisms of Σ-structures induce translations of generalized logics and that these translations are conservative if both Σ-structures in question are formula algebras. In Proposition 2.14 we show that for absolutely free algebras every filter is an i-filter. It follows that the logic defined by the matrices endowed with i-filters is again the same logic as the one obtained from the filter pair. These propositions tie together the various structures arising from a filter pair and establish a coherent picture and can simply be read as such. However, they also have practical applications and offer further intuitions about the notion of filter pair. This is explained in Section 4, and the reader may to skip ahead after reading Section 2, to get some further information and motivation for the content of Section 2.

In Section 3 we introduce morphisms of filter pairs, Definition 3.8. The definition is inspired by [27] where the authors establish a correspondence of certain functors between categories of Σ-algebras and translations of algebraizable logics. Indeed, the definition of morphism can be read as the result of trying to push these results beyond the realm of algebraizable logics. We show that the construction of a logic from a filter pair is functorial and that this functor has a right inverse, which associates to a logic its filter pair of its filters, see Example 2.7. This delineates the scope of the notion of filter pair: Every logic admits a presentation by a filter pair. In fact, this construction satisfies a certain universal property, see Theorem 3.9.

In Section 4 we offer a synthesis and outlook on future work to motivate the notions and results of this article.

1 Preliminaries

Here we fix the basic definitions for the rest of the article.

Definition 1.1 *A signature is a sequence of pairwise disjoint sets* $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$. *If* X *is an* arbitrary *set, let* $Fm_\Sigma(X)$ *denote the set of* Σ*-formulas over* X*, i.e., the free* Σ*-algebra on the set* X*.*

In what follows, $V = \{v_0, v_1, ..., v_n, ...\}$ *will denote a fixed enumerable set (written in a fixed order). Denote* Fm_Σ *(respectively* $Fm_\Sigma[n]$*), the set of* Σ*-formulas over* V *(resp. with* exactly $\{v_0, ..., v_{n-1}\}$ *as the occurring variables).*

Definition 1.2 *Let* Σ *be a signature and* A *be a* Σ*-algebra.*

A Tarskian consequence relation on A *is a relation* $\vdash\ \subseteq \wp(A) \times A$ *which, for every subset* $\Gamma \cup \Delta \cup \{\varphi, \psi\}$ *of* A*, satisfies the following conditions:*

- **Reflexivity** :*If* $\varphi \in \Gamma,\ \Gamma \vdash \varphi$
- **Cut** :*If* $\Gamma \vdash \varphi$ *and for every* $\psi \in \Gamma,\ \Delta \vdash \psi$*, then* $\Delta \vdash \varphi$

- **Monotonicity** :*If* $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash \varphi$*, then* $\Delta \vdash \varphi$
- **Finitarity** :*If* $\Gamma \vdash \varphi$*, then there is a finite subset* Δ *of* Γ *such that* $\Delta \vdash \varphi$*.*
- **Structurality** :*If* $\Gamma \vdash \varphi$ *and* $\sigma : A \to A$ *is a* Σ*-homomorphism (substitution), then* $\sigma[\Gamma] \vdash \sigma(\varphi)$*.*

The notion of (generalized) logic that we consider is:

Definition 1.3 *Let* Σ *be a signature,* X *be a set and* A *a* Σ*-algebra.*

1. *A* logic over the set X *is a pair* $(\Sigma, \vdash)$ *where* $\vdash$ *is a Tarskian consequence relation on the set* $Fm_\Sigma(X)$*.*
2. *A* logic *is a logic over the set* V*.*
3. *A* generalized logic on the Σ-algebra A *is a pair* $(A, \vdash)$ *where* $\vdash$ *is a Tarskian consequence relation on the* Σ*-algebra* A*.*

Definition 1.4 *Let* Σ *be a signature.*

1. *A matrix (over* Σ*) is pair* $\langle M, F\rangle$ *where* M *is a* Σ*-algebra and* F *is subset of* M*.*
2. *Let* $l = (A, \vdash)$ *be a generalized logic on the* Σ*-algebra* A *and let* $\langle M, F\rangle$ *be a matrix. The set* F *is a* l*-filter if the following holds: for every* $\Gamma \cup \{\varphi\} \subseteq A$ *such that* $\Gamma \vdash \varphi$ *and every* Σ*-homomorphism (valuation)* $v : A \to M$*, if* $v[\Gamma] \subseteq F$ *then* $v(\varphi) \in F$*. The pair* $\langle M, F\rangle$ *is then said to be a matrix model of* l*. The set of all matrix models of* l *is denoted by* $Matr_l$*.*

Definition 1.5

1. *Let* L *be a lattice. A element* $a \in L$ *is compact if for every directed subset* $\{d_i\}$ *of* L *we have* $a \leq \bigvee_i d_i \Leftrightarrow \exists i(a \leq d_i)$*.* L *is called algebraic if it is a complete lattice such that every element is a join of compact elements.*
2. *We will denote by Lat (resp. AL), the category of all lattices (resp. algebraic lattices) and order preserving functions.*
3. Σ*-Str stands for the category of all* Σ*-algebras (or* Σ*-structures) and* Σ*-homomorphisms.*

Remark 1.6 *1. Let $l = (A, \vdash)$ be a generalized logic on a Σ-algebra A. We have the map $Fi_l : Obj(\Sigma - Str) \to Obj(Lat)$ such that for any Σ-algebra M, $Fi_l(M) \subseteq \wp(M)$ is the lattice of all l-filters of M. Since $Fi_l(M) \subseteq \wp(M)$ is closed under arbitrary intersections and directed unions, $Fi_l(M)$ is an algebraic lattice where the compact elements are precisely the finitely generated filters. Thus we can restrict the codomain of the above function to the class of all algebraic lattices $Obj(AL)$. Moreover, Fi_l extends to a contravariant functor from the category $\Sigma - Str$ to the category AL sending $f \in hom_{\Sigma-Str}(M, N)$ to $Fi_l(f) = f^{-1}$ (set-theoretic inverse image). This is well-defined, because inverse images of filters are filters again.*

2. Given a Σ-algebra A and a matrix $\langle M, F\rangle$, one can define an increasing, idempotent, monotone and structural, but possibly non-finitary, relation for $\Gamma \cup \{\varphi\} \subseteq A$ by: $\Gamma \vdash_{\langle M,F\rangle} \varphi$ iff for every valuation $v : A \to M$, if $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$. If l denotes the generalized logic thus obtained, then obviously $\langle M, F\rangle$ is an l-matrix[2]

Moreover, given a class $\mathbb{M} \subseteq Matr_l$, the relation $\Gamma \vdash_M \varphi$ is defined if $\Gamma \vdash_{\langle M,F\rangle} \varphi$ for all matrices $\langle M, F\rangle \in \mathbb{M}$.

1.1 Categories of signatures and logics

We provide here a definition of category of logics. The ideas behind it come from [23], [19], [2], [25] and [10].

First, we define the category of signatures with flexible morphisms, $\mathcal{S}_f$. Before defining this category, we introduce the following notation:

If $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ is a signature, then $T(\Sigma) := (F(\Sigma)[n])_{n\in\mathbb{N}}$ is a signature, too (sometimes called the derived signature).

A **flexible** morphism $f \colon \Sigma \to \Sigma'$ is just a usual morphism of signatures $f^\sharp \colon \Sigma \to T(\Sigma')$, i.e. it is a sequence of functions $f^\sharp_n \colon \Sigma_n \to F(\Sigma')[n], n \in \omega$.

For each signature Σ and $n \in \mathbb{N}$, there is a particular flexible morphism j_Σ given by:

$$\begin{array}{rll} (j_\Sigma)_n : & \Sigma_n \to & F(\Sigma)[n] \\ & c_n \mapsto & c_n(v_0, ..., v_{n-1}) \end{array}$$

For each flexible morphism $f : \Sigma \to \Sigma'$, there is a unique function $\check{f} : F(\Sigma) \to F(\Sigma')$, called the extension of f, such that:

(i) $\check{f}(v) = v$, if $v \in V$;

(ii) $\check{f}(c_n(\psi_0, ..., \psi_{n-1})) = f(c_n)(\check{f}(\psi_0), ..., \check{f}(\psi_{n-1}))$, if $c_n \in \Sigma_n, n \in \mathbb{N}$.

[2]In fact l is the strongest (non-finitary) generalized logic on A such that $\langle M, F\rangle$ is a matrix model for that generalized logic.

Definition 1.7 *The category $\mathcal{S}_f$ is the category of signatures and flexible morphisms as defined above. The composition in $\mathcal{S}_f$ is given by $(f' \bullet f'')^\sharp := ((\check{f} \restriction \circ f^\sharp)_n)_{n\in\omega}$. The identity id_Σ in $\mathcal{S}_f$ is given by $(id_\Sigma)^\sharp_n := ((j_\Sigma)_n)_{n\in\omega}$*

Definition 1.8 *If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics, then a flexible translation $f \colon l \to l'$ is a flexible signature morphism $f \colon \Sigma \to \Sigma'$ in $\mathcal{S}_f$ respecting the consequence relation, i.e., for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\check{f}[\Gamma] \vdash' \check{f}(\psi)$.*

The category $\mathcal{L}_f$ is the category of propositional logics and flexible translations as morphisms. Composition and identities are inherited from $\mathcal{S}_f$.

2 Filter Pairs

To motivate the notion of filter pair, we recall from Remark 1.6 that, given a logic $l = (\Sigma, \vdash)$, one can associate to each Σ-algebra M its collection of l-filters $Fi_l(M)$ and that, together with taking inverse image, this constitutes a (contravariant) functor $Fi_l \colon \Sigma\text{-Str} \to AL$, from Σ-structures to algebraic lattices. Notice that for $M \in \Sigma\text{-Str}$ the inclusion $\iota_M : (Fi_l(M), \subseteq) \hookrightarrow (\mathcal{P}(M), \subseteq)$ preserves arbitrary infima and directed suprema, thus, in particular, it is a morphism in the category AL (i.e. it preserves order).

Moreover, given a morphism $h : M \to N$ we have the following commutative diagram:

$$\begin{array}{ccccc}
M & & Fi_l(M) & \xhookrightarrow{\iota_M} & (\mathcal{P}(M), \subseteq) \\
\downarrow h & & \uparrow h^{-1}_{\restriction} & & \uparrow h^{-1} \\
N & & Fi_l(N) & \xhookrightarrow[\iota_N]{} & (\mathcal{P}(N), \subseteq)
\end{array}$$

This collection of data is the motivating example for the notion, and the name, of filter pair.

Definition 2.1 *Let Σ be a signature. A* **finitary filter pair** *over Σ is a pair (G, i), consisting of a contravariant functor $G : \Sigma\text{-}Str \to AL$ and a collection of maps $i = (i_M)_{M \in \Sigma\text{-}Str}$ such that, for any $M \in \Sigma\text{-}Str$, the function $i_M : G(M) \to (\mathcal{P}(M), \subseteq)$ satisfies the following properties:*

1. *For any $M \in \Sigma\text{-}Str$, i_M preserves arbitrary infima (in particular $i_M(\top) = M$) and directed suprema.*

2. *Given a homomorphism $h : M \to N$ of Σ-structures the following diagram commutes:*

$$\begin{array}{ccccc} M & & G(M) & \xrightarrow{i^G_M} & (\mathcal{P}(M);\subseteq) \\ h\downarrow & & G(h)\uparrow & & \uparrow h^{-1} \\ N & & G(N) & \xrightarrow[i^G_N]{} & (\mathcal{P}(N);\subseteq) \end{array}$$

Remark 2.2 *1. Condition* **2.** *says that i is a natural transformation from G to the functor $\wp\colon \Sigma\text{-}Str^{op} \to AL$ sending a Σ-structure to the power set of its underlying set and a homomorphism of Σ-structures to its associated inverse image function.*

2. Frequently, the functor G will preserve arbitrary infima and directed suprema: this occurs whenever the maps i^G_M preserves and reflects *these suprema and infima.*

Remark 2.3 *As the name "finitary filter pair" suggests, there is a more general notion of filter pair. In Proposition 2.4 below we will see how to associate a logic to a finitary filter pair and this logic will be finitary. General filter pairs can yield non-finitary logics. Such filter pairs will be considered in [3].*

As in this article we only consider finitary filter pairs, we will simply call them "filter pair" from now on.

From a filter pair we obtain a consequence relation on every Σ-algebra, i.e. a generalized logic in the sense of Definition 1.3.

Proposition 2.4 *Let (G, i) be a filter pair over the signature Σ and let A a Σ-algebra. Then there is a generalized logic on A, $l^A_G = (A, \vdash^A_G)$, defined as follows:*
Given $\Gamma \cup \{\varphi\} \subseteq A$, define

$$\Gamma \vdash^A_G \varphi \;\; iff \;\; for\ any\ a \in G(A),\ if\ \Gamma \subseteq i^G_A(a)\ then\ \varphi \in i^G_A(a).$$

In particular, when $A = Fm_\Sigma(X)$, for some set X, then we have a induced logic on X, $l^X_G := l^{Fm_\Sigma(X)}_G$. When $X = V$, we just denote this induced logic by l_G.

Proof:

It is easy to see that $\vdash^A_G$ satisfies reflexivity, cut and monotonicity.

The structurality is a consequence of condition **2** (naturality). Indeed, let $\sigma \in \Sigma - str(A, A)$ and $\Gamma \cup \{\varphi\} \subseteq A$ such that $\Gamma \vdash^A_G \varphi$. Consider $a \in G(A)$ such that $\sigma[\Gamma] \subseteq i^G_A(a)$. This implies $\Gamma \subseteq \sigma^{-1}[i^G_A(a)]$. By naturality we have

$\sigma^{-1}[i_A^G(a)] = i_A^G(G(\sigma)(a))$. Therefore $\varphi \in i_A^G(G(\sigma)(a)) = \sigma^{-1}[i_A^G(a)]$ and finally $\sigma(\varphi) \in i_A^G(a)$.

Last, we are going to prove finitarity. Let $\Gamma \cup \{\varphi\} \subseteq A$. Consider the set $S = \{\Gamma' \subseteq A;\ \Gamma' \subseteq_{fin} \Gamma\}$. Note that S is a directed set. Suppose that for every $\Gamma' \in S$, $\Gamma' \nvdash_G^A \varphi$. Then there is an $a \in G(A)$ such that $\Gamma' \subseteq i_A^G(a)$ and $\varphi \notin i_A^G(a)$. Denote $a_{\Gamma'} := \bigwedge\{a \in G(A);\ \Gamma' \subseteq i_A^G(a)\}$; the infimum exists, because algebraic lattices are complete. By condition 1 in the definition of filter pair, the map i_A^G preserves infima, thus $\Gamma' \subseteq i_A^G(a_{\Gamma'})$ and $\varphi \notin i_A^G(a_{\Gamma'})$. We obtain that the set $s := \{a_{\Gamma'};\ \Gamma' \in S\}$ is a directed set.

By condition 1 in the definition of filter pair, i_A^G preserves directed suprema, hence

$$\Gamma = \cup S \subseteq \bigcup_{\Gamma' \in S} i_A^G(a_{\Gamma'}) = i_A(\bigvee s).$$

On the other hand $\varphi \notin \bigcup_{\Gamma' \in S'} i_A^G(a_{\Gamma'}) = i_A^G(\bigvee s)$. Therefore $\Gamma \nvdash_G^A \varphi$. ■

Remark 2.5 *Note that, for each Σ-algebra A, the set of all theories for the generalized logic $l_G^A = (A, \vdash_G^A)$ is the image of $i_A^G : G(A) \to P(A)$: it follows directly from the definition of $\vdash_G^A$ above that for each $a \in A$, $i^G(a)$ is a l_G^Atheory; conversely, if $\Gamma \subseteq A$ is a l_G^A-theory then, by a reasoning analogous to the presented in the proof above, we have $\Gamma = i_A^G(a_\Gamma)$ where $a_\Gamma := \bigwedge\{a \in G(A);\ \Gamma \subseteq i_A^G(a)\}$.*

Remark 2.6 *One can see a filter pair (together with a set X of variables) as a* presentation *of a logic, different in style from the usual presentations by axioms and rules or by matrices. It is clear from the definition that the logic defined in this way does not depend on the values of the filter pair at Σ-algebras other than $Fm_\Sigma(X)$, and indeed it only depends on* the image of the map $G(Fm_\Sigma(X)) \to \wp(Fm_\Sigma(X))$, *as this is exactly the collection of theories of the logic, by definition.*
Thus, just as an algebraic structure can have many different presentations by generators and relations, a logic can have presentations by different filter pairs, each of which can be useful for different purposes.

Example 2.7 *Given a Tarskian logic $l = (\Sigma, \vdash)$, by Remark 2.6, defining $Fi_l(A)$ to be the set of l-filters on a Σ-structure A provides a functor $Fi_l : \Sigma\text{-}Str^{op} \to AL$, and hence a filter pair (Fi_l, i) where and i is the inclusion of filters into all subsets. As filters on the formula algebra are exactly the theories, this shows that every logic admits a presentation by a filter pair.*

Definition 2.8 *For a filter pair (G, i) and a Σ-structure M, the subsets of M lying in the image of i_M are called i-filters.*

Proposition 2.9 below says that, if one obtains a generalized logic from a filter pair as in Proposition 2.4, the sets in the image of i_M, for any other Σ-structure M, are filters for that logic in the sense of Definition 1.4. This justifies the name "i-filter".

Proposition 2.9 can thus be rephrased as saying that the set of i-filters on any Σ-structure is contained in the set of filters. In general this inclusion is strict, see Section 4.2 for a family of examples.

Proposition 2.9 *Let (G, i) be a filter pair, A be a Σ-algebra and $l^A = (\Sigma, \vdash^A)$ the associated generalized logic obtained as in Proposition 2.4. Then for any Σ-algebra M the subsets in the image of i_M are l^A-filters.*

Proof: Let M be a Σ-algebra, $F = i_M(x) \subseteq M$ a subset in the image of i, $\Gamma \cup \{\varphi\} \subseteq A$ formulas with $\Gamma \vdash^A \varphi$ and $v\colon A \to M$ be a Σ-homomorphism with $v[\Gamma] \subseteq F$. We need to show that $\varphi \in F$. For this consider the naturality square

$$\begin{array}{ccccc} A & \quad & G(A) & \xrightarrow{i_A} & (\mathcal{P}(A), \subseteq) \\ {\scriptstyle v}\downarrow & & {\scriptstyle G(v)}\uparrow & & \uparrow{\scriptstyle v^{-1}} \\ M & & G(M) & \xrightarrow[i_M]{} & (\mathcal{P}(M), \subseteq) \end{array}$$

We have $\Gamma \subseteq v^{-1}[F] = v^{-1}[i_M(x)] = i_A(G(v)(x))$. Now the assumption $\Gamma \vdash^A \varphi$ yields $\varphi \in i_A(G(v)(x)) = v^{-1}[F]$, i.e. $v(\varphi) \in F$. ■

Proposition 2.10 *Let (G, i) be a filter pair, A be a Σ-algebra. Define a relation $\vdash^A_* \subseteq \wp(A) \times A$:*

$$\begin{array}{c} \Gamma \vdash^A_* \varphi \ \textit{ iff } \textit{for any } \Sigma\textit{-algebra } M, \textit{ for any } b \in G(M) \\ \textit{and any } \Sigma\textit{-homomorphism } v : A \to M, \\ \textit{if } v[\Gamma] \subseteq i_M(b) \textit{ then } v(\varphi) \in i_M(b) \end{array}$$

then $\vdash^A_*$ is a Tarskian consequence relation on the Σ-algebra A that coincides with the consequence relation $\vdash^A$.

Proof: Suppose $\Gamma \vdash^A_* \varphi$. Then, taking the identity as valuation, one has that if $\Gamma \subseteq i_A(a)$ then $\varphi \in i_A(a)$ for all $a \in G(A)$. Thus, by the very definition of the generalized logic $l^A := (\Sigma, \vdash^A)$, we have $\Gamma \vdash^A \varphi$.

On the other hand suppose $\Gamma \vdash^A \varphi$. Then for any l^A-matrix $\langle M, F\rangle$ and valuation $v : A \to M$ one has that if $v[\Gamma] \subseteq F$ then $v(\varphi) \in F$: this holds in particular whenever $F = i_M(x)$ for some $x \in G(M)$. Since by Proposition 2.9 all subsets in the image of i_M are filters, this implies that the defining condition for $\Gamma \vdash^A_* \varphi$ is satisfied. ■

Remark 2.11 *1. Call a matrix $\langle M, F\rangle$ an i-matrix if the filter F is in the image of i_M. Denoting the collection of all i-matrices by i-Matr, Proposition 2.9 can be subsumed as stating an inclusion i-$Matr \subseteq Matr_l$. Proposition 2.10 then says that, although this inclusion can be strict, the two classes of matrices always define the same logic (by the process given in Remark 1.6).*

2. The inclusions of i-filters into all filters, established by Proposition 2.9, are also easily seen to form a natural transformation $j^G : G \Rightarrow Fi_{l_G}$: For each Σ-structure M, $j^G_M : G(M) \to \mathcal{F}i_{l_G}(M)$ is the unique factorization of $i^G_M : G(M) \to \wp(M)$ through $\iota_M : \mathcal{F}i_{l_G}(M) \hookrightarrow P(M)$. This exhibits (Fi_{l_G}, ι) as a weakly terminal filter pair among all filter pairs presenting the same logic l_G. To make this statement precise, one needs of course a notion of morphism of filter pairs. This will be given in the next section.

The following two propositions establish that, although the notions of i-filter and l^X-filter –where $l^X = (\Sigma, \vdash^X)$ is the logic on the set of variables X– may differ for general algebras, this is not the case *for an absolutely free algebra* $Fm_\Sigma(Z)$, with a set of generators Z possibly different from X.

Proposition 2.12 *Let (G, i) be a filter pair over the signature Σ.*

1. *For any homomorphism of Σ-structures $f\colon A \to B$ and $\Gamma \cup \{\varphi\} \subseteq A$:*

$$\Gamma \vdash^A \varphi \;\Rightarrow\; f[\Gamma] \vdash^B f(\varphi).$$

2. *For any injective map of sets $f : X \rightarrowtail Y$ and $\Gamma \cup \{\varphi\} \subseteq Fm_\Sigma(X)$:*

$$\Gamma \vdash^X \varphi \;\Leftrightarrow\; f[\Gamma] \vdash^Y f(\varphi).$$

3. *$\Gamma \vdash^X \varphi$ iff there are finite sets $X' \subseteq_f X$ and $\Gamma' \subseteq_f \Gamma$ such that $var(\Gamma' \cup \{\varphi\}) \subseteq X'$ and $\Gamma' \vdash^{X'} \varphi$.*

Proof: *1.* Suppose $\Gamma \vdash^A \varphi$. Let $z \in G(B)$ such that $f[\Gamma] \subseteq i_B(z)$. Then $\Gamma \subseteq f^{-1}[i_B(z)] = i_A(G(f)(z))$. Since $\Gamma \vdash^A \varphi$, we have that $\varphi \in i_A(G(f)(z))$. Therefore $f(\varphi) \in i_B(z)$. As z was arbitrary we have $f[\Gamma] \vdash^B f(\varphi)$.

2. Let $f : X \to Y$ be injective. By *1* we have that $\Gamma \vdash^X \varphi \;\Rightarrow\; f[\Gamma] \vdash^Y f(\varphi)$. It remains to prove the converse. We split this proof in cases:
- if $X \neq \emptyset$: since f is injective there is a $g\colon Y \to X$ such that $g \circ f = Id_{Fm_\Sigma(X)}$. Hence $g \circ f[\Gamma] = \Gamma$. Let $z \in G(Fm_\Sigma(X))$ such that $\Gamma \subseteq i^G_X(z)$. Therefore $f[\Gamma] \subseteq g^{-1}[i_X(z)] = i_Y(G(g)(z))$. Since $f[\Gamma] \vdash^Y f(\varphi)$, we have $f(\varphi) \in i^G_Y(G(g)(z)) = g^{-1}[i^G_X(z)]$. Therefore $\varphi = g(f(\varphi)) \in i^G_X(z)$;

- if $X = \emptyset$, but $\Sigma_0 \neq \emptyset$: then $Fm_\Sigma(X) \neq \emptyset$ and there exists a function $g' : Y \to Fm_\Sigma(\emptyset)$ that can be uniquely extended to a Σ-homomorphism $g : Fm_\Sigma(Y) \to Fm_\Sigma(\emptyset)$ and $g \circ f = Id_{Fm_\Sigma(\emptyset)}$. Thus the proof proceeds as in the case just above.
- if $X = \Sigma_0 = \emptyset$: then $Fm_\Sigma(X) = \emptyset$ and the equivalence

$$\Gamma \vdash^X \varphi \;\Leftrightarrow\; f[\Gamma] \vdash^Y f(\varphi).$$

is vacuously satisfied.

3. " $\Rightarrow$ " Since $\vdash^Z$ is finitary, there is a finite set $\Gamma' \subseteq_f \Gamma$ such that $\Gamma' \vdash^Z \varphi$. Consider $Z' = var(\Gamma') \cup var(\varphi)$. Let $a \in G(Fm_\Sigma(Z'))$. Suppose $\Gamma' \subseteq i^G_{Z'}(a)$. We have the inclusion function $j : Z' \hookrightarrow Z$ such that $j[\Gamma'] = \Gamma'$. Due to *2* we have $\Gamma' \vdash^{Z'} \varphi$.

" $\Leftarrow$ " Let $a \in G(Fm_\Sigma(Z))$ such that $\Gamma \subseteq i^G_Z(a)$. By assumption we have that there are $\Gamma' \subseteq_f \Gamma$ and $Z' \subseteq_f Z$ such that $var(\Gamma' \cup \{\varphi\}) \subseteq Z'$ and $\Gamma' \vdash^{Z'} \varphi$. Consider the inclusion function $j : Z' \hookrightarrow Z$. Notice that $j[\Gamma'] = \Gamma'$. By item *2* we have that $\Gamma' \vdash^Z \varphi$, thus $\Gamma \vdash^Z \varphi$. ■

Remark 2.13 *Item 2 of Proposition 2.12 says that for an inclusion $X \subseteq Y$ of sets of variables, the logic $(Fm_\Sigma(Y), \vdash^Y)$ is a* natural extension *of $(Fm_\Sigma(X), \vdash^X)$, i.e. a conservative extension given by extending the set of variables. By [16, Thm. 2.6] finitary logics have a* unique *natural extension to each set of variables. Thus Proposition 2.12.2 tells us that the logic $(Fm_\Sigma(Z), \vdash^Z)$ associated to a filter pair for* any *set of variables Z uniquely determines the logic over every other set of variables. See Section 4.4 for more on this point and what changes in the case of infinitary logics.*

In the same way as for the last two items of Proposition 2.12, the following result depends on the freeness of the Σ-algebra.

Proposition 2.14 *Let Σ be a signature, (G, i^G) be a filter pair on Σ and let l^X be the associated logic on the set X. Suppose that $\Sigma_0 \neq \emptyset$ and that X is an infinite set. Then for any set Z, if $F \in Fi_{l^X}(Fm_\Sigma(Z))$ then there is $a \in G(Fm_\Sigma(Z))$ such that $i^G_Z(a) = F$.*

Proof: Consider the set $S = \{a \in G(Fm_\Sigma(Z));\ F \subseteq i^G_Z(a)\}$. Denote $a_F = \bigwedge S$. Notice that $F \subseteq i^G_Z(a_F)$. We will show that $F = i^G_Z(a_F)$. Suppose that there is $\varphi \in i^G_Z(a_F)$ such that $\varphi \notin F$. We consider two cases: $|Z| \leq |X|$ and $|X| \leq |Z|$ where X is the set of variables over which l^X is defined.

($|Z| \leq |X|$): In this case there is an injective function $f : Z \to X$. By 2.12 we have $F \vdash^Z \varphi$ iff $f[F] \vdash f(\varphi)$. Suppose that $f[F] \vdash f(\varphi)$. Then, since $F \in Fi_{l^X}(Fm_\Sigma(Z))$, we have that for any valuation $v : Fm_\Sigma(X) \to Fm_\Sigma(Z)$

if $v(f[F]) \subseteq F$ then $v(f(\varphi)) \in F$. By hypothesis $\Sigma_0 \neq \emptyset$, then $Fm_\Sigma(Z) \neq \emptyset$ and there is a map $g' : X \to Fm_\Sigma(Z)$ that can be uniquely extended to a Σ-algebra homomorphism $g : Fm_\Sigma(X) \to Fm_\Sigma(Z)$ such that $g \circ f = Id_{Fm_\Sigma(Z)}$. In particular, g is a valuation $g : Fm_\Sigma(X) \to Fm_\Sigma(Z)$ such that $g \circ f[F] = F$ and $g \circ f(\varphi) = \varphi$. Then $\varphi \in F$ which is a contradiction. Therefore $f[F] \nvdash f(\varphi)$. Thus there is $a \in G(F_\Sigma(X))$ such that $f[F] \subseteq i^G_X(a) = i^G_Z(G(f)(a))$. Hence $G(f)(a) \in S$. Thus $a_F \leq G(f)(a)$, since i^G_Z preserves inf, $i^G_Z(a_F) \subseteq i^G_Z(G(f)(a))$. Hence $\varphi \in i^G_Z(G(f)(a))$, and this implies a contradiction. Then $F = i^G_F(a_F)$.

($|X| \leq |Z|$): Suppose that $F \vdash^Z \varphi$. Then by item (3) in Proposition 2.12 there are a finite subset $F' \subseteq F$ and a finite subset $Z' = var(F' \cup \{\varphi\}) \subseteq Z$ such that $F' \vdash^{Z'} \varphi$. Since X is an infinite set there exists a injective map $f : X \rightarrowtail Z$ and a finite subset $X' \subseteq X$ such that $f' := f_\restriction : X' \to Z'$ is a bijection. So there is a map $g : Z \to X$ such that $g' := g_\restriction : Z' \to X'$ is the inverse of f'. Due to items (2) and (3) in 2.12 we have $F' \vdash^Z \varphi \Leftrightarrow f_{F'} \circ g[F'] \vdash^Z f_{F'} \circ g[\varphi] \Leftrightarrow g[F'] \vdash^X g(\varphi)$. Thus, since $F \in Fi_{l^X}(Fm_\Sigma(Z))$, for any valuation $v : X \to Fm_\Sigma(Z)$ we have that if $v(g[F']) \subseteq F$ then $v(g(\varphi)) \in F$. Note that f' can be seen as a valuation and $f'(g'[F']) = F' \subseteq F$. Then $\varphi = f_{F'}(g(\varphi)) \in F$. This implies a contradiction. Hence $F \nvdash^Z \varphi$. Therefore there is $a \in G(Fm_\Sigma(Z))$ such that $F \subseteq i^G_Z(a)$ and $\varphi \notin i^G_Z(a)$. Thus $a \in S$. Thus $a_F \leq a$ and then $i^G_Z(a_F) \subseteq i^G_Z(a)$. However $\varphi \in i^G_Z(a_F)$ and hence $\varphi \in i^G_Z(a)$ which is a contradiction. Finally: $F = i^G_Z(a_F)$. ■

3 The category of filter pairs $\mathcal{F}i$

This section is dedicated to establishing a correspondence between the categories of propositional logics and filter pairs. Influenced by the encoding of flexible morphisms of algebraizable logics given in [27], it is provided an analogous correlation involving generic propositional logics, amplifying the correspondence obtained and opening possible applications to translations in others kinds of "algebraizations" as equivalential, protoalgebraic, truth-equational and others deductive systems in the Leibniz hierarchy.

3.1 A functorial encoding of signature morphisms

We present here the fundamental technical result needed in [27] and in the present section: the functorial encoding of morphisms of signatures.

In what follows the maps $U : \Sigma - Str \to Set$ and $F : Set \to \Sigma - Str$ denote, respectively, the forgetful functor and the free $\Sigma-$algebra functor.

The following notions and results were originally presented in [27]:

Fact 3.1 *Let $\Sigma, \Sigma' \in Obj(\mathcal{S}_f)$. Consider $H : \Sigma' - Str \to \Sigma - Str$ a functor that "commutes over Set" (i.e. $U \circ H = U'$) and, for each set Y, let $\eta_H(Y) : F(Y) \to H(F'(Y))$ be the* unique *Σ-morphism such that $(Y \overset{\sigma_Y}{\to} UF(Y) \overset{U(\eta_Y)}{\to} UHF'(Y)) = (Y \overset{\sigma'_Y}{\to} U'F'(Y))$ (by the universal property of $\sigma_Y : Y \to UF(Y)$, the "inclusion" of variables into its algebra of formulas). Then:*

1. *$(\eta_H(Y))_{Y \in Set}$ is a natural transformation $\eta_H : F \to H \circ F'$.*
2. *For each set Y and each $\psi \in F(Y)$, $Var(\eta_H(Y)(\psi)) \subseteq Var(\psi)$.*
3. *For each $n \in \mathbb{N}$, let $V_n := \{v_0, \cdots, v_{n-1}\} \subseteq V$, if $\eta_H(V_n)$ preserves variables, (i.e., $\forall \psi \in F(V_n)$, $Var(\eta_H(X)(\psi)) = Var(\psi)$), then the mapping $c_n \in \Sigma_n \mapsto \eta_H(V_n)(c_n(v_0, \cdots, v_{n-1})) \in F'(V_n)$ determines a unique $\mathcal{S}_f$-morphism $m_H : \Sigma \to \Sigma'$ such that $\check{m}_H = \eta_H(V)$.*

Definition 3.2 *Let Σ, Σ' be signatures. A* **signature functor** *from Σ to Σ' is a functor $\Sigma - Str \overset{H}{\leftarrow} \Sigma' - Str$ that commutes over Set and such that the natural transformation $\eta^H : F \Rightarrow H \circ F'$ preserves variables.*

Fact 3.3

1. *Let $\Sigma - Str \overset{id}{\to} \Sigma - Str$. Then $\eta_{id_{\Sigma - Str}} = id_F$ and $id_{\Sigma - Str}$ is a signature functor; moreover $m_{id_{\Sigma - Str}} = id_\Sigma \in \mathcal{S}_f(\Sigma, \Sigma)$.*
2. *Let $(\Sigma - Str \overset{H}{\leftarrow} \Sigma' - Str \overset{H'}{\leftarrow} \Sigma'' - Str)$ be functors that commutes over Set. Then $\eta_{H \circ H'} = H(\eta_{H'}) \circ \eta_H$. If H and H' are signature functors, then $H \circ H'$ is a signature functor and, moreover, in this case, $m_{H \circ H'} = m_{H'} \bullet m_H \in \mathcal{S}_f(\Sigma, \Sigma'')$.*

Definition 3.4 Signature morphisms and signature functors: *Given a morphism in $\mathcal{S}_f$, $\Sigma \overset{h}{\to} \Sigma'$, we obtain a functor $\Sigma - Str \overset{h^\star}{\leftarrow} \Sigma' - Str$ in the following way:*

- *For each $M' \in \Sigma' - Str$ denote by $h^\star(M') = M'^h$ the Σ-structure such that*

– $|M'^h| = |M'|$ (structures with same underlying set);
– Let $k \geq 0$ and $c_k \in (\Sigma)_k$, then $h(c_k) \in F(\Sigma')[k]$ is a first-order k-ary term over *Σ' and its interpretation in the Σ'-structure M' is a certain k-ary operation on $|M'|$, $M'^{h(c_k)} : |M'|^k \to |M'|$; considering $(c_k)^{M'^h} := h(c_k)^{M'}$ (it is a k-ary operation on $|M'^h|$).*

- *Let $g \in \Sigma - Str(M', N')$, we define $h^\star(M', g, N') = (M'^h, g, N'^h) \in \Sigma - Str(M'^h, N'^h)$: clearly, the function g determines a Σ-homomorphism from M'^h into N'^h).*

Keeping the notation above, we have the following:

Fact 3.5 *1. By construction, the functor $h^\star : \Sigma' - Str \to \Sigma - Str$ "commutes over Set", i.e., $U \circ h^\star = U'$. It is straightforward that $h^\star$ preserves,* strictly, *the following constructions: substructures, products, directed inductive limits, reduced products, congruences and quotients.*

2. Let $h \in \mathcal{S}_f(\Sigma, \Sigma')$, then for all $Y \subseteq V$, $\eta_{h^\star}(Y) = \check{h}_{\restriction Y} : F(Y) \to F'(Y)^h$. In particular, $\eta_{h^\star}$ preserves variables and $h^\star$ is a signature functor.

Definition 3.6 *Denote by $\mathcal{S}_f^\dagger$ the (non-full) subcategory of the category of all (large) categories*[3] *given by the categories $\Sigma - Str$, for each signature Σ, and the signature functors as morphisms between them.*

Theorem 3.7 *The categories $\mathcal{S}_f$ and $\mathcal{S}_f^\dagger$ are anti-isomorphic. More precisely:*
(a) The mapping $\Sigma \in Obj(\mathcal{S}_f) \mapsto \Sigma - Str \in Obj(\mathcal{S}_f^\dagger)$ is bijective;
(b) Given $\Sigma, \Sigma' \in \mathcal{S}_f$, the mappings $h \in \mathcal{S}_f(\Sigma, \Sigma') \mapsto h^\star \in \mathcal{S}_f^\dagger(\Sigma' - Str, \Sigma - Str)$ and $H \in \mathcal{S}_f^\dagger(\Sigma' - Str, \Sigma - Str) \mapsto m_H \in \mathcal{S}_f(\Sigma, \Sigma')$ are (well-defined) inverse bijections.
(c) $id_\Sigma^\star = id_{\Sigma - Str}$ and $(h' \bullet h)^\star = h^\star \circ h'^\star$;
$m_{id_{\Sigma-Str}} = id_\Sigma$ and $m_{H \circ H'} = m_{H'} \bullet m_H$.

3.2 Filter pairs and Propositional Logics

In this subsection we will define a category of filter pairs and present it as functorial encoding of the category of all (finitary, propositional) logics: in fact, we can represent the category of logics and flexible morphisms as a co-reflexive full subcategory of the category of filter pairs.

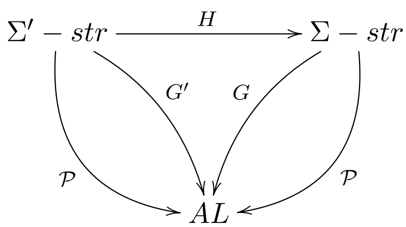
Definition 3.8 The category of Filter Pairs: *The category $\mathcal{F}i$ which is composed by:*

- ***Objects:*** *Filters pairs (G, i^G).*

- ***Morphisms:*** *Let (G, i^G) be a filter pair over a signature Σ and $(G', i^{G'})$ be a filter pair over a signature Σ'. A morphism $(G, i^G) \to (G', i^{G'})$ is a pair (H, j) such that $H : \Sigma' - str \to \Sigma - str$ is a* signature functor *and $j : G' \Rightarrow G \circ H$ is a natural transformation such that given $M' \in Obj(\Sigma' - str)$,*

$$i^G_{H(M')} \circ j_{M'} = i^{G'}_{M'}.$$

[3] I.e., the category whose objects are large categories and the arrows are functors between categories.



• **Identities:** *For each signature* Σ *and each filter pair* (G, i^G) *over* Σ, $Id_{(G,i^G)} := (Id_{\Sigma-Str}, Id_G)$.

• **Composition:** *Given* $(H, j), (H', j') \in Obj\mathcal{F}i$.

$$(H', j') \bullet (H, j) = (H \circ H', j \bullet j')$$

where $(j \bullet j')_{M''} := j_{H'(M'')} \circ j'_{M''}$.

Observe that

$$i^G_{H \circ H'(M'')} \circ ((j \bullet j')_{M''}) = i^{G''}_{M''}.$$

Indeed

$$\begin{array}{rcl} i^G_{H \circ H'(M'')} \circ ((j \bullet j')_{M''}) & = & i^G_{H \circ H'(M'')} \circ (j_{H'(M'')} \circ j'_{M''}) \\ & = & (i^G_{H \circ H'(M'')} \circ j_{H'(M'')}) \circ j'_{M''} \\ & = & i^{G'}_{H'(M'')} \circ j'_{M''} \\ & = & i^{G''}_{M''}. \end{array}$$

It is straitforward to check that the composition is associative and that holds the identity laws.

The correspondence between the objects of $\mathcal{L}_f$ and $\mathcal{F}i$ is obtained through of the Proposition 2.4 and the Example 2.7. Now we provide the correspondence between morphisms of $\mathcal{L}_f$ and of $\mathcal{F}i$.

By 3.1, for any functor $H : \Sigma' - Str \to \Sigma - Str$ such that it is a signature functor, there is a signature morphism $m_H : \Sigma \to \Sigma'$, such that $m_H(c_n) = \eta_H(X)(c_n(x_0, ..., x_{n-1}))$. We consider the functor

$$\begin{array}{lccc} \mathbb{L} : & \mathcal{F}i & \to & \mathcal{L}_f \\ & (G, i^G) & & l_G \\ & \downarrow (H, j) & \mapsto & \downarrow m_H \\ & (G', i^{G'}) & & l_{G'} \end{array}$$

It is clear that the assignments above preserves identities and compositions. It only remains to show that m_H is indeed a translation:

Let $\Gamma \cup \{\varphi\} \subseteq F(X)$ such that $\Gamma \vdash_G \varphi$. Let $a' \in G'(F'(X))$. Suppose that $\check{m}_H[\Gamma] \subseteq i^{G'}_{F'(X)}(a') = i^{G}_{H(F'(X))}(j_{F'(X)}(a'))$. Since (G, i^G) is a filter pair, then $\Gamma \subseteq (\check{m}_H)^{-1} \circ i^{G}_{H(F'(X))}(j_{F'(X)}(a')) = i^{G}_{F(X)} \circ G(\check{m}_H)(j_{F'(X)}(a'))$ and, by the definition of $\Gamma \vdash_G \varphi$, $\varphi \in i^{G}_{F(X)} \circ G(\check{m}_H)(j_{F'(X)}(a')) = (\check{m}_H)^{-1} \circ i^{G}_{H(F'(X))}(j_{F'(X)}(a'))$. Thus $\check{m}_H(\varphi) \in i^{G}_{H(F'(X))}(j_{F'(X)}(a')) = i^{G'}_{F(X)}(a')$. As a' has been taken arbitrary, we conclude that $\check{m}_H[\Gamma] \vdash_{G'} \check{m}_H(\varphi)$. Thus m_H is a $\mathcal{L}_f$-morphism.

In [27] is obtained an accurate categorial relation between the category of algebraizable logics and functors between (algebraic) structures that restrict to quasivarieties. This correspondence has a "gap" when we treat more general propositional logics, and we cover it with a similar construction defining the following functor using the obtained functor by a signature morphisms as in 3.4.

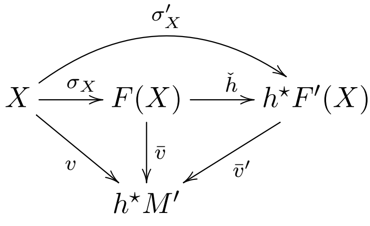
$$\begin{array}{cccc} \mathbb{F}: & \mathcal{L}_f & \to & \mathcal{F}i \\ & l & & (Fi_l, \iota) \\ & h \downarrow & \mapsto & \mathbb{F}(h) \downarrow \\ & l' & & (Fi_{l'}, \iota') \end{array}$$

where $\mathbb{F}(h) = (h^\star, j^\star)$ and the natural transformation $j^\star : Fi_{l'} \Rightarrow Fi_l \circ h^\star$ is given by a family of inclusions, i.e., let $M' \in \Sigma' - str$ and $F' \in Fi_{l'}(M')$, then $j^\star_{M'}(F') := F'$.

Observe that given $l \in \mathcal{L}_f$, $\mathbb{L} \circ \mathbb{F}(l) = \mathbb{L}((Fi_l, \iota)) = l_{Fi_l} = l$. Let $h \in hom_{\mathcal{L}_f}(l, l')$, then $\mathbb{L} \circ \mathbb{F}(h) = \mathbb{L}((h^\star, j^\star)) = m_{h^\star} = h$ and $\mathbb{L} \circ \mathbb{F} = Id_{\mathcal{L}_f}$.

Thus it only remains to check that $j^\star$ is well defined:

Indeed, let $\Gamma \cup \{\varphi\} \subseteq F(X)$ such that $\Gamma \vdash_l \varphi$ and $v : X \to h^\star(M')$ be a map such that $\bar{v}[\Gamma] \subseteq F'$.



Consider σ_X and σ'_X the respective unit of adjunction between the free functors (F, F') and the forgetful functors (U, U') over the categories $(\Sigma - str, \Sigma' - str)$. Consider $\bar{v}' : F'(X) \to M'$ the unique Σ'-homomorphism given by the universal property of the adjunction, such that $\bar{v}' \circ \sigma'_X = v$. As $h^\star(F'(X))$ has the same universe of $F'(X)$, we can see the map $\bar{v}' : h^\star(F'(X)) \to h^\star(M')$ as a Σ-homomorphism, i.e., $\bar{v}' = h^\star(\bar{v}')$. Since $\sigma'_X = \check{h} \circ \sigma_X$, then it holds:

$\bar{v}' \circ \check{h} \circ \sigma_x = v$. Notice that $\bar{v}$ is the unique morphism such that $\bar{v} \circ \sigma_X = v$. Hence $\bar{v}' \circ \check{h} = \bar{v}$. Therefore, $\bar{v}' \circ \check{h}[\Gamma] \subseteq F'$. As $F' \in Fi_{l'}(M')$ and $\check{h}[\Gamma] \vdash_{l'} \check{h}(\varphi)$, then $\bar{v}' \circ \check{h}(\varphi) \subseteq F'$, hence $\bar{v}(\varphi) \in F'$. Since v has been taken arbitrary, $F' \in Fi_l(h^{\star}M')$. Thus $\iota_{h^*M'} \circ j^*_{M'} = \iota'_{M'}$.

On the other hand, in general, $\mathbb{F} \circ \mathbb{L}((G, i^G)) \neq (G, i^G)$.

One can define the following equivalence relation on the class of filter pairs:

$$(G, i^G) \sim (G', i^{G'}) \quad \textit{iff} \quad \textit{Domain}(G) = \textit{Domain}(G') \textit{ and } l_G = l_{G'}$$

where the logics above are defined over the same signature and over the standard set of variables $V = \{v_0, v_1, \cdots\}$ (see Section 2).

This relation means that if two filter pairs are in correspondence, then they define the same logic.

So we have that $\mathbb{F} \circ \mathbb{L}((G, i^G)) = (Fi_{l_G}, \iota)$ and (G, i^G) are in the same class with respect to $\sim$.

The functors $\mathbb{L}$ and $\mathbb{F}$ give us, in a similar way as in [27], a "encoding" for morphisms in $\mathcal{L}_f$. Moreover, they establish that $\mathcal{L}_f$ is isomorphic to a co-reflexive full subcategory of $\mathcal{F}i$:

Theorem 3.9 *(a) The functor $\mathbb{F} : \mathcal{L} \to \mathcal{F}i$ is full, faithful and injective on the objects.*

(b) The functor $\mathbb{F}$ is left adjoint to the functor $\mathbb{L}$. Moreover the components of the counit of this adjunction is given by, for each signature Σ and each filter pair (G, i^G) over Σ:

$$(Id_{\Sigma-Str}, j^G) : (Fi_{l_G}, \iota) \to (G, i^G)$$

where $j^G_M : G(M) \to \mathcal{F}i_{l_G}(M)$ is the unique factorization of $i^G_M : G(M) \to \wp(M)$ through $\iota_M : \mathcal{F}i_{l_G}(M) \hookrightarrow P(M)$, (see Remark 2.11.(2)). Thus for each logic l', j^G induces by composition a (natural) bijection:

$$\mathcal{F}i(\mathbb{F}(l'), (G, i^G)) \;\cong\; \mathcal{L}_f(l', \mathbb{L}(G, i^G)).$$

Proof:

(a) Since $\mathbb{L} \circ \mathbb{F} = Id_{\mathcal{L}_f}$, the functor $\mathbb{F}$ is faithful and injective on objects. Now let $(H, j) : (Fi_l, \iota) \to (Fi_{l'}, \iota')$ be a morphism in $\mathcal{F}i$. Then for each σ'-structure M', $\iota_{H(M')} \circ j_{M'} = \iota'_{M'}$ and $j_{M'}$ is the inclusion $j^*_{M'} : Fi_{l'}(M') \hookrightarrow Fi_l(m^*_H(M'))$. Thus $\mathbb{F}(m_H) = (m^*_H, j^*) = (H, j)$ and $\mathbb{F}$ is a full functor.

(b) By standard results on category theory (see for instance [24]), it is enough to prove that for each filter pair (G, i^G), the morphism

$$(Id_{\Sigma-Str}, j^G) : (Fi_{l_G}, \iota) \to (G, i^G)$$

is the couniversal arrow from the functor $\mathbb{F}$ to the object (G, i^G), since $\mathbb{F} \circ \mathbb{L}((G, i^G)) = \mathbb{F}(l_G) = (Fi_{l_G}, \iota)$.

Let $l' = (\Sigma', \vdash')$ be a logic and let $(H, j) : \mathbb{F}(l') = (Fi_{l'}, \iota') \to (G, i^G)$ be a filter pairs morphism. Suppose that $G : \Sigma - str \to AL$. Then $H : \Sigma - str \to \Sigma' - str$ is a signature functor and, for each $M \in Obj(\Sigma - Str)$, $\iota'_{H(M)} \circ j_M = i^G_M : G(M) \hookrightarrow \wp(M)$. Thus j_M factors uniquely as:

$$(G(M) \xrightarrow{j_M} Fi_{l'}(H(M))) = (G(M) \xrightarrow{j^G_M} Fi_{l_G}(M)) \xrightarrow{j^*_{Id_{\Sigma}(M)}} Fi_{l'}(H(M)))$$

Thus: $(Id_\Sigma, j^G) \bullet \mathbb{F}(m_H) = (Id_\Sigma, j^G) \bullet (m^*_H, j^*) \underset{3.7}{=\!\!\rightarrow} = (H \circ Id_\Sigma, j^*_H \circ j^G)$ $= (H, j)$.

Since $\mathbb{F}$ is a faithful functor, $m_H : l' \to l_G$ is unique $\mathcal{L}_f$-morphism such that the diagram below commutes:

$$\begin{array}{ccc} (Fi_{l'}, \iota') & \xrightarrow{\mathbb{F}(m_H)} & (Fi_{l_G}, \iota) \\ & \searrow^{(H,j)} & \downarrow^{(Id_{\Sigma str}, j^G)} \\ & & (G, i^G) \end{array}$$

finishing the proof.

■

Remark 3.10 *Since $\mathbb{F}$ is a full and faithful left adjoint of $\mathbb{L}$, by a well known result of category theory, the unity of this adjunction is an isomorphism. Moreover it is easy to see that the components of the natural transformation that is the unity of this adjunction is given, for each logic $l \in Obj(\mathcal{L}_f)$, by the identity $id_l : l \to \mathbb{L} \circ \mathbb{F}(l) = l$.*

4 Summary and outlook

In this work we have introduced the notions of (finitary) filter pair and shown how to associate a logic to a filter pair. We further introduced the notion of morphism of filter pairs and saw that the passage from filter pairs to logics becomes a functor. This functor has a section, which in particular shows that every logic admits a presentation by a filter pair.

At the abstract level at which we have introduced these notions and the basic facts about them, it may be hard to get an intuitive understanding and to evaluate where the interest in them lies. In this final section we give outlooks on forthcoming work, indicating the roles of the present results there, and the reasons for our choices of definitions.

In section 4.1 we indicate how our definition of morphism of filter pairs was suggested by the results of [26], and why such a notion is needed to pursue the program of loc. cit. of a representation theory of logics.

The first basic intuition about filter pairs that we offered, is to see a filter pair as a *presentation* of a logic. But in fact from a filter pair (G, i) one gets more than just this logic. One also gets a distinguished set of filters on each algebra, namely the i-filters. Thus a more accurate intuition is to see a filter pair as a presentation of a logic together with a collection of matrices for it. In section 4.2 below, we hint at why this class of matrices can be an interesting extra datum and why a filter pair does in fact contain still more useful information than just the logic and this class of matrices.

In section 4.3 we give an idea of the practical use of Proposition 2.12, and finally in section 4.4 we say how this same Proposition offers yet another intuition about what an (infinitary) filter pair is.

4.1 Filter pairs and representation theory of logics

The introduction of the notions of filter pair and morphism of filter pairs was originally motivated by the programme, initiated in [26], of a representation theory of logics. In this programme, the idea is to study logics through their translations to a well-behaved class of logics, for example to algebraizable logics. The translations are, however, studied on a semantic rather than a syntactic level. In [27] it was shown that translations between algebraizable logics correspond to functors between their associated quasivarieties (together with some additional data). More general, non-algebraizable, logics are no longer captured precisely by their associated classes of algebras - instead one has to consider matrices, i.e. algebras together with filters for the logic.

The same is true for translations between logics. Instead of considering just functors between certain subcategories of Σ-structures, we need to consider a functorial relation between *matrices*, i.e. Σ-structures together with filters, for both logics. For this to yield a translation between logics, it is not necessary to relate *all* matrices of both logics. It is sufficient to consider a big enough class of matrices such that the associated logic is the given one. A filter pair (G, i) provides just such a sufficiently big class of matrices, namely the Σ-algebras together with their i-filters. The notion of morphism of filter pairs is such that it relates the matrices with i-filters on both sides and this is a semantic encoding of translations between general logics.

The functorial encoding of translations between algebraizable logics of [27] is subsumed by the one presented here, but to make this precise, one first needs to introduce *congruence filter pairs*, a class of filter pairs which allows to capture the relation of a logic to its associated class of algebras.

4.2 Congruence filter pairs

A congruence filter pair is a filter pair (G, i), such that the functor G associates to a Σ-algebra the algebraic lattice of congruences relative to a quasivariety $\mathcal{Q} \subseteq \Sigma$-str. This gives a first hint of how filter pairs can express semantical aspects of logics, which was not yet visible from the general setup presented in this work.

One can show that the logics admitting a presentation by a congruence filter pair are precisely the logics admitting an algebraic semantics in the sense of [5, Def. 2.2]. Several properties of such logics can be conveniently expressed through properties of the filter pairs presenting them. For example, a logic presented by a congruence filter pair (G, i) is algebraizable if i is injective, and is truth-equational if i is surjective onto the filters of the associated logic.

This shows why, in spite of logics and i-matrices only being determined by the image of i, it would not be a good idea to identify different filter pairs if they have the same i-images: Then in particular every filter pair (G, i) would be equivalent to a filter pair (G', i') where the transformation i' is injective, namely the filter pair where G' associates to a Σ-algebra its lattice of i-filters and i' is the inclusion of these into the power set. But in the context of congruence filter pairs, injectivity is a meaningful extra property, indicating algebraizability of the logic. This is an example of how a filter pair can be still more than a presentation of a logic together with a class of matrices.

We give a further example justifying our interest in the relation of i-filters and general filters. We have seen in Proposition 2.9 that i-filters are always filters for the associated logic, and have said that the image of i need not always be the set of all filters. Indeed, by what we have said about congruence filter pairs above, any logic with an algebraic semantics which is not truth-equational provides an example of this phenomenon. One can show that for a congruence filter pair the i-filters are always equationally defined. Since i-filters are closed under intersection, one obtains a closure operator which to any filter of the logic associates the smallest i-filter containing it. This gives a method to turn any filter into an equationally defined filter.

Congruence filter pairs have proven to be a rich field of study. In particular, the relation of the Leibniz operator to the natural transformation i is an interesting new feature not present for general filter pairs.

4.3 Horn filter pairs and Craig interpolation

Horn filter pairs are a generalization of congruence filter pairs, which allow to encode not just algebraic semantics of a logic but also semantics in a class of first order structures axiomatized by universal Horn sentences. In upcoming work we will give a criterion for when amalgamation in this class of structures implies the Craig interpolation property of the associated logic. This criterion subsumes, among other things, the cases of algebraizable logics and the left variable inclusion companions of [6].

As amalgamation statements are concerned with the sets of variables occurring in a formula, the proof for this criterion involves the logics associated to a filter pair over several different sets of variables, and relies on Proposition 2.12.2. Also Proposition 2.14 is crucial for these results. As these propositions are valid for arbitrary filter pairs, and besides their usefulness are part of the basic picture, we decided to include them in the present work.

4.4 Infinitary filter pairs and natural extensions

Finitary filter pairs do by design always present finitary logics. In the definitions of algebraic lattice and directed colimit, which enter into the definition of the notion of finitary filter pair, there are implicit occurrences of the cardinal $\aleph_0$. Replacing these occurrences by some other regular cardinal κ, we obtain the notion of κ-filter pair. Analogously to finitary filter pairs, a κ-filter pair can be seen as a presentation of a logic of cardinality at most κ. Here a logic is said to be of cardinality at most κ if, whenever $\Gamma \vdash \varphi$, there exists a subset $\Gamma' \subseteq \Gamma$ of cardinality smaller than κ such that $\Gamma' \vdash \varphi$.

Proposition 2.12.2 continues to hold for κ-filter pairs. It can be rephrased as saying that given sets of variables $X \subseteq Y$, the logic associated to the filter pair over the set of variables Y is a conservative extension of the logic over the set of variables X. Such a conservative extension obtained by enlarging the set of variables is called a *natural extension*; the availability of natural extensions plays a role in several technical arguments in propositional logic.

It was noticed by Cintula and Noguera in [16] that a certain standard construction of natural extensions actually only works under restrictions on the cardinality of the logic. Further cardinality conditions on uniqueness were given and the question was raised whether there always exists a unique natural extension of a given logic to a given set of variables. The question was solved by Přenosil in [29]: Existence is always guaranteed, uniqueness fails in a certain cardinality range. Proposition 2.12.2, in the case of κ-filter pairs, is another existence proof and it coincides with one of Přenosil's constructions.

This throws a new light on Proposition 2.12.2: For finitary logics, by [16, Thm. 2.6] there is a unique natural extension to any infinite set of variables

- so Proposition 2.12.2 says that the logic associated to a filter pair over one set of variables uniquely determines the associated logics over all other sets of variables.

In the infinitary case, when uniqueness fails, this offers another perspective on what a κ-filter pair is: It is a presentation of a logic, together with choices of natural extensions to all bigger sets of variables. These considerations will be presented in the forthcoming work [3].

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