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$SP3B$ as an Extension of C_1

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Abstract

In [2] the author presents two paraconsistent three-valued logics in which neither the explosion principle nor the principle of non-contradiction hold. We study here one of these logics, $SP3B$, which is defined in terms of three connectives $\neg$, $\wedge$, $\vee$. We define a non-primitive implication that preserves tautologies under Modus Ponens, we prove that the set of connectives in $SP3B$ and the set of connectives in the paraconsistent logic CG'_3 are functionally equivalent and prove a weak form of the substitution theorem for $SP3B$. We also provide an axiomatization of $SP3B$ that contains properly logic C_1.

Keywords: non-monotonic reasoning, paraconsistent logic, multi-valued-logic.

Introduction

The law of non-contradiction seems to represent such a natural tool in the way we think and construct arguments that it would be, at least at first thought, impossible to conceive the idea of debating its validity. This fundamental character of the law, however does not seem to hold with the same strength as once did. In [3] the authors argue that it is possible to reject this principle and they show how to proceed; they also note that some of the ideas about logical non-apriorism have been already presented in works by Von Neumann, Hilary Putnam and Newton da Costa, in particular they present the case of quantum Mechanics, a theory that does not seem to adapt well to classical logic, but to a logic defined in terms of a-posteriori considerations. In the foreword of [16] Graham Priest talks of cultures where contradictions are interpreted in such a way that they become consistent, or they become two different alternative truths, such considerations are discussed in Conze's book about the law of non-contradiction, written in 1930 and recently translated by Holger R. Heine [16]. [7] explores interpretations of contradictions at the level of semantics but also at the level of metaphysics, and it poses one question: what does it mean to say that a contradiction is real? Some other works where the importance of the law of non-contradiciton is discussed are [17, 18].

Arguments in favor of rejecting the law of non-contradiction have been supported more recently by the research done on paraconsistent logics and the applications they have encountered, in particular, in artificial intelligence. Paraconsistent logics accept inconsistencies without presenting the problem that once a contradiction has been derived, then any proposition follows as a consequence, as is the case of classical logic.

The importance of non-monotonic reasoning in the study of knowledge representation, an important aspect of artificial intelligence, has been partially responsible for the relevance taken by paraconsistent logics in the field of mathematical logic. Two major classes can successfully be used to model non-monotonic reasoning. On one side the constructive intermediate logics, like intuitionism and Gödel logic G_3 [15], and on the other side, the paraconsistent logics like C_ω [4], and G'_3 [12]. These logics can provide mathematical bases to define semantics in knowledge representation, for example, G_3 is adequate to express the stable model semantics, also called semantics of answer sets [15] which is one of the main semantics in non-monotonic reasoning. A similar result shows that the paraconsistent logic G'_3 can express the p-stable semantics, an alternative to the stable semantics that in some sense is closer to the semantics defined by classical logic [14]. G'_3 was originally defined as a three-valued logic with the same truth tables as those for logic G_3 except in one value for the negation connective. Further properties of G'_3 were studied in [11] where an axiomatization of G'_3 is presented, that is, a soundness and completeness theorem is presented for G'_3 and a Hilbert style axiomatic system defined there. Among other results, it is also shown that G'_3 can express logic G_3.

Apparently there is no general consensus as to what a definite definition of paraconsistent logic must be, but it is universally accepted that one property must be present in any paraconsistent logic, namely, the logic must reject the explosion principle, which means that the formula $(p \wedge \neg p) \to q$ should not be valid. Logics like C_ω [4], $\mathbb{Z}$, G'_3, CG'_3 [12], C_1 [5] are paraconsistent since the explosion principle is not valid in any of them. More recently, Beziau [2] defines two 3-valued paraconsistent logics, for which the formula $\neg(p \wedge \neg p)$ is not a tautology, and argues that a paraconsistent logic that has this property: the rejection of the principle of non-contradiciton, is in some sense stronger than those where the formula $\neg(p \wedge \neg p)$ is valid. In [2] a logic that rejects both, the explosion principle and the principle of non-contradiction, is called a genuine paraconsistent logic. In this sense $\mathbb{Z}$, G'_3, CG'_3 are not genuine [12], but C_1 [5] is.

In this work, we deal with $SP3B$, one of the genuine paraconsistent logics defined in [2]; it has three primitive connectives: $\neg$, $\wedge$, $\vee$. The disjunction is defined as the maximum of the two values, as is usual in other logics, however the conjunction is not always the least of the two values. A non-primitive implication connective can be defined in many ways so that Modus Ponens preserves tautologies. One such definition, consistent with the implication in logics G_3, G'_3, and CG'_3, is provided.

Our main interest in this work is to provide a convenient examination of $SP3B$ that allows us to explore its relationships with other paraconsistent logics, in particular with logic C_1 [5], which has certain maximality property: it cannot be extended to a paraconsistent logic where the substitution property is valid [13]. Our contributions are the following:

1. In sections 2 and 3, we prove that C_1 is properly contained in $SP3B$, and we also provide an axiomatization of $SP3B$ that extends the axiomatic system that defines C_1.

2. In section 4, we explore the relationship of $SP3B$ with the paraconsistent 3-valued logic CG'_3 [12]; both logics have two designated values, we prove that any function expressed in terms of the connectives of any of these two logics can be expressed in terms of the connectives of the other, i.e., the two sets of connectives express the same functions. As a consequence, a tautology in any of these logics translates into a tautology in the other logic.

3. In section 5, we also provide a weak form of the substitution property for $SP3B$, [2] shows that the regular substitution property does not hold in $SP3B$.

Further research has been done in logic CG'_3, for example [9] presents a characterization in terms of Kripke systems of CG'_3, and [8] presents an axiomatization of it. The work presented in [6] studies families of many-valued logics related to Gödel logic G, considers adding a Lukasiewicz type negation to some of these logics and also studies Kripke semantics for some of these structures. These results and the relationship presented here between these two logics open new opportunities in the study of logic $SP3B$.

1 Background

In this section, we present two of the more common ways of defining a logic, and provide examples.

1.1 Multi-valued logics

A way to define a logic is with the use of truth values and interpretations. Multi-valued systems generalize the idea of using truth tables that are used to determine the validity of formulas in classical logic. It has been suggested that multi-valued systems should not count as logics; on the other hand pioneers such as Lukasiewicz considered such multi-valued systems as alternatives to the classical framework. Like other authors do, we prefer to give to multi-valued systems the benefit of the doubt about their status as logics.

The core of a multi-valued system is its *domain* of values D, where some of such values are special and identified as *designated*. Connectives (e.g. $\wedge$, $\vee$, $\rightarrow$, $\neg$) are then introduced as operators over D according to the particular definition of the logic. An *interpretation* is a function $I\colon \mathcal{L} \rightarrow D$ that maps atoms to elements in the domain. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in D, and then evaluating the resulting expression in terms of the connectives of the logic. A formula is said to be a *tautology* if, for every possible interpretation, the formula evaluates to a designated value. The most simple example of a multi-valued logic is classical logic where: $D = \{0, 1\}$, 1 is the unique designated value, and connectives are defined through the usual basic truth tables.

Not all multi-valued logics have to have the four connectives mentioned before, in fact classical logic can be defined in terms of two of those connectives $\neg, \wedge$ (primitive connectives), and the other two (non-primitive) can be defined in terms of $\neg, \wedge$. In case of a logic having the implication connective, it is desirable that it preserves tautologies, in the sense that if $x, x \rightarrow y$ are tautologies, then y is also a tautology. This restriction enforces the validity of Modus Ponens in the logic.

Since we will be working with several logics, we will use subindices next to the connectives to specify to which logic they correspond, for example $\neg_{G_3}$ corresponds to the connective $\neg$ of logic G_3. In those cases where the given logic is understood from the context, we drop such subindexes.

1.1.1 The logic $SP3B$

Now we review an interesting genuine three-valued paraconsistent logic called $SP3B$. This logic is defined in [2]. $SP3B$ logic is a 3-valued logic with truth values in the domain $D = \{0, 1, 2\}$ where 1 and 2 are the designated values. The connectives $\wedge$, $\vee$, and $\neg$ in $SP3B$ are defined according to the truth tables given in Table 1.

$\wedge$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$\neg x$
0	2
1	1
2	0

Table 1: Truth tables of connectives $\wedge$, $\vee$, and $\neg$ in $SP3B$.

1.1.2 The logics G'_3 and CG'_3

Logic G'_3 was first presented in [14], it is a 3-valued paraconsistent logic with one designated value. In [11] the authors present an axiomatization of G'_3 by providing a Hilbert system with a soundness and completeness theorem for it. We present here its definition as a 3-valued logic by means of the truth tables of its connectives, with 2 as designated value. Table 2 shows the truth tables of connectives $\neg_{G'_3}$ and $\to_{G'_3}$. We observe that $\neg_{G'_3}$ varies in one of its values from $\neg_{G_3}$, the negation of the three-valued Gödel logic G_3. The implication for these two logics have the same truth tables. Conjunction and disjunction for G'_3 are defined, just as in all other well known logics, as the min and max functions respectively, namely $\alpha \wedge \beta = min(\alpha, \beta)$, $\quad \alpha \vee y = max(\alpha, \beta)$.

$\to_{G'_3}$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

x	$\neg_{G'_3} x$
0	2
1	2
2	0

x	$\neg_{G_3} x$
0	2
1	0
2	0

Table 2: Truth tables of connectives $\to$, $\neg$ in G'_3 and $\neg$ in G_3.

Logic CG'_3 is defined by the same truth tables as G'_3 but 1 and 2 are designated values.

1.2 Hilbert style proof systems

In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms (which is usually assumed to be closed under substitution). This set of axioms specifies, so to speak, the 'kernel' of the logic. The actual logic is obtained when this 'kernel' is closed with respect to some given inference rules which include Modus Ponens. We will use this approach of Hilbert systems, and all logics in this paper will have only Modus Ponens as inference rule. Given a theory T, we use $T \vdash_X \alpha$ to denote the fact that the formula α can be derived from the axioms of the logic X and the formulas contained in T by a sequence of applications of Modus Ponens[1]. For

[1]We drop the subscript X in $\vdash_X$ when the given logic is understood from the context.

any pair of theories T and U, we use $T \vdash_X U$ to state the fact that $T \vdash_X \alpha$ for every formula $\alpha \in U$.

As examples of a Hilbert style system we present two logics.

C_ω [4] is defined by the following set of axioms:

Pos1 $\alpha \to (\beta \to \alpha)$

Pos2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Pos3 $(\alpha \wedge \beta) \to \alpha$

Pos4 $(\alpha \wedge \beta) \to \beta$

Pos5 $\alpha \to (\beta \to (\alpha \wedge \beta))$

Pos6 $\alpha \to (\alpha \vee \beta)$

Pos7 $\beta \to (\alpha \vee \beta)$

Pos8 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

C_ω**1** $\alpha \vee \neg\alpha$

C_ω**2** $\neg\neg\alpha \to \alpha$

Note that the first eight axioms somewhat constrain the meaning of the $\to$, $\wedge$ and $\vee$ connectives to match our usual intuition. It is a well known result that in any logic satisfying **Pos1** and **Pos2**, and with Modus Ponens as its unique inference rule, the Deduction Theorem holds [10].

The set consisting of the first eight axioms of the list above defines Positive Logic.

Logic C_1 is defined by adding to the axioms set of C_ω the following two axioms to which we will refer as $\neg_1$ and $\neg_2$ respectively:

($\neg_1$): $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$,

($\neg_2$): $\alpha^\circ \wedge \beta^\circ \to (\alpha \wedge \beta)^\circ \wedge (\alpha \vee \beta)^\circ \wedge (\alpha \to \beta)^\circ$, where $\beta^\circ = \neg(\beta \wedge \neg\beta)$.

The first of these two axioms tells us that if β is well-behaved then reduction at absurdum is recovered. Modus Ponens is the only inference rule of C_1.

2 An axiomatic system for $SP3B$

In this section, we start by defining a non-primitive implication connective for logic $SP3B$ which will be consistent with the implication of logics G'_3 and CG'_3. Then we will define a new logic in terms of a list of axioms and Modus Ponens as its unique inference rule in order to prove that the theorems of this new logic are precisely the tautologies of $SP3B$ providing so a soundness and completeness theorem for $SP3B$. This new axiomatic system is called $AXSP3B$.

Definition 2.1 *For formulas $\alpha, \beta \in SP3B$ we define the implication connective in $SP3B$ as follows*

$$\alpha \rightarrow_{SP3B} \beta := (\neg(\alpha \wedge \alpha) \vee \beta) \vee ((\alpha \wedge \neg\alpha) \wedge (\beta \wedge \neg\beta))$$

Tables 10 and 11 in Appendix B show that the true values of this formula coincide with those of the implications in Gödel logic G_3 and in logic G'_3. From these tables it follows that if $\alpha \rightarrow \beta$ and α are tautologies then β is also a tautology, since if β could take the only non-designated value, which is zero, then $\alpha \rightarrow \beta$ would be a tautology only if α takes the value zero, in which case α would not be a tautology.

Next, we present a set of axioms that defines our new logic $AXSP3B$, there is a total of 21. Following the idea presented in [10], we define a list of necessary formulas as axioms from which the tautologies of $SP3B$ can be deduced when using Modus Ponens as inference rule for this axiomatic system. We do not prove the independence of the formulas in the system, but later we use it to provide another axiomatic system that contains logic C_1 and is also complete for $SP3B$. The proof of a soundness and completeness theorem for $SP3B$ and $AXSP3B$ appears in Appendix A.

Pos1 $\alpha \rightarrow (\beta \rightarrow \alpha)$

Pos2 $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

Pos3 $(\alpha \wedge \beta) \rightarrow \alpha$

Pos8 $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$

PB1 $\neg(\neg\alpha \wedge \neg\alpha) \rightarrow \alpha$

PB2 $(\neg(\alpha \wedge \alpha)) \vee (\alpha \wedge \neg\alpha) \vee (\neg(\neg\alpha \wedge \neg\alpha))$

PB3 $\alpha \rightarrow \alpha$

PB4 $(\beta \wedge \neg\beta) \rightarrow (\neg\beta \wedge \neg\neg\beta)$

PB5 $\neg(\beta \wedge \beta) \rightarrow \neg(\neg\neg\beta \wedge \neg\neg\beta)$

PB6 $\neg(\beta \wedge \beta) \wedge \neg(\alpha \wedge \alpha) \rightarrow \neg((\beta \vee \alpha) \wedge (\beta \vee \alpha))$

PB7 $(\beta \wedge \neg\beta) \wedge \neg(\alpha \wedge \alpha) \rightarrow (\beta \vee \alpha) \wedge \neg(\beta \vee \alpha)$

PB8 $\neg(\beta \wedge \beta) \wedge (\alpha \wedge \neg\alpha) \to (\beta \vee \alpha) \wedge \neg(\beta \vee \alpha)$

PB9 $(\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \to (\beta \vee \alpha) \wedge \neg(\beta \vee \alpha)$

PB10 $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\alpha \wedge \alpha) \to \neg(\neg(\beta \vee \alpha) \wedge \neg(\beta \vee \alpha))$

PB11 $\neg(\neg\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \vee \alpha) \wedge \neg(\beta \vee \alpha))$

PB12 $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\neg\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \vee \alpha) \wedge \neg(\beta \vee \alpha))$

PB13 $\neg(\beta \wedge \beta) \wedge \neg(\alpha \wedge \alpha) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$

PB14 $\neg(\beta \wedge \beta) \wedge (\alpha \wedge \neg\alpha) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$

PB15 $(\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$

PB16 $(\beta \wedge \neg\beta) \wedge (\neg(\neg\alpha \wedge \neg\alpha)) \to (\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha)$

PB17 $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\neg\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$

As it is well known **Pos1**, **Pos2** and Modus Ponens as a unique inference rule entail the validity of the deduction theorem in $AXSP3B$.

Our main result regarding logic $AXSP3B$ is the following.

Theorem 2.2 *(soundness and completeness). The set of theorems of $AXSP3B$ is the same as the set of tautologies of $SP3B$.*

Proof. See Appendix A. ■

3 $SP3B$ as an extension of C_1

The family of logics C_n has been proposed as the base of inconsistent but non-trivial theories [5]. In particular, C_1 has been widely studied [13, 5] and one of its main features, as mentioned earlier, is that any paraconsistent extension of it does not accept the replacement property. C_1 is properly contained in $SP3B$ since all the axioms that define C_1 are tautologies in $SP3B$, however in $SP3B$ the formula $\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta$ is a tautology, but it is not a theorem in C_1.

We present now an axiomatic system that extends C_1 and for which all tautologies of $SP3B$ are theorems, besides all formulas in this system are tautologies of $SP3B$; therefore what we obtain is an axiomatization of $SP3B$ that contains all axioms that define C_1.

Our family of axioms is denoted by $AX3B$ and consists of the following:

1. All axioms of C_1.
2. $\alpha \to \neg\neg\alpha$.
3. $(\alpha \wedge \alpha)^\circ$ where $\beta^\circ = \neg(\beta \wedge \neg\beta)$.
4. **(De Morgan Law):** $\neg(\alpha \vee \beta) \to \neg\alpha \wedge \neg\beta$ and $\neg\alpha \wedge \neg\beta \to \neg(\alpha \vee \beta)$.
5. **(Y):** $\neg\alpha^\circ \wedge \neg\beta^\circ \to (\alpha \wedge \beta)^\circ$.
6. **(YY):** $\neg\beta^\circ \wedge (\alpha \wedge \beta)^\circ \to \neg\alpha$.

Next, we will prove that this axiomatic system is in fact an axiomatization for logic $SP3B$. In order to do that, we first prove that each of the formulas that define the system $AXSP3B$ is proved in $AX3B$.

We start by listing some properties valid in logic C_1 that will be used in what follows, for details see [2, 13, 5].

Lemma 3.1 *In logic C_1 we have:*

1. $\beta^\circ, \alpha \to \beta \vdash \neg\beta \to \neg\alpha$
2. $\beta^\circ, \alpha \to \neg\beta \vdash \beta \to \neg\alpha$
3. $\beta^\circ, \neg\alpha \to \beta \vdash \neg\beta \to \alpha$
4. $\beta^\circ, \neg\alpha \to \neg\beta \vdash \beta \to \alpha$
5. $\vdash (\alpha^\circ)^\circ$

Lemma 3.2 *Reductio ad absurdum in logic C_1:*

1. $(\Gamma \cup \{\alpha\} \vdash \beta^\circ,\ \ \Gamma \cup \{\alpha\} \vdash \beta,\ \ \Gamma \cup \{\alpha\} \vdash \neg\beta) \to \Gamma \vdash \neg\alpha$
2. $(\Gamma \cup \{\neg\alpha\} \vdash \beta^\circ,\ \ \Gamma \cup \{\neg\alpha\} \vdash \beta,\ \ \Gamma \cup \{\neg\alpha\} \vdash \neg\beta) \to \Gamma \vdash \alpha$

Lemma 3.3 *The following formula is valid in logic C_1:*

$$\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta$$

Now we provide one by one the proofs of the axioms in the list that define the system $AXSP3B$ from our new system $AX3B$:

PB1: $\neg(\neg\alpha \wedge \neg\alpha) \to \alpha$
Proof. $\neg\alpha \to (\neg\alpha \wedge \neg\alpha)$ (positive logic),
$(\neg\alpha \wedge \neg\alpha)^\circ$ (axiom),
$\neg(\neg\alpha \wedge \neg\alpha) \to \neg\neg\alpha$ (Lemma 3.1, item 1).

Now, applying double negation and transitivity we obtain:
$\neg(\neg\alpha \wedge \neg\alpha) \rightarrow \alpha$. ■

PB2: $\neg(\alpha \wedge \alpha) \vee (\alpha \wedge \neg\alpha) \vee \neg(\neg\alpha \wedge \neg\alpha)$
Proof. Let us call γ the formula we want to prove and assume
$\neg\gamma \rightarrow \neg\neg(\alpha \wedge \alpha) \wedge \neg(\alpha \wedge \neg\alpha) \wedge \neg\neg(\neg\alpha \wedge \neg\alpha)$ (**De Morgan Law**).
Then by using basic logic, double negation and transitivity:
$\neg\gamma \rightarrow \alpha$,
$\neg\gamma \rightarrow (\alpha)^\circ$,
$\neg\gamma \rightarrow \neg\alpha$.
Now by Lemma 3.2, we obtain $\vdash \neg\neg\gamma$ and the conclusion follows from the double negation axiom. ■

PB3: $\alpha \rightarrow \alpha$
Proof. It follows from **Pos1** and **Pos2**. ■

PB4: $(\beta \wedge \neg\beta) \rightarrow (\neg\beta \wedge \neg\neg\beta)$
Proof. It follows from the axiom $\alpha \rightarrow \neg\neg\alpha$. ■

PB5: $\neg(\beta \wedge \beta) \rightarrow \neg(\neg\neg\beta \wedge \neg\neg\beta)$
Proof. We have from double negation and basic properties $(\neg\neg\beta \wedge \neg\neg\beta)) \rightarrow (\beta \wedge \beta)$. Also we have $(\beta \wedge \beta)^\circ$. Hence, the result follows by Lemma 3.1, item 1. ■

PB6: $\neg(\beta \wedge \beta) \wedge \neg(\alpha \wedge \alpha) \rightarrow \neg((\beta \vee \alpha) \wedge (\beta \vee \alpha))$
Proof. From basic logic, we have $(\beta \vee \alpha) \rightarrow (\beta \wedge \beta) \vee (\alpha \wedge \alpha)$ and therefore $((\beta \vee \alpha) \wedge (\beta \vee \alpha)) \rightarrow (\beta \wedge \beta) \vee (\alpha \wedge \alpha)$. Since we have $(\beta \wedge \beta)^\circ$ and $(\alpha \wedge \alpha)^\circ$ we have $(\beta \wedge \beta)^\circ \vee (\alpha \wedge \alpha)$ according to axiom $\neg_2$ of C_1. Then, the conclusion follows by applying Lemma 3.1, item 1. ■

PB7: $(\beta \wedge \neg\beta) \wedge \neg(\alpha \wedge \alpha) \rightarrow ((\beta \vee \alpha) \wedge \neg(\beta \vee \alpha))$
Proof. Since we have $\beta \rightarrow (\beta \vee \alpha)$ from positive logic, it is enough to prove $(\beta \wedge \neg\beta) \wedge \neg(\alpha \wedge \alpha) \rightarrow (\neg\beta \wedge \neg\alpha)$ and then the result follows from the **De Morgan Law** we have. We also have $(\alpha \wedge \alpha)^\circ$ and $\alpha \rightarrow (\alpha \wedge \alpha)$ then, by Lemma 3.1, item 1 we obtain $\neg(\alpha \wedge \alpha) \rightarrow \neg\alpha)$. ■

PB8: This formula is equivalent to **PB7**.

PB9: $(\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \rightarrow (\beta \vee \alpha) \wedge \neg(\beta \vee \alpha)$
Proof. It follows from the next two facts:
$(\beta \wedge \neg\beta) \rightarrow (\beta \vee \alpha)$ and $(\neg\beta \wedge \neg\alpha) \rightarrow \neg(\beta \vee \alpha)$. ■

PB10: $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\alpha \wedge \alpha) \to \neg(\neg(\beta \vee \alpha) \wedge (\neg(\beta \vee \alpha))$
Proof. We will prove $\neg(\neg\beta \wedge \neg\beta) \to \neg(\neg(\beta \vee \alpha) \wedge (\neg(\beta \vee \alpha))$.
By **De Morgan Law** $(\neg(\beta\vee\alpha)\wedge(\neg(\beta\vee\alpha) \to \neg\beta\wedge\neg\alpha$, but $\neg\beta\wedge\neg\alpha \to \neg\beta\wedge\neg\beta$. Since we have $(\neg\beta\wedge\neg\beta)^\circ$ we can apply Lemma 3.1, item 1 to obtain the desired relation. ■

PB11: $\neg(\neg\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \vee \alpha) \wedge (\neg(\beta \vee \alpha))$
Proof. The proof is the same as for **PB10**. ■

PB12: $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\neg\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \vee \alpha) \wedge (\neg(\beta \vee \alpha))$
Proof. The proof is the same as for **PB10**. ■

PB13: $\neg(\beta \wedge \beta) \wedge \neg(\alpha \wedge \alpha) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$
Proof. According to **PB6** it is enough to prove:
$\neg((\beta \vee \alpha) \wedge (\beta \vee \alpha)) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$.
Since $((\beta \wedge \alpha) \wedge (\beta \wedge \alpha)) \to ((\beta \vee \alpha) \wedge (\beta \vee \alpha))$ and we have $((\beta \vee \alpha) \wedge (\beta \vee \alpha))^\circ$ the result follows from Lemma 3.1, item 1. ■

PB14: $\neg(\beta \wedge \beta) \wedge (\alpha \wedge \neg\alpha) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$
Proof. It is enough to prove the formula $\neg(\beta \wedge \beta) \to \neg((\beta \wedge \alpha) \wedge (\beta \wedge \alpha))$. From positive logic, we have $((\beta \wedge \alpha) \wedge (\beta \wedge \alpha)) \to (\beta \wedge \beta)$. Since we have $(\beta \wedge \beta)^\circ$, we only need to apply part 1 of Lemma 3.1. ■

PB15: $(\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$
Proof. First we prove that the formula $\eta = \neg(\alpha\wedge\beta)\wedge\neg(\neg(\alpha\wedge\neg\alpha)\vee\neg(\beta\wedge\neg\beta))$ is a bottom particle. In order to do this, we show that the three formulas $(\alpha \wedge \beta)$, $\neg(\alpha \wedge \beta)$, $(\alpha \wedge \beta)^\circ$ follow from η. From our formula and using **De Morgan Law**, we obtain $\neg\neg(\alpha \wedge \neg\alpha) \wedge \neg\neg(\beta \wedge \neg\beta)$ and by using one of our double negation axioms we obtain $(\alpha\wedge\neg\alpha)\wedge(\beta\wedge\neg\beta)$, and from here we obtain $(\alpha \wedge \beta)$ which is the first formula. The second formula is a conjunct in η. We prove now the third formula; from the second conjunct of η and by using **De Morgan Law**: $\neg(\neg(\alpha\wedge\neg\alpha))\wedge\neg(\neg(\beta\wedge\neg\beta))$, and the result follows from axiom (**Y**).

Next, we prove the formula $\gamma = \neg(\beta \wedge \alpha) \to (\beta)^\circ \vee (\alpha)^\circ$. Equivalently, according to the deduction theorem we need to prove $\neg(\beta \wedge \alpha) \vdash (\beta)^\circ \vee (\alpha)^\circ$, but since $\eta = \neg(\alpha\wedge\beta)\wedge\neg(\neg(\alpha\wedge\neg\alpha)\vee\neg(\beta\wedge\neg\beta))$ is a bottom particle, it proves any formula, and by applying Lemma 3.2 part 2 we obtain $\neg(\beta\wedge\alpha) \vdash (\beta)^\circ\vee(\alpha)^\circ$.

Next we complete the proof of **PB15**: from formula γ and positive logic $(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha)) \to (\beta)^\circ \vee (\alpha)^\circ$. Now, by Lemma 3.1 part 5 and axiom $\neg_2$ we know that $((\beta)^\circ \vee (\alpha)^\circ)^\circ$ is a theorem in logic C_1 so we can apply Lemma 3.1

part 1 to obtain $\neg((\beta)^\circ \vee (\alpha)^\circ) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$ therefore by one of **De Morgan Laws** this translates into $\neg(\beta)^\circ \wedge \neg(\alpha)^\circ \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$. Now, by using the equivalence $\alpha \leftrightarrow \neg\neg\alpha$ we conclude $((\beta \wedge \neg\beta) \wedge (\alpha \wedge \neg\alpha)) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$ as desired. ∎

PB16: $((\beta \wedge \neg\beta) \wedge (\neg(\neg\alpha \wedge \neg\alpha)) \to (\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha)$
Proof. From axiom (**YY**), the deduction theorem and positive logic we have $\neg(\beta)^\circ \vdash (\alpha \wedge \beta)^\circ \to (\neg\alpha \wedge \neg\alpha)$. Now, by using Lemma 3.1 part 1 along with axiom $(\alpha \wedge \alpha)^\circ$ we obtain $\neg(\beta)^\circ \vdash \neg(\neg\alpha \wedge \neg\alpha) \to \neg((\alpha \wedge \beta)^\circ)$ then the result follows by removing double negations. ∎

PB17: $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\neg\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$
Proof. From Lemma 3.3 $\neg(\beta \wedge \alpha) \to (\neg\beta \vee \neg\alpha)$ and properties of positive logic we obtain $(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha)) \to ((\neg\beta \wedge \neg\beta) \vee (\neg\alpha \wedge \neg\alpha))$. Now by axiom $(\alpha \wedge \alpha)^\circ$ and results of logic C_1, the formula $((\neg\beta \wedge \neg\beta) \vee (\neg\alpha \wedge \neg\alpha))^\circ$ is a theorem, so we can apply Lemma 3.1 part 1 to the previous formula to get $\neg((\neg\beta \wedge \neg\beta) \vee (\neg\alpha \wedge \neg\alpha)) \to \neg((\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$. Then by using one of De Morgan Laws on the left hand side of last formula we obtain our result $\neg(\neg\beta \wedge \neg\beta) \wedge \neg(\neg\alpha \wedge \neg\alpha) \to \neg(\neg(\beta \wedge \alpha) \wedge \neg(\beta \wedge \alpha))$. ∎

We are now in position to prove that $SP3B$ has an axiomatic version that extends logic C_1

Theorem 3.4 *(Soundness and completeness) Each theorem in $AX3B$ is a tautology in $SP3B$ and each tautology in $SP3B$ is a theorem in $AX3B$.*

Proof. The result follows from the fact that the system $AX3B$ proves all axioms that define system $AXSP3B$ and therefore all tautologies in $SP3B$ are theorems in $AX3B$ according to Theorem 2.2. Conversely, all axioms in the system $AX3B$ are tautologies, as can be proved straightforward. ∎

4 Functional equivalences between logics CG'_3 and $SP3B$

We proceed to prove that the sets of connectives of logics CG'_3 and $SP3B$ generate the same functions. In order to do this, we express the connectives of CG'_3 in terms of those of $SP3B$ and vice-versa.

4.1 $SP3B$ expresses CG'_3

Table 3 shows how each connective in CG'_3 can be expressed in terms of the connectives of $SP3B$. The truth-tables that confirm these facts are presented in Appendix B, section 1.

Connectives in CG'_3	Formula in $SP3B$
$\neg x$	$(\neg x \wedge x) \vee (\neg x)$
$x \wedge y$	$(x \wedge y) \wedge (x \vee y)$
$x \vee y$	$x \vee y$
$x \to y$	$(\neg(x \wedge x) \vee y) \vee ((x \wedge \neg x) \wedge (y \wedge \neg y))$

Table 3: Table of transformations

We state as a theorem the following result that indicates that any function expressed in terms of the connectives of CG'_3 can be expressed in terms of the connectives of $SP3B$.

Theorem 4.1 *Any function expressed in terms of the connectives of* CG'_3 *can be expressed in terms of the connectives of* $SP3B$.

Proof. Follows from tables presented in Appendix B, section 1. ■

4.2 CG'_3 expresses $SP3B$

Table 4 shows how each connective in $SP3B$ can be expressed in terms of the connectives of CG'_3. The truth-tables that confirm these facts are presented in Appendix B, section 2.

Connectives in $SP3B$	Formula in CG'_3
$\neg x$	$(x \wedge \neg x) \vee (x \to (\neg x \wedge \neg\neg x))$
$x \wedge y$	$[\sim\sim (x \wedge \neg x) \wedge \sim\sim (y \wedge \neg y)] \vee (x \wedge y)$
	where $\sim x = x \to (\neg x \wedge \neg\neg x)$
$x \vee y$	$x \vee y$
$x \to y$	$x \to y$

Table 4: Table of transformations

The following result follows from these facts.

Theorem 4.2 *Any function expressed in terms of the connectives of* $SP3B$ *can be expressed in terms of the connectives of* CG'_3.

Proof. Follows from tables presented in Appendix B, section 2. ■

4.3 Transformations of tautologies

According to the facts presented, we can state the following result:

Corollary 4.3 *Under the transformations provided by Table 3 and Table 4, a tautology in CG'_3 translates into a tautology in $SP3B$ and a tautology in $SP3B$ translates into a tautology in CG'_3.*

The previous results show that each of these logics can be expressed in terms of the other; however it is natural to ask whether they are comparable. Our next result shows that in fact none of these logics is stronger than the other; no every tautology in $SP3B$ is a tautology in CG'_3, and vice-versa no every tautology in CG'_3 is a tautology in $SP3B$.

Theorem 4.4 *Logics $SP3B$ and CG'_3 are not comparable.*

Proof. It is not hard to see that the formula $\neg(x \wedge \neg x)$ is a tautology in CG'_3. As we already know the principle of non-contradiction does not hold in $SP3B$. Therefore $SP3B$ is not stronger than CG'_3. On the other hand the formula $\neg(y \wedge \neg y) \rightarrow ((x \rightarrow y) \rightarrow ((x \rightarrow \neg y) \rightarrow \neg x))$ is a tautology in $SP3B$, but it is not a tautology in CG'_3 as it can be seen with the assignation $x = 2$, $y = 1$. Therefore CG'_3 is not stronger than $SP3B$. ■

5 Substitution property

The replacement theorem is an important and desirable feature in any logic, particularly if one wants to study logics that remain as close as possible to classical logic. Logic C_1, an extension of da Costa logic C_ω [4] has the property that it cannot be extended to a paraconsistent logic where the replacement property remains valid [2, 13]. It is not hard to see that $SP3B$ is a parconsistent extension of C_1 [2], hence it does not possess the substitution property, however a weaker form of that property is valid as we show next. In [2, 13] different ways of extending C_1 to paraconsistent logics are considered.

Definition 5.1 *For formulas x and y in $SP3B$, we define the following relation: $x \sim y$ iff $\models x \leftrightarrow y$, i.e., $x \leftrightarrow y$ is a tautology in $SP3B$.*

Table 5 shows that $x \leftrightarrow y$ is a tautology only when both formulas have the same truth values, or one of them takes the value 1 and the other one takes the value 2.

x	y	$x \leftrightarrow y$
0	0	2
0	1	0
0	2	0
1	0	0
1	1	2
1	2	1
2	0	0
2	1	1
2	2	2

Table 5: Truth table for the connective $x \leftrightarrow y$

Theorem 5.2 *The relation $\sim$ is an equivalence relation.*

Proof. The symmetry and the reflexivity follow from the rows where the two variables have the same truth values, and from the symmetry of the formula $x \leftrightarrow y$. The transitivity follows from the fact that the family $\{\{0\}, \{1, 2\}\}$ is a partition of the set of truth values. ■

Let us define the connective $\Leftrightarrow$ as follows: $x \Leftrightarrow y$ means $(x \leftrightarrow y) \wedge (\neg x \leftrightarrow \neg y)$.

Lemma 5.3 $\models x \Leftrightarrow y$ *if and only if formulas x and y have the same truth values for any interpretation.*

Proof. Table 6 proves the proposition. ■

x	y	$x \leftrightarrow y$	$\neg x$	$\neg y$	$\neg x \leftrightarrow \neg y$	$(x \leftrightarrow y) \wedge (\neg x \leftrightarrow \neg y)$
0	0	2	2	2	2	2
0	1	0	2	1	1	0
0	2	0	2	0	0	0
1	0	0	1	2	1	0
1	1	2	1	1	2	2
1	2	1	1	0	0	0
2	0	0	0	2	0	0
2	1	1	0	1	0	0
2	2	2	0	0	2	2

Table 6: Truth table for the connective $x \Leftrightarrow y$

This connective allows us to have a substitution theorem for logic $SP3B$ when using the symbol $\leftrightarrow$ as defined before. We need the following definition.

Definition 5.4 $\varphi[\frac{\psi}{\rho}] = \begin{cases} \varphi & \textit{if } \varphi \textit{ is atomic and different from } \rho \\ \psi & \textit{if } \varphi = \rho \end{cases}$

In the case φ is not atomic then $\varphi = \varphi_1 \Box \varphi_2$ (where $\Box$ is any of the binary connectives) or $\varphi = \neg\varphi_1$.

For the first case we define $\varphi_1 \Box \varphi_2[\frac{\psi}{\rho}] = \varphi_1[\frac{\psi}{\rho}] \Box \varphi_2[\frac{\psi}{\rho}]$.

For the second case we define $\neg(\varphi_1)[\frac{\psi}{\rho}] = \neg\varphi_1[\frac{\psi}{\rho}]$.

Finalli, we present a weak version of the substitution theorem for $SP3B$.

Theorem 5.5 $\models \psi_1 \Leftrightarrow \psi_2$ *then* $\models \varphi[\frac{\psi_1}{\rho}] \Leftrightarrow \varphi[\frac{\psi_2}{\rho}]$.

Proof. The proof is done by induction on the length of φ.

1. If $\varphi = \rho$ then for each i, $\varphi[\frac{\psi_i}{\rho}] = \varphi$ and the result follows from the induction hypothesis.

2. If φ is an atom different from ρ then there is no substitution to be done and the result follows.

3. If $\varphi = \varphi_1 \Box \varphi_2$ then by induction hypothesis $\models \varphi_1[\frac{\psi_1}{\rho}] \Leftrightarrow \varphi_1[\frac{\psi_2}{\rho}]$ and $\models \varphi_2[\frac{\psi_1}{\rho}] \Leftrightarrow \varphi_2[\frac{\psi_2}{\rho}]$. By Lemma 5.3, we know that any interpretation gives the same truth values to $\varphi_i[\frac{\psi_1}{\rho}]$ and $\varphi_i[\frac{\psi_2}{\rho}]$ and we also know that the truth values of $(\varphi_1 \Box \varphi_2)[\frac{\psi_2}{\rho}]$ depend on those truth values solely, hence the result follows.

4. If $\varphi = \neg\varphi_1$, then under any interpretation the truth values of $\varphi_1[\frac{\psi_1}{\rho}]$ are the same as those for $\varphi_1[\frac{\psi_2}{\rho}]$ by hypothesis, therefore as in the previous case, the truth values of $\neg\varphi_1[\frac{\psi_1}{\rho}]$ are the same as those for $\neg\varphi_1[\frac{\psi_2}{\rho}]$. Then it follows that $\models \neg\varphi_1[\frac{\psi_1}{\rho}] \Leftrightarrow \neg\varphi_1[\frac{\psi_2}{\rho}]$.

■

6 Further remarks

In Section 2, we defined an implication connective for $SP3B$ in terms of its original connectives. There are many ways of defining an implication for $SP3B$ in such a way that it preserves tautologies, i.e., if $\alpha \to \beta$ and α are tautologies then β is also tautology. One of these definitions is provided by the formula $a \mapsto b : \neg(a \wedge a) \vee b$, which reproduces the implication of three-valued logic PAC, a maximal paraconsistent logic studied in [1]. Conjunction and disjunction in

PAC are defined by the *min* and *max* functions respectively. The negation in PAC is the same as in $SP3B$ and the implication of PAC is shown in Table 7. PAC has as designated value 1 and 2. Proofs of Theorem 2.2 and 3.4 do not depend on the particular definition of the implication and the developments of their proofs are still the same. Therefore we have the following corollary.

Corollary 6.1 *If we define $\mapsto$ as the implication for $SP3B$ then Theorem 2.2 and 3.4 are still valid.*

$\mapsto$	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

Table 7: Truth tables of connectives $\to$ in PAC.

7 Conclusions and future work

We have proved that the sets of connectives of $SP3B$ and in CG'_3 are equivalent; each function expressed by one set of connectives can be expressed by the other. In particular this means that logic CG'_3 can be expressed in terms of logic $SP3B$ and vice-versa. We provided a weak form of the substitution theorem for $SP3B$ and extended the axiomatic system of C_1 to an axiomatization of $SP3B$, which in particular shows that $SP3B$ is stronger than C_1. Logic $SP3B$ has been introduced recently and there are many questions one can ask about it, like for example: to which extent can we import properties of CG'_3 to $SP3B$? In particular, can we characterize logic $SP3B$ in terms of a Kripke system, given that such a representation has been provided for logic CG'_3? The research aimed at answering these and other questions is the subject of future work.

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A Soundness and completeness proof

In this section, we present the equivalence between the tautologies of $SP3B$ and the theorems of $AXSP3B$.

A.1 Soundness

We provide the result that assures that the axiomatic system presented in Section 2 is in fact an axiomatic version of $SP3B$. The first part consists in proving that every tautology in $SP3B$ is a theorem in $AXSP3B$. We present it next:

Theorem A.1 *(Soundness) Any formula in $AXSP3B$ that is a theorem is a tautology in $SP3B$.*

Proof. The proof is done by induction on the length of the proof of the theorem. First, by means of truth-tables one can check that all axioms of $AXSP3B$ are tautologies.

Let A be a theorem and let $\alpha_1, \alpha_2, \ldots, \alpha_n = A$ be its proof. We assume A is not an axiom, otherwise we are done, then there are two previous steps α_k, α_m such that $\alpha_m = \alpha_k \to A$, by induction hypothesis α_k and α_m are tautologies, then the result follows from the the fact that the implication $\to_{SP3B}$ preserves tautologies as noted after Definition 2.1. ■

A.2 Completeness

The idea of the proof has its roots in Kalmars's proof of completeness for classical logic [10]. Such proof was generalized in the setting of 3-valued logics for G'_3 [14].

Now we proceed to prove that $AXSP3B$ is an axiomatization of $SP3B$. We need some definitions and a lemma.

Definition A.2 *Given a formula $\varphi \in SP3B$ and a 3-valuation ν on $SP3B$, we define a new formula φ_ν according to the following:*

$$\varphi_\nu = \neg(\neg\varphi \wedge \neg\varphi) \textit{ if } \nu(\varphi) = 2$$

$$\varphi_\nu = \varphi \wedge \neg\varphi \textit{ if } \nu(\varphi) = 1$$

$$\varphi_\nu = \neg(\varphi \wedge \varphi) \textit{ if } \nu(\varphi) = 0$$

Definition A.3 *For a set of atoms* Δ *and a 3-valuation* ν*, we define* $\Delta_\nu := \{P_\nu | P \in \Delta\}$.

In order to prove a completeness theorem for $SP3B$, we need Lemma A.4.

Lemma A.4 *Given a formula* A*, whose set of atoms is* Δ*, and a 3-valuation* ν*, the following relation holds:* $\Delta_\nu \vdash A_\nu$.

Proof. We use induction on the number of connectives. We observe that the result is true if A is an atom.

1. Let us suppose that A has the form $\neg B$.
   (a) By hypothesis $\Delta_\nu \vdash B_\nu$.
   If $\nu(B) = 2$ then $\Delta_\nu \vdash \neg(\neg B \wedge \neg B)$. Since $\nu(A) = \nu(\neg B) = 0$, we need to prove that $\Delta_\nu \vdash \neg(A \wedge A)$, this is $\Delta_\nu \vdash \neg(\neg B \wedge \neg B)$. This follows from Axiom **PB3**.
   (b) If $\nu(B) = 1$, $\Delta_\nu \vdash B \wedge \neg B$. Since $\nu(A) = \nu(\neg B) = 1$, we need to prove that $\Delta_\nu \vdash \neg B \wedge \neg\neg B$. This follows from Axiom **PB4**.
   (c) If $\nu(B) = 0$, $\Delta_\nu \vdash \neg(B \wedge B)$. Since $\nu(A) = \nu(\neg B) = 2$, we need to prove that $\Delta_\nu \vdash \neg(\neg A \wedge \neg A)$, i.e., we need to prove that $\Delta_\nu \vdash \neg(\neg\neg B \wedge \neg\neg B)$. This follows from Axiom **PB5**.

2. Let us suppose that A is $B \vee C$.
   (a) If $\nu(A) = 0$ then $\nu(B) = \nu(C) = 0$. $\Delta_\nu \vdash B_\nu, C_\nu$, $\Delta_\nu \vdash \neg(B \wedge B), \neg(C \wedge C)$, we want to prove that $\Delta_\nu \vdash A_\nu$, i.e., $\Delta_\nu \vdash \neg((B \vee C) \wedge (B \vee C))$. This follows from Axiom **PB6**.
   (b) Let $\nu(A) = 1$, $\nu(B) = 1$, $\nu(C) = 0$.
   If $\Delta_\nu \vdash B_\nu$, $\Delta_\nu \vdash C_\nu$ then $\Delta_\nu \vdash B \wedge \neg B$, $\Delta_\nu \vdash \neg(C \wedge C)$.
   We want to prove that $\Delta_\nu \vdash (B \vee C) \wedge \neg(B \vee C)$. This follows from Axiom **PB7**.
   (c) Let $\nu(A) = 1$, $\nu(B) = 0$, $\nu(C) = 1$.
   If $\Delta_\nu \vdash B_\nu, C_\nu$ then $\Delta_\nu \vdash \neg(B \wedge B)$, $\Delta_\nu \vdash (C \wedge \neg C)$.
   We want to prove that $\Delta_\nu \vdash A_\nu$, i.e., we want to prove that $\Delta_\nu \vdash (B \vee C) \wedge \neg(B \vee C)$. This follows from Axiom **PB8**.
   (d) Let $\nu(A) = \nu(B) = \nu(C) = 1$.
   $\Delta_\nu \vdash B_\nu, C_\nu$, $\Delta_\nu \vdash B \wedge \neg B$, $\Delta_\nu \vdash (C \wedge \neg C)$.
   We want to prove that $\Delta_\nu \vdash A_\nu$, i.e., $\Delta_\nu \vdash (B \vee C) \wedge \neg(B \vee C)$. This follows from Axiom **PB9**.

(e) Let $\nu(A) = \nu(B) = 2$, $\nu(C) = 0$.
$\Delta_\nu \vdash \neg(\neg B \wedge \neg B)$, $\Delta_\nu \vdash \neg(C \wedge C)$.
We want to prove that $\Delta_\nu \vdash A_\nu$, i.e., $\Delta_\nu \vdash \neg(\neg(B \vee C) \wedge \neg(B \vee C))$.
This follows from Axiom **PB10**.

(f) Let $\nu(A) = \nu(B) = 2$, $\nu(C) = 1$.
$\Delta_\nu \vdash \neg(\neg B \wedge \neg B)$, $\Delta_\nu \vdash C \wedge \neg C$.
We want to prove that $\Delta_\nu \vdash A_\nu$, i.e., $\Delta_\nu \vdash \neg(\neg(B \vee C) \wedge \neg(B \vee C))$.
This follows from Axiom **PB11**.

(g) Let $\nu(A) = \nu(B) = \nu(C)$.
$\Delta_\nu \vdash B_\nu, C_\nu$, i.e., $\Delta_\nu \vdash \neg(\neg B \wedge \neg B)$, $\Delta_\nu \vdash \neg(\neg C \wedge \neg C)$.
We want to prove that $\Delta_\nu \vdash \neg(\neg(B \vee C) \wedge \neg(B \vee C))$. This follows from Axiom **PB12**.

3. Let us suppose that A is $B \wedge C$.

(a) Let $\nu(A) = \nu(B) = \nu(C) = 0$. By hypothesis $\Delta_\nu \vdash \neg(B \wedge B)$, $\Delta_\nu \vdash \neg(C \wedge C)$, we want to prove that $\Delta_\nu \vdash \neg(A \wedge A)$, i.e., $\Delta_\nu \vdash \neg((B \wedge C) \wedge (B \wedge C))$. This follows from Axiom **PB13**.

(b) Let $\nu(A) = \nu(B) = 0$, $\nu(C) = 1$.
Let $\Delta_\nu \vdash \neg(B \wedge B)$, $\Delta_\nu \vdash C \wedge \neg C$.
We want to prove that $\Delta_\nu \vdash \neg(A \wedge A)$, i.e.,
$\Delta_\nu \vdash \neg((B \wedge C) \wedge (B \wedge C))$. This follows from Axiom **PB14**.

(c) Let $\nu(A) = 2$, $\nu(B) = \nu(C) = 1$.
Let $\Delta_\nu \vdash B \wedge \neg B$, $\Delta_\nu \vdash C \wedge \neg C$.
We want to prove that $\Delta_\nu \vdash \neg(\neg A \wedge \neg A)$, i.e.,
$\Delta_\nu \vdash \neg(\neg(A \wedge B) \wedge \neg(A \wedge B))$. This follows from Axiom **PB15**.

(d) Let $\nu(A) = 1$, $\nu(B) = 1$, $\nu(C) = 2$.
Let $\Delta_\nu \vdash B_\nu, C_\nu$, $\quad \Delta_\nu \vdash B \wedge \neg B$ and $\Delta_\nu \vdash \neg(\neg C \wedge \neg C)$.
We want to prove that $\Delta_\nu \vdash (A \wedge \neg A)$, i.e.,
$\Delta_\nu \vdash (B \wedge C) \wedge \neg(B \wedge C)$. This follows from Axiom **PB16**.

(e) Let $\nu(A) = \nu(B) = \nu(C)$.
Let $\Delta_\nu \vdash B_\nu, C_\nu$, i.e., $\Delta_\nu \vdash \neg(\neg B \wedge \neg B), \neg(\neg C \wedge \neg C)$.
We want to prove that $\Delta_\nu \vdash \neg(\neg A \wedge \neg A)$, i.e.,
$\Delta_\nu \vdash \neg(\neg(B \wedge C) \wedge \neg(B \wedge C))$. This follows from Axiom **PB17**.

■

Before we present the completeness theorem we establish the following easy result which follows from **Pos8**.

Lemma A.5 *If ϕ_1, ϕ_2, ϕ_3 are formulas and $\phi_1 \vdash A, \quad \phi_2 \vdash A, \quad \phi_3 \vdash A$ then $\phi_1 \vee \phi_2 \vee \phi_3 \vdash A$.*

Finally, we present the completeness part of our main result.

Theorem A.6 *(Completeness) If A is a tautology in $SP3B$ then A is a theorem in the axiomatic system $AXSP3B$.*

Proof. Let us assume that A is a tautology in $SP3B$ whose set of atomic formulas is Δ. By Lemma A.4, $\Delta_\nu \vdash A_\nu$ for any 3-valuation ν. Therefore we have $\Delta_\nu \vdash A \wedge \neg A$ or $\Delta_\nu \vdash \neg(\neg A \wedge \neg A)$ according to $\nu(A) = 1$ or $\nu(A) = 2$. Now by **Pos3** $(A \wedge \neg A) \to A$ and by **PB1** $\neg(\neg A \wedge \neg A) \to A$ we conclude that $\Delta_\nu \vdash A$. Now let a be any atomic formula in Δ and let us define $\Gamma = \Delta \setminus \{a\}$. The previous lines tell us that $\Gamma_\nu, \neg(a \wedge a) \vdash A$, $\Gamma_\nu, a \wedge \neg a \vdash A$ and $\Gamma_\nu, \neg(\neg a \wedge \neg a) \vdash A$. Then according to Lemma A.5 and Axiom **PB2**, we obtain $\Gamma_\nu \vdash A$. By applying the deduction theorem $|\Delta|$ steps we conclude that $\vdash A$. ■

Corollary A.7 *The axioms that define positive logic and C_ω logic are theorems in $AXSP3B$.*

Proof. It is not difficult to see that **Pos1**, **Pos2**,. . .,**Pos8**, C_ω**1**, C_ω**2** are tautologies in $SP3B$, then the result follows from Theorem A.6. ■

A.3 Constructing a proof for a given tautology

Lemma A.4 and Theorem A.6 provide a method to construct a proof for a given tautology in terms of the axioms of $AXSP3B$, as we show next with an example.

Example A.8 *The formula $p \vee (\neg(p \wedge \neg p))$ is a tautology in $SP3B$ which is not hard to check. This formula is a disjunction, we put $\eta = \neg(p \wedge \neg p)$. We prove that this formula is a theorem by using Lemma A.4. Let ν be a valuation such that $\nu(p) = 0$, then $\nu(\eta) = 2$. According to case 2 (e) of Lemma A.4 $\neg(p \wedge p) \vdash \neg(p \wedge p)$ and $\neg(p \wedge p) \vdash \neg(\neg\eta \wedge \neg\eta)$. According to Axiom **PB10** $\neg(\neg\eta \wedge \neg\eta)) \wedge \neg(p \wedge p) \to \neg(\neg(\eta \vee p) \wedge \neg(\eta \vee p))$. According to Axiom **PB1** $\neg(\neg(\eta \vee p) \wedge \neg(\eta \vee p)) \to (\eta \vee p)$. Thus we obtain $\neg(p \wedge p) \vdash p \vee (\neg(p \wedge \neg p)$.*

*Now let ν be a valuation such that $\nu(p) = 1$, then $\nu(\eta) = 0$ and $\nu(p \vee \eta) = 1$. According to Lemma A.4 case 2 (c) $(p \wedge \neg p) \vdash (p \wedge \neg p)$ and $(p \wedge \neg p) \vdash \neg(\eta \wedge \eta)$. By Axiom **PB8** we have $(\neg(\eta \wedge \eta) \wedge (p \wedge \neg p)) \to (\eta \vee p) \wedge \neg((\eta \vee p))$. And by Axiom **Pos3** we have $(\eta \vee p) \wedge \neg((\eta \vee p)) \to (\eta \vee p)$. So we obtain $p \wedge \neg p \vdash \neg(p \wedge \neg p) \vee p$.*

Now, let ν *be a valuation such that* $\nu(p) = 2$*, then* $\nu(\eta) = 2$ *and* $\nu(p \vee \eta) = 2$*. According to Lemma A.4 case 2(g),* $\neg(\neg p \wedge \neg p) \vdash \neg(\neg p \wedge \neg p)$ *and* $\neg(\neg p \wedge \neg p) \vdash \neg(\neg \eta \wedge \neg \eta)$*. According to Axiom* ***PB12****,*
$\neg(\neg p \wedge \neg p) \wedge \neg(\neg \eta \wedge \neg \eta) \to \neg(\neg(p \vee \eta) \wedge \neg(p \vee \eta))$*. Now applying Axiom* ***PB1*** $\neg(\neg(p \vee \eta) \wedge \neg(p \vee \eta)) \to (p \vee \eta)$*. Thus we obtain* $\neg(\neg p \wedge \neg p) \vdash p \vee (\neg(p \vee \neg p))$*. According to Lemma A.5, taking the last row from each of the three cases we conclude* $(\neg(p \wedge p)) \vee (p \wedge \neg p) \vee (\neg(\neg p \wedge \neg p)) \vdash p \vee (\neg(p \wedge \neg p)$*. Since the left hand side of this relation is Axiom* ***PB2****, it follows that* $p \vee (\neg(p \wedge \neg p)$ *is a theorem in* $AXSP3B$*.*

B Connectives of CG'_3 and $SP3B$

Logics $SP3B$ and CG'_3 have 1 and 2 as designated values, although this fact is not relevant in this section, our goal in this paper is to start looking into similarities between these two logics.

B.1 Expressing the connectives of CG'_3 in terms of the connectives of $SP3B$

1. Table 8 and Table 9 show that the negation and conjunction of logic CG'_3 can be defined in terms of the connectives of logic $SP3B$.

$(\neg$	x	$\wedge$	$x)$	$\vee$	$(\neg$	$x)$
2	0	0	0	**2**	2	0
1	1	2	1	**2**	1	1
0	2	0	2	**0**	0	2

$\neg_{CG'_3} x$	x
2	0
2	1
0	2

Table 8: $\neg_{CG'_3} x$ is expressed in terms of the connectives of $SP3B$ as $(\neg x \wedge x) \vee (\neg x)$.

$(x$	$\wedge$	$y)$	$\wedge$	$(x$	$\vee$	$y)$
0	0	0	**0**	0	0	0
1	0	0	**0**	1	1	0
2	0	0	**0**	2	2	0
0	0	1	**0**	0	1	1
1	2	1	**1**	1	1	1
2	1	1	**1**	2	2	1
0	0	2	**0**	0	2	2
1	1	2	**1**	1	2	2
2	2	2	**2**	2	2	2

x	$\wedge_{CG'_3}$	y
0	**0**	0
1	**0**	0
2	**0**	0
0	**0**	1
1	**1**	1
2	**1**	1
0	**0**	2
1	**1**	2
2	**2**	2

Table 9: $(x \wedge_{CG'_3} y)$ is expressed in terms of the connectives of $SP3B$ as $(x \wedge y) \wedge (x \vee y)$.

2. The next formula expresses the implication connective of CG'_3 in terms of the connectives of $SP3B$:

$$(\neg(x \wedge x) \vee y) \vee ((x \wedge \neg x) \wedge (y \wedge \neg y))$$

In order to see this, we exhibit the two truth tables (Tables 10 and 11).

α	β	$\alpha \wedge \alpha$	$\neg(\alpha \wedge \alpha)$	$\neg(\alpha \wedge \alpha) \vee \beta$	$\alpha \wedge \neg\alpha$	$\beta \wedge \neg\beta$	$(\alpha \wedge \neg\alpha) \wedge (\beta \wedge \neg\beta)$
0	0	0	2	2	0	0	0
1	0	2	0	0	2	0	0
2	0	2	0	0	0	0	0
0	1	0	2	2	0	2	0
1	1	2	0	1	2	2	2
2	1	2	0	1	0	2	0
0	2	0	2	2	0	0	2
1	2	2	0	2	2	0	0
2	2	2	0	2	0	0	0

Table 10: Truth table of $\alpha \rightarrow_{SP3B} \beta$ (first part)

α	β	$(\neg(\alpha \wedge \alpha) \vee \beta) \vee ((\alpha \wedge \neg\alpha) \wedge (\beta \wedge \neg\beta))$
0	0	2
1	0	0
2	0	0
0	1	2
1	1	2
2	1	1
0	2	2
1	2	2
2	2	2

Table 11: Truth table of $\alpha \rightarrow_{SP3B} \beta$ in terms of 3 primitive connectives (second part)

According to Table 11, we can define a non-primitive connective in $SP3B$, in terms of the other $SP3B$ connectives, whose truth table is the same as the truth table for the implication of CG'_3:

$$x \rightarrow_{SP3B} y := (\neg(x \wedge x) \vee y) \vee ((x \wedge \neg x) \wedge (y \wedge \neg y))$$

B.2 Expressing the connectives of $SP3B$ in terms of the connectives of CG'_3

1. Table 12 shows that the $SP3B$ negation can be expressed in terms of the connectives of CG'_3: $\neg_{SP3B} = (x \wedge \neg x) \vee (x \rightarrow (\neg x \wedge \neg\neg x))$, where all the connectives are CG'_3-connectives. The formula in the last column represents the $SP3B$ negation.

x	$\neg x$	$\neg\neg x$	$x \wedge \neg x$	$\neg x \wedge \neg\neg x$	$x \rightarrow (\neg x \wedge \neg\neg x)$	H
0	2	0	0	0	2	2
1	2	0	1	0	0	1
2	0	2	0	0	0	0

Table 12: H is the formula $(\neg x \wedge \neg\neg x) \vee (x \rightarrow (\neg x \wedge \neg\neg x))$.

2. Table 13 shows that the $SP3B$ conjunction can be expressed in terms of the connectives of G_3 and CG'_3 by means of the formula

$$[\sim\sim (x \wedge \neg x) \wedge \sim\sim (y \wedge \neg y)] \vee (x \wedge y)$$

where $\sim$ is the negation of G_3, $\neg$, $\vee$ and $\wedge$ are the negation, disjunction and conjunction of CG'_3 respectively. The connective $\wedge_{SP3B}$ is fully expressed solely in terms of CG'_3 connectives when we use the equivalence $\sim x = x \rightarrow (\neg x \wedge \neg\neg x)$.

[$\sim$	$\sim$	(x	$\wedge$	$\neg$	x)	$\wedge$	$\sim$	$\sim$	(y	$\wedge$	$\neg$	y)]	$\vee$	(x	$\wedge$	y)
0	2	0	0	2	0	0	0	2	0	0	2	0	**0**	0	0	0
2	0	1	1	2	1	0	0	2	0	0	2	0	**0**	1	0	0
0	2	2	0	0	2	0	0	2	0	0	2	0	**0**	2	0	0
0	2	0	0	2	0	0	2	0	1	1	2	1	**0**	0	0	1
2	0	1	1	2	1	2	2	0	1	1	2	1	**2**	1	1	1
0	2	2	0	0	2	0	2	0	1	1	2	1	**1**	2	1	1
0	2	0	0	2	0	0	0	2	2	0	0	2	**0**	0	0	2
2	0	1	1	2	1	0	0	2	2	0	0	2	**1**	1	1	2
0	2	2	0	0	2	0	0	2	2	0	0	2	**2**	2	2	2

x	$\wedge_{SP3B}$	y
0	**0**	0
1	**0**	0
2	**0**	0
0	**0**	1
1	**2**	1
2	**1**	1
0	**0**	2
1	**1**	2
2	**2**	2

Table 13: $[\sim\sim(x \wedge \neg x) \wedge \sim\sim(y \wedge \neg y)] \vee (x \wedge y)$ equivalent to $\wedge_{SP3B}$.

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