

Paraconsistent Logic as Model Building

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Abstract

The terms “model” and “model-building” have been used to characterize the field of formal philosophy, to evaluate philosophy’s and philosophical logic’s progress and to define philosophical logic itself. A model is an idealization, in the sense of being a deliberate simplification of something relatively complex in which several important aspects are left aside, but also in the sense of being a view too perfect or excellent, not found in reality, of this thing. Paraconsistent logic is a branch of philosophical logic. It is however not clear how paraconsistent logic can be seen as model-building. What exactly is modeled? In this paper I adopt the perspective of looking at a particular instance of paraconsistent logic—paranormal modal logic—which might be seen as a model of a specific kind of agent: inductive agents. After introducing what I call the high-level and low-level models of inductive agents, I analyze the extent to which the above-mentioned idealizing features of model-building appear in paranormal modal logic and how they affect its philosophical significance.

Keywords: model-building, paraconsistent logic, paranormal modal logic, inductive agent, plausibility.

1 Introduction

There is a recent trend in philosophy of science to use the terms “model” and “model-building” to identify a specific type of theoretical activity that some scientists engage in [7] [22] [24]. They have also been used to characterize the field of formal philosophy [8] [21] [4], to evaluate philosophy’s and philosophical logic’s progress [24] and to define philosophical logic itself:

What is philosophical logic? [...] We say that philosophical logic covers all significant uses of mathematical modelling in philosophy.

[...] in philosophical logic, we take a particular philosophical subject of interest, such as a part of natural language whose logical structure we wish understand better, or the metaphysical relation of part to whole, or the norms governing beliefs and actions; then we describe a mathematical structure that we take to represent important features of this subject; and we investigate the subject by investigating that mathematical structure” (pg. 1, in [10])

Timothy Williamson [24] defines a model of something as a hypothetical example of it, in the sense of a mathematical description of such a thing which might not, in all respects, correspond to any of its actual examples. The advantages of model-building in philosophy are put by him as follows:

Philosophy can never be reduced to mathematics. But we can often produce mathematical models of fragments of philosophy and, when we can, we should. No doubt the models usually involve wild idealizations. It is still progress if we can agree what consequences an idea has in one very simple case. Many ideas in philosophy do not withstand even that very elementary scrutiny, because the attempt to construct a non trivial model reveals a hidden structural incoherence in the idea itself. By the same token, an idea that does not collapse in a toy model has at least something going for it. (pg. 291, in [23])

In addition to referring to the use of mathematical models in philosophy, Williamson mentions the eminently idealistic character of a model. According to Sven Hansson [8], there are at least two possible ways that a model can be seen as an idealization: it may be an idealization in the sense of being a deliberate simplification of something relatively complex in which several important aspects are left aside, or it may be an idealization in the sense of being a view too perfect or excellent, not found in reality, of this thing.

Many instances of philosophical logic can be seen as idealizations in these two senses. The various versions of epistemic logic, for example, can be seen as models of different types of epistemic agents. While such models are idealizing-simplifying—they leave out several crucial aspects of the epistemic agents we find in the world—they are also idealizing-perfecting—they represent certain characteristics as satisfying standards of rationality much higher than what real agents can satisfy (such as the so-called omniscience principle: if β logically follows from α and $K_a(\alpha)$, then $K_a(\beta)$, where $K_a(\alpha)$ means that agent a knows that α).

Williamson elaborates on the advantages of this idealizing-perfecting feature of epistemic logic as follows:

Standard epistemic logic treats agents as logically omniscient: the structure of its models presupposes that if one knows some propositions, one also knows any other proposition they entail. [...] Such models ignore the computational limits of actual agents. Even if two mathematical formulas are logically equivalent, we may accept one but not the other because we are unaware of their equivalence; mathematics is difficult. However, idealizing away such computational limits is not just a convenient oversimplification. One may be interested in the epistemological effects of our perceptual limits: our eyesight is imperfect; our powers of discrimination by sight are limited. Since ignorance may result from either perceptual or computational limits, we must separate the two effects. A good way to do that is by studying models where the agent resembles a short-sighted ideal logician, with perceptual limits but no computational limits, whose ignorance therefore derives only from the former. For that purpose, the structure of standard models of epistemic logic is just right [...] More generally, model-building allows us to isolate one factor from others that in practice always accompany it. (pg. 168, in [24])

Concerning the idealizing-simplifying feature, as Williamson points out in the earlier quotation, it can be useful insofar as it serves to test the internal coherence of the basic approach embodied in the model. However, things are a little more dramatic; the idealizing-simplifying feature of model building is something from which formal philosophy and philosophical logic cannot escape:

Philosophical or scientific model-making is always a trade-off between simplicity and faithfulness to the original. In philosophy, the subject-matter is typically so complex that an attempt to cover all aspects will entangle the model and make it useless. A reasonably simple model will have to leave out some philosophically relevant features. (pg. 146, in [8])

Paraconsistent logics are usually seen as a branch of philosophical logic. They are traditionally defined as logics in which the *ex falso quodlibet* principle (also called the principle of explosion: from a contradiction of the form $\alpha \wedge \neg\alpha$ one can conclude anything) is not satisfied or, alternatively, logics able to formalize inconsistent but non-trivial theories [3]. Put in this way, however, it is not clear how paraconsistent logic can be characterized as model building. What exactly is modeled? One might say: inconsistent but non-trivial theories. Although an acceptable answer, it might be replied that this is a too general object of modeling. It is also too abstract. In general, we build models to

understand concrete things, in a very broad sense of the term, which we find in the world.

Perhaps a more insightful perspective of paraconsistent logic as model building can be obtained by looking at particular instances of paraconsistent logic. For example, some paraconsistent logics have been developed inside an epistemic context in which paraconsistency is seen as an aspect or feature of a specific kind of epistemic agent. Such is the case of paranormal modal logic [17] [19] [20], which has been introduced as an attempt to formalize or conceptually explain the notion pragmatic probability [2] or inductive plausibility. Further philosophical motivations rely on the problem of inductive inconsistencies [9] [16] [6] [1] and the connection it might be shown to have with two different but complementary approaches to induction [1] [18].

Concerning the philosophical analysis of the notion of inductive plausibility, a key result of paranormal modal logic is the following axiom:

$$\text{K2: } (\neg\alpha)? \leftrightarrow \neg(\alpha?)$$

, where $\alpha?$ means that α is (credulously) plausible. Despite the counterintuitiveness of the left-to-right side of K2 (that $\neg\alpha$ is plausible implies the implausibility of α), it is a distinguishing feature of the notion of inductive plausibility, in fact one of the features that distinguishes it from the notion of possibility.

Paranormal modal logic can be seen an attempt to model inductive-plausible agents. Insofar as the notions of plausibility and induction are connected with the notion of rationality—every rational agent is also an inductive-plausible agent—paranormal modal logic can be seen as an attempt to partially model rational agents. Since paraconsistency (as well as its complementary notion of paracompleteness [13]) has emerged as a necessary feature of the resulting model, it might be argued that paraconsistency is an intrinsic feature of rational agents. Such is the philosophical significance of seeing paranormal modal logic from the perspective of model building.

My purpose in this paper is to introduce and analyze paranormal modal logic from the perspective of model building. In the next section I will introduce paranormal modal logic from the standpoint of what I call a high-level model of inductive agents. In Section 3, I introduce the logic proper, which corresponds to the low-level modelling of inductive agents. In section 4, I show how from this model building perspective we can philosophically justify the left-to-right side of axiom K2 and the right-to-left side of a twin axiom related to the other plausibility notion which emerges from the model. Finally, in the last section, I will analyze the extent to which the above-mentioned idealizing features of model building appear in paranormal modal logic and how they affect its philosophical significance.

2 The High-Level Model

Rational agents are able to perform many tasks. They make plans, they act in the world, they have beliefs and desires, they make moral judgments ... and they make inductive inferences, in the Carnapian sense of rational non-deductive inferences [2]. Beside being planning agents, epistemic agents, moral agents, etc., they are therefore also inductive agents. Paranormal modal logic is an attempt to model inductive agents; it is also an attempt to partially model rational agents.

I shall be following Williamson's definition of model:

[...] a model of something is a hypothetical example of it. Thus a model of predator-prey interaction is a hypothetical example of predator-prey interaction. The point of the qualification "hypothetical" is that the example is presented by an explicit description in general terms, rather than by pointing to an actual case. For instance, one writes down differential equations for the changing population sizes of the two species, rather than saying "the changing numbers of foxes and rabbits in Victorian Sussex." The description picks out a type of case, rather than one particular case: for instance, the type of any predator-prey interaction that obeys the given differential equations. (pg. 160, in [24])

A model of inductive agent is then a hypothetical example of it, in the sense of a (preferably) mathematical description of one such an agent which might not, in all respects, correspond to any of its actual examples; it is thus an idealization of real inductive agents. This has the advantage (along with the clarity which comes from the mathematical side of the description) of making "direct study of the model easier than direct study of the phenomenon itself." (pg. 160, in [24])

While a low-level model of something is a fully mathematical description of a hypothetical example of it, a high-level model is an informal, or only partially mathematized description which in some sense complements the low-level model. In this section I will provide a model for inductive agents from this high-level perspective; it contains the basic concepts and philosophical motivations for the development of paranormal modal logic. The low-level model will be described in the next section.

Independently of the internal structure of inductive inferences, they have some general features. A very interesting one is this: due to their being non-truth-preserving and ampliative, it is possible that from the same consistent set of statements Δ one infers both α and $\neg\alpha$. In other words, contradictions are a very likely outcome of the use of inductive inferences [9] [16] [6] [1]. An

inductive agent is therefore potentially a contradictory agent. The question then arises: how to deal with these contradictions?

Let Δ be the set of beliefs of inductive agent a and Σ a mechanism of inductive inferences to be applied to the members of Δ (it might be simply a set of inductive inferential rules); I name the deductive closure of each consistent set of conclusions obtained from Δ an inductive extension. The cases where contradictions are obtained from the application of Σ to Δ lead to more than one inductive extension. In these cases, we have at least two options at our disposal: to ignore contradictions and recognize as sound only those inductive conclusions belonging to the intersection of all extensions, or to take contradictions seriously and accept as authentic inductive conclusions all statements belonging to the union of all extensions. While the first option is a strict or skeptical approach which requires a great deal to accept an inductive conclusion as sound, the second is a tolerant or credulous approach which requires just the minimum to accept a statement as an authentic inductive conclusion. Agent a has therefore two different ways or approaches to look at contradictory conclusions; I call them the skeptical and credulous approaches to induction.¹

Now, supposing that a can effectively infer new conclusions from Δ with the help of Σ , it seems natural for a to qualify such conclusions so to distinguish them from the members of Δ as well as from conclusions deductively obtained from it. The common attitude in philosophy has been to use some probability notion to do the job. In order to distinguish such concept of probability from other probability notions, in special from his notion of logical probability, Carnap [2] used the term “pragmatical probability”. I shall use the less controversial term “plausibility” or “inductive plausibility”. What I have called so far inductive conclusions are thus the same as plausible conclusions, or still pragmatically probable conclusions. Besides being an inductive agent, a is thus also a plausible agent or, as I shall call it, a plausible-inductive agent.

Notice that according the general idea being presented here, we cannot speak of inductive or plausible conclusions *per se*. Instead, we must speak of inductive or plausible conclusions according to this or that approach: when

¹The distinction between a skeptical approach and a credulous one is of course not new; it has been extensively used in the nonmonotonic literature, for instance [15] [14]. What is new however is its being used to name two general approaches to inductive reasoning, which are in fact the end-product of a conceptual analysis to the notion of induction which takes seriously the phenomenon of inductive inconsistencies [18]. As far as the philosophical literature is concerned, even though the existence of these two approaches to induction has not been explicitly acknowledged, it is possible to identify isolated uses of them in several discussions related to the problem of inductive inconsistencies. It can be shown for instance how some of the main solutions given to the lottery paradox [12] can be seen as instances either of a skeptical approach or of a credulous one, and that when we recognize these approaches as complementary instead of competing, the whole controversy regarding the proper solution to the lottery paradox is dissolved [18].

α is true in all inductive extensions we say that α is plausible according to a skeptical approach, and when α is true in at least one extension we say that α is plausible according to a credulous approach. Trivially then, the skeptical and credulous approaches work as evaluation functions which assess in different ways the truthfulness of plausible statements, giving rise in fact to two plausibility notions: what I call skeptical plausibility and credulous plausibility.

From a general point of view, we can say that the credulous and skeptical approaches represent, respectively, minimizing and maximizing strategies of truth assessing. If a adopts a credulous position, for example, she will not require too much to accept statement α as plausible. If we use 1 to represent truth and 0 to represent falsehood, this can be restated by saying that she will somehow try to maximize or bring close to 1 the truth-value of plausible statements. On the other hand, if a adopts a skeptical position she will be more demanding in the matter of accepting α as plausible, which means that she will try to minimize or bring close to 0 the truth-value of plausible statements. Therefore, while adopting a skeptical position means to be strict in the matter of accepting something as true, in our case the plausibility of sentences, adopting a credulous position means to be tolerant, not so demanding in the matter of taking something as truth.

What has been said so far can be quite fairly represented with the aid of a Kripkean semantic framework. First of all, each inductive extension might be naturally associated with a possible world of a special kind, named plausible world; every plausible world is a possible world, but not vice versa. Second, representing the notions of skeptical and credulous plausibility with the help of the modal operators ! and ? used in a post-fixed notation— $\alpha!$ means “ α is plausible according to a skeptical approach” or “ α is skeptically plausible” and $\alpha?$ “ α is plausible according to a credulous approach” or “ α is credulously plausible”—we have that while ! is interpreted alike to traditional modal operator \Box , ? is interpreted alike to \Diamond . In other words, $\alpha!$ is true iff α is true in all plausible worlds, and $\alpha?$ is true iff α is true in at least one plausible world².

Since the key semantic notion here is the notion of plausible world, there will be important conceptual differences between a logic of plausibility so conceived and the logic of possibility and necessity as formalized, say, by S5. But there will be important relations too. For instance, since every plausible world is a possible world, the following relations between the notions of necessity, possibility, skeptical plausibility and credulous plausibility hold [1]:

²I am for the time being neglecting the accessibility relation component; it shall be taken into account by the low-level model.

$$\Box\alpha \rightarrow \alpha!$$

$$\alpha! \rightarrow \alpha?$$

$$\alpha? \rightarrow \Diamond\alpha$$

In other words, necessity implies skeptical plausibility, skeptical plausibility implies credulous plausibility and credulous plausibility implies possibility. The notions of inductive plausibility are therefore in between the notions of necessity and possibility. As one might expect, plausible or rational beliefs are weaker than necessary truths (be them logical or metaphysical), but stronger than mere possible truths.

More important however is that this logic of plausibility seems to have the same formal structure as traditional modal logic, so that the task of building a logic of plausibility would be reduced, one might think, to the task of deciding which one of the normal modal systems, say, is more adequate to our needs.³ This in fact would be so if it were not for the following fact: having a skeptical and a credulous approach to evaluate the truth value of plausible formulas causes the negation operator to behave in a way that traditional modal logic simply cannot handle. This is materialized in the following pair of axioms:

$$\text{K2. } (\neg\alpha)? \leftrightarrow \neg(\alpha?)$$

$$\text{K3. } (\neg\alpha)! \leftrightarrow \neg(\alpha!)$$

, which are completely at odds with the standard interpretation of \Diamond and \Box . While the right-to-left side of K2 and left-to-right side of K3

$$\neg(\alpha?) \rightarrow (\neg\alpha)?$$

$$(\neg\alpha)! \rightarrow \neg(\alpha!)$$

are in accordance with traditional interpretation of \Diamond and \Box

$$\neg\Diamond\alpha \rightarrow \Diamond\neg\alpha$$

$$\Box\neg\alpha \rightarrow \neg\Box\alpha$$

, their respective left-to-right side and right-to-left side

$$\text{K2}_l: (\neg\alpha)? \rightarrow \neg(\alpha?)$$

$$\text{K3}_r: \neg(\alpha!) \rightarrow (\neg\alpha)!$$

³A normal modal logic is modal logic which has K ($\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$) valid principle and modus ponens, generalization (from α conclude $\Box\alpha$) and the rule of uniform substitution as valid inference rules [11].

are not. The following formulas are clearly false (in any modal system):

$$\diamond\neg\alpha \rightarrow \neg\diamond\alpha$$

$$\neg\square\alpha \rightarrow \square\neg\alpha$$

This is the main formal difference between the notions of necessity and skeptical plausibility on one side, and possibility and credulous plausibility on the other. My purpose in Section 4 is to show how the high-level model that I have just described can be used to philosophically justify K2 and K3 and their odd sides K2_l and K3_r. Before that, however, I will introduce, in Section 3, what I am calling the low-level model, that is to say, paranormal modal logic proper; I will also mention something about the relations that exist between paranormal modal logic and traditional normal modal logic.

3 The Low-Level Model

The low-level model I will present here corresponds basically to the proof-theoretical and semantic presentations of paranormal modal logic [19] [20]. They are revealing as to the essential features of our two plausibility notions. While the skeptical plausibility is a paracomplete notion, the credulous plausibility is a paraconsistent one. More specifically, while in connection with credulous plausibility ? the negation symbol \neg behaves paraconsistently—we might have both $\alpha?$ and $\neg(\alpha?)$ —, in connection with ! it behaves paracompletely—it might be that neither $\alpha!$ nor $\neg(\alpha!)$ are true. This is at the root of axioms K2 and K3. Because its paraconsistency and paracompleteness depend on the modality attached to the formula, I call \neg a modality-dependent paranormal negation.

They are also revealing as to the relations that exist between the plausibility and traditional modality. From the semantic point of view, paranormal modal logic uses a model-theoretical structure as closest as possible to Kripkean structures used in traditional modal logic. In special, the semantic model uses the same three elements of traditional (propositional) modal logic: a set of worlds, in this case called plausible words, an accessibility relation and a truth valuation function.⁴ This has a couple of advantages.

First of all, paranormal modal logic could now be seen not as an individual logic, but as a family of logics, akin to the family of normal modal logics. In the same way that normal modal system K can be extended into D, T, B, S4, S5, etc., the most basic paranormal modal logic K_? can be extended into

⁴In [19] and [20] this paranormal modal logic was presented inside a general framework in which a wide range of logics, including classical logic and traditional normal modal logic, can be defined.

corresponding paranormal modal systems: add $\alpha! \rightarrow \alpha?$ (axiom $D_?$) to $K_?$ and you have system $D_?$; add $\alpha! \rightarrow \alpha$ (axiom $T_?$) to $K_?$ and you have system $T_?$; add $\alpha \rightarrow \alpha?!$ (axiom $B_?$) to $T_?$ and you have system $B_?$; add $\alpha! \rightarrow \alpha!!$ (axiom $4_?$) to $T_?$ and you have system $S4_?$; add axiom $B_?$ to $S4_?$ and you have $S5_?$.

Second, as one would expect, from the semantic point of view the differences between these paranormal systems are the same as between their normal counterparts: while the accessibility relation of $K_?$ has no restrictions at all, $D_?$'s models are serial, $T_?$'s are reflexive, $B_?$'s are reflexive and symmetric, $S4_?$'s are reflexive and transitive and $S5_?$'s are reflexive, transitive and symmetric. But more important than this, and this is the third and last point, it is possible to show that these paranormal modal systems are equivalent to their normal counterpart: $K_?$ is equivalent to K , $D_?$ to D , and so on and so forth.

3.1 Language and Calculus

Here I present the simplest paranormal modal system corresponding in traditional modal logic to system K . I call it $K_?$.

Definition 3.1 *Let P be a set propositional symbols. The paranormal modal language $\mathfrak{S}_?$ is defined as follows:*

- (i) *If p is a propositional symbol of P , then $p \in \mathfrak{S}_?$;*
- (ii) *If $\alpha, \beta \in \mathfrak{S}_?$, then $\neg\alpha, \alpha \rightarrow \beta, \alpha \wedge \beta, \alpha \vee \beta \in \mathfrak{S}_?$;*
- (iii) *If $\alpha \in \mathfrak{S}_?$, then $\alpha!, \alpha? \in \mathfrak{S}_?$;*
- (iv) *Nothing else belongs to $\mathfrak{S}_?$.*

Symbols \rightarrow, \wedge and \vee are interpreted according to their usual meaning. As I have said, \neg represents what I have named paranormal modality-dependent negation: regarding the elements of P , \neg behaves classically, but in connection with $!$ -marked and $?$ -marked formulae it behaves para-completely and para-consistently, respectively.

Definition 3.2 *Let $\alpha, \beta \in \mathfrak{S}_?$ be any formulae of $\mathfrak{S}_?$. We define the derived symbols \leftrightarrow, \perp and \sim as follows:*

- (i) $\alpha \leftrightarrow \beta =_{def} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$;
- (ii) $\perp =_{def} p \wedge \neg p$, where $p \in P$ is an arbitrary propositional symbol;
- (iii) $\sim \alpha =_{def} \alpha \rightarrow \perp$.

Symbols \leftrightarrow and \perp are taken in accordance with their usual meaning. \sim is a derived symbol meant to represent classical negation.

Definition 3.3 *Let $\alpha \in \mathfrak{S}_?$ be a formula. We say that α is $?$ -free iff $?$ does not occur in α and that α is $!$ -free iff $!$ does not occur in α . If α is both $?$ -free and $!$ -free we call it a $?!$ -free formula.*

Definition 3.4 *The axiomatic of paranormal modal logic $K_?$ is as follows:*

Positive Classical Axioms

$$P1: \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$P2: (\alpha \rightarrow (\beta \rightarrow \varphi)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \varphi))$$

$$P3: \alpha \wedge \beta \rightarrow \alpha$$

$$P4: \alpha \wedge \beta \rightarrow \beta$$

$$P5: \alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$$

$$P6: \alpha \rightarrow \alpha \vee \beta$$

$$P7: \beta \rightarrow \alpha \vee \beta$$

$$P8: (\alpha \rightarrow \beta) \rightarrow ((\varphi \rightarrow \beta) \rightarrow (\alpha \vee \varphi \rightarrow \beta))$$

$$P9: ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$$

Paranormal Classical Axioms

$$A1: (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha) \quad \text{wherein } \beta \text{ is } ?\text{-free and } \alpha \text{ is } !\text{-free}$$

$$A2: \neg\alpha \rightarrow (\alpha \rightarrow \beta) \quad \text{wherein } \alpha \text{ is } ?\text{-free}$$

$$A3: \alpha \vee \neg\alpha \quad \text{wherein } \alpha \text{ is } !\text{-free}$$

Non-Positive Additional Classical Axioms

$$N1: \neg(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \neg\beta)$$

$$N2: \neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$$

$$N3: \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$$

$$N4: \neg\neg\alpha \leftrightarrow \alpha$$

Paranormal Modal Axioms

$$K1: \alpha? \leftrightarrow \sim((\sim\alpha)!)$$

$$K2: (\neg\alpha)! \leftrightarrow \neg(\alpha!)$$

$$K3: (\neg\alpha)? \leftrightarrow \neg(\alpha?)$$

Modal Axioms

$$K_?: (\alpha \rightarrow \beta)! \rightarrow (\alpha! \rightarrow \beta!)$$

Rules of Inference

$$MP: \alpha, \alpha \rightarrow \beta / \beta$$

$$N_?: \alpha / \alpha!$$

The calculus of paranormal modal logic is a modal extension of positive classical logic (axioms P1-P9). Schemas of formula A1-A3 correspond to the negative axioms of classical logic, restricted in such a way as to take into account the paraconsistent and paracomplete behavior of ? and !, respectively. Trivially enough, since these restrictions apply only to non-modal formulas, the non modal fragment of paranormal modal logic behaves exactly like in classical logic. Axiom schemas N1-N4 are meant to restore the deductive power of paranormal modal logic weakened by the restrictions of A1-A3.

The paranormal modal axioms K1-K3 set the basic properties of the modal operators ! and ?. K1 states that in connection with classical negation ~, ? and ! are the dual operators of each other. K2 and K3 state that the skeptical

plausibility of $\neg\alpha$ is equivalent to the skeptical implausibility of α , and that the credulous plausibility of $\neg\alpha$ is equivalent to the credulous implausibility of α , respectively. $K_?$ is the paranormal equivalent to normal modal logic's axiom K and, finally, MP is the rule of *modus ponens* and $N_?$ is the paranormal rule of necessitation.

Due to K1, one might think that ? or ! could be introduced as definitions and that either K2 or K3 could be obtained as theorems. Unfortunately this is not so, the reason being the non-standard behavior of \rightarrow , which shall become clear when we present the semantics in the next subsection.

Letting $A, B \subseteq \mathfrak{S}_?$ be two sets of formulas, A representing the set of global premises and B the set of local premises, $\alpha \in \mathfrak{S}_?$ a formula and $L_?$ a paranormal modal logic, we define α 's being $L_?$ -deducted from A and B (in symbols: $A + B \vdash_{L_?} \alpha$) in the standard way.⁵ If $B = \emptyset$ we write $A \vdash_{L_?} \alpha$ and if $A = B = \emptyset$, we write $\vdash_{L_?} \alpha$, in which case we say that α is a $L_?$ -theorem. In the case of $K_?$ for example, we say that α is $K_?$ -deducted from A and B (in symbols: $A + B \vdash_{K_?} \alpha$) or α is a $K_?$ -theorem (in symbols: $A + K \vdash_{k_?} \alpha$).

3.2 Semantics

Here I present the semantics of paranormal modal logic. The two definitions that follow define a semantic structure identical to the one we use to evaluate formulas in normal modal logic. The difference is the function that makes use of this structure to recursively attribute true or false to all sorts of formulas, to be defined in Definition 3.7

Definition 3.5 *A frame F is a duple $\langle W, R \rangle$ where W is a non-empty set of entities called (plausible) words and R is a binary relation on W called accessibility relation.*

Definition 3.6 *A model M is a triple $\langle W, R, v \rangle$ where $F = \langle W, R \rangle$ is a frame and v is function mapping pairs composed by elements of P and elements of W to truth-values 0 and 1. We say that model M is based on F and that $w \in W$ is a world of M .*

Definition 3.7 *The max-min modal valuations are functions Ω and \mathcal{U} which, given a model $M = \langle W, R, v \rangle$ and a world $w \in W$, map formulas of $\mathfrak{S}_?$ to truth-values 0 and 1. Ω and \mathcal{U} are defined as follows:*

⁵From an axiomatic point of view, the difference between global and local premises is that the rule of necessitation can be applied only to those formulas derived exclusively from the set of global premises. For a textbook-style presentation of modal logic that uses global and local premises in the definition of the relations of deductibility and logical consequence see [5].

- (i) $\Omega_{M,w}(p) = \mathcal{U}_{M,w}(p) = 1$ iff $v_w(p) = 1$;
- (ii) $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mathcal{U}_{M,w}(\alpha) = 0$;
- (iii) $\mathcal{U}_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$;
- (iv) $\Omega_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$;
- (v) $\mathcal{U}_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\mathcal{U}_{M,w}(\beta) = 1$;
- (vi) $\Omega_{M,w}(\alpha \wedge \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 1$ and $\Omega_{M,w}(\beta) = 1$;
- (vii) $\mathcal{U}_{M,w}(\alpha \wedge \beta) = 1$ iff $\mathcal{U}_{M,w}(\alpha) = 1$ and $\mathcal{U}_{M,w}(\beta) = 1$;
- (viii) $\Omega_{M,w}(\alpha \vee \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 1$ or $\Omega_{M,w}(\beta) = 1$;
- (ix) $\mathcal{U}_{M,w}(\alpha \vee \beta) = 1$ iff $\mathcal{U}_{M,w}(\alpha) = 1$ or $\mathcal{U}_{M,w}(\beta) = 1$;
- (x) $\Omega_{M,w}(\alpha?) = 1$ iff, for some $w' \in W$ such that wRw' , $\Omega_{M,w'}(\alpha) = 1$;
- (xi) $\mathcal{U}_{M,w}(\alpha?) = 1$ iff, for all $w' \in W$ such that wRw' , $\mathcal{U}_{M,w'}(\alpha) = 1$;
- (xii) $\Omega_{M,w}(\alpha!) = 1$ iff, for all $w' \in W$ such that wRw' , $\Omega_{M,w'}(\alpha) = 1$;
- (xiii) $\mathcal{U}_{M,w}(\alpha!) = 1$ iff, for some $w' \in W$ such that wRw' , $\mathcal{U}_{M,w'}(\alpha) = 1$.

Ω and \mathcal{U} are functions that, depending on the modal operator attached to the formula, evaluate it either skeptically or credulously (if the formula is a non-modal one, the evaluation is, we can say, neutral.) They represent therefore, depending on the formula being evaluated, our skeptical and credulous approaches to induction. In what follows I shall try to elucidate some key aspects of Ω and \mathcal{U} .⁶

While Ω , which is the function I use in the definition of the notion of logical consequence, gives the intended meaning of ! and ?, \mathcal{U} does the opposite and evaluates !-formulas credulously and ?-formulas skeptically. The need for this “anomalous” interpretation has to do with the way Ω and \mathcal{U} deal with negation, which is materialized in items (ii) and (iii). The rationale behind these rules can be seen through a simple interpretation of the meaning of functions Ω and \mathcal{U} .

Consider $\neg(\alpha?)$. Since Ω evaluates it credulously, \mathcal{U} evaluates $\alpha?$ skeptically. We can therefore read $\Omega_{M,w}(\neg(\alpha?)) = 1$ as “according to a credulous position that tries to maximize the truthfulness of formulas, ‘not [α is plausible]’ is true.” This can be said to be equivalent to “the task of maximizing the truthfulness of ‘not [α is plausible]’ was successful,” which in its turn is the same as “the task of maximizing the falsehood of ‘ α is plausible’ was successful.”

Now consider $\mathcal{U}_{M,w}(\alpha?) = 0$. It can be read as “according to a position that tries to minimize the truthfulness of formulas, ‘ α is plausible’ is not true,” which in its turn is equivalent to “the task of minimizing the truth of ‘ α is plausible’ failed,” which is pretty the same as “the task of maximizing the falsehood of ‘ α is plausible’ was successful.” That is why $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mathcal{U}_{M,w}(\alpha) = 0$.

⁶It should be said that a full explanation of the intricacies of this unusual way to evaluate modal formulas would need much more space. For a full account of this and other aspects of paranormal modal logic, including a first-order formulation, see [19] and [20].

The same can be done to show that $\mathcal{U}_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$. Thus we see why in order to evaluate $\neg(\alpha?)$ credulously and $\neg(\alpha!)$ skeptically we need to evaluate $\alpha?$ skeptically and $\alpha!$ credulously.

Alike to the axiomatic presented in the last section, there is a strong parallel between paranormal modal logic's semantics and traditional normal modal logic's one. As I said, the notion of semantic model is exactly the same; what changes is the way we use the model to evaluate the truth-value of formulas. Roughly speaking, we just have, in addition to the usual way to evaluate modal formulas (which is, except for the asymmetry of (iv), captured by Ω), another function intended to reverse the position taken by Ω . This additional function is essential for the proper treatment of negation in (ii) and (iii), which is the key of the non-standard behavior of $!$ and $?$. Aside that, other paranormal modal systems are obtained in the standard way by restricting the accessibility relation.

Definition 3.8 *Let $M = \langle W, R, v \rangle$ be a model, $w \in W$ a world of M and $\alpha \in \mathfrak{S}_?$ a formula: α is satisfied by M at w (in symbols: $M, w \Vdash \alpha$) iff $\Omega_{M,w}(\alpha) = 1$; α is satisfied by M (in symbols: $M \Vdash \alpha$) iff, for all $w' \in W$, $M, w' \Vdash \alpha$.*

From the two notions above I define, in the customary way, the notion of logical consequence. Let $A, B \subseteq \mathfrak{S}_?$ be two sets of formulae, A representing the global premises and B the local ones, and $\alpha \in \mathfrak{S}_?$ a formula. If α is a logical consequence of A and B (all classes of models being considered) we say that α is a $K_?$ -logical consequence of A (in symbols: $A + B \models_{k?} \alpha$). In the case $A = B = \emptyset$, we say that α is $K_?$ -valid (in symbols: $\models_{k?} \alpha$).

Theorem 3.9 *$K_?$ is sound and complete (in symbols: Let $A, B \subseteq \mathfrak{S}_?$ be two sets of formulas and $\alpha \in \mathfrak{S}_?$ a formula. $A + B \vdash_{K?} \alpha$ iff $A + B \models_{K?} \alpha$).⁷*

3.3 Normal and Paranormal Modal Logic: K and $K_?$

In this subsection I show how, from a formal point of view, paranormal modal logic relates with normal modal logic. Let \mathfrak{S}_\diamond be the language of propositional normal modal logic, \diamond being taken either as a primitive or as a derived symbol, and \vdash_K and \models_K the relations of deduction and logical consequence of modal logic K .

Definition 3.10 *I define function $\Phi: \mathfrak{S}_\diamond \rightarrow \mathfrak{S}_?$ as follows:*

- (i) $\Phi(p) = p$, where $p \in P$;
- (ii) $\Phi(\neg\alpha) = \sim \Phi(\alpha)$;

⁷The proof of this and all other theorems mentioned here can be found in [19] and [20].

- (iii) $\Phi(\alpha \oplus \beta) = \Phi(\alpha) \oplus \Phi(\beta)$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$;
- (iv) $\Phi(\diamond\alpha) = \Phi(\alpha)?$;
- (v) $\Phi(\Box\alpha) = \Phi(\alpha)!$.

Theorem 3.11 *Let $\alpha \in \mathfrak{S}_?$. If $\vdash_K \alpha$, then $\vdash_{K?} \Phi(\alpha)$.*

The meaning of theorem 3.11 is that, when taken in conjunction with \sim , paranormal modalities ! and ? are indistinguishable from normal modalities \Box and \diamond . This of course implies that the full expressive power of traditional normal modal logic is contained in paranormal modal logic. Theorems 3.15-3.18 below show a stronger result.

Definition 3.12 *I define functions Π and Π of the form: $\mathfrak{S}_? \rightarrow \mathfrak{S}_\diamond$ as follows:*

- (i) $\Pi(p) = \Pi(p) = p$;
- (ii) $\Pi(\alpha?) = \diamond\Pi(\alpha)$;
- (iii) $\Pi(\alpha?) = \Box\Pi(\alpha)$;
- (iv) $\Pi(\alpha!) = \Box\Pi(\alpha)$;
- (v) $\Pi(\alpha!) = \diamond\Pi(\alpha)$;
- (vi) $\Pi(\neg\alpha) = \neg\Pi(\alpha)$;
- (vii) $\Pi(\neg\alpha) = \neg\Pi(\alpha)$;
- (viii) $\Pi(\alpha \oplus \beta) = \Pi(\alpha) \oplus \Pi(\beta)$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$;
- (ix) $\Pi(\alpha \oplus \beta) = \Pi(\alpha) \oplus \Pi(\beta)$, where $\oplus \in \{\wedge, \vee\}$;
- (x) $\Pi(\alpha \rightarrow \beta) = \Pi(\alpha) \rightarrow \Pi(\beta)$.

Definition 3.13 *I define functions Δ and ∇ of the form: $\mathfrak{S}_\diamond \rightarrow \mathfrak{S}_?$ as follows:*

- (i) $\Delta(p) = \nabla(p) = p$;
- (ii) $\Delta(\diamond\alpha) = \Delta(\alpha)?$;
- (iii) $\nabla(\diamond\alpha) = \nabla(\alpha)!$;
- (iv) $\Delta(\Box\alpha) = \Delta(\alpha)!$;
- (v) $\nabla(\Box\alpha) = \nabla(\alpha)?$;
- (vi) $\Delta(\neg\alpha) = \neg\nabla(\alpha)$;
- (vii) $\nabla(\neg\alpha) = \neg\Delta(\alpha)$;
- (viii) $\Delta(\alpha \oplus \beta) = \Delta(\alpha) \oplus \Delta(\beta)$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$;
- (ix) $\nabla(\alpha \oplus \beta) = \nabla(\alpha) \oplus \nabla(\beta)$, where $\oplus \in \{\wedge, \vee\}$;
- (x) $\nabla(\alpha \rightarrow \beta) = \Delta(\alpha) \rightarrow \nabla(\beta)$.

Definition 3.14 *Let $A \subseteq \mathfrak{S}_?$ and $B \subseteq \mathfrak{S}_\diamond$.*

- (i) $\Pi(A) = \{\Pi(\alpha) | \alpha \in A\}$;
- (ii) $\Pi(A) = \{\Pi(\alpha) | \alpha \in A\}$;
- (iii) $\Delta(B) = \{\Delta(\alpha) | \alpha \in B\}$;
- (iv) $\nabla(B) = \{\nabla(\alpha) | \alpha \in B\}$.

For the two theorems below, let $A, B \subseteq \mathfrak{S}_\diamond$ and $\alpha \in \mathfrak{S}_\diamond$.

Theorem 3.15 $A + B \vdash_K \alpha$ iff $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\alpha)$.

Theorem 3.16 $A + B \models_K \alpha$ iff $\Delta(A) + \Delta(B) \models_{K?} \Delta(\alpha)$.

For the two theorems below, let $A, B \subseteq \mathfrak{S}_?$ and $\alpha \in \mathfrak{S}_?$.

Theorem 3.17 $A + B \vdash_{K?} \alpha$ iff $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$.

Theorem 3.18 $A + B \models_{K?} \alpha$ iff $\Pi(A) + \Pi(B) \models_K \Pi(\alpha)$.

Definitions 3.12 and 3.13 formalize a translation that comes naturally when we look at the semantics of normal and paranormal modal logics. Using them, theorems 3.15-3.18 state that, according to this translation, both normal and paranormal logics can be fully embedded inside each other. The implications of this are obvious. For example, since formulas resulting from the application of Δ can be seen as abbreviations inside \mathfrak{S}_\diamond , it might be said that there is a formal paraconsistent and paracomplete inferential relation (in addition to a conceptual one) based on a truly paranormal modality-dependent negation inside normal modal logic.

4 The High-Level Model and the Philosophy of Plausibility

In this section I will use the high-level model to explain why having a skeptical and a credulous approach to evaluate the truth value of plausible formulas causes the negation operator to behave in a way that traditional modal logic cannot handle. As a consequence of this, I will also philosophically justify axioms K2 and K3:

$$\text{K2. } (\neg\alpha)? \leftrightarrow \neg(\alpha?)$$

$$\text{K3. } (\neg\alpha)! \leftrightarrow \neg(\alpha!)$$

To start with, let me examine how the notion of implausibility would be represented inside this high-level model. First of all, for all intends and purposes, the notion of implausibility might be simply seen as the negation of plausibility, so that “ α is implausible” can be taken as an abbreviation to “it is not the case that α is plausible”. But since here the concept of plausibility is taken obligatorily according either to a credulous view or to a skeptical view, the same should be done to all notions derived from it, in special the notion of implausibility. Therefore we shall have something like (I) and (II) below:

(I) it is not the case that α is plausible (α is implausible) according to a skeptical position.

(II) it is not the case that α is plausible (α is implausible) according to a credulous position.

According to our notation, (I) and (II) are trivially represented as $\neg(\alpha!)$ and $\neg(\alpha?)$, respectively.

Note however that there is an ambiguity in the reading of these two sentences. Are we negating the plausibility of α according to such and such approach? Or are we negating, according to that approach, the plausibility of α ? This can be better seen with the help of brackets, where (i) and (ii) below correspond to each one of the two possible ways we can read (I) and (II):

(i) it is not the case that [α is plausible according to a skeptical (credulous) position].

(ii) [it is not the case that α is plausible] according to a skeptical (credulous) position.

In the skeptical case, for example, while (i) means that we were not able to take “ α is plausible” as truth according to a rigid, strict posture, (ii) means that we *did* succeed in the task of attributing “true” to sentence “ α is not plausible” according to that posture. Similarly for the credulous case: while (i) means that adopting a tolerant posture concerning truth-assignment we were not able to classify “ α is plausible” as true, (ii) says is that “ α is not plausible” is true according to that posture.

Now, (i) clearly involves a negation pretty much alike the negation of traditional modal logic: (i) is true iff α is false in at least one world, in the case of the skeptical approach; and iff α is false in all worlds, in the case of the credulous one. Regarding (ii), however, the situation seems to be quite different: instead of denying that α is plausible according to a specific position, (ii) is in fact classifying the whole sentence “it is not the case that α is plausible” as true according to a specific position. We can therefore see the negation involved in (ii) as meaning something like this: it is not the case according to the position from which a given statement is uttered. As I shall try to show below, this reading allows us to philosophically justify axioms K2 and K3.

According to what I have already explained, to evaluate “ α is not plausible” according to a skeptical position means to be very strict, requiring the maximum we can to classify “ α is not plausible” as true; and to evaluate “ α is not plausible” according to a credulous position means to be tolerant, requiring the minimum we can to classify “ α is not plausible” as true. Given the semantic framework sketched in Section 2, clearly to require the *maximum*

we can to classify “ α is not plausible” as true means to require α to be false in all plausible worlds, and to require the *minimum* we can to classify “ α is not plausible” as true is tantamount to requiring α to be false in at least one world. This means that the skeptical version of (ii), or in symbols $\neg(\alpha!)$, is true iff α is false in all plausible worlds; and the credulous version of (ii), or in symbols $\neg(\alpha?)$, is true iff α is false in at least one plausible world.

As already mentioned, a trivial presupposition present in analyses such as the one I am doing here is that the notion of implausibility is to be analyzed, represented or described in terms of the concepts of negation and plausibility. As consequence of that, it can be claimed that a fundamental step in the task of formally disambiguating statements (I) and (II) involves having two different negations, one for each reading of (I) and (II). Letting \sim refer to the negation involved in (i) and \neg to the negation involved in (ii), the first reading of (I) and (II) might be formally represented as $\sim(\alpha!)$ and $\sim(\alpha?)$, respectively, and the second reading of (I) and (II) as $\neg(\alpha!)$ and $\neg(\alpha?)$, respectively. While $\sim(\alpha!)$ and $\sim(\alpha?)$ are interpreted in exactly the same way as $\sim\Box\alpha$ and $\sim\Diamond\alpha$ of traditional modal logic, respectively, \neg has a different, non-classical behavior, according to which $\neg(\alpha!)$ is true iff α is false in all plausible worlds, and $\neg(\alpha?)$ is true iff α is false in at least one plausible world. From a general perspective, $\neg\alpha$ means “it is not the case that α according to the position from which it is being uttered.”

About the relations between these two negations, it is easy to see that neither $\neg\alpha \rightarrow \sim\alpha$ nor $\sim\alpha \rightarrow \neg\alpha$ are generally valid: even though $\neg(\alpha!) \rightarrow \sim(\alpha!)$ holds, $\neg(\alpha?) \rightarrow \sim(\alpha?)$ is not valid; and even though $\sim(\alpha?) \rightarrow \neg(\alpha?)$ holds, $\sim(\alpha!) \rightarrow \neg(\alpha!)$ is not valid.

One might ask now how $(\neg\alpha)!$ and $(\neg\alpha)?$ are to be analyzed. Well, according to the way I am reading \neg , $(\neg\alpha)!$ shall mean something like “it is skeptically plausible that [it is not the case that α according to the position from which it is being uttered]”. But α is being uttered according to no position at all (the skeptical reading is being applied to the whole of $\neg\alpha$). Therefore \neg must in this case behave in the usual way: $\neg\alpha$ is true iff α is false. We have thus as follows: $(\neg\alpha)!$ is true iff α is false in all plausible worlds and $(\neg\alpha)?$ is true iff α is false in at least one plausible world. This however is the same evaluation which, we have agreed above, should be given to $\neg(\alpha!)$ and $\neg(\alpha?)$ in order to account for the second reading of (I) and (II). Therefore $\neg(\alpha!)$ is semantically equivalent to $(\neg\alpha)!$ and $(\neg\alpha)?$ is semantically equivalent to $\neg(\alpha?)$, or in symbols: $\neg(\alpha!) \leftrightarrow (\neg\alpha)!$ and $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$. Needless to say, these are exactly axioms K2 and K3.

5 Concluding Remarks: Paranormal Modal Logic as Model Building

Following Williamson [24], I have taken model-building as the enterprise of building hypothetical examples of something, or as providing a (preferably) mathematical description of one such a thing which might not, in all respects, correspond to any actual example of it. A model might therefore be both perfecting-idealizing and simplifying-idealizing; we clearly see these two idealizing aspects present in the high-level and low-level models I introduced.

Let me start by the high-level model. Clearly enough, an inductive agent is much more than the ridiculously simple description I have made in Section 2. The characterization I made in terms of a set of beliefs Δ and a mechanism of inference Σ leave an absurd number of things out of the analysis. Which kind of beliefs Δ has? What exactly are the inferences contained in Σ ? How they interact with other elements which we know are present in a rational agent, such as emotions, prejudices, moral assessments, etc.? What are their roles in the interaction that we know a has with other agents?

These questions of course are not answered. And they are deliberately not answered. It was a conscious choice to describe an example of an inductive agent which neglects many important aspects of such an agent. It is an idealization which enormously simplifies the very concept of inductive agent; in fact, it simplifies it so much that the resulting product is hypothetical in the sense of being effectively very distant from the real instances of inductive agents we find in the world.

As I have mentioned in the introductory section, this simplifying-idealizing aspect of models is really not an option: to cover all aspects of inductive agents would make the model unfeasible; we have no choice but to leave out relevant features of the subject-matter.

Besides this, however, there are advantages in offering such a absurdly simple description of inductive agents. By focusing on just two aspects of such an agent—a set of beliefs Δ and a mechanism of inference Σ —we were able through a quite straightforward reasoning to reach important conclusions. First, that the conclusions obtained from the application of Σ to Δ have to be seen according to two different approaches. As a consequence of that, the plausibility label which inductive agent a attaches to such conclusions are really two labels, one representing a skeptical notion of plausibility and other a credulous notion of plausibility. Finally, these two plausibility notions have two distinguishing features: while the first is paracomplete, the second is paraconsistent.

Paranormal logic as a low-level model centers around these three conclusions. It is therefore a model which explores a very tiny, albeit important aspect of inductive agents. And this is where it lays its merit as a model. By

isolating a very limited aspect of inductive agents, it is capable of revealing some of their distinguishing features and pointing out relations that they eventually have with other aspects of rational agents. For example, it revealed that inductive agents internally have two negations, one of them being classical and other paranormal-modality-dependent, that there is an equivalence in terms of classical negation between the two plausibility notions (axiom K1), that the plausibility of $\neg\alpha$ is equivalent to the implausibility of α (K2 and K3), etc.

This has directly to do with Williamson's claim that direct study of the model is easier than direct study of the phenomenon itself. However, he attributes that to the "mathematical clarity of the description" (pg. 160, in [24]).⁸ But here the fact that it is easier to study the model (reaching, as a consequence of this, relevant conclusions about the nature of inductive agents) than to study actual inductive agents comes not only from the mathematical aspect of the model, but also from this very essential feature of models, that they are idealizations which enormously simplify the subject-matter.

It might also reveal the relations that exist between inductive agents and other "avatars" of rational agents, such as epistemic agents and moral agents. In philosophical logic these agents are modeled, respectively, by epistemic and deontic logics, which in general are formally structured in accordance with traditional normal modal logic. The theorems of Subsection 3.3 show that there is an intimate connection between paranormal modal logic and traditional normal modal logic. Consequently, there is also an intimate connection in terms of models between inductive agents and epistemic and moral agents, for example.

As far as the perfecting-idealization side is concerned, the fragmentation between different inductive extensions is completely artificial and unreal. That our inductive agent is aware of such a partition is completely artificial, involving a kind of cognitive capability that we do not find in real inductive agents. Since the classification of statements in skeptically and credulously plausible depends of this fragmentation, this division is also a perfecting-idealization. And see that my use of the word "aware" is appropriate, for to say that α is skeptically plausible, for instance, means that for the agent implicit in our language, α is in the intersection of all extensions. This is of course made explicit in the semantics of paranormal modal logic.

From the more syntactic and proof-theoretical point of view, our low-level model also exhibits important aspects of this perfecting-idealizing side of models. First, it also contains a version of the principle of logical omniscience: if $\alpha!$ and β logically follows from α , then $\beta!$; and if $\alpha?$ and β logically follows from α , then $\beta?$. Second, the very idea that inductive agents are able to dis-

⁸Here is the full quote: "The mathematical clarity of the description helps make direct study of the model easier than direct study of the phenomenon itself." (pg. 160, in [24])

tinguish between two concepts of plausibility is unreal; perhaps some people do make such a distinction when reaching inductive conclusions, but certainly most people do not. I accept that the argumentation given with the help of my high-level model is convincing enough to show that inductive agents do have two plausibility notions, but this is an idealizing version of inductive agents. Third, that inductive agents do see the credulous plausibility of $\neg\alpha$ as implying the credulous implausibility of α and the skeptical implausibility of α as implying the skeptical plausibility of $\neg\alpha$

$$\text{K2}_l: (\neg\alpha)? \rightarrow \neg(\alpha?)$$

$$\text{K3}_r: \neg(\alpha!) \rightarrow (\neg\alpha)!$$

is also unreal. Only perfecting-idealized agents have this ability.

References

- [1] A. Buchsbaum and T. Pequeno. A Logical Expression of Reasoning. *Synthese*, 154: 431–466, 2007.
- [2] R. Carnap. *Logical Foundations of Probability*. Chicago: University of Chicago Press, 1950.
- [3] N. da Costa. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, 15: 497–510, 1974.
- [4] P. Engel. Formal Methods in Philosophy. In *Proceedings of the 6th European Congress of Analytic Philosophy*, T. Czarnecki et al. (editors), London: College Publications, 2010.
- [5] M. Fitting. Basic Modal Logic. In *Handbook of Logic in Artificial Intelligence and Logic Programming, Vol. 1, Logical Foundations*, D. Gabbay, D. Hogger and J. Robinson (editors), 368–448. Oxford: Oxford University Press, 1993.
- [6] D. Gabbay and A. Hunter. Making Inconsistency Respectable: A Logical Framework for Inconsistency in Reasoning. In *Foundations of Artificial Intelligence Research*, P. Jorrand and J. Kelemen (editors), 19–32. New York: Springer-Verlag, 1991.
- [7] P. Godfrey-Smith. The Strategy of Model-Based Science. *Biology and Philosophy*, 21: 725–740, 2006.
- [8] S. Hansson. Formalization in Philosophy. *The Bulletin of Symbolic Logic*, 6: 162–175, 2000.

- [9] C. Hempel. Inductive inconsistencies. *Synthese*, 12: 439–469, 1960.
- [10] L. Horsten and R. Pettigrew. Mathematical Methods in Philosophy. In *The Continuum Companion to Philosophical Logic*, L. Horsten and R. Pettigrew (editors), 14–26. London: Continuum Publishing, 2011.
- [11] G. Hughes and M. Cresswell. *A New Introduction to Modal Logic*. New York: Routledge, 1996.
- [12] H. Kyburg. *Probability and the Logic of Rational Belief*. Middletown: Wesleyan University Press, 1961.
- [13] A. Loparc and N. da Costa. Paraconsistency, Paracompleteness and Valuations. *Logique et Analyse*, 106: 119–131, 1984.
- [14] D. Makinson. General Patterns in Nonmonotonic Reasoning. In *Handbook of Logic in Artificial Intelligence and Logic Programming, Vol. 3, Nonmonotonic Reasoning and Uncertain Reasoning*, D. Gabbay, D. Hogger and J. Robinson (editors), 35–110. Oxford: Oxford University Press, 1994.
- [15] D. McDermott. Non-monotonic logic II. *Journal of the Association for Computing Machinery*, 29: 33–56, 1982.
- [16] D. Perlis. On the Consistency of Commonsense Reasoning. *Computational Intelligence*, 2: 180–190, 1987.
- [17] R. Silvestre. Modality, Paraconsistency and Paracompleteness. In *Advances in Modal Logic, Volume 6*, G. Governatori, I. Hodkinson and Y. Venema (editors), 449–467. Noosa: College Publications, 2006.
- [18] R. Silvestre. Ambiguidades Indutivas, Paraconsistncia, Paracompletude e as Duas Abordagens da Induo. *Manuscrito*, 30: 101–134, 2007.
- [19] R. Silvestre. Paranormal Modal Logic - Part I: The System $K_?$ and the Foundations of the Logic of Skeptical and Credulous Plausibility. *Logic and Logical Philosophy*, 21: 65–95, 2012.
- [20] R. Silvestre. Paranormal Modal Logic - Part II: $K_?$, K and Classical Logic and other Paranormal Modal Systems. *Logic and Logical Philosophy*, 22: 89–130, 2013.
- [21] J. Van Benthem. Logic in Philosophy. In *Philosophy of Logic*, D. Jacquette (editor), 65–100. Amsterdam: North-Holland, 2006.

- [22] M. Weisberg. *Simulation and Similarity: Using Models to Understand the World*. Oxford: Oxford University Press, 2013.
- [23] T. Williamson. *The Philosophy of Philosophy*. Oxford: Wiley-Blackwell, 2007.
- [24] T. Williamson. Model-Building in Philosophy. In *Philosophy's Future: The Problem of Philosophical Progress*, R. Blackford and D. Broderick (editors), 159–172. Hoboken: Wiley, 2017.

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