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Compactification of Zero-Dimensional Topological Spaces and Abstract Logics

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Abstract

In this paper we give a characterization of compact topological spaces, in special, of compact zero-dimensional spaces (Theorems 1 and 2), in terms of ultrafilter convergence (Definition 1). Motivated by this characterization, we introduce a general method of compactification, which, in the zero-dimensional case, preserves zero-dimensionality. Since zero-dimensional spaces are uniformizable, the method also yields the respective Cauchy completion. In general case this compactification can be interpreted as "completion" with respect to ultrafilter convergence. We also prove that the continuous functions with values in compact zero-dimensional spaces possess an extension property with respect to the mentioned compactification. Some applications to Model Theory are given, specially to the compactification of a logic.

Keywords: zero-dimensional topological spaces, compactness and compactification, ultrafilter convergence, abstract logics.

Introduction

In 1989 the Colombian Professor Xavier Caicedo, one of the most renowned specialists in mathematical logic related to topological aspects of model theory, paid a visit to IMECC (Institute of Mathematics, Statistics and Scientific Computing) of Unicamp (University of Campinas), Brazil, for three months. He is a professor at Los Andes University in Bogotá, Columbia. At that time, he was invited by my doctoral adviser, Antonio Mario Sette, a professor at Unicamp, to carry out a joint research with IMECC, and one of his activities there was to teach some seminars on "Uniform topological methods in model theory".

The topological methods in model theory of first order logic were introduced mainly by Alfred Tarski (see [10], p. 713), who defined the *elementary topology* in structure spaces of the same kind of similarity via a closure operator. Nowdays, this topology is given by a *clopen* basis consisting of the $Mod(\varphi)$ classes for each sentence φ of the language $L_{\omega\omega}$ corresponding to the type of similarity. Notoriously, from this topology, which can also be defined by other traditional languages of logic, it follows that topological compactness is equivalent to logical compactness.

Professor Caicedo started his seminars showing that this elementary topology is in fact a uniform topology in which the structure spaces are, in a natural way, totally bounded, directing the discussion to his research on logical equivalents of uniform properties such as uniform continuity of functions between structure spaces [3].

The main idea that originated our research, in the context of Professor Caicedo's seminar, was that the *Cauchy completeness* of a compact uniform space, namely, "all Cauchy nets converge", must also have a logical equivalent in model theory, and our conjecture was the following: given a Cauchy net of structures, it converges, up to an elementary equivalence, to the ultraproduct of the net.

This fact was established by me, Antonio Mario Sette and Daniele Mundici, by the time the latter was a Professor at University of Milan, Italy, and had already visited UNICAMP several times before. This result was published in [6], and independently by Caicedo in [3].

The subsequent work, which is the subject of this paper, led to the following directions: 1) to find a topological characterization for the compactness of zero-dimensional spaces, namely, the ones with a clopen basis, and its Cauchy completeness. Since its topology is uniform, this topological characterization, as we will see, involves a notion of convergence through ultrafilters, the U-convergence, which allows us to find a topological version of Łoś's ultraproduct theorem; 2) to develop a general method of compactification using local ultrafilters (as we define in section 3) so that, in the case of zero-dimensional spaces, the resulting compact space is still zero-dimensional; and 3) to apply the obtained results to the process of "compactification of abstract logics" by compactfying the corresponding structure spaces via the developed method. Preliminary versions of this work were published in [4], [5] and [6].

1 U-Convergence and Compactness

A zero-dimensional (z-d) space X is a topological space which admits a basis of clopen (i.e. open-closed) sets. It is clear that the collection of clopen sets of a z-d space is a basis of the space that is closed under finite intersections and complements. In what follows, if X is a z-d space, we will suppose that $\mathcal{B}$ is a basis of clopen sets of X which is closed under finite intersections and complements (in this case obviously $\mathcal{B}$ contains X and the empty set $\emptyset$).

The topology of a z-d space admits an uniform structure (see [8]) that generates it given by the following subbasis of uniformity: $\{\mathcal{U}_V / V \in \mathcal{B}\}$, where, for every $V \in \mathcal{B}$, $\mathcal{U}_V = \{(x,y) \in X \times X / x \in V \iff y \in V\}$. In particular, for every $V \in \mathcal{B}$ there exists $W \in \mathcal{B}$, which can be taken as V itself, such that $\mathcal{U}_W \circ \mathcal{U}_W \subseteq \mathcal{U}_V$. This sub-basis has the following property that, besides ensuring that the space topology is generated by this uniformity, shows that the resulting uniform space is totally bounded: for every $x \in X$ and every $V \in \mathcal{B}$,

$$\mathcal{U}_V[x] = \{y \in X / (x,y) \in \mathcal{U}_V\} = \begin{cases} V & , \quad \text{if } x \in V \\ V^c & , \quad \text{if } x \notin V. \end{cases}$$

(V^c denotes the complement of V in X).

The fact that X is totally bounded can be proved with the following arguments: for every $n \geq 1$, we define $S_n = \{\sigma = (\varepsilon_1, \ldots, \varepsilon_n) / \varepsilon_i = 0, 1\}$, then, given $V_1, \ldots, V_n \in \mathcal{B}$ we can decompose

$$X = \bigcup_{\sigma \in S_n} (V_1^{\varepsilon_1} \cap \ldots \cap V_n^{\varepsilon_n}), \text{ where } V_i^{\varepsilon_i} = \begin{cases} V_i & , \quad \text{if } \varepsilon_i = 1 \\ V_i^c & , \quad \text{if } \varepsilon_i = 0 \end{cases}$$

therefore, taking, for every $\sigma \in S_n$, $x_\sigma \in V_1^{\varepsilon_1} \cap \ldots \cap V_n^{\varepsilon_n}$ (if it is not empty), we have that $X = \bigcup_{\sigma \in S_n} (\mathcal{U}_{V_1}[x_\sigma] \cap \ldots \cap \mathcal{U}_{V_n}[x_\sigma])$.

For these uniform spaces, then, being complete, in the sense that every Cauchy net converges, is equivalent to being compact (see [8], pp. 198–9).

This paper has four parts: the first one gives a characterization of compact spaces (Theorem 1), giving greater emphasis to compact z-d spaces, in terms of convergence modulo ultrafilters or U-convergence for short, which we will soon define. It is interesting to observe that Theorem 1 gives us a topological version of Łoś's ultraproduct theorem. The second part introduces a method of general compactification that preserves zero-dimensionality in the case of z-d spaces and gives the respective completeness. In this section we will analyze the case of big spaces with small topologies, specially important for the applications to Model Theory. In the third part we will study the functorial properties of the built compactification and we will prove an extension theorem for certain continuous functions (Theorem 12). And in the fourh part we will apply the obtained results to the case of structure spaces, of Model Theory, equipped with the elementary topology given by a formal language. The emphasis on z-d spaces through this paper is due to these applications.

These are important examples of z-d spaces: the rationals $\mathbb{Q}$ with the induced topology from the line, the space of orders of a formally real field with the Harrison topology (see [5]), and the non-arquimedian topological vector spaces, among others.

In this paper, if I is a non-empty set, then a *ultrafilter* over I is a collection $U \subseteq \mathcal{P}(I)$ such that:

a) $\emptyset \notin U$

b) $A, B \in U$ implies $A \cap B \in U$

c) $A \in U$, $A \subseteq B$ implies $B \in U$

d) for all $A \subseteq I$: $A \in U$ or $A^c \in U$.

Definition 1. Let X be a topological space and $\mathcal{B}$ be a basis of X:

1. U-convergence: Let $(x_i)_{i\in I}$ be a family in X and U be an ultrafilter over I; we define $\lim_U x_i$ as the set of the $x \in X$ such that for all $V \in \mathcal{B}$ with $x \in V$, $\{i \in I \,/\, x_i \in V\} \in U$ (see [1]).

2. Let $(D, \leq)$ be a directed set; a *net* in X is any family $(x_i)_{i\in D}$ of elements of X.

3. An ultrafilter U over a directed set D is called *free* if it contains all the subsets $A_k = \{i \in D \,/\, i \geq k\}$, for $k \in D$. (The notion of free ultrafilter over a directed set generalizes the one of non-principal ultrafilter over the natural numbers $\mathbb{N}$; it can be seen that $\{A_k\}_{k\in D}$ has the Finite Intersection Property (FIP)).

Definition 2. Let X be a z-d space:

1. Let $(x_i)_{i\in I}$ be a family in X and U be an ultrafilter over I: we define $\lim_U x_i$ as the set of $x \in X$ such that for every $V \in \mathcal{B}$ there exists $A \in U$ such that for every $i \in A$, $(x, x_i) \in \mathcal{U}_V$, or equivalently, if for every $V \in \mathcal{B}$, $\{i \in I/(x, x_i) \in \mathcal{U}_V\} \in U$. (This definition is equivalent, for z-d spaces, to the usual U-convergence given in Definition 1).

2. A net $(x_i)_{i\in D}$ is called a *Cauchy net* is for every $V \in \mathcal{B}$ there exists $k \in D$ such that for every $i, j \geq k$, $(x_i, x_j) \in \mathcal{U}_V$

3. Let $(x_i)_{i\in D}$ be a net in X; we define $\lim_i x_i$ as the set of every $x \in X$ such that for every $V \in \mathcal{B}$ there exists $k \in D$ such that for every $i \geq k$, $(x, x_i) \in \mathcal{U}_V$.

If X is a Hausdorff space, we can easily prove that the limits $\lim_U$ and $\lim_i$, if they exist, are unique.

The following lemma is essencial for the proof of our characterization of compactness of z-d spaces, and it is due, fundamentally, to D. Mundici and A.M. Sette, in the case of spaces of structures in Model Theory (see [4], p. 20).

Convergence Lemma. Let X be a z-d space, $(x_i)_{i\in D}$ be a Cauchy net in X and U be a free ultrafilter over D, then $\lim_U x_i = \lim_i x_i$ and, in consequence, $\lim_U x_i$ independs of the ultrafilter U. (It is observed that the previous equality is a set equality which can be empty).

Proof. a) Let $x \in \lim_i x_i$ and let $V \in \mathcal{B}$ with $x \in V$; we have that there exists $k \in D$ such that for every $i \geq k$, $(x, x_i) \in \mathcal{U}_V$, i.e., $x_i \in \mathcal{U}_V[x] = V$, hence $A_k = \{i \in D/i \geq k\} \subseteq \{i \in D/x_i \in V\}$, therefore, as U is free we have that $\{i \in D/x_i \in V\} \in U$, i.e., $x \in \lim_U x_i$.

b) Let $x \in \lim_U x_i$. So, for each $V \in \mathcal{B}$, there exists $A_V \in U$ such that for $i \in A_V$, $(x, x_i) \in \mathcal{U}_V$. However, since $(x_i)_{i\in D}$ is a Cauchy net, then for each $V \in \mathcal{B}$, there exists $k_V \in D$ such that for all $i, j \geq k_V$, $(x_i, x_j) \in \mathcal{U}_V$.

Let $V \in \mathcal{B}$ and consider $W \in \mathcal{B}$ such that $\mathcal{U}_W \circ \mathcal{U}_W \subseteq \mathcal{U}_V$, then there exist A_W and k_W as above, hence, since U is free, $Z = A_W \cap A_{k_W} \in U$ (in particular, $Z \neq \emptyset$).

Let $k \in Z$ and $i \geq k$. Then, on the one hand, $(x, x_k) \in \mathcal{U}_W$ since $k \in A_W$. On the other hand, $i, k \geq k_W$ implies that $(x_k, x_i) \in \mathcal{U}_W$. So $(x, x_i) \in \mathcal{U}_W \circ \mathcal{U}_W \subseteq \mathcal{U}_V$. Therefore $x \in \lim_i x_i$. ■

Theorem 1. The following statements are equivalent (for z-d spaces):

i) For every family $(x_i)_{i\in I}$ and every ultrafilter U over I, $\lim_U x_i \neq \emptyset$. In this case, if $x \in \lim_U x_i$, we have that for every $V \in \mathcal{B}$:

$$x \in V \iff \{i \in I/x_i \in V\} \in U.$$

ii) For every Cauchy net $(x_i)_{i\in D}$ and every free ultrafilter U over D, $\lim_U x_i \neq \emptyset$. In this case, if $x \in \lim_U x_i$, we have that for every $V \in \mathcal{B}$:

$$x \in V \iff \{i \in D/x_i \in V\} \in U$$

iii) The space X is complete.

iv) The space X is compact.

Proof. (i $\Rightarrow$ ii): Obvious.
(ii $\Rightarrow$ iii): From (ii) and from the Convergence Lemma every Cauchy net $(x_i)_{i\in D}$ converges to every $x \in \lim_i x_i$.
(iii $\Rightarrow$ iv): It follows directly from the fact that X is a totally bounded uniform space.
(iv $\Rightarrow$ i): Suppose that for every $x \in X$ there exists $V_x \in \mathcal{B}$ such that $x \in V_x$ and $\{i \in I / x_i \in V_x\} \notin U$. Obviously, $\{V_x\}_{x\in X}$ is an open covering of the space X. Hence, since X is compact, there exist $x_1, \ldots, x_n \in X$ such that $\{V_{x_k}\}$ is still a covering of X.

Claim. $\{i \in I / x_i \in V_{x_1}\} \cup \ldots \cup \{i \in I / x_i \in V_{x_n}\} = I$.

Indeed, given $i \in I$, $x_i \in X = \bigcup_{k=1}^{n} V_{x_k}$, then there exists $k \in \{1, \ldots, n\}$ such that $x_i \in V_{x_k}$, i.e., $i \in \{i \in I / x_i \in V_{x_k}\}$.

Therefore, since $I \in U$ and U is a maximal filter, there exists $k \in \{1, \ldots, n\}$ such that $\{i \in I / x_i \in V_{x_k}\} \in U$, a contradiction. Hence, $\lim_U x_i \neq \emptyset$.

Let $x \in \lim_U x_i$ and $V \in \mathcal{B}$. If $x \in V$, then, from the definition, $\{i \in I / x_i \in V\} \in U$. If $x \notin V$, then $x \in V^c \in \mathcal{B}$. So, from the definition, $\{i \in I / x_i \in V^c\} \in U$, i.e., $\{i \in I / x_i \in V\} \notin U$. ■

Now we give a characterization theorem of compactness of any topological space in terms of U-convergence. As a consequence, we obtain a new method of compactification.

Theorem 2. Let X be any topological space, then the following statements are equivalent:

i) The space X is compact.

ii) For every family $(x_i)_{i\in I}$ and every ultrafilter U over I, $\lim_U x_i \neq \emptyset$.

iii) For every net $(x_i)_{i\in D}$ and every ultrafilter U over D, $\lim_U x_i \neq \emptyset$.

Proof. (i $\Rightarrow$ ii): It is contained in the proof of (iv $\Rightarrow$ i) of Theorem 1, in which the fact of X be a z-d space is not used.
(ii $\Rightarrow$ iii): Obvious.
(iii $\Rightarrow$ i): Let $\{F_j\}_{j\in J}$ be a family of closed sets of X with the FIP. Let $K = \mathcal{P}_\omega(J)$ be the collection of finite subsets of J, and, for every $\Delta \in K$ let $x_\Delta \in \bigcap_{j\in\Delta} F_j$. Consider the family $(x_\Delta)_{\Delta\in K}$; this family is a net with a natural order given by the inclusion.

For every $\Delta \in K$, let $A_\Delta = \{\Delta' \in K / \Delta \subseteq \Delta'\}$ (see Definition 1, part 3), then $\{A_\Delta\}_{\Delta\in K}$ have the FIP. Let U be an ultrafilter over K such that $\{A_\Delta\}_{\Delta\in K} \subseteq U$; then U is free and, from the hypothesis, there exists $x \in \lim_U x_\Delta$.

Claim. $x \in \bigcap_{j \in J} F_j$.

Indeed, let $j \in J$ and suppose that $x \notin F_j$, then $x \in F_j^c$ which is open, hence, there exists $V \in \mathcal{B}$ with $x \in V$ and $V \cap F_j = \emptyset$. But because $x \in \lim_U x_\Delta$, then for that V, $\{\Delta \in K / x_\Delta \in V\} \in U$; besides, $A_{\{j\}} \in U$, in particular, $\{\Delta \in K / x_\Delta \in V\} \cap A_{\{j\}} \neq \emptyset$, i.e., there exists $\Delta \in K$ with $j \in \Delta$ such that $x_\Delta \in V$. On the other hand, $x_\Delta \in \bigcap_{k \in \Delta} F_k$, in particular, $x_\Delta \in F_j$, i.e., $V \cap F_j \neq \emptyset$, a contradiction. ■

2 A General Method of Compactification

The following is a method of compactification of every topological space without any separation condition. In the zero-dimensional case, it preserves the zero-dimensionality of the space.

Let X be any topological space and $\mathcal{B}$ be a basis of X closed for finite intersections and containing X and $\emptyset$ (the topology generated by $\mathcal{B}$ itself can be considered).

Let γX be the collection of every pair (K, U) where $K \subseteq X$ and U an ultrafilter over K. We define over γX the following basis: $\mathcal{B}^* = \{V^* / V \in \mathcal{B}\}$ where $V^* = \{(K, U) \in \gamma X / K \cap V \in U\}$. This basis is closed for finite intersections, it satisfies $X^* = \gamma X$, $\emptyset^* = \emptyset$, $(V \cup W)^* = V^* \cup W^*$ and $(V \cap W)^* = V^* \cap W^*$, and, in the zero-dimensional case, it is closed by complements, since $(V^*)^c = (V^c)^*$. It is observed that the topology of γX strongly depends of the basis $\mathcal{B}$ considered in X.

The pairs (K, U) with $K \subseteq X$ and U an ultrafilter over K can be called *local ultrafilters over X*.

Let us consider the following injective map $h : X \longrightarrow \gamma X$ given by $h(x) = (\{x\}, \{\{x\}\})$. It can be easily verified that: (a) $h(x) \in V^* \Leftrightarrow x \in V$, (b) h is a embedding of X in γX because $V^* \cap h[X] = h[V]$, and (c) $h[X]$ is dense in γX since that if $(K, U) \in V^*$, there exists $x \in K$ such that $h(x) \in V^*$.

Theorem 3. γX is compact.

Proof. Let $\{V_i^{*c}\}_{i \in I}$ be a collection of basic closed sets with the Finite Intersection Property (FIP). We have to prove that $\bigcap_{i \in I} V_i^{*c} \neq \emptyset$.

Claim 1. $\{V_i^c\}_{i \in I}$ has the FIP.

Indeed, if $i_1, \ldots, i_n \in I$, there exists $(K, U) \in V_{i_1}^{*c} \cap \ldots \cap V_{i_n}^{*c}$, i.e., $K \cap (V_{i_1}^c \cap \ldots \cap V_{i_n}^c) \in U$, hence, there exists $x \in K$ such that $x \in V_{i_1}^c \cap \ldots \cap V_{i_n}^c$.

Let $M = \bigcup_{i \in I} V_i^c$, then there exists an ultrafilter W over M that contains $\{V_i^c\}_{i \in I}$.

Claim 2. $(M, W) \in \bigcap_{i \in I} V_i^{*c}$.

It suffices to observe that for every $i \in I$, $M \cap V_i^c = V_i^c \in W$. ■

It can be observed that if X is compact and $\{V_i^{*c}\}_{i \in I}$ is a family of basic closed sets in γX with the FIP, then we can choose $x \in X$ such that $h(x) \in \bigcap_{i \in I} V_i^{*c}$.

Indeed, the family $\{V_i^c\}_{i \in I}$ has the FIP in X, then, because X is compact, there exists $x \in \bigcap_{i \in I} V_i^c$, hence, $h(x) \in \bigcap_{i \in I} V_i^{*c}$.

Sometimes, like in the applications to Model Theory, for example (see last section), it is necessary to consider big topological spaces with small topologies, i.e., topological spaces where the collection of elements is a proper class and the topology is a collection of subclasses that can be parametrized by a set. This kind of collection are usually called *small classes*.

In a space of this kind, i.e., with a small topology, several topological concepts can be defined in the usual way. For example, the concepts of interior and closure can be defined because they involve only unions and intersections of small collection of open and closed sets, respectively. The compactness property involves only coverings by small families of open sets or small families of open sets with the FIP, and convergence can be studied using nets defined on directed sets (not proper classes). In particular, in the case of uniform structures defined from small basis, the notion of Cauchy completeness involves only the convergence of small Cauchy nets.

Intuitively, the small classes are sufficiently elementary to be accepted without problems from the point of view of the foundations of set theory. Obviously, the usual set theories of Zermelo-Fraenkel (**ZF**) or Von Neumann-Bernays-Gödel (**NBG**) do not allow to fundament such collections. However, a theory which is sufficiently strong and relatively consistent with **ZF** + {there exists at least one (strong) inaccessible cardinal θ} was suggested by Levy (see [9]). In this theory, a natural model for such system could be obtained for example, interpreting "class" by "set" and "set" by "set of cardinality $< \theta$".

Here we will suppose that our "Levy's set theory" contains the following axiom of choice for collections of small classes: if $\{A_i\}_{i \in I}$ is a collection of small non-empty classes, i.e., I is a set, then there exists a set A that contains an unique element of each class of the collection. This version will be denoted by AC^w.

Observe that Theorems 1 and 2 above can be also extended for big topological spaces with small topologies. For example, in the proof of the part (iii $\Rightarrow$ i) of Theorem 2, if $\{F_j\}_{j \in J}$ is a small family of closed sets of X with the FIP, the choice, for each finite $\Delta \subseteq J$ of an element $x_\Delta \in \bigcap_{j\in\Delta} F_j$ can be justified as an application of the axiom AC^w.

Next we will properly modify the definition of γX in the case of X be a big space with a small basis $\mathcal{B}$ of open sets, and we will reformulate the proof that γX is compact using the axiom AC^w. It can be observed that in the case that X is a set we only need the Ultrafilter Theorem.

Let X be a proper class and $\mathcal{B}$ be a small basis of open sets in X (closed for finite intersections and containing $\emptyset$ and X). We define

$$\gamma X = \{(K,U) \ / \ K \text{ is a } \textbf{set} \text{ contained in } X \text{ and } U \text{ is an ultrafilter over } K\},$$

and for every $V \in \mathcal{B}$, we define $V^* = \{(K,U) \in \gamma X / K \cap V \in U\}$.

Theorem 4. For a proper class X and a small basis $\mathcal{B}$ over X, γX is compact.

Proof. Let $\{V_i^{*c}\}_{i\in I}$ be a collection of basic closed sets with the FIP (note that we can suppose $I \subseteq \mathcal{B}$, thus I is a set). We need to prove that $\bigcap_{i\in I} V_i^{*c} \neq \emptyset$.

Let $J = \mathcal{P}_\omega(I)$ (finite parts of I) and, for every $\Delta \in J$ we choose (based on the axiom AC^w) an element $(K_\Delta, U_\Delta) \in \bigcap_{i\in\Delta} V_i^{*c}$. We have that $\{K_\Delta\}_{\Delta\in J}$ is a collection of sets and $K = \bigcup_{\Delta\in J} K_\Delta$ is a set.

Claim 1. The family $\{K \cap V_i^c\}_{i\in I}$ has the FIP.

Indeed, let $\Delta \in J$. Then, since $(K_\Delta, U_\Delta) \in \bigcap_{i\in\Delta} V_i^{*c}$ we have that for $i \in \Delta$, $K_\Delta \cap V_i \notin U_\Delta$, i.e., $K_\Delta \cap V_i^c \in U_\Delta$, hence $K_\Delta \cap (\bigcap_{i\in\Delta} V_i^c) \in U_\Delta$, in particular, $K_\Delta \cap (\bigcap_{i\in\Delta} V_i^c) \neq \emptyset$, therefore $\bigcap_{i\in\Delta}(K \cap V_i^c) = K \cap (\bigcap_{i\in\Delta} V_i^c) \neq \emptyset$, because $K_\Delta \subseteq K$.

Let $M = \bigcup_{i\in I}(K \cap V_i^c)$, then M is a set and there exists an ultrafilter W over M such that $\{K \cap V_i^c\}_{i\in I} \subseteq W$.

Claim 2. $(M, W) \in \bigcap_{i\in I} V_i^{*c}$.

Indeed, let $j \in I$. Then, since $K \cap V_j^c \in W$ and $M \cap V_j^c = \bigcup_{i\in I}(K \cap V_i^c) \cap V_j^c = \bigcup_{i\in I}(K \cap V_i^c) \cap (K \cap V_j^c) = K \cap V_j^c$, we have that $M \cap V_j^c \in W$, i.e., $M \cap V_j \notin W$. Then, $(M,W) \in V_j^{*c}$. Therefore, $(M,W) \in \bigcap_{i\in I} V_i^{*c}$. ■

Next we will see that γX can be considered as the completion of X with respect to U-convergence, where every family of elements of X U-converges to some limit in γX (Theorem 5), and every point of γX is a U-limit of a family of elements of X (Theorem 6).

Theorem 5. Every family of elements of X U-converges in γX.

Proof. Let $(x_i)_{i\in I}$ be a family in X and U be an ultrafilter over I (the family may be understood as a function $x : I \to X$ such that for every $i \in I$, $x(i) = x_i$).

Let K_x be the image set of the family, i.e., $K_x = x[I]$, and $U_x = \{A \subseteq K_x / x^{-1}[A] \in U\}$ (it can be easily proved that U_x is an ultrafilter over K_x).

Then, identifying x_i ($\in X$) with $h(x_i)$ ($\in \gamma X$), we have that, in γX: $(K_x, U_x) \in \lim_U x_i$.

Indeed, observing that the facts $x_i \in V$ and $h(x_i) \in V^*$ are equivalent, we need to prove that for every V^* with $(K_x, U_x) \in V^*$, $\{i \in I / x_i \in V\} \in U$.

If $(K_x, U_x) \in V^*$, then $K_x \cap V \in U_x$, hence, by definition, $x^{-1}[K_x \cap V] \in U$, but $x^{-1}[K_x \cap V] = \{i \in I / x_i \in V\}$, therefore, $\{i \in I / x_i \in V\} \in U$. ■

Theorem 6. If $(K, U) \in \gamma X$, then, considering K as a family in X whose elements are parameterized by themselves, we have that $\lim_U K = cl\{(K, U)\}$ (cl denotes the closure in γX). In particular, $(K, U) \in \lim_U K$, i.e., every element of γX is U-limit of a family of elements of X.

Proof. $(A, W) \in \lim_U K \Leftrightarrow$ for every V^* with $(A, W) \in V^*$: $\{x \in K / x \in V\} \in U$, but $\{x \in K / x \in V\} \in U \Leftrightarrow K \cap V \in U \Leftrightarrow (K, U) \in V^* \Leftrightarrow V^* \cap \{(K, U)\} \neq \emptyset$. Hence, $(A, W) \in \lim_U K \Leftrightarrow$ for every V^* with $(A, W) \in V^*$: $V^* \cap \{(K, U)\} \neq \emptyset \Leftrightarrow (A, W) \in cl\{(K, U)\}$. ■

We have, then, that γX is a compactification of X in any case, zero-dimensional or not. In the zero-dimensional case it is also the completion of the underlying uniform space. In the general case, as we already saw, γX can be considered as the completion of X with respect of U-convergence. It shall be observed that γX is never a Hausdorff space, even if X is so, and that if X is compact, in general, γX does not reduce to X, as the next two theorems will show.

Let $(K, U) \in \gamma X$ and $J \subseteq U$. Then, we define $U\lceil J = \{A \cap J / A \in U\}$. It can be proved that $U\lceil J$ is an ultrafilter over J that trivially satisfies: for every $A \subseteq K$, $A \in U$ if and only if $A \cap J \in U\lceil J$. Let $(K, U) \in \gamma X$, we say that U is an *uniform ultrafilter* over K if for every $A \in U$, $|A| = |K|$ (the bars denote the cardinality of the set). Let (K_1, U_1) and (K_2, U_2) in γX. We define $(K_1, U_1) \equiv (K_2, U_2)$ if for every $V \in \mathcal{B}$, $(K_1, U_1) \in V^*$ if and only if $(K_2, U_2) \in V^*$, i.e., (K_1, U_1) and (K_2, U_2) are indistinguishable by open sets of γX.

Theorem 7. For every $(K, U) \in \gamma X$ and every $J \in U$, we have that $(K, U) \equiv (J, U\lceil J)$; in particular, if $x \in K$ and U_x is the principal ultrafilter (over K) generated by x, then, $(K, U_x) \equiv (\{x\}, \{\{x\}\})$.

Proof. (i) Let $(K, U) \in V^*$, i.e., $K \cap V \in U$, and suppose $J \cap V \notin U\lceil J$, then, for every $A \in U$, $J \cap V \neq A \cap J$, in particular, for $A = K \cap V$ we have that $J \cap V \neq K \cap V \cap J = J \cap V$ (since $J \subseteq K$), a contradiction, hence $(J, U\lceil J) \in V^*$.

(ii) Let $(J, U\lceil J) \in V^*$, i.e., $J \cap V \in U\lceil J$, then there exists $A \in U$ such that $J \cap V = A \cap J$; suppose that $K \cap V \notin U$, then $K \cap V^c \in U$, hence, $K \cap V^c \cap J \in U\lceil J$, i.e., $J \cap V^c \in U\lceil J$. Therefore, $J \cap V \notin U\lceil J$, a contradiction, in consequence, $(K, U) \in V^*$.

For the case (K, U_x), it is enough to take $J = \{x\}$, so $U_x\lceil J = \{\{x\}\}$. ∎

Theorem 8. Let K be an infinite set contained in X and U be a non-principal ultrafilter over K, then, there exist $J \in U$ and W an uniform ultrafilter over J such that $(K, U) \equiv (J, W)$.

Proof. Let $\alpha = \min\{|A|/A \in U\}$ and let $J \in U$ be any set with $|J| = \alpha$, then, from Theorem 7, $(K, U) \equiv (J, U\lceil J)$ (if U is principal, then $\alpha = 1$ and $J = \{x\}$ with $x \in K$, so $U\lceil J = \{\{x\}\}$, which is uniform, hence, the case that interests us is when U is non-principal).

Claim. $U\lceil J$ is uniform, i.e., for every $B \in U\lceil J$, $|B| = \alpha$.

Indeed, if $B \in U\lceil J$, then there exists $A \in U$ such that $B = A \cap J$. We have that $|B| = |A \cap J| \leq |J| = \alpha$. On the other hand, since $A, J \in U$, we have that $B \in U$. So $|B| \geq \alpha$ since α is the minimum. Therefore, $|B| = \alpha$. ∎

It is observed that Theorems 7 and 8 reduce the local principal ultrafilters to elements of X and the local non-principal ultrafilters to uniform ultrafilters.

3 Functorial Properties

Next we will define a functor $*$ of the category of the topological spaces $\langle X, \mathcal{B}\rangle$ with a distinguished basis in the category of the compact spaces also with a distinguished basis. A morphism in this category is a continuous function $f : \langle X, \mathcal{B}\rangle \longrightarrow \langle Y, \mathcal{C}\rangle$ that preserves the respective bases, i.e., if $W \in \mathcal{C}$, then $f^{-1}[W] \in \mathcal{B}$. We say in this case that f is *s-continuous* (strongly continuous).

For every space X we define, as above, $X^* = \gamma X$, and if $f : \langle X, \mathcal{B}\rangle \longrightarrow \langle Y, \mathcal{C}\rangle$ is s-continuous, we define $f^* : \langle \gamma X, \mathcal{B}^*\rangle \longrightarrow \langle \gamma Y, \mathcal{C}^*\rangle$ in a way that the following diagram comutes (h and k are the canonical imersions):

$$\begin{array}{ccccc} & X & \xrightarrow{f} & Y & \\ h & \downarrow & & \downarrow & k \\ & \gamma X & \xrightarrow{f^*} & \gamma Y & \end{array}$$

If $(K,U) \in \gamma X$, we define $f^*(K,U) = (K_f, U_f)$ where $K_f = f[K]$ and $U_f = \{A \subseteq f[K]/K \cap f^{-1}[A] \in U\}$.

f^* has the following properties: (a) U_f is an ultrafilter over K_f, (b) $f^* \circ h = k \circ f$, i.e., $f^* \lceil X = f$, (c), $(id_X)^* = id_{X^*}$, (d) $U_{g\circ f} = (U_f)_g$, so $(g \circ f)^* = g^* \circ f^*$.

Theorem 9. f^* is s-continuous.

Proof. Let $W^* = \{(K', U') \in \gamma Y / K' \cap W \in U'\} \in \mathcal{C}^*$, we will prove that $(f^*)^{-1}[W^*] = (f^{-1}[W])^*$. Indeed, $(f^*)^{-1}[W^*] = \{(K,U) \in \gamma X / f^*(K,U) \in W^*\} = \{(K,U) \in \gamma X/(K_f, U_f) \in W^*\} = \{(K,U) \in \gamma X / f[K] \cap W \in U_f\} = \{(K,U) \in \gamma X / K \cap f^{-1}[f[K] \cap W] \in U\} = \{(K,U) \in \gamma X / K \cap f^{-1}[W] \in U\} = (f^{-1}[W])^* \in \mathcal{B}^*$. ■

In analogy to the definition of s-continuous function we can define the notion of s-open function: we say that $f : \langle X, \mathcal{B}\rangle \longrightarrow \langle Y, \mathcal{C}\rangle$ is *s-open* if for every $V \in \mathcal{B}$ we have that $f[V] \in \mathcal{C}$. It is imediate that every s-open funtion is open. A special kind of funtions in the mentioned category is the following: $f : \langle X, \mathcal{B}\rangle \longrightarrow \langle Y, \mathcal{C}\rangle$ is called an *almost-homeomorphism* if f is surjective, s-continuous, s-open, and for every $V \in \mathcal{B}$, $f^{-1}[f[V]] = V$. The following theorem will show that the functor $*$ preserves almost-homeomorphisms.

Theorem 10. If $f : \langle X, \mathcal{B}\rangle \longrightarrow \langle Y, \mathcal{C}\rangle$ is an almost-homeomorphism, then $f^* : \langle \gamma X, \mathcal{B}^*\rangle \longrightarrow \langle \gamma Y, \mathcal{C}^*\rangle$ is also an almost-homeomorphism

Proof. From Theorem 9, f^* is s-continuous. We will prove now that f^* is s-open. Let $V^* \in \mathcal{B}^*$, we will prove that $f^*[V^*] = f[V]^*$.

$\subseteq$: Let $(B,W) \in f^*[V^*]$, then there exists $(K,U) \in V^*$ such that $(B,W) = f^*(K,U) = (K_f, U_f)$, i.e., $B = K_f = f[K]$ and $W = U_f = \{A \subseteq f[K]/K \cap f^{-1}[A] \in U\}$.

Suppose that $(B,W) \notin f[V]^*$, hence, $f[K] \cap f[V] = B \cap f[V] \notin W$, i.e., $K \cap f^{-1}[f[K] \cap f[V]] \notin U$, therefore $K \cap f^{-1}[f[K]] \cap f^{-1}[f[V]] \notin U$, but $K \cap V \subseteq K \cap f^{-1}[f[K]] \cap f^{-1}[f[V]]$, hence, $K \cap V \notin U$, a contradition, because $(K,U) \in V^*$.

$\supseteq$: Let $(B,W) \in f[V]^*$, i.e., $B \cap f[V] \in W$. We will build $(K,U) \in V^*$ such that $f^*(K,U) = (B,W)$.

Since $B \subseteq Y$ and f is surjective, there exists $K (= f^{-1}[B]) \subseteq X$ such that $f[K] = B$.

Claim. There exists an ultrafilter U over K such that $W = \{A \subseteq B/f^{-1}[A] \in U\}$.

Indeed, since $K = f^{-1}[B]$, then the restriction $f\lceil K : K \to B$ is surjective. So, from the fact that W is an ultrafilter over B, we have that $W^f = \{f^{-1}[A]/A \in W\}$ is a filter basis over K. Let U be any ultrafilter over K such that $U \supseteq W^f$ and let $F = \{A \subseteq B/f^{-1}[A] \in U\}$. We will prove that $F = W$.

(a) If $A \in F$, then $f^{-1}[A] \in U$, hence, since U extends W^f, there exists $A' \in W$ such that $f^{-1}[A'] \subseteq f^{-1}[A]$. Therefore, since f is surjective, $A' = f[f^{-1}[A']] \subseteq f[f^{-1}[A]] = A$, hence $A \in W$.

(b) If $A \in W$, then $f^{-1}[A] \in W^f \subseteq U$, hence $f^{-1}[A] \in U$, therefore $A \in F$.

We will prove now that $f^*(K,U) = (B,W)$. Indeed, $f^*(K,U) = (K_f, U_f)$ where $K_f = f[K] = B$ and $U_f = \{A \subseteq f[K]/K \cap f^{-1}[A] \in U\} = \{A \subseteq B/f^{-1}[B] \cap f^{-1}[A] \in U\} = \{A \subseteq B/f^{-1}[B \cap A] \in U\} = \{A \subseteq B/f^{-1}[A] \in U\} = W$.

In fact, $(K,U) \in V^*$ because we have that $f^{-1}[B \cap f[V]] \in U$, i.e., $f^{-1}[B] \cap f^{-1}[f[V]] \in U$ since $B \cap f[V] \in W = \{A \subseteq B/f^{-1}[A] \in U\}$. Therefore, since $f^{-1}[B] = K$ and f is an almost-homeomorphism we have that $K \cap V \in U$.

f^* is obviously surjective. Finally, if $V^* \in \mathcal{B}^*$, then $(f^*)^{-1}[f^*[V^*]] = (f^*)^{-1}[f[V]^*] = (f^{-1}[f[V]])^* = V^*$. In consequence, f^* is an almost-homeomorphism. ∎

Almost-homeomorphisms have surprising properties not only from a topological but also from a logical point of view. A detailed study of its properties will be done in another paper.

We will finish this construction with the results refering to the extension property of the built compactification.

Lemma. If X is a z-d space and $(K,U) \in \gamma X$, then, in X:

$$\lim{}_U K = \cap\{V \in \mathcal{B}/K \cap V \in U\}.$$

Proof. i) If $x \in \lim_U K$, then for every $V \in \mathcal{B}$ with $x \in V$, $K \cap V \in U$. Suppose that there exists $W \in \mathcal{B}$ with $K \cap W \in U$ such that $x \notin W$, in this case, $x \in W^c \in \mathcal{B}$. Hence $K \cap W^c \in U$, i.e., $K \cap W \notin U$, a contradiction, therefore, $x \in \cap\{V \in \mathcal{B}/K \cap V \in U\}$.

ii) Suppose that $x \in \cap\{V \in \mathcal{B}/K \cap V \in U\}$ and $x \notin \lim_U K$, then there exists $W \in \mathcal{B}$ with $x \in W$ such that $K \cap W \notin U$, i.e., $K \cap W^c \in U$. Hence, since $W^c \in \mathcal{B}$ we have that $x \in W^c$, a contradiction, therefore, $x \in \lim_U K$. ∎

Observe that this lemma, in the case that X is compact, gives another proof of the part (iv $\Rightarrow$ i) of Theorem 1 since the family $\{V \in \mathcal{B}/K \cap V \in U\}$ is a family of closed sets with the FIP.

Theorem 11. If X is a compact z-d space, then there exists an application $g : \gamma X \to X$ that satisfies the following properties:

i) g is s-continuous and can be defined in such a way that $g\lceil X = id_X$ (identifying X with $h[X]$, in particular, X is a retraction of γX.

ii) g is s-open.

iii) For every $V \in \mathcal{B}$: $g^{-1}[g[V^*]] = V^*$.

iv) g is a proper map, i.e., g is closed and if $K \subseteq X$ is compact, then $g^{-1}[K]$ is compact in γX.

v) γX and X have the induced and coinduced by g topologies, respectively.

Proof. Let $(K, U) \in \gamma X$. Then from Theorem 6, $(K, U) \in \lim_U K$. Besides, from the previous Lemma, there exists $x \in X$ such that $x \in \lim_U K$. Define $g(K, U) = x$ and observe that the choice of x is, initially, arbitrary in $\lim_U K$. Thus, it is possible to have more than one function g satisfying conditions (i)–(v). In the case that $(K, U) = (\{z\}, \{\{z\}\})$ with $z \in X$ we have that $\cap\{V \in \mathcal{B}/\{z\} \cap V \in \{\{z\}\}\} = \cap\{V \in \mathcal{B}/z \in V\} = \{z\}$, hence, we define $g(K, U) = z$ satisfying the demand of $g\lceil X = id_X$.

i) In order to prove the s-continuity of g we will prove that for every $V \in \mathcal{B}$: $g^{-1}[V] = V^*$.

   If $(K, U) \in g^{-1}[V]$, then $x = g(K, U) \in V$, but since $x \in \lim_U K$ we have that, for that V, $K \cap V \in U$, i.e., $(K, U) \in V^*$.

   If $(K, U) \in V^*$, then, from the definition, $x = g(K, U) \in \lim_U K = \cap\{W \in \mathcal{B}/K \cap W \in U\}$, in particular, from the fact that $K \cap V \in U$ we have $x \in V$. Therefore, $(K, U) \in g^{-1}[V]$.

ii) From (i) we have that if $V \in \mathcal{B}$, $g[V^*] = g[g^{-1}[V]] = V$ since g is (obviously) surjective. Therefore, g is s-open.

iii) It is an immediate consequence from (i) and (ii) that for every $V \in \mathcal{B}$: $g^{-1}[g[V^*]] = g^{-1}[V] = V^*$.

iv) The fact that g is closed will be proved in two steps.

   a) For every $V \in \mathcal{B}$, $g[V^{*c}]^c \in \mathcal{B}$: indeed, we will prove that $g[V^{*c}] = V^c$; it results from the proof of part (i) that $V^* = g^{-1}[V]$, so $V^{*c} = g^{-1}[V]^c = g^{-1}[V^c]$, hence, $g[V^{*c}] = g[g^{-1}[V^c]] = V^c$ because g is surjective.

b) $g[\bigcap\limits_{i\in I} V_i^{*c}] = \bigcap\limits_{i\in I} g[V_i^{*c}]$: indeed, $\bigcap\limits_{i\in I} V_i^{*c} = \bigcap\limits_{i\in I} g^{-1}[V_i^c] = g^{-1}[\bigcap\limits_{i\in I} V_i^c]$, hence, $g[\bigcap\limits_{i\in I} V_i^{*c}] = \bigcap\limits_{i\in I} V_i^c = \bigcap\limits_{i\in I} g[V_i^{*c}]$.

The part (a) proves that the image of a basic closed set by g is a basic closed set, and the part (b) proves that such property can be extended to arbitrary closed sets. Therefore, g is closed.

Let $K \subseteq X$ be compact, and suppose that $g^{-1}[K] \subseteq \bigcup\limits_{i\in I} V_i^*$, then $K = g[g^{-1}[K]] \subseteq g[\bigcup\limits_{i\in I} V_i^*] = \bigcup\limits_{i\in I} g[V_i^*] = \bigcup\limits_{i\in I} V_i$, hence since K is compact, there exist $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup\limits_{k=1}^{n} V_{i_k}$, therefore, $g^{-1}[K] \subseteq \bigcup\limits_{k=1}^{n} g^{-1}[V_{i_k}] = \bigcup\limits_{k=1}^{n} V_{i_k}^*$, i.e., $g^{-1}[K]$ is compact.

v) It suffices to prove that the bases of γX and X are the bases induced and coinduced by g, respectively.

Indeed, since for every $V \in \mathcal{B}$ we have that $g^{-1}[V] = V^*$, then $\mathcal{B}^* = \{V^*/V \in \mathcal{B}\} = \{g^{-1}[V]/V \in \mathcal{B}\} =$ induced basis by g; analogously, $\mathcal{B} = \{V/V^* \in \mathcal{B}^*\} = \{V/g^{-1}[V] \in \mathcal{B}^*\} =$ coinduced basis by g. ■

The part (v) of the previous theorem asserts that X is a quotient of γX, in the case that X is a compact z-d space. In fact, the function g built in this part is an almost-homeomorphism, and it can be proved that the properties (iv) and (v) of Theorem 11 are satisfied by every almost-homeomorphism.

Theorem 12. If $f : X \to Y$ is s-continuous (respectively almost-homeomorphism) with Y a compact z-d space, then there exists a s-continuous (respectively almost-homeomorphism) function $\tilde{f} : \gamma X \to Y$ such that $\tilde{f}\lceil X = f$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} & X & \xrightarrow{f} & Y & & \\ h & \downarrow & & \downarrow & k & \\ & \gamma X & \xrightarrow{f^*} & \gamma Y & \xrightarrow{g} & Y \end{array}$$

Defining $\tilde{f} = g \circ f^*$ we have that $\tilde{f}$ is s-continuous, respectively almost-homeomorphism, from Theorems 9, 10 and 11, and that if $x \in X$, $\tilde{f}(x) = g(f^*(x)) = g(f(x)) = f(x)$, i.e., $\tilde{f}\lceil X = f$. Note that $\tilde{f}$ is not necessarily unique provided that g is not necessarily unique. ■

4 Application to Model Theory

In this section, an abstract logic is a pair $\langle L, \models_L \rangle$, or simply L, in which L is an application defined over the collection of vocabularies or similarity types T such that for every $\tau \in T$, L^τ is the class of sentences of L of type τ, and $\models_L \subseteq \bigcup_\tau (St^\tau \times L^\tau)$ is the truth relation of L, where St^τ is the collection of structures of type τ (see [7]). We will suppose that L is small, in the sense that, for every $\tau \in T$, the collectiom of sentences L^τ is a set, not a proper class.

Here, L will be an extension of the elementary or first order logic $L_{\omega\omega}$ in the following sense. If for every $\varphi \in L^\tau$, $Mod_L(\varphi) = \{\mathcal{A} \in St^\tau / \mathcal{A} \models_L \varphi\}$, then the following sentences are satiefied:

(a) *Atomic sentences property*: for every type τ and every atomic sentence $\varphi \in L^\tau_{\omega\omega}$, there exists $\psi \in L^\tau$ such that $Mod_L(\psi) = Mod_{L_{\omega\omega}}(\varphi)$.

(b) *Negation property*: for every type τ and every sentence $\varphi \in L^\tau$, there exists $\psi \in L^\tau$ such that $Mod_L(\psi) = Mod_L(\varphi)^c$.

(c) *Conjunction property*: for every type τ and every sentences $\varphi, \psi \in L^\tau$, there exists $\theta \in L^\tau$ such that $Mod_L(\theta) = Mod_L(\varphi) \cap Mod_L(\psi)$.

(d) *Existencial quantifier property*: if $c \notin \tau$ (c is a constant symbol), then, for every sentence $\varphi \in L^{\tau \cup \{c\}}$, there exists $\varphi \in L^\tau$ such that for every structure $\mathcal{A} = \langle A, \ldots \rangle \in St^\tau$: $\mathcal{A} \models_L \psi \Leftrightarrow$ for some $a \in A$, $\langle \mathcal{A}, a \rangle \models_L \varphi$.

The properties (a), (b) and (c) ensure that the collection $\mathcal{B}^\tau = \{Mod_L(\varphi) / \varphi \in L^\tau\}$ is a basis to the zero-dimensional topology over St^τ, closed under finite intersections and complements. We will denote by $St^\tau(L)$ the space St^τ equipped with this topology. It is observed that for every τ, $St^\tau(L)$ is a big space with a small topology, since St^τ is a proper class and L is small.

The logical compactness of L is usually formulated in the following way: for every type τ, if $\Sigma \subseteq L^\tau$ is such that every finite subcollection has a model (in St^τ), so Σ has a model (in St^τ). Equivalently, if for every finite subcollection $\Delta \subseteq \Sigma$, $Mod_L(\Delta) = \bigcap_{\varphi \in \Delta} Mod_L(\varphi) \neq \emptyset$, then $Mod_L(\Sigma) = \bigcap_{\varphi \in \Sigma} Mod_L(\varphi) \neq \emptyset$. We will see that the second version means the topological compactness of each space $St^\tau(L)$ in terms of families of basic closed sets with the FIP.

Theorem 1 which characterizes the compacity of a z-d space in terms of the U-convergence adopts the following form in the case of the logic L.

Theorem 13. Let L be a small abstract logic. Then the following statements are equivalent:

i) L is compact.

ii) For every type τ every family $\{\mathcal{A}_i\}_{i\in I} \subseteq St^\tau$ and every ultrafilter U over I, $\lim_U \mathcal{A}_i \neq \emptyset$, and if $\mathcal{A} \in \lim_U \mathcal{A}_i$ we have that for every $\varphi \in L^\tau$:

$$\mathcal{A} \models_L \varphi \Leftrightarrow \{i \in I/\mathcal{A}_i \models_L \varphi\} \in U.$$

■

It is observed that the statement (ii) of Theorem 13 is an abstract version and a generalization, for compact logics, of the Łoś's Theorem of ultraproducts restricted to sentences. It is always valid in $L_{\omega\omega}$ because, in this case, $\Pi_U \mathcal{A}_i \in \lim_U \mathcal{A}_i$. Therefore, the statement(i) of Theorem 1 can be considered a topological version of Łoś's Theorem. It is important to point that, since that the spaces $St^\tau(L)$ are totally bounded, the version of Łoś's Theorem given on Theorem 13(ii) is the statement that such spaces are Cauchy-complete.

We shall also mention that the proof of (iii $\Rightarrow$ i) on Theorem 2 is just an adaptation of the known proof of the compactness of $L_{\omega\omega}$ from Łoś's Theorem. In it, as we have already mentioned, the axiom AC^w is explicitly used.

Our compactification method can be applied to $St^\tau(L)$ spaces in the following manner:

For every $\tau \in T$, we define $CSt^\tau = \{(K,U)/K \subseteq St^\tau$ is a **set** of structures and U is an ultrafilter over $K\}$, and for every $\varphi \in L^\tau$ we define $Mod_L^*(\varphi) = \{(K,U) \in CSt^\tau/K \cap Mod_L(\varphi) \in U\} = \{(K,U) \in CSt^\tau/\{\mathcal{A} \in K/\mathcal{A} \models_L \varphi\} \in U\}$.

$Mod_L^*(\varphi)$ can be considered as a collection of "generalized models" of φ. From this point of view we can define the "truth" of the sentence φ in (K,U) as

$$(K,U) \Vdash_L \varphi \Leftrightarrow (K,U) \in Mod_L^*(\varphi) \Leftrightarrow \{\mathcal{A} \in K/\mathcal{A} \models_L \varphi\} \in U,$$

therefore, (K,U) behaves as an ultraproduct of K modulo U.

The collection $\{Mod_L^*(\varphi)/\varphi \in L^\tau\}$ is a basis of clopen sets, for a z-d and compact (small) topology over CSt^τ, which is closed for finite intersections and complements. St^τ is a dense subspace of CSt^τ by the embedding $h : St^\tau \to CSt^\tau$ given by $h(\mathcal{A}) = (\{\mathcal{A}\}, \{\{\mathcal{A}\}\})$. In addition, for every $\mathcal{A} \in St^\tau$ and every $\varphi \in L^\tau$ we have $h(\mathcal{A}) \Vdash_L \varphi \Leftrightarrow \mathcal{A} \vDash_L \varphi$. Therefore, the semantics of CSt^τ is an extension at level of sentences, of the semantics of St^τ.

With respect to this new semantics, the logic L is compact in the following sense: given $\Sigma \subseteq L^\tau$, if every finite subset of Σ has a generalized model (i.e., in CSt^τ), then, Σ has a generalized model. We did, therefore, compactify the logic L by extending the semantics.

The following are examples of s-continuous or s-open functions:

1. $h : St^\tau \to CSt^\tau$ is s-continuous because for every $\varphi \in L^\tau$ we have that

$$h^{-1}[Mod_L^*(\varphi)] = Mod_L^*(\varphi) \cap St^\tau = Mod_L(\varphi).$$

2. If $L_1 \leq L_2$ (i.e., for every $\varphi \in L_1^\tau$, there exists $\psi \in L_2^\tau$ such that $Mod_{L_2}(\psi) = Mod_{L_1}(\varphi)$), then, the identity $I : St^\tau(L_2) \to St^\tau(L_1)$ is s-continuous, and $I : St^\tau(L_1) \to St^\tau(L_2)$ is s-open. Moreover, any of these properties characterizes the fact that $L_1 \leq L_2$.

3. Let $c \notin \tau$ be a constant, then the function $F : St^{\tau\cup\{c\}} \to St^\tau$ given by $F(\langle \mathcal{A}, a\rangle) = \mathcal{A}$ is s-open. Indeed, the existencial quantifier property (d) states, in other words, that for every $\varphi \in St^{\tau\cup\{c\}}$ there exists $\psi \in St^\tau$ such that $F[Mod_L(\varphi)] = Mod_L(\psi)$.

We shall finish discussing the problem of the extension of the notion of "satisfaction" of well formed formulas to CSt^τ, but before this we will show an application of the new semantics to the elucidation of an old problem in the foundantions of mathematics: the distinction between the "arbitrarily big" and the "infinite", which is, ultimately, closely related to the not always clear distinction between the potential infinite and the actual infinite. The argument we will present was suggested by A. M. Sette (personal communication).

Theorem 14. There exists a "set-object" which is finite but arbitrarily big.

Proof. Let $St^\emptyset$ be the space of the structures where $\emptyset$ is the empty similarity type, i.e., the only symbol allowed in the structures of St^τ is the equality $=$. Therefore, $St^\emptyset$ can be identified as the universe the the sets.

Let us consider the logic $L = L_{\omega\omega}(Q_0)$ where Q_0 is the cardinal quantifier which interpretation in a structure $\mathcal{A} = \langle A, \ldots, \rangle$ is the following: $\mathcal{A} \models_L (Q_0 x)\varphi(x) \Leftrightarrow |\{a \in A/\mathcal{A} \models_L \varphi[a]\}| \geq \aleph_0$. In L, the fact that a set A, i.e., a structure of $St^\emptyset$, is "finite" can be expressed; indeed, $|A| < \aleph_0 \Leftrightarrow A \models_L \neg(Q_0 x)(x = x)$. On the other hand, for every $n < \aleph_0$, the fact that $|A| \geq n$ can also be expressed in L; indeed, $|A| \geq n \Leftrightarrow A \models_L \exists^{\geq n}$, where $\exists^{\geq n}$ is the sentence $(\exists x_1)\ldots(\exists x_n) \bigwedge\limits_{i<j\leq n} (x_i \neq x_j)$.

It can be observed that if A is a set, then $|A| \geq n$ for every $n < \aleph_0$ implies that $|A| \geq \aleph_0$, which is virtually in contradiction with $|A| < \aleph_0$. It means that the collection $\Sigma = \{\exists^{\geq n}/n < \aleph_0\} \cup \{\neg(Q_0 x)(x = x)\}$ has no model in $St^\emptyset$, which is a consequence of the compactness theorem of $L_{\omega\omega}$ applied to the first set. However, we will show that Σ has a generalized model in $CSt^\emptyset$. In fact, the existence of such model is ensured by the compactness of CSt^τ because every finite subset of Σ has a model which still is in $St^\emptyset$.

Next we will build a concrete model of Σ. For every $n < \aleph_0$ let $A_n \in St^\emptyset$ such that $|A_n| = n$, and let $K = \{A_n/n < \aleph_0\}$. Obviously, for every ultrafilter U over K we have that $(K, U) \Vdash_L (\neg Q_0 x)(x = x)$ because $\{A \in K/A \models_L \neg(Q_0 x)(x = x)\} = \{A \in K/|A| < \aleph_0\} = K \in U$.

We will build now an ultrafilter W over K such that for every $n < \aleph_0$, $(K, W) \Vdash_L \exists^{\geq n}$: for every $n < \aleph_0$ let $M_n = \{A \in K/|A| \geq n\}$, then the family

$\{M_n\}$ has the FIP; let W be an ultrafilter that contains the family $\{M_n\}$ (in fact, W is non-principal), then, for every $n < \aleph_0$, $(K, W) \Vdash_L \exists^{\geq n}$ because $\{A \in K/A \models_L \exists^{\geq n}\} = \{A \in K/|A| \geq n\} = M_n \in W$.

We have then that (K, W) is a generalized model of Σ and so it is the wanted "set-object". ■

The previous theorem shows, among other things, that the relation between two concepts, for example, of "being finite" and of "being arbitrarily big", depends of the considered logic. Even more, the existence of generalized models shows that the concept of "set" itself is not caught by any of the considered languages, although the quality of having some cardinality is. Metalinguistically, based on the Theorems 7 and 8 above, it is possible to assign a cardinality to the models (K, U) on the following way: $|(K, U)| = \min\{|A|/\langle A, \ldots\rangle \in U\}$. Back to the previous theorem we have that $L_{\omega\omega}$ allows to "reach" the actual infinite from the potential infinite, although, $L_{\omega\omega}(Q_0)$ separates both concepts. We must comment that non-standart models of Arithmetics also manage to relate those two concepts creating the notion fo "hyperfinite". It would be interesting to make a comparative study.

It remains, then, to discuss the extension of the notion of "satisfaction" for CSt^τ. The notion of "truth" for sentences was defined, as we already saw, in a natural way because the topology of these spaces is defined from them, although, the notion of "satisfaction" involves formulas with free variables and the problem of extending the semantics for this case is not trivial and consists in finding the ontological domain of these variables in relation to the new models.

A first step is to redefine the notion of abstract logic as a pair $\langle L, \Vdash_L\rangle$ where, in this case, $\Vdash_L \subseteq \bigcup_\tau (CSt^\tau \times L^\tau)$ is the new truth relation of L. Replacing $Mod_L(\varphi)$ for $Mod^*_L(\varphi)$ in the clauses (a), (b) and (c) given in the beginning of this section, we have that they are satisfied due to the zero-dimensionality of the CSt^τ spaces.

The property (d) of the existencial quantifier implicitly has the concept of formulas with free variables. Let us see: a n-ary formula $\varphi(c_1, \ldots, c_n)$ of L^τ is a sentence of $L^{\tau\cup\{c_1,\ldots,c_n\}}$ where $c_1, \ldots, c_n \notin \tau$ are constants. It allows us to define the concept of *satisfaction* for φ in L^τ from the concept of truth in $L^{\tau\cup\{c_1,\ldots,c_n\}}$: if $\mathcal{A} = \langle A, \ldots\rangle \in St^\tau$ and $a_1, \ldots, a_n \in A$, then

$$\mathcal{A} \models_L \varphi[a_1, \ldots, a_n] \Leftrightarrow \langle \mathcal{A}, a_1, \ldots, a_n\rangle \models_L \varphi(c_1, \ldots, c_n).$$

In these terms, the property (d) of the existencial quantifier adopts the following form: for every formula $\varphi(c)$ in L^τ, there exists $\psi \in L^\tau$, which can be denoted by $(\exists x)\varphi(x)$, such that for every $\mathcal{A} \in St^\tau$,

$$\mathcal{A} \models_L (\exists x)\varphi(x) \Leftrightarrow \text{ there exists } a \in A \text{ such that } \langle \mathcal{A}, a\rangle \models_L \varphi(c).$$

Using the function $F : St^{\tau\cup\{c\}} \to St^{\tau}$ defined by $F(\langle\mathcal{A}, a\rangle) = \mathcal{A}$ and observing that for $\mathcal{A} = \langle A, \ldots\rangle$, $F^{-1}[\mathcal{A}] = \{\langle\mathcal{A}, a\rangle \in St^{\tau\cup\{c\}}/a \in A\}$, this last equivalence can be replaced by:

$$\mathcal{A} \models_L (\exists x)\varphi(x) \Leftrightarrow \text{ there exists } \langle\mathcal{A}, a\rangle \in F^{-1}[\mathcal{A}] \text{ such that } \langle\mathcal{A}, a\rangle \models_L \varphi(c).$$

This last version can be adapted to the CSt^{τ} spaces applying the functor $*$ to the function F providing the following formulation to the property (d):

Existencial Quantifier Property for CSt^{τ} spaces: if $c \notin \tau$, then, for every sentence $\varphi \in L^{\tau\cup\{c\}}$, there exists $\psi \in L^{\tau}$ such that for every $(M, W) \in CSt^{\tau}$,

$$(M, W) \Vdash_L \psi \Leftrightarrow \text{ there exists } (K, U) \in (F^*)^{-1}[(M, W)]$$

$$\text{such that } (K, U) \Vdash_L \varphi,$$

or, equivalently,

$$(M, W) \Vdash_L (\exists x)\varphi(x) \Leftrightarrow \text{ there exists } (K, U) \in (F^*)^{-1}[(M, W)]$$

$$\text{such that } (K, U) \Vdash_L \varphi(c).$$

On the other hand, the expression $(K, U) \Vdash_L \varphi(c)$, for $(K, U) \in CSt^{\tau\cup\{c\}}$, means $K \cap Mod_L(\varphi(c)) \in U$, i.e., $\{\langle\mathcal{A}, a\rangle \in K/\langle\mathcal{A}, a\rangle \models_L \varphi(c)\} \in U$.

If we parameterize K as $K = \{\langle\mathcal{A}_i, a_i\rangle/i \in I\}$ and we identify U with an untrafilter over I, then, we have that the involved elements a_i determine a function $f \in \Pi_{i\in I}A_i$ such that $f(i) = a_i \in A_i$, and, therefore,

$$(K, U) \Vdash_L \varphi(c) \Leftrightarrow \{i \in I/\langle\mathcal{A}_i, f(i)\rangle \models_L \varphi(c)\} \in U.$$

It can be easily proved that if $g \in \Pi_{i\in I}A_i$ is such that $g \sim_U f$, i.e., $\{i \in I/g(i) = f(i)\} \in U$, then we also have

$$(K, U) \Vdash_L \varphi(c) \Leftrightarrow \{i \in I/\langle\mathcal{A}_i, g(i)\rangle \models_L \varphi(c)\} \in U.$$

It means that the constant c admits an interpretation in (K, U) through the equivalence class $\overline{f}$ of f modulo U, and we can formulate the desired notion of "satisfaction" in the following way:

$$(K, U) \Vdash_L \varphi(c) \text{ in } \overline{f} \in \Pi_U K \overset{(\Delta)}{\Leftrightarrow} \{i \in I/\langle\mathcal{A}_i, f(i)\rangle \models_L \varphi(c)\} \in U.$$

It can be observed that the equivalence (Δ) cannot be written in the following way: $(K, U) \Vdash_L \varphi(c)$ in $\overline{f} \Leftrightarrow \langle\Pi_U K, \overline{f}\rangle \models_L \varphi(c)$, because φ is not necessarily a first order formula.

In a last analysis and making the appropriate identifications, the existencial quantifier property for CSt^τ spaces admits the following reformulation: if $K = \{\mathcal{A}_i\}_{i\in I}$ and U is an ultrafilter over I, then, for every $\varphi(c)$,

$$(K,U) \Vdash_L (\exists x)\varphi(x) \overset{(\Delta\Delta)}{\Longleftrightarrow} \text{there exists } \overline{f} \in \Pi_U K \text{ such that}$$

$$\{i \in I/\langle \mathcal{A}_I, f(i)\rangle \models_L \varphi(c)\} \in U,$$

which, from the previous observation, cannot be rewritten as $(K,U) \Vdash_L (\exists x)\varphi(x) \Leftrightarrow$ there exists $\overline{f} \in \Pi_U K$ such that $\langle \Pi_U K, \overline{f}\rangle \models_L \varphi(c)$. We see, then, not necessarily $(K,U) \equiv_L \Pi_U K$. Although, the question if $|(K,U)| = |\Pi_U K|$ is interesting here.

We conjecture here that if for every $(K,U) \in CSt^\tau(L)$, $(K,U) \equiv_L \Pi_U K$, then, $L \equiv L_{\omega\omega}$.

The equivalences (Δ) and $(\Delta\Delta)$ constitute our formulation of the relation of "satisfaction" for the semantics of the CSt^τ spaces. Now their model theory can be developed.

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