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A Brief Historical Survey on Hyperstructures in Algebra and Logic

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Abstract

Hyperstructures (also known as hyperalgebras, or non-deterministic algebras) has been studied from different standpoints over the last several decades. However, the relationship and connections between the broad amount of results on the subject found in the literature is not evident. This is due to the different approaches taken by the diverse authors in order to generalize the concepts and constructions from the theory of standard algebras. In addition, there exists an apparent lack of communication between the main areas of knowledge in which this topic has been studied, namely: Mathematics, Computer Science and Logic. This brief survey aims to cover part of the historical development of the theory of hyperalgebras, presenting the main approaches of important concepts on the hyperstructures theory from the point of view of universal algebra, linking them with algebraic semantics of logic systems.

Keywords: hyperstructures, multialgebras, non-deterministic algebras, non-deterministic matrices, swap structures.

Introduction

The aim of this short analytic-historical essay is to bring together the main definitions of algebraic character from the theory of hyperstructures, and discuss the scope of the application of this vast theory to the framework of the algebraic semantics for logic systems. Original results, other than these analyzed and linked here, may be found in [13], [21] and [33]. A good reference for the theory of hyperstructures and its aplications to Mathematics and Computer Science can be found in [26].

In ordinary algebras the concept of *operation* is a fundamental. It can be generalized to *multioperation* and consequently leads to the emergence of *multialgebras*. This generalization was already made and as far as is known the first to idealize it for group theory was the French mathematician Frédéric Marty, in 1934, with the publication of the paper "Sur une généralisation de la notion de groupe" [48].

An operation is a relation that manipulate elements of a set and returns a value that is in another set. A *multioperation* (or *hyperoperation*) is a generalization of an operation when it returns a set of values instead of a single value. The class of structures composed by a set and at least one multioperation is what we call of *algebraic hyperstructure*. Multialgebras (or hyperalgebras) are a kind of hyperstructures as well as hypergroups, hyperrings, hyperlattices and so on.

The hyperstructures theory was studied from many points of view and applied to several areas of Mathematics, Computer Science and Logic. In the realm of the Logic, multialgebras were used as semantics for logical systems. More recently, matrix semantics based on multialgebras were considered by Arnon Avron and his collaborators under the name of *non-deterministic matrices* (or *Nmatrices*) and used them to characterize logics, in particular, some paraconsistent logics in the class of *Logics of Formal Inconsistency* – **LFI**s (see for instance [2]). Semantics based on multialgebras for **LFI**s called *swap structures* were also proposed by Carnielli and Coniglio [13]. More recently, swap structures for some modal systems were developed by Golzio and Coniglio in [33, 23], while Coniglio, Figallo-Orellano and Golzio have shown that the swap structures associated to algebraizable logics coincide with the corresponding quasi-varieties of algebras [21].

The development of new results and concepts in the theory of multialgebras requires a careful research about what has been done in the literature on this subject and this task is not very easy due to the different approaches proposed in the literature to treat the same concept. For instance, *hyperlattice*, that is a lattice with multioperations, was introduced by M. Benado under the name of *multistructure* [8], by J. Morgado [56] as *reticuloide*, and most of the authors use the terms hyperlattice or *multilattice*.

Moreover, the names "multialgebra" and "hyperalgebra" often are used with the same significance, but in 1950 G. Pickert [63] called it "structures". Already, a different approach, closer to relational systems, was given by B. Jonsson and A. Tarski, which used the name *complex algebra* [40, 41]. Many of these authors have developed their concepts independently, hence building a bridge between them is not a trivial work.

Since we consider important to establish a relationship between logic and algebra in the framework of hyperstructures, in this paper we will present a historical and analytic background about the main hyperstructures studied in literature linking them with semantics of logical systems via non-deterministic matrices and swap structures.

1 Hypergroups

The study of hyperstructures began with the presentation of the paper entitled "Sur une généralisation de la notion de groupe", in 1934 by the French mathematician Frédéric Marty in the 8º Congress of Scandinavian Mathematicians [48]. In this paper, Marty presents the notion of *hypergroups* (or *multigroups*) from the analysis of their properties. But, due to his premature death, Marty only published two papers related to his concept of hypergroups.

> 1934 was the year that Frédéric Marty defined the hypergroup [48]. This happened in connection with his thesis on meromorphic functions, which was written under the direction of Paul Montel. Unfortunately F. Marty died young, during the Second World War, when his airplane was shot down over the Baltic Sea, while he was going on a mission to Finland. In the duration of his short life (1911-1940), F. Marty studied properties and applications of the hypergroups in two more communications [49, 50].
> [53, p. 19]

In 1937, H. S. Wall [77] and M. Krasner [43] also gave their respective definitions of hypergroup.

As R. Bayon and N. Lygeros highlight in [5, p. 821], the origin of the hyperstructures is still not completely known but the Marty's definition of hypergroup this led to the emergence of several works related to multialgebras. We will quote some of them in the course of this paper.

In [48], Frédéric Marty introduced the following definition of hypergroup:

Definition 1.1 *Let be a set of elements, non-empty, with four combination laws: AB, BA, $\frac{A}{B|}$ and $\frac{A}{|B}$, each of which may have several determinations; the first two are associative. If C is a determination of AB we write $AB \supset C$, (AB contains C). We will say that the family is a hypergroup if the two divisions are related to multiplication by the following relations: $\frac{A}{B|} \supset C \leftrightarrows BC \supset A$ and $\frac{A}{|B} \supset C \leftrightarrows CB \supset A$*

In the above definition, $\frac{A}{B|}$ represents the division on the right and $\frac{A}{|B}$ represents the division on the left.

The definition of Marty is equivalent to the following definition of Krasner [43]:

Definition 1.2 *A set H organized by a composition law $a.b$ of each pair $a, b \in H$ is called hypergroup with regard to this composition law if* *(i)* $a.b$ *is a non-empty subset of* H*;*

(ii) $(a.b).c = a(b.c)$ *(associative law);*

(iii) *For each pair* $a, c \in H$ *there is* $x \in H$ *such that* $c \in a.x$ *and there is* $x' \in H$ *such that* $c \in x'.a$.

Also Christos G. Massouros, in his paper [52, p. 7] shows how to derive the property of regenerativity[1] from original definition of F. Marty and later on, in [53, 54], the authors present one definition of hypergroup equivalent to the definition of F. Marty.

Remark 1.3 *Many mathematicians in several countries contributed to the studies of the hypergroups theory. In particular, the scientific group A.H.A. (Algebraic Hyperstructures and Applications) of the Democritus University of Thrace in Greece. One of the first books dedicated to hypergroups was written by P. Corsini in 1993 [25]. As mentioned in the Introduction, a good reference for applications of hyperstructures is [26].*

In the following, we will present another important definition in the history of the hyperstructures: the *hyperlattice* definition.

2 Hyperlattices

The concept of hyperlattice was introduced by the Romanian algebraist Mihail Benado in 1953 in the paper "Asupra unei generalizări a noţiunii de structură" [8]. In this work, Benado presents two equivalent definitions of hyperlattice and also some examples.

Despite the introduction of the concept already appears in his paper from 1953, several authors considers Benado's paper "Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. II. Théorie des multistructures" [9], published in 1955 as being the initial reference for the hyperlattice theory. Benado called *théorie des multistructures* to what is known as *hyperlattice theory.* In this section our definition will be based on [9].

Definition 2.1 *A multistructure*[2] *(now known as multilattice or hyperlattice) is any partially ordered set (poset)* P *that satisfies the following principles:*

$M1)$ *Let* $a, b \in P$*; if there is* $\Omega \in P$ *such that* $\Omega \geq a$ *and* $\Omega \geq b$*, then there is also an* $M \in P$ *such that* $M \leq \Omega$*,* $M \geq a$ *and* $M \geq b$*, and the conditions* $x \leq M$*,* $x \geq a$ *and* $x \geq b$ *imply* $x = M$.

[1] Let $\langle H, \cdot\rangle$ be a hypergroup and $a \in H$, the *regenerativity* is the property: $a.H = H.a = H$.

[2] In this paper, we use the name multistructure/hyperstructure to denote the class of all multialgebras such as hypergroups, hyperlattices, and so on.

$M2$) *Let $a, b \in P$; if there is $\omega \in P$ such that $\omega \leq a$ and $\omega \leq b$, then there is also a $d \in P$ such that $d \geq \omega$, $d \leq a$ and $d \leq b$, and the conditions $y \geq d$, $y \leq a$ and $y \leq b$ imply $y = d$.*

Benado uses the notation $(a \vee b)_\Omega$ to denote all M of $(M1)$, that is $(a \vee b)_\Omega = \{M : M$ is minimal in $\{x : x \geq a, x \geq b\}$ and $M \leq \Omega\}$ and he uses the notation $(a \wedge b)_\omega$ to denote all d of $(M2)$, that is $(a \wedge b)_\omega = \{d : d$ is maximal in $\{y : y \leq a, y \leq b\}$ and $d \geq \omega\}$.

> [...] I see that the meaning and the validity of several fundamental principles of the structures theory do not depend on the fact that in the structure S, $a \vee b$ and $a \wedge b$ are respectively the upper and lower bounds [12][3] of elements $a, b \in S$, but it depends *only* on the fact that $a \vee b$ and $a \wedge b$ are respectively a *minimal* element [12] between all $x \in S$, such that $\Omega \geq x \geq a, b$ and a *maximal* element [12] between all $y \in S$, such that $\omega \leq y \leq a, b$ (here Ω, ω are arbitrary, but fixed) [9, p. 309, our translation].

In this citation, the author makes clear the difference between lattices and hyperlattices. The main difference between Benado's definition and the usual definition of supremum (infimum, respectively) is that the minimal upper bounds (maximal lower bounds, resp.) are considered instead of the minimum (the maximum, resp.).

Still in the paper [9], Benado presents also his definition of *submultistructure* (or *subhyperlattice* as we call here), see it below:

Definition 2.2 *Let $\mathcal{M}$ be any multistructure, a non-empty subset $\mathcal{R}$ of $\mathcal{M}$ is a submultistructure of $\mathcal{M}$ if $\mathcal{R}$ satisfies the following conditions:*

$1'$) *If $a, b \in \mathcal{R}$ and there is at least an $\Omega \in \mathcal{R}$ such that $\Omega \geq a$ and $\Omega \geq b$, then $\mathcal{R} \cap (a \vee b)_\Omega \neq \emptyset$.*

$2'$) *If $a, b \in \mathcal{R}$ and there is at least an $\omega \in \mathcal{R}$ such that $\omega \leq a$ and $\omega \leq b$, then $\mathcal{R} \cap (a \wedge b)_\omega \neq \emptyset$.*

Classically, a subset $\mathcal{R}'$ is a sublattice of a lattice $\mathcal{R}$ if $\mathcal{R}'$ is closed under the operations $\vee$ and $\wedge$, that is, for every $a, b \in \mathcal{R}'$, $(a \vee b) \in \mathcal{R}'$ and $(a \wedge b) \in \mathcal{R}'$. Similarly, the definition of Benado provides that to $\mathcal{R}$ be a submultistructure of $\mathcal{M}$, for every $a, b \in \mathcal{R}$, $\mathcal{R}$ must have at least one element that satisfies the condition $(M1)$ and at least one element that satisfies the condition $(M2)$ of the definition of multistructure (Definition 2.1).

[3] By [12], Benado refers to "*N. Bourbaki*, Théorie des ensembles (fasc. de résultats), Paris, Hermann, 1939".

Benado [9] defines a submultistructure $\mathcal{R}$ of $\mathcal{M}$ as being *closed* if, for every $a, b \in \mathcal{R}$ of $(1')$ (Definition 2.2) we have that $(a \vee b)_\Omega \subseteq \mathcal{R}$ and for each $a, b \in \mathcal{R}$ of $(2')$ (Definition 2.2) we have that $(a \wedge b)_\omega \subseteq \mathcal{R}$.

In [9, p. 321], we also found the following definition of multilattice:

Definition 2.3 *A* multistructure *is a non-empty set R with two operations $\wedge$ and $\vee$*[4] *satisfying the following axioms:*

$MI)$ *Let $a, b \in R$, if $a \vee b \neq \emptyset$ $(a \wedge b \neq \emptyset)$ and $b \vee a \neq \emptyset$ $(b \wedge a \neq \emptyset)$. Then:*

- $I')$ $a \vee b = b \vee a$;
- $I'')$ $a \wedge b = b \wedge a$.

$MII)$ *Let $a, b, c \in R$, if $a \vee b \neq \emptyset$ and $(a \vee b) \vee c \neq \emptyset$ $(a \wedge b \neq \emptyset$ and $(a \wedge b) \wedge c \neq \emptyset)$ and if $b \vee c \neq \emptyset$ and $a \vee (b \vee c) \neq \emptyset$ $(b \wedge c \neq \emptyset$ and $a \wedge (b \wedge c) \neq \emptyset)$, then*

- $II')$ *for each $M \in (a \vee b) \vee c$ there is an $M' \in a \vee (b \vee c)$ such that $M \vee M' \neq \emptyset$ and $M \vee M' = M$;*
- $II'')$ *for each $d \in (a \wedge b) \wedge c$ there is a $d' \in a \wedge (b \wedge c)$ such that $d \wedge d' \neq \emptyset$ and $d \wedge d' = d$.*

$MIII)$ *Let $a, b \in R$, if $a \vee b \neq \emptyset$ $(a \wedge b \neq \emptyset)$ and if $a \wedge (a \vee b) \neq \emptyset$ $(a \vee (a \wedge b) \neq \emptyset)$ then*

- $III')$ $a \wedge (a \vee b) = a$;
- $III'')$ $a \vee (a \wedge b) = a$.

$MIV)$ *For each $a \in R$ we have $a \vee a \neq \emptyset$ and $a \wedge a \neq \emptyset$.*

$MV)$ *Let $a, b, c \in R$ such that $a = b$ and if $c \vee a \neq \emptyset$ $(c \wedge a \neq \emptyset)$ and if $c \vee b \neq \emptyset$ $(c \wedge b \neq \emptyset)$ then*

- $V')$ $c \vee a = c \vee b$;
- $V'')$ $c \wedge a = c \wedge b$.

$MVI)$ *Let $a, b \in R$ such that $a \vee b \neq \emptyset$ $(a \wedge b \neq \emptyset)$ and let $M, M' \in a \vee b$ $(d, d' \in a \wedge b)$ such that $M \vee M' \neq \emptyset$ $(d \wedge d' \neq \emptyset)$ then if $M \neq M'$ $(d \neq d')$ we have $M'' \neq M, M'$ $(d'' \neq d, d')$ for each $M'' \in M \vee M'$ $(d'' \in d \wedge d')$.*

[4] According to Benado [9] these operations are not necessarily universal and unambiguous, that is, there are $a, b \in R$, such that if $a \vee b \neq \emptyset$ $(a \wedge b \neq \emptyset)$, then the set $a \vee b$ $(a \wedge b)$ may has at least two distinct elements.

Axiom (MI) refers to commutativity, while axiom (MII) is a kind of partial associativity. Axiom $(MIII)$, which refers to the absorption laws, is called *reduction* by Marty. Axiom (MIV) only says that $a \vee a$ and $a \wedge a$ are not empty (for every $a \in R$), while axiom (MV) guarantees the equalities $c \vee a = c \vee b$ and $c \wedge a = c \wedge b$ in the case of $a = b$.

Benado [9] also showed that (MI)-(MVI) of Definition 2.3 and $(M1)$-$(M2)$ of Definition 2.1 are equivalent.

Hyperlattices have also been studied by other authors such as D.J. Hansen [35], that presents an alternative to the axiomatization given by Benado for characterization of a hyperlattice. The motivation of Hansen is to avoid partial associativity in Benado's definition. The new axiomatic of Hansen only validates the axioms (MI) and $(MIII)$ above and adds three new axioms.

Also in [51], we found an alternative definition of hyperlattice which aims to eliminate some disadvantages generated by generalized associativity in the definitions of hyperlattice given by Benado and Hansen. Among the disadvantages cited by the authors, we have a non natural generalization of the associative property and the fact that such properties do not allow definition of submultilattice similar to usual definition of sublattice. So, [51] introduce a new algebraic structure of hyperlattice with a weaker associative property.

After Benado and before Hansen, the Brazilian mathematician Antonio Antunes Mario Sette, with the aim of obtaining an algebraic semantics for the logic C_ω,[5] introduced the concept of hyperlattice C_ω [70] in his Master's thesis (1971) supervised by Newton da Costa.

To introduce the concept of hyperlattice C_ω, Sette used the definition of hyperlattice presented by José Morgado in the book "Introdução à Teoria dos Reticulados" [56]. Morgado calls his hyperlattices of "reticuloides" and he uses the concepts of "supremoide" and "infimoide" in his definition.

Definition 2.4 *[56, 70] A* hyperlattice (reticuloide) *is a system $\langle R, \leqslant \rangle$ consisting of a set $R \neq \emptyset$ and a quasi-order $\leq$ (relation only reflexive and transitive) such that, for all $a, b \in R$,*

$$a \bigtriangleup b \neq \emptyset \neq a \bigtriangledown b$$

*where $a \bigtriangleup b$ (*infimoide*) is the set of all infimum of the pair $(a, b) \in R^2$ and $a \bigtriangledown b$ (*supremoide*) is the set of all supremum of the pair $(a, b) \in R^2$.*

Remark 2.5 *Note that in a partially ordered system, by the antisymmetric property, we can show the uniqueness of the supremum (infimum) (if it exists),*

[5] C_ω is a paraconsistent logic introduced by Newton da Costa as a kind of limit of his well-known hierarchy C_n $(1 \leqslant n \leqslant \omega)$ of paraconsistent systems, see [27, 28].

but this result is not obtained in the quasi-ordered systems.[6] *Since it is possible to have more than one supremum (infimum), it is legitimate to define the supremoide (infimoide) as the set of all suprema (infima).*

Morgado also presents another reticuloide definition:

Definition 2.6 *[56] A* hyperlattice (reticuloide) *is a system* $\langle R, \triangle, \triangledown \rangle$ *consisting of a set* $R \neq \emptyset$ *and two operations* $\triangle, \triangledown : R \times R \longrightarrow (\mathcal{P}(R) - \{\emptyset\})$ *such that, for all* $a, b, c \in R$*, the following conditions are satisfied:*

h1) $a \triangledown b = b \triangledown a$;

h2) If $x \in a \triangledown b$ *and* $y \in b \triangledown c$*, then* $x \triangledown c = a \triangledown y$;

h3) If $x \in a \triangledown b$*, then* $a \in a \triangle x$.

h4) $a \triangle b = b \triangle a$;

h5) If $x \in a \triangle b$ *and* $y \in b \triangle c$*, then* $x \triangle c = a \triangle y$;

h6) If $x \in a \triangle b$*, then* $a \in a \triangledown x$;

The definitions of Morgado seems to be more intuitive (more similar that usual lattice definition) than those by Benado. In Definition 2.4, what changes in relation to the usual definition is the loss of uniqueness of the supremum and of the infimum. And in Definition 2.6, similar to the usual case, the items ($h1$) and ($h4$) correspond to generalization of the commutativity, the items ($h2$) and ($h5$) are a kind of associativity and the items ($h3$) and ($h6$) are a kind of generalization of the absorption property.

In Morgado's book and in the Sette's dissertation we found only some considerations about the hyperlattices, a definition of an *implicative hyperlattice* and a definition of the hyperlattice C_ω. Sette, in the concluding remarks, notes that algebraization of inconsistent (meaning 'paraconsistent') formal systems can lead us to the consideration of *hypersystems*, that could be a generalization of reticuloides (hyperlattices). However, he did not give any other information about the development of such hypersystems. The generalization of the idea of hypergroup and hyperlattice led to the emergence of *hyperalgebras/multialgebras*.

These more general structures will be the topic of the next section.

[6] That is the systems composed by a set and a quasi-order.

3 Multialgebras

In general, multialgebras (also known as hyperalgebras) are algebras such that the operations can return, for a given entry, a set of values instead of a single value. The origin of multialgebras is a little obscure, because of a very large number of papers came from Marty's paper in 1934 [48].

Some author, for example [76], consider as seminal work the two papers: "Algebras with Operators. Part I" [40] and "Algebras with Operators" [41], both published by Bjarni Jonsson and Alfred Tarski.

In these papers, Jonsson and Tarski introduce the concept of *complex algebra* and they prove the representation of Boolean algebras with operators by means of these algebras. However, the term "complex algebra" has several meanings in the literature and is usually found in authors such as [18] and [10], with the name of *full complex algebra.* See the complex algebra definition [40, p. 933] below:

Definition 3.1 *A complex algebra of a relational structure*[7] $U = \langle U, \{R_i\}_{i\in I}\rangle$ *is defined by:*

$$U^+ = \langle \mathcal{P}(U), \{R_i^+\}_{i\in I}\rangle$$

such that,

(i) U^+ *is a Boolean algebra with operators;*[8]

(ii) Let $R_i \subseteq U^{n+1}$*; then* $R_i^+ : \mathcal{P}(U)^n \longrightarrow \mathcal{P}(U)$ *is an operation such that* $R_i^+(X_0, X_1, \ldots, X_{n-1}) = \{y \in U : (x_0, x_1, \ldots, x_{n-1}, y) \in R_i,$ *for* $x_0 \in X_0, x_1 \in X_1, \ldots, x_{n-1} \in X_{n-1}\}$.

Following Marty and Benados' line, that is, by defining hyperalgebras from multioperations, we can say that the origin of multialgebras can be found in the paper "Bemerkungen zum Homomorphiebegriff" (Comments to the concept of homomorphism) of Günter Pickert, published in 1950 [63]. The goal of the author in this paper was to define homomorphism from structures, but what he calls "structure" is what we call multialgebra.

In 1958, and probably independently, the concept of multialgebra was introduced by P. Brunovský in "O zovšeobecnených algebraických systémoch" (A generalization of algebraic systems) [12]. Brunovský's definitions of multioperation and of multialgebra are the following:

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[7] In [40, p. 933], the authors use the term *algebra in the wider sense.*

[8] A *Boolean algebra with operators (BAO)* is an algebra $\langle A, \{f_i\}_{i\in I}\rangle$ such that A is a Boolean algebra and each f_i is an operator over A, that is, an operation that is additive (distributive on the usual Boolean addition) on each of its arguments.

Definition 3.2 *An* n-ary generalized operation f_α *in the set A is a function, such that for all sequences of elements $n = n(\alpha)$ in A, assigns any subset of the set A.*

Definition 3.3 *Let be A a set and F any set of generalized operations in A. The set A with the set F will be called a* multialgebra *of A.*

Brunovský cites the Benado multilattices as an example of multialgebras.

Note that the definition of Brunovský is quite similar to the definition of more recent authors like Hansoul [36], Schweigert[69] and Ameri and Rosenberg [1]. Namely:

Definition 3.4 *Let A be a non-empty set, a multioperation (or* hyperoperation*) (n-ary) σ on A is a function $\sigma : A^n \to (\mathcal{P}(A) - \{\emptyset\})$, such that n is a positive integer.*

Definition 3.5 *A multialgebra (or* hyperalgebra*) is a pair $\langle A, \{\sigma_i\}_{i\in I}\rangle$, such that A is a non-empty set and $\{\sigma_i\}_{i\in I}$ is a family of multioperations on A.*

The main difference between the Definitions 3.2 and 3.4 is that, in the former, the value of a multioperation can be empty.

Multialgebras can be defined as relational structures with a composition of relations of arbitrary arity. Properties of multialgebras seen as relational systems can be found in [64]. In the next section, some important concepts in the multialgebras theory will be recalled.

3.1 Homomorphisms of multialgebras and other concepts

There are in the literature several generalizations of the notion of homomorphism for multialgebras. We observe that Marty, in his paper [49], already presented a concept of homomorphism for hypergroups: "[...] a representation of a hypergroup over (or in) other is a homomorphism if the image of a determination of the product is the determination of the product of the images." [49, p. 636, our translation].

The definition of Marty means that given two hypergroups[9] $\langle G_1, \cdot_1\rangle$ and $\langle G_2, \cdot_2\rangle$, a function h from G_1 to G_2 is a *homomorphism of hypergroups* if, for all x, a and b in G_1,

$$x \in a._1 b \Rightarrow h(x) \in (h(a)._2 h(b)).$$

Marty says, in the same paper, that a *isomorphism between hypergroups* is a homomorphism such that the correspondence (in the definition above) is

[9] According to the definition of Marty [48].

biunivocal. The author also remarks the necessity to distinguish degrees of homomorphism and he presents his definition of a *quasi isomorphism*, which is basically a kind of surjective homomorphism between hypergroups.

For the generalized notion of the multialgebras, again we can say that Brunovský, in [12], was the first to introduce a definition of homomorphism for multialgebras:[10]

Definition 3.6 *Let $\mathcal{A} = \langle A, F\rangle$ and $\mathcal{B} = \langle B, G\rangle$ be two multialgebras of the same type*[11] *and let h be a function from A to B. Then, h is said to be a homomorphism if, for all n-ary generalized operations $f_\alpha \in F$ and for any sequence of elements $x_1, \ldots, x_n$ in A, it is true:*

$$h[f_\alpha(x_1, \ldots, x_n)] = g_\alpha[h(x_1), \ldots, h(x_n)].$$

In the literature, however, there are several definitions of homomorphism between multialgebras. In 1979, Francis Maurice Nolan, in his Ph.D. Thesis [57], introduced five definitions of multialgebras homomorphisms and constructed a category to each one. The definitions of Nolan are the following:

Definition 3.7 *Let $\mathcal{A} = \langle A, F\rangle$ and $\mathcal{B} = \langle B, F'\rangle$ be two multialgebras of the same type and let h be a function from A to B,*

- *h is a full homomorphism between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ if, for every $f' \in F'$, for every $a \in A$ and for any sequence $b_1, \ldots, b_n \in B$, $h(a) \in f'(b_1, \ldots, b_n)$ iff there are $a_1, \ldots, a_n \in A$ such that $a \in f(a_1, \ldots, a_n)$ and $h(a_i) = b_i$ for every i such that $1 \leq i \leq n$.*

- *h is a weak homomorphism between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ if, for every $f \in F$ and for any sequence $a_1, \ldots, a_n \in A$, $h(f(a_1, \ldots, a_n)) \subseteq f'(h(a_1), \ldots, h(a_n))$.*

- *h is a strong homomorphism between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ if, for every $f \in F$ and for any sequence $a_1, \ldots, a_n \in A$, $h(f(a_1, \ldots, a_n)) = f'(h(a_1), \ldots, h(a_n))$.*

- *h is a bimorphism between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ if, for every $f' \in F'$, for every $a \in A$ and for any sequence $h(a_1), \ldots, h(a_n) \in h[A]$, $h(a) \in f'(h(a_1), \ldots, h(a_n))$ iff $a \in f(a_1, \ldots, a_n)$.*

[10] Multialgebras in the sense of Definitions 3.2 and 3.3.

[11] Let $\mathcal{A} = \langle A, F\rangle$ and $\mathcal{B} = \langle B, G\rangle$ be two multialgebras such that F and G are their respective sets of n-ary generalized operations. We say that the multialgebras $\mathcal{A}$ and $\mathcal{B}$ are of the *same type* if it is possible to establish a biunivocal correspondence between the n-ary generalized operations of A and the n-ary generalized operations of B, such that for each operation $f_\alpha \in F$, the operation $g_\alpha \in G$ corresponding to it, will be n-ary with the same n.

- *h is an absolute homomorphism between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ if, for every $f' \in F'$, for every $b \in B$ and for any sequence $b_1, \ldots, b_n \in B$, $b \in f'(b_1, \ldots, b_n)$ iff there are $a, a_1, \ldots, a_n \in A$ such that $a \in f(a_1, \ldots, a_n)$, $h(a) = b$ and $h(a_i) = b_i$ for every i such that $1 \leq i \leq n$.*

Similarly to the notion of homomorphism, other concepts such as congruence, submultialgebra, direct product and so on, can also be defined in the context of multialgebras. We will talk briefly about the origin of some of them.

The concept of congruence in the framework of multialgebras was firstly considered by Schweigert, in the paper called "Congruence relations of multialgebras", published in 1985 [69]. The goal of Schweigert was to find a suitable concept of variety of multialgebras. In his paper, Schweigert claims that the Birkhoff theorem[12] is valid for multialgebras. However, the author simply claims (without any rigorous proofs) that the demonstration is similar to the case for the standard algebras, by adapting the definitions and proofs. The (big) problem with this kind of assertion is that, as we saw before, in multialgebras we have many possibilities to define concepts such as homomorphism, submultialgebras, congruences, and many others. Being so, the ambiguity present in Schweigert's argument is not a minor issue. This is why the proof of a Birkhoff representation theorem for multialgebras in general still remains as an open problem.

The concept of congruence of multialgebras was studied in detail in the paper "Congruences of multialgebras" of Reza Ameri and Ivo G. Rosenberg [1]. Other concepts such as identities and direct limit of multialgebras can be found on the Ph.D. Thesis by Cosmin Pelea [62].

In the next section, we will briefly discuss an important result for the multialgebras theory.

3.2 Representation theorem

The result known as *representation theorem* for multialgebras was firstly studied by G. Grätzer in [34] and by H. Höft and P. Howard in [38]. The representation theorem is one of the most important results on multialgebras theory. Intuitively speaking, it proves that the study of multialgebras is a natural extension of the theory of universal algebra. This theorem was introduced by G. Grätzer, in the paper entitled "A representation theorem for multialgebras" [34]. It is important to notice that this theorem does not apply to multialgebras with multioperations in the sense of Definition 3.2, because that multialgebras are closer to relational systems than to universal algebra.

[12] All multialgebra can be embedded into a product of directly irreducible multialgebras.

The representation theorem of Grätzer uses the concept of *concrete multialgebra*, see the definition and the theorem below:

Definition 3.8 *Let $\mathcal{A}$ be an algebra with domain A and a collection of operations F, and let θ be an equivalence relation on A. A* concrete multialgebra *is a multialgebra $\mathcal{A}/\theta$ which consists of the following elements:*

(i) a set A/θ of the equivalence classes, such that if $a \in A$, then a/θ is the equivalence class represented by a; and

(ii) a set of n-ary multioperations (in the sense of Definition 3.4), such that if $f \in F$, then each multioperation is defined by: $f(a_1/\theta, \ldots, a_n/\theta) = \{f(b_1, \ldots, b_n)/\theta \ : \ b_1 \in a_1/\theta, \ldots, b_n \in a_n/\theta\}$.

Theorem 3.9 *Every multialgebra is concrete.*

Proof. In [34, p. 453]. ■

This representation theorem states that *any* multialgebra is a concrete one. Thus, for example, if $\mathcal{A}$ is a multialgebra with a single binary multioperation $\cdot$, then there is an algebra $\mathcal{B}$ with a binary operation $\cdot$ and with a equivalence relation θ such that $\mathcal{B}/\theta$ (defined as above, namely: $a/\theta{\cdot}b/\theta = \{(a'{\cdot}b')/\theta \ : \ a\theta a'$ and $b\theta b'\}$) is isomorphic (as multialgebras) to $\mathcal{A}$. So, every multialgebra has an algebra that represents it.

Grätzer, also in his paper of 1962, lists some problems that arise naturally from the representation theorem. For instance, is the theorem equivalent to the axiom of choice? (Problem 1). This problem was solved by H. Höft and P. E. Howard in the paper "Representing multi-algebras by algebras, the axiom of choice, and the axiom of dependent choice" [38]. In this paper, the authors showed that the axiom of choice is equivalent to the representation theorem for multialgebras.

Before Grätzer, Bjarni Jonsson and Alfred Tarski, in [40, p. 933], also introduced a representation theorem for complex algebras (defined briefly in Section 3). But, we present here the representation theorem of Grätzer instead of the theorem of Jonsson and Tarski given that this survey is focused on multialgebras seen as algebras with multioperations (following Marty's line), instead of Jonsson and Tarski perspective.

4 Hyperrings and hyperfields.

In this section, we will briefly discuss the origin and definition of other hyperstructures, namely the hyperrings and hyperfields.

A hyperring is a generalization of a ring where one of the operations is a hyperoperation. Similarly, a hyperfield is a hyperstructure that generalizes the usual concept of field in the above sense.

The concept of *hyperfield* was introduced by Marc Krasner in [44, 45] in connection with his work on valued fields. See below:

Definition 4.1 *A* hyperfield $\langle C, +, \cdot\rangle$ *is a set* C *with a operation* $(\cdot) : C\times C \to C$ *and with a multioperation* $(+) : C \times C \to \mathcal{P}(C)$.

According to the established, the use of this multioperation can be extended to subsets of C *as following:* $A + B = \bigcup(a + b)$, *for* $a \in A$, $b \in B$, $a + B = \{a\} + B$ *and* $A + b = A + \{b\}$. *The structure* $\langle C, +, \cdot\rangle$ *satisfies the following properties:*

(i) Properties of the operation $(\cdot)$*:*

(a) C *is a multiplicative semigroup*[13] *with respect to operation* $(\cdot)$ *and has a bilaterally absorbing element*[14] *denoted by* 0*;*

(b) $C-\{0\}$ *is a group with respect to operation* $\cdot$ *and the identity element is denoted by* 1.

(ii) Properties of the multioperation $(+)$*:*

(a) For every $a, b \in C$, $a + b = b + a$ *(commutativity)*

(b) For every $a, b, c \in C$, $(a + b) + c = a + (b + c)$ *(associativity)*[15]

(c) For every $a \in C$, *there exists one and only one* $a' \in C$, *such that* $0 \in a + a'$ *(inverse element)*[16]

(d) For every $a, b, c \in C$, *if* $c \in a+b$ *then* $b \in c+(-a)$ *(quasi subtration)*

(iii) Properties of distributivity:

(a) For every $a, b, c \in C$, $c \cdot (a + b) = c \cdot a + c \cdot b$

(b) For every $a, b, c \in C$, $(a + b) \cdot c = a \cdot c + b \cdot c$

The concept of *hyperring* was introduced in 1941, by Robert S. Pate in the paper entitled "Rings with multiple-valued operations" [60]. The main

[13] See [65] for more about definitions and properties of standard groups/fields.

[14] Let $\langle S, \cdot\rangle$ be a system composed by a set S and a binary operation $\cdot$, a *bilaterally absorbing element* on $\langle S, \cdot\rangle$ is an element z such that, for every s in S, $z \cdot s = s \cdot z = z$.

[15] This property, as well as properties (iii) below, uses the natural notion of compositions of multioperations. Thus, $(a+b)+c$ denotes the set $\bigcup\{d+c \ : \ d \in a+b\}$. A similar meaning is assumed for the set $a + (b + c)$.

[16] a' will be denoted by $-a$.

difference between the hyperrings and the usual rings is that in a hyperring, the addition is not necessarily unique.

When modifying an axiom of Krasner's definition of of hyperfield, we can also get a hyperring definition. So, according to Krasner [44, 45], we have that a *hyperring* is a structure $\langle C, +, \cdot \rangle$ composed by a set C, an operation $(\cdot) : C \times C \to C$ and a multioperation $(+) : C \times C \to \mathcal{P}(C)$. The structure $\langle C, +, \cdot \rangle$ satisfy the properties of the items (ii) and (iii) and the property (i) (of the Definition 4.1) is replaced by the following property:

(i)' C is a multiplicative semigroup with bilaterally absorbing element 0.

Krasner also defines a *subhyperring* as being a subset C' of a hyperring C such that C' is closed for the multioperation $(+)$, for the operation $(\cdot)$ and for the inverse element, that is, if $a, b \in C'$ then $a + b \subseteq C'$, $a \cdot b \in C'$ and $-a \in C'$.

In the next section, we will introduce the definition of other kind of hyperstructure, obtained by replacing some axioms (as the axiom of associativity and commutativity) by weaker versions.

5 H_v-structures

The concept of *H_v-structure* was introduced by Thomas Vougiouklis in his paper "The fundamental relation in hyperrings. The general hyperfield" [71]. In the quote below, Vougiouklis explains his motivations:

> This paper concludes with the definition of a new class of hyperstructures, more general than the known ones, introduced by the author at the Fourth International Congress on Algebraic Hyperstructures and Applications (AHA). The motivation to introduce this class was the uniting elements procedure [2][17] [71, p. 210, our translation].

In the quote, the *uniting elements procedure* is a method that allows you to put in the same class two or more elements. Vougiouklis [74] claims that, by means of hyperstructures, this method leads to structures with additional properties.

The H_v-structures are weaker generalizations of some hyperstructures, for example hypergroups, hyperrings, hyperlattices, and so on. In the H_v-structures, some axioms are replaced by corresponding weaker versions. See, for example, the definition of a H_v-group from [71]:

[17] By [2] the author refers to his paper of 1989 written together with Piergiulio Corsini: "From groupoids to groups through hypergroups" [24].

Definition 5.1 *A H_v-group $\langle G, \cdot\rangle$ is a set G equipped with a multioperation $(\cdot) : G \times G \to (\mathcal{P}(G) - \{\emptyset\})$ that satisfies the following axioms:*

1. *For every $a, b, c \in G$, $((a \cdot b) \cdot c) \cap (a \cdot (b \cdot c)) \neq \emptyset$ [weak associativity]*
2. *For every $a \in G$, $a \cdot G = G \cdot a = G$.*

In the definition above, the non-empty intersection, called by him *weak associativity*, substitutes the equality of the usual associativity.

In the same paper, Thomas Vougiouklis claims that structures like the H_v-groups, H_v-rings and H_v-fields and so on, can be defined similarly, by replacing the associative law for its weaker version (1) and the commutative law by the following weaker version:

$$(a \cdot b) \cap (b \cdot a) \neq \emptyset \text{ (for every } a, b \in G).$$

The definition of the H_v-groups motivated the emergence of several papers related to H_v-structures in recent years, as for instance [75], [72], [29], [30], [73] and [74].

With relation to the application of hyperstructures, a lot of work have been done in several areas of Mathematics (pure and applied) like in algebra, geometry, topology, graph theory, probability theory, theory of automata, fuzzy theory, and so on. However, this is not the only way to apply hyperstructures. In the next section, we will talk about other application of hyperstructures, strongly related to the non-deterministic notion.

6 Non-deterministic matrices

At the introduction of this paper, we already highlight the strong relationship between non-deterministic matrices and hyperstructures. Now, we will talk about the origin of the concept of non-deterministic matrix.

Non-deterministic matrix is a generalization of the usual concept of many-valued matrix[18]. The idea of logical matrices was used by Pierce [61] and by Schröder [68]. They applied truth-tables for manipulating logical problems, but both had as focus only classical logic. This notion can be formalized as follows:

Definition 6.1 *Let Σ be a signature and Ξ a set of variables. A* logical matrix *for a propositional language $L(\Sigma, \Xi)$ is a pair $\mathcal{M} = \langle \mathcal{A}, D\rangle$ such that $\mathcal{A} = \langle A, \sigma_{\mathcal{A}}\rangle$ is an algebra over Σ with domain A, and D is a subset of A. The elements of D are called designated elements.*

[18]Additional information about many-valued matrices can be found at [67] and [47].

Logical matrices are used to give a natural semantics for propositional logics and they also play an important role in the general techniques of algebraization of logics introduced by W. Blok and D. Pigozzi [6, 7]. However, the logical matrices can be used not only like semantics for bivalent logics (as in the case of classical propositional logic), but also for many-valued logics, as is the case of some modal logics and many-valued logics in general.

Although several propositional logics can be characterized semantically using a many-valued logical matrix [46], many of them can only be characterized by infinite matrices, and such matrices do not constitute a good decision procedure for these logics. So, an alternative solution is the use of non-deterministic matrices.

In 1962, Nicholas Rescher [66] used non-deterministic matrices under the name of *quasi-truth-functional systems*. The same concept was also applied by John Kearns in 1981 [42] and by Yury Ivlev in 1988 [39]. We believe that all these authors gave a definition of non-deterministic matrix independently. Non-deterministic matrices were also studied in [3] by Arnon Avron and Iddo Lev. Although they were not the first to introduce this concept, it was, for the first time, well-formalized and quite disseminated and applied in several articles by Avron and his colaborators. It is worth noting that the name *non-deterministic matrices* or *Nmatrices* in short, was introduced Avron and Lev. The rest of this section we will recall some of their definitions.

Avron and Zamansky [4] motivated by the conflict between the truth-functionality principle and the non truth-functional character of information present in the real world, use non-deterministic matrices to weaken this principle. A simple application example of non-determinism can be given. This example is based on Example "Linguistic ambiguity" in [4, p. 3].

In the natural language, the word "or" can have two meanings: an *inclusive* and other *exclusive*. For example:

(1) My father is, right now, playing soccer in Brazil or in Japan.

(2) I'm going to buy either the pair of shoes light blue or the pair of shoes dark blue.

In the item (1), the disjunction "or" is exclusive, because a person cannot be in two places at the same time, but in the item (2) the disjunction "or" is inclusive, because if I'm with a doubt about which pair of shoes to buy, I can buy the two pair of shoes.

The problem associated with the use of "or" is because, in many cases, we can't distinguish if the "or" in question is inclusive or if it is exclusive. However, even in these cases, we would like to be able to infer something from what was said, and this can be done through the non-deterministic matrices, as displayed below:

		$\vee$
1	1	{1,0}
1	0	{1}
0	1	{1}
0	0	{0}

Now, we present the formal definitions of non-deterministic matrix, valuation and consequence relation over non-deterministic matrices from [3, p. 536]. From now on, Σ will denote a propositional signature, while $L(\Sigma)$ is the propositional language over Σ generated by a given set of propositional variables.

Definition 6.2 *A* non-deterministic matrix *(or in short, an* Nmatrix*) for a propositional language $L(\Sigma)$ is an ordered triple $\mathcal{M} = \langle V, D, O\rangle$, such that:*

(i) V is a non-empty set (of truth-values*),*

(ii) D is a proper and non-empty subset of V called set of designated truth values *and*

(iii) O assigns a function $\tilde{c} : V^n \to (\mathcal{P}(V) - \{\emptyset\})$ for any n-ary connective c in Σ.

Note that, in the above definition, the function $\tilde{c}$ is a multioperation and so the non-deterministic matrices are multialgebras.

One of the main features of the non-deterministic matrices is that the truth value of a complex formula can be chosen non-deterministically from a non-empty set of options. In [3] we also found a definition of valuation and of semantics consequence in the non-deterministic matrices theory:

Definition 6.3 *A valuation in an Nmatrix $\mathcal{M} = \langle V, D, O\rangle$ is a function $v : L(\Sigma) \to V$ such that, for every n-ary connective c in Σ, $\alpha_1, \ldots, \alpha_n \in L(\Sigma)$ and $n \in \mathbb{N}$, the following condition is satisfied:*

$$v(c(\alpha_1, \ldots, \alpha_n)) \in \tilde{c}(v(\alpha_1), \ldots, v(\alpha_n)).$$

Definition 6.4 *Let $\Delta \cup \{\alpha\} \subseteq L(\Sigma)$. Then $\Delta \vDash_{\mathcal{M}} \alpha$ if, for every valuation v in an Nmatrix $\mathcal{M} = \langle V, D, O\rangle$,*

$$v[\Delta] \subseteq D \quad \textit{implies} \quad v(\alpha) \in D.$$

In particular, if $\Delta = \emptyset$, then $\vDash_{\mathcal{M}} \alpha$ if, for every valuation v in $\mathcal{M}$, $v(\alpha) \in D$.

Besides several results has been obtained by Avron and his collaborators, providing semantics and proof systems for several non-classical logics, the link between non-deterministic matrices and multialgebras from the point of view of universal algebra has not been very studied yet. In the next section, some recent results and perspectives along these lines will be discussed.

7 Multialgebras and algebraic semantics

As it was mentioned throughout this text, the formal study of multialgebras from an algebraic perspective is not so immediate, since there are several possibilities to generalize each of the concepts to a multivalued environment.

In particular, the possibility of defining an algebraic theory of multialgebras for logics along the same lines of the so-called abstract algebraic logic (see, for instance, [32]) is an open question which deserves to be investigated. For instance, as it is well-known, several logics in the hierarchy of the Logics of Formal Inconsistency (**LFI**s), proposed in [16, 15], cannot be semantically characterized by a single finite matrix. Moreover, they lie outside the scope of the usual techniques of algebraization of logics such as Blok and Pigozzi's method (see [11]). Different kinds of semantical tools were introduced in order to deal with such systems: non-truth-functional bivaluations, possible-translations semantics, and non-deterministic matrices (or Nmatrices), obtaining so decision procedures for these logics.

However, the problem of finding an algebraic counterpart for this kind of logic, in a precise sense still to be determined, remains open. Looking for elements to clarify this problem, in [13, Chapter 6] it was proposed a semantics based on an special kind of multialgebra called *swap structure* which generalizes the characterization results of **LFI**s by means of finite non-deterministic matrices due to Avron (see [2]). Moreover, the swap structures semantics allows soundness and completeness theorems by means of a very natural generalization of the well-known Lindenbaum-Tarski process. It was also applied to non-normal modal logics in [23] and [33]). Morever, when this technique is applied to logics which are already algebraizable in the sense of Blok and Pigozzi (see [7]), the class of algebras associated to these logics are recovered as (deterministic) swap structures. This suggests interesting possibilities for dealing with non-algebraizable logics by means of multialgebraic semantics.

In the basic examples, swap structures have been proposed as being multialgebras whose elements are triples in a particular Boolean algebra, such that some unary multioperations change of place (swap) some components of their inputs. The elements (triples) of a given swap structure are called *snapshots*[19].

[19]This terminology is inspired by its use in computer systems to refer to states.

The intend idea of a snapshot is to give a semantical description of a formula not just in terms of a single value in a algebra (as it is usually done in algebraic semantics), but by means of a triple of such values (in general, by means of a n-uple of terms). Each swap structure determines a non-deterministic matrix in a natural way. From this, a consequence relation over swap structures is induced by means of Avron-Lev notion of valuation over non-deterministic matrices.

In [13] it was introduced the notion of swap structures for **mbC**, the weakest system in the hierarchy of **LFI**s, as well as for some axiomatic extensions of it. The logic **mbC** is defined as follows:

Definition 7.1 *The logic* ***mbC****, defined over signature* $\Sigma = \{\wedge, \vee, \to, \neg, \circ\}$*, is obtained from the* classical positive logic – $\mathbf{CPL}^+$ *by adding the following axiom schemas:*

$$\alpha \vee \neg\alpha \qquad (\mathbf{Ax10})$$

$$\circ\alpha \to \Big(\alpha \to \big(\neg\alpha \to \beta\big)\Big) \qquad (\mathbf{bc1})$$

It is easy to see (by using semantical arguments) that the negation $\neg$ of **mbC** is paraconsistent, that is: there are formulas α and β such that β does not follows in **mbC** from the contradiction $\{\alpha, \neg\alpha\}$. However, the *consistency* connective $\circ$ is such that every β is derivable in **mbC** from $\{\alpha, \neg\alpha, \circ\alpha\}$. This being so, **mbC** is an **LFI** (see [16, 15, 13]).

Definition 7.2 *Let* $\mathcal{A} = \langle A, \wedge, \vee, \to\rangle$ *be a Boolean algebra* $\mathcal{A}$ *with domain* A*. The universe of swap structures for* ***mbC*** *over* $\mathcal{A}$ *is the set* $\mathrm{B}_{\mathcal{A}}^{\boldsymbol{mbC}} = \{(z_1, z_2, z_3) \in A^3 \ : \ z_1 \vee z_2 = 1 \ \text{ and } \ z_1 \wedge z_2 \wedge z_3 = 0\}$.

Definition 7.3 *Let* $\mathcal{A} = \langle A, \wedge, \vee, \to\rangle$ *be a Boolean algebra* $\mathcal{A}$ *with domain* A*, and let* $B \subseteq \mathrm{B}_{\mathcal{A}}^{\boldsymbol{mbC}}$*. A* swap structure for **mbC** over $\mathcal{A}$ *is any multialgebra* $\mathcal{B} = \langle B, \tilde{\wedge}, \tilde{\vee}, \tilde{\to}, \tilde{\neg}, \tilde{\circ}\rangle$ *over* Σ *such that the multioperations satisfy the following, for every* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *in* B*:*

(i) $\emptyset \neq (a_1, a_2, a_3)\tilde{\#}(b_1, b_2, b_3) \subseteq \{(c_1, c_2, c_3) \in B : \ c_1 = a_1 \# b_1\}$*, for* $\# \in \{\wedge, \vee, \to\}$*;*

(ii) $\emptyset \neq \tilde{\neg}(a_1, a_2, a_3) \subseteq \{(c_1, c_2, c_3) \in B : \ c_1 = a_2\}$*;*

(iii) $\emptyset \neq \tilde{\circ}(a_1, a_2, a_3) \subseteq \{(c_1, c_2, c_3) \in B : \ c_1 = a_3\}$*.*

The idea behind swap structures for **mbC** is that a triple (c_1, c_2, c_3) in such structure represents a (composite) truth-value in which c_1 represents the truth-value of a formula α, while c_2 and c_3 represent a possible truth-value for $\neg\alpha$ and $\circ\alpha$, respectively. This being so, the definition of the multioperatios are justified. Moreover, the definition of the universe $\mathrm{B}_{\mathcal{A}}^{\mathbf{mbC}}$ can be explained in terms of the axioms of **mbC**. Indeed, the requirement $z_1 \vee z_2 = 1$ is justified in terms of axiom (**Ax10**): given a snapshot $z = (z_1, z_2, z_3)$ then any $w = (w_1, w_2, w_3)$ in $\tilde{\neg} z$ is such that $w_1 = z_2$, by definition. Then, any snapshot (u_1, u_2, u_3) in $z \,\tilde{\vee}\, w$ is such that $u_1 = z_1 \vee w_1 = z_1 \vee z_2$. Since the first coordinate z_1 of the snapshot z carries on the semantical information about a formula α, then u_1 represents the truth-value of $\alpha \vee \neg\alpha$, and this should be 1 because of axiom (**Ax10**). Analogously, the condition $z_1 \wedge z_2 \wedge z_3 = 0$ is a consequence of axiom (**bc1**).

In [33] and [23] swap structures were introduced for some modal systems in which the snapshots (triples) for any formula α represents, intuitively, the truth-values for α, $\Box\alpha$ and $\Box{\sim}\alpha$.

The first example of modal swap structures was the the modal system **Tm**, originally introduced in [39] and studied in [19], [20] and [59].

Definition 7.4 *The system* ***Tm*** *over* $\Sigma' = \{\sim, \supset, \Box\}$ *is given by the following axiom schemes:*

$$\alpha \supset (\beta \supset \alpha) \quad (Ax1)$$

$$(\alpha \supset (\beta \supset \sigma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \sigma)) \quad (Ax2)$$

$$({\sim}\beta \supset {\sim}\alpha) \supset (({\sim}\beta \supset \alpha) \supset \beta) \quad (Ax3)$$

$$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta) \quad (K)$$

$$\Box(\alpha \supset \beta) \supset (\Box{\sim}\beta \supset \Box{\sim}\alpha) \quad (K1)$$

$${\sim}\Box{\sim}(\alpha \supset \beta) \supset (\Box\alpha \supset {\sim}\Box{\sim}\beta) \quad (K2)$$

$$\Box{\sim}\alpha \supset \Box(\alpha \supset \beta) \quad (M1)$$

$$\Box\beta \supset \Box(\alpha \supset \beta) \quad (M2)$$

$$\Box{\sim}(\alpha \supset \beta) \supset \Box{\sim}\beta \quad (M3)$$

$$\Box{\sim}(\alpha \supset \beta) \supset \Box\alpha \quad (M4)$$

$$\Box\alpha \supset \alpha \quad (T)$$

$$\Box\alpha \supset \Box{\sim}{\sim}\alpha \quad (DN1)$$

$$\Box{\sim}{\sim}\alpha \supset \Box\alpha \quad (DN2)$$

together with the following inference rule:

$$\frac{\alpha, \alpha \supset \beta}{\beta} \quad (MP)$$

Observe that axioms (Ax1)-(Ax3) plus MP constitute an axiomatization of classical logic over signature $\{\sim, \supset\}$.

Definition 7.5 (Swap structures for Tm) *Let*

$$\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$$

be a Boolean algebra and let

$$\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}} = \{(a_1, a_2, a_3) \in A^3 \ : \ a_2 \leq a_1 \quad and \quad a_1 \wedge a_3 = 0\}.$$

A swap structure for $\boldsymbol{Tm}$ *over* $\mathcal{A}$ *is any multialgebra*

$$\mathcal{B} = \langle B, \tilde{\supset}, \tilde{\sim}, \tilde{\Box}\rangle$$

over $\Sigma' = \{\sim, \supset, \Box\}$ *such that* $B \subseteq \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ *and the multioperations satisfy the following, for every* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *in* B*:*

(i) $(a_1, a_2, a_3)\tilde{\supset}(b_1, b_2, b_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_1 \rightarrow b_1, c_3 = a_2 \wedge b_3 \quad and \quad a_3 \vee b_2 \leq c_2 \leq (a_1 \rightarrow b_1) \wedge (a_2 \rightarrow b_2) \wedge (b_3 \rightarrow a_3)\}$*;*

(ii) $\tilde{\sim}(a_1, a_2, a_3) = \{(\neg a_1, a_3, a_2)\}$*;*[20]

(iii) $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2\}$*.*

After define swap structures for some modal logics, in [33] and [23] was proved soundness and completeness theorems of the Hilbert calculi defining these non-normal modal systems with respect to such non-deterministic matrix semantics.

The swap structures semantics presented in the previous definitions was based on multialgebras since the given logics are not algebraizable in the classical sense. Being so, multialgebras arise as a natural alternative to algebras.

In [21] the same techniques were applied to algebraizable logics which are characterized by a single 3-valued logical matrix. It has been seen that the algebras associated to these logics were recovered as special cases of swap structures, obtaining so an interesting relationship with the twist-structures semantics (in the sense of [58]). Moreover, the dual Kalman functor for swap structures is indeed a generalization of the original construction of Kalman applied to 3-valued logics (see [17]). This connection suggest that swap structures can be seen as non-deterministic twist structures.

[20]Here, $\neg$ denotes the boolean complement of $\mathcal{A}$, it should not be confused with the paraconsistent negation of **mbC**, since they are different contexts.

8 Final remarks

Hyperstructures can be studied and applied in several research areas such as algebra, geometry, computer science and logic. Historically, the studies on hyperstructures were developed independently by several researchers and with different purposes and motivations. Making a link between all these developments is not an easy task to be done in a single paper: moreover, it was not the purpose of the present paper. As already mentioned in the introduction, the aim of this paper was to offer a brief survey of the main definitions found in the algebraic study of hyperstructures, as well as to discuss their possible applications to a new theory of algebraic semantics for non-classical logics.

Many results can still be obtained with respect to the algebraic theory of multialgebras, but one requirement for this task is the prior knowledge (at least superficially) of what has already been done. For instance, the relationship between non-deterministic matrices and hypergroups was unexplored so far and despite the study of multialgebras as semantics for formal systems is on the rise, usually this theory is developed independently from the hyperstructures theory. This paper intends to present some of the possibilities of algebraic developments relating multialgebra and logic, besides contributing as a source of bibliographical references in this field of research.

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