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A Topological Approach for the Notion of Quasi Topology Structure

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Abstract

In this paper, we define and study the quasi topology structure (QTS) in the frame of a topological space. The basic cognitive elements postulated by the definition of this structure are analyzed in the frame of topological elements of a general topological space. Consequently, we study some particular topological spaces especially some types of lattices that can host the structure of quasi topology. The notion of quasi topology structure in topological spaces is consistent.

Keywords: Quasi topology structure, topological space, openness, closure, lattice, filter, ideal

1 Introduction

The notion of quasi topology structure (QTS) was founded by Jean-Pierre Desclés in [3],[4],[5],[6]. It was defined as a cognitive model with applications in several fields of sciences beginning with linguistics, especially formal semantics and continuing with anthropology, sociology and laws as humanities and continuing with computer science at least in the area of image processing. That is the reason for an analysis of quasi topology structure as mathematical model, in other word to build the mathematical model associated to the quasi topology structure.

This model must be a mathematical model encoding the idea that a set as member of an abstract space can have a *strict interior* and a *large interior* a *closure* and a *large closure*. It follows, obviously, that this set can be provided with an *internal boundary* and an *external boundary*. These ideas comes from the linguistic expressions of the space, the linguistic expression of the time, even from basic notion in law as "legal" opposite to "illegal" or from the social notion of "inhabitant of a city". The strict interior and the strict exterior are subsets of a given set. The closure and the large closure are supersets of it and the internal boundary and the external boundary are defined by mean of previous notions. Their definitions are based on the structure of the host space of the given set.

These are the cognitive elements of the structure of quasi topology.

From the technical point of view, it is a question of "approximate categorization ", the term "approximate" being taken in its large meaning.

In the mathematical literature, several approaches modeling the notions of strict interior and large closure where defined and studied. We can recall between them at least "fuzzy sets"[14], "rough sets"[11], "locology "[15]. The first two approaches generated several derivative models as well as several applications in different fields of artificial intelligence. The third one has been had developments in non-classical logic. From the mathematical point of view, fuzzy sets are based on the extension of the classical notion of membership function. Rough sets and the locology are based on algebraic relations. The notions of "interior" and "exterior" from classical topology stand for "lower approximation" and "upper approximation" in rough sets. In locology, these notions are replaced by "heart" and "shadow". There are two aspects to be highlighted about these approaches: The "conceptual metaphor" which relates "interior" of the topology to "lower approximation" of rough sets and to the "heart" of locology have not the same "mathematical meaning". For the "interior" versus "closure" in topology the basic privileged idea is that one on continuity / separation. For "lower approximation" versus "upper approximation" there is an algebraic relation defining them. As for the locology, it is also an algebraic relation but with some additional conditions. The limits of the applicative power of these mentioned above approaches was proved by using examples from social sciences and humanities, particularly from linguistic. The quasi topology structure comes to cover these limits. Some examples of applications were presented in [10], [6]. These examples come from various fields, so with a different cognitive background. The quasi topology structure encodes more cognitive features related to "interior", "exterior" and "boundary" of a set than rough sets and locology, namely, the existence of two types of "interior", two types of "closure" and, therefore, two types of "boundaries". A unified mathematical structure related to the quasi topology has not yet been defined. The set theory conditions for a structure to be a quasi topology are directly outcome from real examples. Theses conditions are established in [4]. These conditions model the cognitive ideas of thick boundary, internal boundary and external boundary of a set from an general space (a space of sets in a set theory which is not provided with any other algebraic or topological structure whatever it may be). The name of quasi topology is given because the first well known structure modeling the idea of border or boundary was the classical topology. A general idea is to immerse the Desclés's based set

structure in a general topology or in an algebraic structure in order to give it a mathematical status. Moreover, the mathematical status is needed in order to have a pattern for a computational model for applications. The goal of this paper is to anchor the quasi topology structure in the frame of a topological space, in other words to study the types of topological spaces capable to host this structure. In addition to that, we try to model a space in which each of its subset can acquire a structure of quasi topology. This space, if any will be a quasi-topological space. Starting from the definition of a quasi topology structure, we analyze its definitional conditions from the point of view of their modeling firstly in a general topological space and then, in some particular topological spaces as lattices and metric spaces.
The study of this structure has at least two reasons:

1. A mathematical structure susceptible to be adapted to computational goals.

2. A computational implementation could serve in several types of applications, in experimental sciences or in humanities.

The structure of this paper is as follows: Section 2 presents the definition of the quasi topology structure starting from a topological space. Section 3 recalls the basic notions of topology. Section 4 presents the QTS anchored in some types of topological spaces. Section 5 presents the conclusions.

2 Quasi topology structure (QTS)

Definition 2.1 *[10] Let $\langle X, O\rangle$ be a topological space where X denotes the space and O denote the topology. We say that a set E from this space is structured by a quasi topology or it has a quasi topology structure (QTS) if there exists two open sets O_1 and O_2 of O, and F_1 and F_2 two closed sets such that:*

$$O_2 \subset O_1 \subseteq E \subseteq F_1 \subset F_2 \tag{2.1}$$

with:

$$O_1 \text{ is the biggest open set contained in } E\text{, that is } O_1 = Int(E) \tag{2.2}$$

$$F_1 \text{ is the smallest closed set containing } E\text{, that is } F_1 = Cl(E) \tag{2.3}$$

$$O_2 \text{ is the biggest open set strictly contained in } O_1 \tag{2.4}$$

$$F_2 \text{ is is the smallest closed set strictly containing } F_1 \tag{2.5}$$

The set O_2 is said to be the strict interior *of E; the set O_1 is the* large interior *of E.*

The set F_2 *is said to be the* large closure *of* E *and the set* F_1 *the* strict closure *of* E.

The internal boundary, *the* external boundary, *the* strict boundary *and the* large boundary *of* E *are defined by:*

$$\textit{Int-bound}(E) = F_1 - O_2 \tag{2.6}$$

$$\textit{Ext-bound}(E) = F_2 - O_1 \tag{2.7}$$

$$\textit{Large-bound}(E) = \textit{Int-bound}(E) \cup \textit{Ext-bound}(E) \tag{2.8}$$

$$\textit{Bound}(E) = Cl(E) - Int(E) = F_1 - O_1 \tag{2.9}$$

The above definition is presented in an intuitive way in the Figure 1.

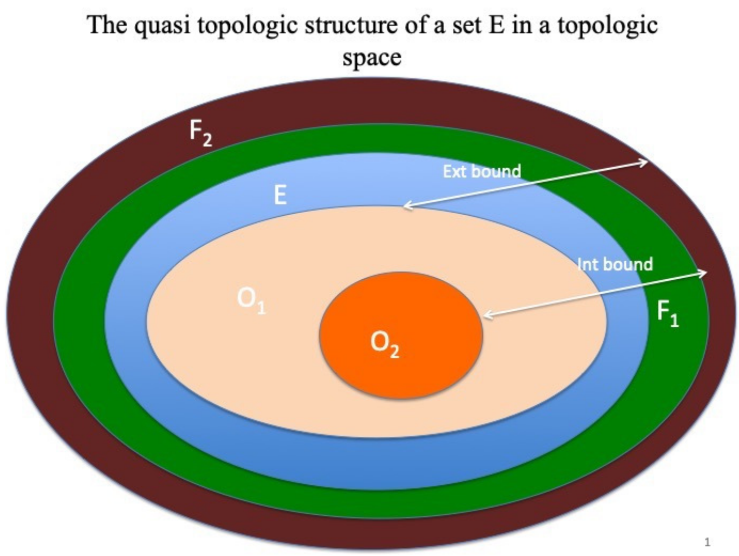

Figure 1: The quasi topology structure of a set E in a topological space

Remark 2.2 *By analyzing conditions* (2.2), (2.3), (2.4), (2.5) *of the definition 2.1, we remark that one must capture two elements in the quasi topology structure coming from the space on which this structure is anchored, namely:*

- *(a) The two strict inclusions between* O_2 *and* O_1 *and between* F_1 *and* F_2 *respectively;*
- *(b) To ensure the biggest open and the smallest closed.*

To manage (a), we must work with the strict inclusion that is not a transitive relation. To manage (b), one of the solutions is to work with spaces X in which there are sets with an infimum and a supremum.

3 Basic notions of general topology

Before analyzing and presenting the topological approach of a quasi topology, we recall some basic notions of general topology [1],[2]. Despite their equivalence, we need to recall at least the four equivalent definitions of a topological space in order to highlight which is the most "salient feature" from the cognitive point of view that can be used in a mathematical modeling. Moreover we need investigate how these features are found in the structure of quasi topology.

Definition 3.1 *(via open sets) A topological space is a pair $\langle X, T\rangle$ of a set X together with a collection T of subsets of X satisfying:*

1. *The empty set and X are in T.*
2. *The union of any collection of sets in T is also in T.*
3. *The intersection of any pair of sets in T is also in T.*

The sets in T are the open sets. In this definition the salient feature is open set.

Definition 3.2 *(via closed sets) A topological space is a pair $\langle X, T\rangle$ of a set X together with a collection T of subsets of X satisfying:*

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2. *The intersection of any collection of sets in T is also in T.*
3. *The union of any pair of sets in T is also in T.*

The sets in T are the closed sets. In this definition the salient feature is closed set.

Definition 3.3 *(via interior operator) - Kuratowski operator) A topological space is a pair $\langle X, Int\rangle$ of a set X together with an interior operator Int from power set of X into power set of X, $Int : \mathcal{P}(X) \to \mathcal{P}(X)$ such that:*

1. *$Int(A) \subseteq A$ for all $A \subseteq X$*
2. *$Int(A \cap B) = Int(A) \cap Int(B)$ for any two subsets A and B of X (preserving finite intersection)*

3. Int(X) = X, Int(∅) = ∅

4. Int(Int(A)) = Int (A) for any subset A of X (idempotency of interior operator)

In this definition the salient feature is the idempotency of the interior operator.

Definition 3.4 *(via closure operator) - Kuratowski operator) A topological space is a pair ⟨X, Int⟩ of a set X together with a closure operator Int from power set of X into power set of X, CL : $\mathcal{P}(X) \to \mathcal{P}(X)$ such that:*

1. A ⊆Cl(A)for all A ⊆ X

2. Cl(A∪B) = Cl(A)∪Cl(B) for any two subsets A and B of X (preserving finite union)

3. ClX) = X, Cl(∅) = ∅

4. Cl(Cl(A)) = Cl(A) for any subset A of X (idempotency of closure operator)

In this definition the salient feature is the idempotency of the closure operator.

These four previous definitions are equivalent.

4 A quasi topology structure defined by the notion of topology

From the Definition 2.1, we notice that a QTS structure needs two cognitive elements for its definition. If we take as a base a topological space as first cognitive elements and something else as second element we say we are in a quasi topology structure one topology based. In order to model the openess in conditions (2.2) and (2.4) of the Definition 2.1, a natural idea is that a quasi-topology structure can be defined on a space X using either a single topology or two topologies.

4.1 Quasi topology structure one topology based

In Figure 2, it is presented the first quasi topology structure from [4]. It is inspired from the Desclés's theory of times and aspects in a grammar of a language. From the mathematical point of view, this quasi topology is based on the order relation induced on the real line $\mathbb{R}$ and by order relation un integers. In fact, on the basis of this quasi topology, there are two cognitive elements expressed inside the fundamentals of mathematics:

- The fundamental classification In (integer numbers) $\subset \mathbb{R}$ (real numbers);
- The order relation on $\mathbb{R}$.

However, it's about one and the same topology.

Example 1 *(Quasi topology one topology based. [3],[4]) In Figure2, there is one of the first examples of Desclés of a QTS. We analyse from the mathematical point of view this example, in order to establish the types of sets E that can receive a QTS structure generated by a topology in the space of real numbers* $\mathbb{R}$.

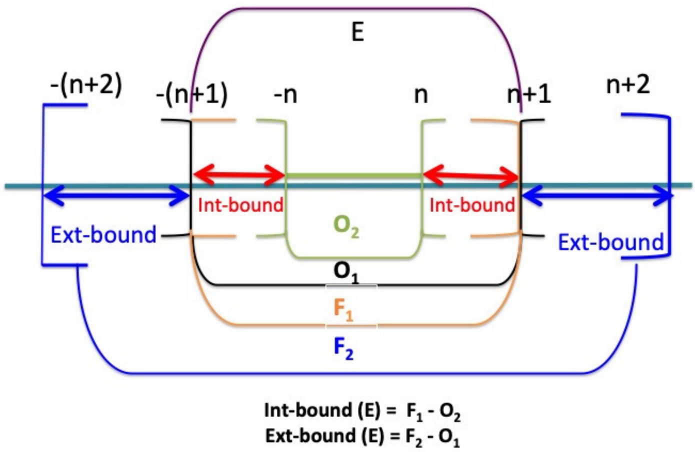

Figure 2: A quasi topology structure of the set E in the space $\mathbb{R}$ with the topology of intervals

As we can see in Figure 2, the space X is the real line $\mathbb{R}$. *The topology* T_1 *is given by the integer bordered intervals of the form* $]-n, n[$ *as open sets. So,* $X = \langle \mathbb{R}, T_1 \rangle$. *It is easy to see that* T_1 *is a topology on* $\mathbb{R}$ *if one accepts that* $]-0, 0[= \emptyset$. *We also remark that this topology is a topology based on an order relation, namely order relation on real numbers. What kind of sets can have a structure of quasi topology in this space? As we can see in the Figure 2, we have the following cases:*

1. *If* $E =]-(n+1), (n+1)[$ *(E is an open interval bounded by opposite natural numbers):*

$]-n, n[\ \subset\ ]-(n+1), (n+1)[\ = E \subset [-(n+1), (n+1)] \subset [-(n+2), (n+2)]$
(this is the case represented in Figure 2)

2. *If* $E = [-(n+1), (n+1)]$*(E is a closed interval bounded by opposite natural numbers):*
$]-n, n[\ \subset\ ]-(n+1), (n+1)[\ \subset E = [-(n+1), (n+1)] \subset [-(n+2), (n+2)]$

3. *If* $E =]-n, m[$ *ou* $E = [-n, m]$ *with* $n + 1 \leq m$ *(E is an open or a closed interval bounded at left by a negative natural and at right by a positive one in the relation* $n + 1 \leq m$*):*
$]-(n-1), (n-1)[\ \subset\ ]-n, n[\ \subset E \subset [-m, m] \subset [-(m+1), (m+1)]$

4. *If* $E =]-n, m[$ *ou* $E = [-n, m]$ *with* $n + 1 > m$*:*
$]-(m-1), (m-1)[\ \subset\ ]-m, m[\ \subset E \subset [-n, n] \subset [-(n+1), (n+1)]$

5. *If* $E' = E \cup \{r_1, r_2, \ldots, r_k\}$ *where* $r_1, r_2, \ldots, r_k \notin E$ *and they are real numbers. We can easily see that we can find a* O_1 *and a* O_2 *from E if and only if there is at least a natural number n, such that, n and* $-n$ *are between* $r_1, r_2, \ldots, r_k$*. For* $F_1 = [-n_{max}, n_{max}]$ *where* $n_{max} = max\{n, m, int(\mid r_1 \mid) + 1, int(\mid r_2 \mid) + 1, \ldots, int(\mid r_k \mid) + 1\}$ *and* $int(\mid r_i \mid)$ *is the integer part of absolute value of* r_i*.*

6. *If* $E' =]-r, r'[$ *where r and* r' *are not naturals, we take* $E =]-n, m[$*, where* $n = int(r+1)$ *and* $m = int(r') + 1$

7. *Contrariwise, an interval of the type* $]n, m[$ *cannot have a quasi topologic structure in the space* $\mathbb{R}$ *endowed with the topology* T_1*.*

So, $X = \langle \mathbb{R}, T_1 \rangle$ *cannot be organized as a quasi-topological space but there are some types of set E having a QTS structure. Thus, the notion of quasi topology structure is consistent in the case of this space.*

Remark 4.1 *Instead to take the integers as the second cognitive element to define the topology* T_1*, we can take another subcategory of real numbers, namely, rational numbers. However, the form of open sets of* T_1 *is very particular compared with a common interval* $]a, b[$*.*

Remark 4.2 *On the real line* $\mathbb{R}$ *endowed with the usual topology of intervals induced by the order relation between two real numbers, we cannot define the open* O_2 *under the condition* (2.4)*. In order to have the condition* (2.4) *fulfilled we need another element as, for example, to endow the topological space with a distance. That is to work in a metric space.*

4.2 Quasi topology structure in metric spaces

On $\mathbb{R}$, as topological space, the structure of quasi topology can be instanciated because of the total order of real numbers. For each open interval we can find two other open intervals playing the role of O_1 et O_2 with respect of conditions (2.2) and (2.4) respectively. On $\mathbb{R}^2$, with the topology of disks as open sets, for each disk, we can find two other disks playing the role of O_1 and O_2. It is the same on $\mathbb{R}^n$ with the topology of n-dimensional balls. An euclidean complete space can be organized as a quasi-topological space with the meaning that each of its nontrivial set can receive a QTS structure.

4.3 Quasi topology structure in lattices

To ensure condition (a) from *Remark* 2.2 the topological space must be an *ordered space.*

4.3.1 Basic lattice theory definitions [17]

Definition 4.3 *(Order set; Preordered set; Ordered topology) A set X is a ordered set under the relation $\leq$ if the following statements hold for all a, b, c in X:*

1. *$a \leq a$ (reflexivity)*
2. *If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)*
3. *If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)*

A set X is a preordered set if and only if only the reflexivity and the transitivity are fulfilled.

The set X is said to be a totally ordered set, if and only if the following additional condition is fulfilled: either $a \leq b$ or $b \leq a$, for all a and b in X.

An ordered topology is a topology that can be defined on a totally ordered set. We study some topologies on a ordered space. We take as an ordered space a *lattice.*

Definition 4.4 *(Ideal; Filter) A subset I of a partially ordered set $(X, \leq)$ is an ideal, if the following conditions hold:*

1. *I is non-empty*
2. *For every x in I and every y in X with $y \leq x$ implies that y is in I. (I is a lower set)*

3. *For every x, y in I, there is some element z in I, such that $x \leq z$ and $y \leq z$. (I is a directed set).*

A subset F of a partially ordered set (X, $\leq$) is a filter if the following conditions hold:

1. *F is non-empty*
2. *For every x in F and y in X with $x \leq y$ implies that y is in F. (F is an upper set, or upward closed)*
3. *For every x, y in F, there is some element z in F such that $z \leq x$ and $z \leq y$ (F is downward directed).*

Definition 4.5 *(Lattice) An algebra (L,$\vee$,$\wedge$) is called a lattice if L is a nonempty set, $\wedge$ and $\vee$ are binary operations on L, both $\wedge$ and $\vee$ are idempotent, commutative, and associative, and they satisfy the absorption law.*

A lattice can be defined by mean of the notion of preordered set.

Definition 4.6 *(Lattice-ordered set). A lattice L is a preordered set (L, $\leq$) in which each two-element subset (a, b) has an infimum, denoted inf (a, b), and a supremum, denoted sup (a, b). There is a natural relationship between lattice-ordered sets and lattices. In fact, a lattice (L, $\vee$, $\wedge$) is obtained from a lattice-ordered poset (L, $\leq$) by defining $a \wedge b$ = inf (a, b) and $a \vee b$ = sup (a, b) for any a, $b \in L$. Also, from a lattice (L, $\vee$, $\wedge$), one may obtain a lattice-ordered set (L, $\leq$) by setting $a \leq b$ in L if and only if $a = a \wedge b$. One obtains the same lattice-ordered set (L, $\leq$) from the given lattice by setting $a \leq b$ in L if and only if $a \vee b = b$. (In other words, one may prove that for any lattice, (L, $\vee$, $\wedge$), and for any two members a and b of L, $a \wedge b = b$ if and only if $a = a \vee b$.) A complete lattice is a partially ordered set in which all subsets have both a supremum (least upper bound) and an infimum (greatest lower bound).*

4.3.2 Quasi topology structure in a complete lattice

In a complete lattice L, the topology T_{int} is the topology of open intervals. An open interval is a set of the form $O_{ab} = \{x \in X \ / \ a < x < b\}$. For a set E, we define:
sup E = $\{x \ / \text{ for all } y \in E, y \leq x\}$ (the set of all upper bounds of E)
and
inf E = $\{x \ / \text{ for all } y \textit{ in } E, x \leq y\}$ (the set of all lower bounds of E).

Starting from a set E, O_1, O_2 can be chosen with regard to T_{int} as:
$O_1 = \{x \in X$ / for all $y \in \inf E$ and for all $z \in \sup E$, $y < x < z\}$
$O_2 = \{x \in X$ / for all $y \in \inf O_1$ and for all $z \in \sup O_1$, $y < x < z\}$

In the same way:
$F_1 = \{x \in X$ / for all $y \in E$, $y \leq x$ and for all x like this, there are no $z \in E$ such that $y \leq z \leq x\}$
$F_2 = \{x \in X$ / for all $y \in F_1$, $y \leq x$ and for all x like this, there are no $z \in F_1$ such that $y \leq z \leq x\}$

In what follows, we take some types of lattices and bring out sets having a QTS structure and sets without QTS structure. We apply this QTS in the lattice of the Figure 3 by giving the QTS structure of three types of sets, namely E_1, E_2 and E_3.

Example 2 *In the Figure 3, it is represented a complete lattice if we consider all the vertices on the same level to be equal. The set E is considered at a time as E_1 (in red), as E_2 (in blue) and as E_3 (in green).*

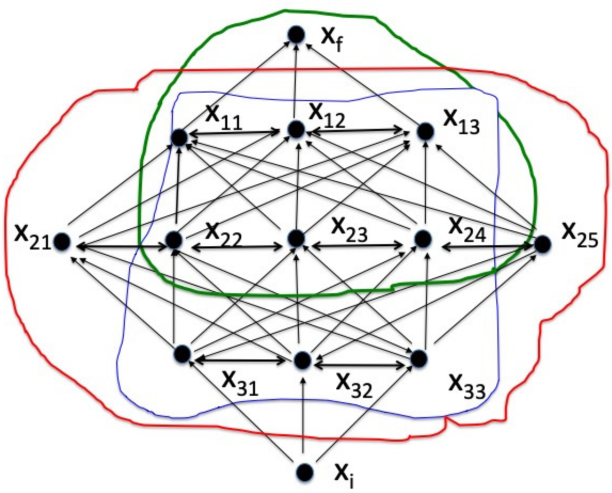

Figure 3: A quasi topology structure of the set E in the space of a complete lattice

For $E_1 = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{33}\}$, $O_1 = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{33}\}$,

*$O_2 = \{x_{31}, x_{32}, x_{33}\}$, $F_1 = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{33}\} = E$ and $F_2 = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{33}, x_f\}$.
So we have: $O_2 \subset O_1 \subset E = F_1 \subset F_2$*

*For $E_2 = \{x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}\}$, $O_1 = \{x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}\}$, $O_2 = \{x_{31}, x_{32}, x_{33}\}$, $F_1 = \{x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}\}$ and $F_2 = \{x_f, x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_f.\}$.
So we have: $O_2 \subset O_1 \subset E = F_1 \subset F_2$.*

*For $E_3 = \{x_f, x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}\}$, $O_1 = \{x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}\}$, $O_2 = \{x_{22}, x_{23}, x_{24}\}$, $F_1 = \{x_f, x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}\} = E$, $F_2 = \{x_f, x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{24}\} = E$. So we have: $O_2 \subset O_1 \subset E = F_1 = F_2$.
So, E_3 has no quasi topology structure in this topology.*

Remark 4.7 *In a lattice as total space, even it is a complete lattice we cannot define a topology based on filters or, dually by ideals. For filters, that is because of the fact that the union of two filters is not always a filter. For ideals, that is because the union of two ideals is not always an ideal. Now, the idea is to study the conditions, if any, under which the ideals and the filters represent the open sets and, correspondingly, closed sets, respectively in a lattice viewed as topological space. For this goal, we give Example 3 and Example 4.*

Example 3 *Let us consider the lattice L in Figure 4 that is a lattice with an infimum. This is the space X. As open sets we consider all the filters having the first element x_i, that is all the filters of the form $Fil = \{x_i\} \cup \{x_{jk} / x_i \to^* x_{jk}\}$ where $\to^*$ is the transitive closure of $\to$. These filters are the filters constructed by starting from xi and the path $x_1, x_{11}, x_{12}, \ldots$ going at the right till the path $x_1, x_{1m1}, x_{2m2}, \ldots$ and as far as possible down if the lattice is infinite or till the last level, if the lattice is finite. Let F denote the set of all these filters. We can easily verify that (L, F) is a topological space if and only if the empty set is considered as an filter of type Fil. The closed set corresponding to an open set is the set obtained by adding to the open set all the vertices on the following level after the last one in relation with vertices on the last level. For example, for the filter of this type $Fil_1 = \{x_i, x_{11}, x_{12}\}$ as open set, the closed set is the ideal $Id_2 = \{x_i, x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{2m2}\}$*

*For $E = \{x_i, x_{11}, x_{12}, x_{21}, x_{22}\}$, we obtain $O_1 = \{x_i, x_{11}, x_{12}\}$; $O_2 = \{x_i\}$; $F_1 = \{x_i, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, \ldots, x_{3m3}\}$; $F_2 = \{x_i, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, \ldots, x_{3m3}, x_{41}, x_{42}, \ldots, x_{4m4}\}$.
So, $O_2 \subset O_1 \subset E \subset F_1 \subset F_2$.*

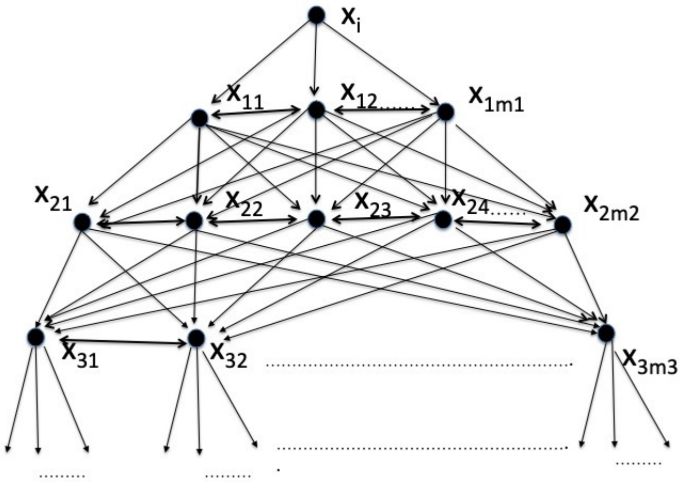

Figure 4: Lattice with an infimum

Remark 4.8 *1. The lattice L can be viewed as a topological space (L, F) if we accept the empty set to be a filter. Formally, this represents a contradiction of the axiom 1. from the Definition 4.4. In some applications, we can accept this statement.*

2. Only its subsets being the structure of a filter here above can be described as having the structure of quasi topology.

Example 4 *Let us consider the lattice L in Figure 5 that is a lattice with a supremum. As in Example 3, from the lattice in Figure 5, we consider ideals of the form* $Id = \{x_f\} \cup \{x_{jk} \ / \ x_{jk} \rightarrow^* x_f\}$ *Let I be the set of all these ideals. In the same way, as in the Example 3, we can organize this lattice by an ideal topology (L, I) and we can find sets with a quasi topology structure.*

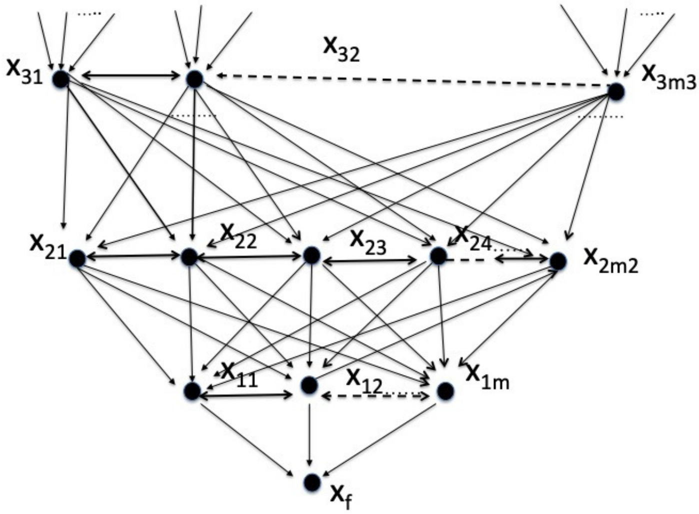

Figure 5: Lattice with a supremum

Example 5 *In the Figure 6 there is another type of lattice. This lattice is related with the mathematical model of the Logic of Determination of Objects (LDO) [7]. As finite lattice, it is also used in some transportation problems in operational research. However, it is a particular lattice. We study this lattice as a topological space and, then the sets receiving a quasi topology structure.*

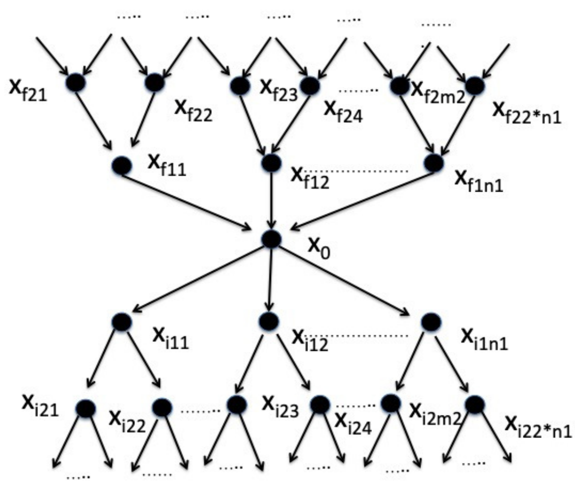

Figure 6: The LDO's lattice model

The particularities of this lattice are:

In the upper side L_{up} of the vertex x_0 we can define ideals while in its lower side L_{low}, filters. The first level in the upper side and, equaly, in the lower side has n1 vertices, the second level 2 × n1, and so on the k level has $2^{k-1} \times n1$ vertices. As in Example 3 and Example 4, we can organize the upper side of the lattice L_{up} with the ideal topology (L_{up} , F) from Example 4 and the lower side L_{low} with the filter topology (L_{low}, I). Let us denote by L_{up-low} a lattice of this form.

On L_{up-low}, we define a couple (Id, Fil), formed by an ideal Id and a filter Fil as follow:

Definition 4.9 *A pair (Id, Fil) is formed by an ideal of the form Id =$\{x_{jk}$ / $x_{jk} \rightarrow^* x_f\} \cup \{x_0\}$ and by a filter Fil of the form Fil = $\{x_i\} \cup \{ x_{jk}$ / $x_0 \rightarrow^* x_{jk}\}$. The lattice by L_{up-low} endowed with the two topologies (L_{up}, I) and (L_{low}, F) can be seen as a topological space (L_{up-low}, (L_{up} ,I), (L_{low}, F)). But these two topologies are disjoints with the meaning of the open sets are defined only on the upper side of the lattice and the closed sets on both the upper side and the lower side of a lattice. The only connection is by mean of open and a closed containing both the vertex x_0 .*

Definition 4.10 *We can define a hybrid topology T_{hyb} in the sense that the open sets are the ideals and the closed sets the formed by the pairs ideal – filter of the form (Id, Fil) here above. Only condition to add is to consider that the empty set is equaly a filter and an ideal. In this case, the closed set corresponding to a filter Fil as an open set is the pair (Id, Fil) where the ideal Id has the same number on vertex on the same level as the filter Fil. For example, the filter of final vertices x_{f11} and x_0, that is $\ldots x_{f21}\ x_{f22}\ x_{f11}\ x_0$ has several closed sets and namely $\ldots x_{f21}\ x_{f22}\ x_{f11}\ x_0\ x_{i11}\ x_{i21}\ x_{i22} \ldots$ or $\ldots x_{f21}\ x_{f22}\ x_{f11}$ $x_0\ x_{i12}\ x_{i23}\ x_{i23} \ldots$ and so on, $\ldots x_{f21}\ x_{f22}\ x_{f11}\ x_0\ x_{i1n1}\ x_{i2m2}\ x_{i22*n1} \ldots$*

Remark 4.11 *In this case, the only violation of general topology axioms is the fact that an open set can accept several closed sets.*

From all considerations here above, it result the following theorem:

Theorem 4.12 *In the lattice Lup-low viewed as topological space (L_{up-low}, T_{hyb}), there is a type of sets with a quasi topologic structure. This type is E = $\{ x/x \in Id \cup Fil \}$ of the pair (Fil, Id)$\}$ where Fil has the same number of vertices as Id on the same level.*

Example 6 *For the lattice in Figure 6, let be E =$\{x_{f21}, x_{f22}, x_{f11}, x_0, x_{i11}, x_{i21}, x_{i22}\}$. Then $O_1 = \{x_{f21}, x_{f22}, x_{f11}, x_0\}$; $O_2 = \{x_{f11}, x_0\}$; $F_1 = \{x_{f21}, x_{f22}, x_{f11}, x_0, x_{i11}, x_{i21}, x_{i22}, x_{31}, x_{32}, x_{33}, x_{34}\}$; $F_2 = \{x_{f21}, x_{f22}, x_{f11}, x_0, x_{i11}, x_{i21}, x_{i22}, x_{i31}, x_{i32}, x_{i33}, x_{i34}, x_{i41}, x_{i42}, x_{i43}, x_{i44}, x_{i45}, x_{i46}, x_{i47}, x_{i48}\}$.*

4.4 A quasi topology in a space endowed with two topologies

Let X be a space endowed with two topologies T_1 and T_2 such that $T_2 \subset T_1$, that is each open set in T_2 is a open set of T_1 and T_1 contains at least an open not belonging to T_2 . In order to define a quasi topology structure, it is necessary and sufficient the following conditions:
There are open sets O_1 and O_2 , $O_1 \in T_1$ and $O_2 \in T_2$, $O_2 \in T_2$ such that $O_2 \subset O_1$ and O_2 is the largest with this strictly inclusion property; There are closed sets, F_1 and F_2, $F_1 \in T_1$ and $F_1 \notin$, T_2, $O_2 \in T_2$ such that $F_1 \subset F_2$ and F_1 it is the largest with this strictly inclusion property. We must find out which type of topological spaces can ensure in a not trivial way these conditions, it means there are sets of type E receiving a quasi topologic structure. In a complete lattice to capture these constraints one can define two topologies T_1 and T_2 using the relation $\rightarrow$ for T_1 and the relation $\rightarrow^2$, this last having the meaning that $x \rightarrow^2 y$ if and only if between x and y there is only one intermediate vertex.

4.5 Conclusions

The general conclusion is that if we want to keep all axioms (2.1) – (2.5) for the quasi topology structure (QTS), we must take into account particular topological spaces. We analyzed some types of particular lattices and instanciated some QTS anchored in them. Therefore, the notion of quasi topology structure in topological spaces is consistent. Other conclusions are:

1. The definitional features of the QTS show that we still stay in the framework of classical set theory but one highlights the idea of relation of "biggest" and of "smallest" between the elements of a set and between the elements of its power set.

2. None of the previous examples of topological spaces can be organized as a quasi topological space, except metric spaces.

3. We can start from a more general definition of the QTS. Comparing with the Definition 2.1 where the universe is a topological space $\langle X, O \rangle$, we can put $\langle X, T \rangle$ where T is either a topology or a pair (T_1, T_2) of topologies.

4. The QTS can be modeled in a non-trivial manner and in a consistency mode either by ordered discrete topologies or in metric spaces as topological spaces.

5. If sets, O_1 and O_2 and F_1 and F_2 respectively are not related with the ideas of continuity and separation that are the main ideas in classical topology. Therefore, we can try to model a quasi topologic structure and,

eventually a quasi topological space by means of spaces endowed with algebraic relations as, for example, the approximation spaces proposed by rough sets theory. This will be the topic of the next paper.

References

[1] N. Bourbaki,*Elements of Mathematics, General Topology*, Hermann Editeurs des Sciences et des Arts, Paris, France and Addison-Wesley Publishing Company Reading Massachusetts, USA, 1966.

[2] K. Kuratowski,*Topology*, Volume I, Academic Press, 1966.

[3] J.P. Descles, *Schemes et topologie (de Kant à la sémantique cognitive)*, Colloque Histoire des mathematiques et sciences sociales EHESS, 15–17 novembre, Lyon, 2007.

[4] J.P. Desclés, *Schèmes sémantico-cognitifs et schèmes topologiques*, http://lalic.paris-sorbonne.fr/PUBLICATIONS/2009/schemes_quasi_Lyon_09.pdf, 2009.

[5] J.P. Desclés, *Du trimorphe aux frontières quasi-topologiques*, Ateliers d'anthropologie, 37, Laboratoire d'ethnologie et de sociologie comparative, Université de Nanterre, hhtp:// ateliers.revues.org 2012.

[6] J.P. Desclés, Z. Guentcheva, *Quasi-Topological Representations (QTR) of spatial places and spatio-temporal movements in Natural Languages*, Space in Language, Giovanna et Lenci, Meini, Rovali (éds), Edizioni ETS, (2010), 213–233.

[7] J.P. Desclés, A. Pascu, *Logic of Determination of Objects (LDO) : How to Articulate "Extension" with "Intension" and "Objects" with "Concepts"*, Logica Universalis, Springer, vol. **5** nr **1**, (2011), 75–89.

[8] J.P. Desclés, A.C. Pascu, C. Jouis, *The Logic of Typical and Atypical Instances (LTA)*, Proceedings of the Twenty-Sixth International Florida Artificial Intelligence Research Society Conference, FLAIRS13, Miami, USA. AAAI Press , (2013), 321–326.

[9] J.P. Desclés, A.C. Pascu, *Logique de la Détermination des Objets (LDO); structuration topologique et quasi-topologique des extensions*, Conference La logique en question / Logic in Question, Paris-Sorbonne, 2016.

[10] J.P. Desclés, A.C. Pascu, I. Biskri, *A Quasi-Topologic Structure of Extensions in the Logic of Typical and Atypical Objects (LTA) and Logic of*

Determination of Objects (LDO), Proceedings of the 31th FLAIRS Conference. AAAI Press, 2017.

[11] Z. Pawlak, *Rough Sets* Internation Journal of Computer and Information Sciences, **11(5)** (1982), 341-356.

[12] T.B. Irwinski, *Algebraic approach to rough sets*, Bulletin of the Polish Academy of Science and Mathematics, (1987), 35, p. 673–683.

[13] P. Pomykala, *Approximation operations in approximation space*, Bulletin of the Polish Academy of Science, Mathematics, 1(987), **35**, 653-662.

[14] L. Zadeh, *Fuzzy sets* Information and Control, **8** (1965), 338–353.

[15] M. De Glas, *Sortir de l'enfer cantorien.* Intellectica, 2010, http://intellectica.org/SiteArchives/actuels/n51/51-7-MicheldeGlas.pdf

[16] C. Cornelis, M. De Cock, A.M. Radzikowska, *Fuzzy Rough Sets: from Theory to Practice*, (2008), http://www.cwi.ugent.be/Chris/frs2.pdf.

[17] G. Birkhoff, *Lattice Theory*, American Mathematical Theory, 1967

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