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Description and Disjunction: Reflections on an Argument of Thomas Forster

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Abstract

After some stage-setting in Section 1, Section 2 presents a proof offered by Thomas Forster (in his book *Reasoning About Theoretical Entities*) to show that procedure offered for eliminating definite descriptions from a certain range of formulas always yields a description-free equivalent for any given description-containing formula. (This equivalent amounts to a formula in which the description in question has been given the broadest possible scope.) In Section 3, we show that the inductive case of disjunction in this proof (by induction on formula complexity) does not go through as claimed, and that the result itself is not correct. In Section 4 we look at some similarities and contrasts between Forster's proposed elimination procedure and one emerging more directly from one prominent strand – the binary quantifier approach – in the Russellian legacy. This leads us, on a more positive note, to a few observations about a class of truth-functions intimately connected with that range of contexts in which the descriptive binary quantifier is "scope-indifferent" – the falsity-preserving functions – from which we pass, by way of conclusion, to a corrective reformulation suggested by that discussion for the description-as-terms treatment of Forster's discussion.

Keywords: Definite descriptions, disjunction, Boolean clones

Introduction

We focus on one aspect of a treatment of definite descriptions to be found in Chapter 2 of Forster [9], familiarity with which will not be assumed. The relevant passages will be quoted in full, and in any case, it is a treatment of a kind which might independently occur to anyone curious about, as we might put it, variations on a theme of Russell – whether or not they were pursuing the project Forster sets himself in the main body of that work. Indeed, after the present introductory section, no further mention of that project will be made.

Our main concern will be to draw attention to a problem in this treatment (Section 2), and then to look into some issues raised by a variant approach avoiding the problem in question (Section 3). But, just to give the background for those wondering about it, we include in this section a very brief summary, before getting down to the business of outlining that treatment and a difficulty that it faces. Other readers can skip straight to Section 1 for the needed exposition of [9], without missing anything relied on later.

Forster's main concern in [9] is in interpretations as recursive (or structure-respecting) translations from one language into another, in the main cases achieving this recursivity by, as he puts it, commuting with the quantifiers and connectives. On p. 13 of this book he writes: "An interpretation of $\mathcal{L}_1$ into $\mathcal{L}_2$ that sends equality to equality is an *implementation.* All other interpretations are *simulations.*" His main focus is on the latter and of the status, from the perspective of $\mathcal{L}_2$, as 'virtual objects', of the would-be referents of the terms of $\mathcal{L}_1$ which are translated away in the case of a simulation. As a warm-up exercise, Forster presents a Russell-inspired interpretation which counts as an implementation rather than a simulation: definite descriptions, as terms, are translated away as we pass from a language containing to a language lacking them, but there is no tendency to think of what they might denote (in the case of *proper* definite descriptions) as merely virtual objects.[1] It is with one aspect of this initial exercise, of considerable interest in its own right, that the present discussion is concerned, rather than with the general program it is being used to introduce. The argument referred to in our title is an inductive argument (given as the proof of Forster's Theorem 2.1) which, we will be suggesting, goes wrong in a fairly subtle way, especially when compared with a similar but slightly different argument coming out of the broadly Russellian tradition (Section 3). Indeed one might draw from that comparison the conclusion that the spirit of Forster's treatment is on the right lines, even though its exact articulation goes off track.

There may be good reason for readers of [9] to get clear about how this interpretation (in Forster's specialized sense) is meant to work, since it is the only interpretation result which is worked out in full detail in the book. We shall see in Sections 1 and 2 below, that although the idea of Foster's Theorem 2.1 is ingenious – restricting attention there to a class of formulas for which the contrast between narrow and broad scope for descriptions is (in effect) claimed to make no difference, permitting a relatively uniform elimination-of-descriptions recipe – the details actually supplied leave something to be desired. Further, definite descriptions are appealed to 'implicitly' throughout [9], both to define

[1]He we use the special terminology of *denoting* as opposed to *referring to* employed in Russell [30] to avoid the suggestion that it is the ('logically proper') name–bearer relation of reference that is involved.

function symbols (on p. 22) and in the characterisation of virtual entities (on p. 30). Given that the latter are meant to be Forster's main object of study, it would be good if this case study worked out well, though the question of whether or not the difficulty we find with his account of definite descriptions in any way vitiates the overall project, we leave to those with a special interest in that project to determine. Forster's own view would seem to be that the details to which we will be attending have no such significance for that project, since at the start of Chapter 2 of [9] he writes: "Readers who are completely confident about Russell's theory of descriptions can probably safely skip the rest of this chapter."[2] As already emphasized, however, we are concerned in the following two sections not with the general project but precisely with the details of the treatment provided of descriptions, and the instructive mistake arising in the course of that treatment. Section 1 summarizes Forster's exposition in Chapter 2 of [9], and includes some incidental critical commentary on it. Section 2 presents the main problem – of considerable interest in its own right – with Forster's treatment, and, as already mentioned, there are some comparative remarks on what might be called a more mainstream contemporary descendant (the binary quantifier approach) of the Russellian tradition in Section 3.

1 Descriptions

On p. 12 of [9], before the chapter – Chapter 2, pp. 17–25 – on definite descriptions begins, Forster writes that "Following Russell, we use upside-down iotas for singular descriptions," but to make life easier let us follow Prior [26], or,

[2]We might add here: most readers regarding themselves as *completely* confident about Russell's theory should probably have a look at Kuhn [20] and Kripke [18]. Forster himself has some discussion going beyond the received wisdom on Russell, writing on p. 18: "If one wants to think of Russell's account of ι-terms as a thesis about the logical structure of ordinary language, this compels one to say that the (top level) logical structure of 'the King of France is bald' is an existential quantification. According to this view, ι-terms are syntactic sugar and do not really denote. However, we are not obliged to adhere to this purist view: in any situation in which an ι-term can legitimately be introduced there is a *canonical* implementation for it." This talk of introducing ι-terms only when legitimate suggests the Hilbert–Bernays approach (see Gratzl [13] for references and discussion and Gratzl [14] for aspects of what the 'purist' view involves) but on the following page Forster tells us that it is not a path he intends to go down, saying instead that "[t]he obvious question of what one is to make of formulæ containing embedded definite descriptions whose introduction has not been authorised by a theorem in the style $\exists! x\phi(x)$ is largely sidestepped here by theorem 2.1". We return to the theorem in question in the following section. Apropos of the non-purist approach, to which Forster here implies he is in general (if not for present purposes) sympathetic, we should mention that talk in the text of eliminating descriptions (as terms) is not itself intended to suggest that it would be a good idea to 'get rid of' such terms, but simply to refer to the provision of description-free equivalents for sentences containing them.

for that matter, the more recent Neale [22], or Kuhn [20], and use a simple uninverted iota ("ι") for this purpose.[3] We use this notation even in giving direct quotations from Forster, such as the following (from p. 19 – and indeed did so already in note 2):

> Suppose we have a language $\mathcal{L}$ which does not contain singular descriptions. We want to show how to interpret in $\mathcal{L}$ the language $\mathcal{L}^\iota$, the result of adding singular descriptions to $\mathcal{L}$.
>
> **Definition 2.1** Let $\mathcal{I}$ be the interpretation $\mathcal{L}^\iota \longrightarrow \mathcal{L}$ defined as follows. The recursive clauses are that $\mathcal{I}$ commutes with quantifiers and connectives. Atomic and negatomic formulas are sent to themselves unless they contain singular descriptions. Let us consider this case in more detail.
>
> If 'x' is free in ϕ then for atomic or negatomic Ψ with 'y' free, $\mathcal{I}$ of $\Psi[\iota x_{\bullet}\phi(x)/y]$ is
>
> $$(\exists x)(\phi(x) \wedge \Psi[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x)).$$

Forster goes on to say what this becomes in the case that Ψ is an equation, and to emphasize that the order in which we expand the ι-terms in $\phi(\iota x(A(x), \iota y B(y)))$ does not matter in the sense that the results will be (logically[4]) equivalent. Presumably the above definition should be modified in order to take into account multiple ι-terms replacing variables (playing the 'y' role) within the same atomic formula; this and several other complications, mentioned in the final paragraph of this section, do not affect the main problem raised in Section 2.[5] Next, on p. 20*f.* we have the following passage,[6] in which

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[3]Since the regular iota is no longer used for denoting unit sets, no confusion is in danger of arising. We will return to "$\imath$" for authenticity when quoting from and discussing Russell directly in Section 3, in which there also appears a Digression with further details on notational matters.

[4]Since Forster makes no mention of non-classical logic, the presumption should be that we use classical predicate logic when formulas of $\mathcal{L}$ are concerned and stay as close to classical predicate logic as possible for $\mathcal{L}^\iota$ – certainly not deviating from classical propositional logic, even if some free-logical restrictions were to be imposed on universal instantiation (or $\forall$-elimination) moves occasioned by the presence of ι-terms.

[5]The exact formation rules of the language Forster calls $\mathcal{L}^\iota$ are not stated, but clearly its formulas are not just those satisfying the restrictive conditions of Theorem 2.1 below, which do not allow, for example, quantifiers binding variables within ι-terms, whereas any use of definite descriptions to understand function symbols, such as that mentioned in Section as appearing on p. 22 of [9] – and particularly emphasized Smiley [32] – requires us to be able to say such things as: $\forall x(x < f(x))$. When Forster explains this on p. 22, he stresses that "nothing in theorem 2.1 depends on ϕ containing no free variables", which can then be bound by outlying quantifiers: it's just that the simple elimination-of-ι-terms recipe provided by Theorem 2.1 does not deliver a formula $\mathcal{I}(\alpha)$ from a formula α containing such outlying quantifiers. Perhaps Forster envisages extending $\mathcal{I}$ in such a way that $\mathcal{I}(\forall v\beta) = \forall v\mathcal{I}(\beta)$.

[6]Naturally, we are forced to reproduce here the duplicate numbering used by Forster, with

we take its opening reference to 'induction on $\mathcal{L}$', to be to induction on the complexity of formulas of $\mathcal{L}$, for some suitable measure of complexity (an issue to which we return below):

> We can now prove by induction on $\mathcal{L}$ that:
>
> **Theorem 2.1** *Suppose ϕ is built up from atomics and negatomics by means of $\wedge$, $\vee$, $\forall$, and $\exists$. Suppose also that 'x' is free in ϕ and no variable free in ϕ is bound by any quantifier in Ψ. Then for any Ψ with 'y' free, $\mathcal{I}$ of $\Psi[\iota x.\phi(x)/y]$ is logically equivalent to*
>
> $$(\exists x)(\phi(x) \wedge \Psi[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$$
>
> *Proof*:
>
> This is proved by induction on Ψ. The base case (Ψ atomic or negatomic) follows immediately from definition 2.1. The inductive steps for the quantifiers and connectives are as follows.
>
> Ψ is $\Psi_1 \vee \Psi_2$
>
> Suppose 'x' is free in ϕ and no variable free in ϕ is bound by any quantifier in Ψ_1 or Ψ_2. By induction hypothesis $\Psi_1[\iota x.\phi(x)/y]$ is equivalent to $(\exists x)(\phi(x) \wedge \Psi_1[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$ and $\Psi_2[\iota x.\phi(x)/y]$ is equivalent to $(\exists x)(\phi(x) \wedge \Psi_2[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$.
>
> Therefore $(\Psi_1 \vee \Psi_2)[\iota x.\phi(x)/y]$ is equivalent to:
>
> $$(\exists x_1)(\phi(x_1) \wedge \Psi_1[x_1/y] \wedge (\forall y)(\phi(y) \rightarrow y = x_1)) \vee$$
> $$(\exists x_2)(\phi(x_2) \wedge \Psi_2[x_2/y] \wedge (\forall y)(\phi(y) \rightarrow y = x_2))$$
>
> By the uniqueness condition any witness to '$\exists x_1$' must also satisfy the first and third clauses of the second disjunct so the formula simplifies to
>
> $$(\exists x)(\phi(x) \wedge (\Psi_1 \vee \Psi_2)[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$$
>
> as desired.

The proof continues with the inductive cases presented by conjunction, the universal quantifier, and the existential quantifier – the last case not being treated in detail but being said to be similar to that of the universal quantifier "but easier". ([9], p. 32. Incidentally, as Forster explains after the proof, the reason for the restriction in the formulation of the Theorem, that "no variable free in ϕ is bound by any quantifier in Ψ" is that such variables would then become free variables in $\mathcal{I}\big(\Psi[\iota x.\phi(x)/y]\big)$.) However, since the problematic inductive case, as we shall see, is that concerning disjunction, which is covered in the quoted passage, we can stop there. We can look more closely at this case

the "2.1" as the number for the above definition and for the theorem below (and indeed for the section in which both appear).

in the following section, before which there are three preliminary matters to attend to. The first concerns the slash notation, in such things as "$\Psi[\iota x\phi(x)/y]$" and "$\Psi[x/y]$"; we are taking it – nothing is explicitly said about this in [9] – that in general for a term t (which in the current cases is either an ι-expression or a variable) $\Psi[t/y]$ is to be the result of replacing all occurrences of y in Ψ by the term t. Whether this "all" should be replaced by the word "any" to make room for the option that there are no such occurrences is a further issue to return to in Section 2.[7] The second preliminary point is that dividing up the existence and uniqueness parts of the ι-free translations and having the Ψ part sit between them makes the discussion harder to follow, so we will do all this at the front of the translation,[8] replacing:

$$(\exists x)(\phi(x) \wedge \Psi[x/y] \wedge (\forall y)(\phi(y) \to y = x))$$

with

$$(\exists x)(\phi(x) \wedge (\forall y)(\phi(y) \to y = x) \wedge \Psi[x/y]).$$

This allows us to abbreviate further, when convenient, to:

$$\exists x(\phi!(x) \wedge \Psi[x/y]).$$

Here we use a notation from Kripke [19], p. 33, for example, in which the "!" is shifted from making a uniqueness quantifier – "$\exists! x(Fx)$" for "there exists exactly one thing which is F" – to making a uniqueness-demanding predicate – "$\exists x(F!x)$" for "there exists something which is uniquely F". We have taken the further liberty of dropping the unnecessary parentheses around "$\exists x$" that appear in Forster's presentation.[9]

Thirdly, we need to attend to a cluster of issues about the inductive organization of the proof of Theorem 2.1, the first part of which we saw above. The 'suitable measure of complexity' alluded to before the result was quoted would appear to be the number of occurrences of $\wedge$, $\vee$, $\forall$, and $\exists$ used in the construction of the formula in question. But what is the formula in question? Peculiarly, the statement of the result begins by asking us to suppose that ϕ is constructed from atomic and negated atomic formulas by means of these four logical devices. But then the proof of the result is described as proceeding by

[7] See the discussion of 'dummying in' there.

[8] This is what Forster does on the first page of Chapter 2. It is not clear why he later changes it to the ordering we are now changing back from.

[9] Incidentally, Kripke [19] also uses the parenthesized existential quantifier notation. However, it should be recalled that the material in question was presented – in that same notation – on the original 1973 occasion of the lectures concerned, and uses the "(x)" notation for the universal quantifier, where the parentheses are integral to the notation, making them desirable in the $\exists$ case for parity.

induction on (the complexity of) Ψ, rather than of ϕ. Since that is how the proof does indeed proceed, we might do well to take the initial supposition on the construction of ϕ to be a typographical slip, and replace it with Ψ – since no corresponding assumptions about ϕ seem to be called on in the proof at all. On the other hand, possibly Forster intends both Ψ and ϕ to be constructed only using the devices in question. Note in passing that either way, Definition 2.1 does not succeed in defining an interpretation of $\mathcal{L}^\iota$ in $\mathcal{L}$ as it stands, since $\mathcal{I}$ has only been defined for a fragment of $\mathcal{L}^\iota$ conforming to whatever restrictions Forster may have had in mind. (On this see the second half of note 11 below.) Indeed, why do we have the upper case "Ψ" alongside the lower case "ϕ" anyway? Both are schematic letters for syntactic objects of the same kind – (possibly open) formulas – after all. A lower case "ψ" did appear on the first page (p. 17) of the chapter on descriptions, with a reference to the expression "$\phi((\iota x)(\psi(x)))$" (which, in the notation used in the passages quoted above, becomes "$\phi(\iota x\,.\,\psi(x))$"), and here as well as the change of case, the roles of ϕ and ψ are the other way around, the latter rather than the former being used to construct the ι-term. Perhaps this explains the later confusion about whether the proof of Theorem 2.1 proceeds by induction on the construction of ϕ or of Ψ.[10]

2 The Disjunction Issue

We come now to the main flaw we want to focus on in the proof quoted in the preceding section of Theorem 2.1. Forster remarks of it ([9], p. 22), that "this inductive proof will not work for $\neg$ and $\rightarrow$" and that accordingly formulas constructed with their aid "are not covered by this result." We return to this comment in Section 3, here emphasizing instead that in fact there is already a problem about $\vee$, a problem of a kind somewhat reminiscent of those aired in

[10]Another somewhat murky aspect of at least the exposition of the inductive proof of Theorem 2.1 concerns the basis case. It is to this that the remark mentioned above (between quoting Definition 2.1 and Theorem 2.1) is addressed, concerning the insignificance of the order in which we expand the ι-terms in $\phi(\iota x\,.\,A(x), \iota y\,.\,B(y))$. And not explicitly attended to at all is the fact ι-terms may themselves contain ι-terms in which they bind a variable, which may in turn... (This is rather surprising in view of the explicit attention given on p. 19 of [9] to the non-equivalence of $\exists!x\exists!y\phi(x,y)$ and $\exists!y\exists!x\phi(x,y)$.) Some complications arising for such descriptions within descriptions are canvassed in Kuhn [20]; they turn out to be problematic for the use of an duplicated iota-term used to indicate the scope of the description, to be found in Whitehead and Russell [37]: see the following section. Kripke [18] also notes that multiple occurrences of such terms provide one among several interesting potential obstacles to showing that such terms can always be eliminated, though according to Grabmayer et al. [12] these obstacles are merely apparent. Let us set aside all such complications here, since whatever is said about them, the difficulty raised in the following section needs only a single occurrence of one ι-term to illustrate it.

A. N. Prior's discussion of his modal logic Q (from [26], Chapter 5, esp. p. 49), where the problem arises specifically in modal contexts).

For a counterexample to Theorem 2.1, using (invisible "=" aside) only monadic predicate letters F, G, H, take $\Psi(y)$ as $Gy \vee \exists z(Hz)$ and for our ι-term $\iota x.Fx$. So $\mathcal{I}\big(\Psi[\iota x.\phi(x)/y]\big)$ is $\mathcal{I}\big(G\iota x.Fx \vee \exists z(Hz)\big)$, which, since $\mathcal{I}$ commutes with the connectives, is $\mathcal{I}\big(G\iota x.Fx\big) \vee \mathcal{I}\big(\exists z(Hz)\big)$, i.e., $\mathcal{I}\big(G\iota x.Fx\big) \vee \exists z(Hz)$, which is to say, the formula $(*)$ on the left below.[11] On the other hand, according to Theorem 2.1, $\mathcal{I}\big(\Psi[\iota x.\phi(x)/y]\big)$ should be logically equivalent to $\exists x(\phi!(x) \wedge \Psi[x/y])$, giving us in the present case the formula $(**)$ on the right below:

$$(*)\quad \exists x(F!x \wedge Gx) \vee \exists z(Hz) \qquad\qquad (**)\quad \exists x(F!x \wedge (Gx \vee \exists z(Hz)))$$

Since $(*)$ and $(**)$ are evidently not equivalent – the former following from the truth of its second disjunct regardless of whether there is a unique F (be it G or otherwise) – Forster's Theorem 2.1 is not correct.

Such examples suggest the following diagnosis of the fault in the inductive case for Ψ being of the form $\Psi_1 \vee \Psi_2$. Using now the more concise notation of the above discussion, the proof continued by suggesting that the inductive hypothesis allowed us to conclude that $\Psi_1[\iota x.\phi(x)/y]$ and $\Psi_2[\iota x.\phi(x)/y]$ were respectively equivalent to $\exists x(\phi!x \wedge \Psi_1(x))$ and $\exists x(\phi!x \wedge \Psi_2(x))$, and the argument continued from here to note the equivalence of the disjunction of these two to $\exists x(\phi!x \wedge (\Psi_1(x) \vee \Psi_2(x)))$, the desired conclusion. But the appeal to the inductive hypothesis is over-hasty, since for the original formula $\Psi = \Psi_1 \vee \Psi_2$ to fall under the conditions of the Theorem, in respect of having a free occurrence of the variable y in it, y does not need to be free in each of the disjuncts – only in at least one of them. In the above $(*)/(**)$ example, our disjunctive Ψ had y free in the first disjunct ("Gy") but not the second ("$\exists z(Hz)$"), so when occurrences of y are replaced by the ι-term (in our example, $\iota x.Fx$) this term finds its way only into the first disjunct, and not the second, blocking the collapsing of the two $\exists x(\phi!x \wedge \_\_)$ prefixes into a single one. (This does not cause a similar difficulty for the case of $\Psi = \Psi_1 \wedge \Psi_2$, as the interested reader

[11] The phrase "commutes with the connectives" here is adapted from Forster's Definition 2.1 quoted in the previous section: "$\mathcal{I}$ commutes with quantifiers and connectives," in which the reference to connectives is presumably just to $\wedge$ and $\vee$, since $\rightarrow$ has been banned altogether and $\neg$ confined to pre-atomic position. (And the reference to quantifiers similarly excludes any quantifier binding a variable in the ι-term.) These restrictions mean that $\mathcal{I}$ is not quite what the start of Definition 2.1 suggests it is: a (total) function from the set of $\mathcal{L}^\iota$ formulas into $\mathcal{L}$. After saying that formulas involving $\rightarrow$ and (non-preatomic) $\neg$ "are not covered by this result," (= Theorem 2.1), Forster writes (p. 22) "However, every formula is classically equivalent to one that is," which may be intended to show that the fragment of $\mathcal{L}^\iota$ which constitutes the actual domain of $\mathcal{I}$ contains a formula equivalent to any given $\mathcal{L}^\iota$, though the status of the 'no variable free in ϕ is bound by any quantifier in Ψ' restriction remains unclear in this connection.

is encouraged to verify,[12] or for the inductive case presented by the universal quantifier. More interestingly, there is no problem for the inductive case of the existential quantifier. In particular, it does not face the same obstacle as that for $\vee$, even though existentially quantified formulas are often thought of as tantamount to – possibly infinite – disjunctions of their instances, on the presumption that everything in the domain under discussion has a term denoting it. This is because the above problem for $\vee$ arises from a 'mixed disjunction' in which the variable y occurs in one but not the other disjunct, while the disjunction-of-instances just envisaged has y free in all disjuncts if it has y free in any of them. There is something quite special, among disjunctions, about the – infinite or otherwise – disjunctions which the analogy between $\exists$ and $\vee$ invites us to consider.[13])

These diagnostic remarks may provoke the reaction that it cannot matter whether we explicitly insist that when we write $\Psi(y)$ as $\Psi_1 \vee \Psi_2$ the latter is itself $\Psi_1(y) \vee \Psi_2(y)$ with the parenthesized variable indicating an actual occurrence in the subformula in question, since if there is no such occurrence to begin with we can always 'dummy one in', introducing a vacuous occurrence keeping the formula equivalent to the original y-less formula. For example, we could replace a Ψ_2 not containing any such free occurrence with the formula $\Psi_2 \wedge y = y$, or, plucking a monadic predicate letter J out of the air, rewrite Ψ_2 as $(\Psi_2 \wedge Jy) \vee \Psi_2$, or dually, as $(\Psi_2 \vee Jy) \wedge \Psi_2$.[14] Thus, it might be thought, it does not matter whether we understand notations like $\Psi_i(y)$, for $i = 1, 2$, as, on the one hand, allowing for a free occurrence of "y" free in Ψ_i ready to be replaced by the ι-term or, on the other hand, as requiring that there be at least one such occurring free: dummying in allows us to boost "zero occurrences" to "one occurrence" with no change in content.

Replying to this reaction requires attention to the particular proposal for how the dummying-in is to proceed, because different ways of introducing inessential occurrences of "y" have different consequences (vis-à-vis Theorem 2.1) when these occurrences are replaced by ι-terms. From the $(*)/(**)$ example above, however, it is clear enough *a priori* that a hitch will arise somewhere or other in the procedure. For example, trading in Ψ_2 for $\Psi_2 \wedge y = y$ means that the results of replacing all free occurrences of y by $\iota x.\phi(x)$ in the original

[12] If, instead of the Russellian gloss on $G(\iota x.Fx)$ as $\exists x(F!x \wedge Gx)$, we went for the (as we might say) dual interpretation suggested in Bacon [1] – as $\forall x(F!x \to Gx)$ – then the difficulties would be reversed: now the problem arises from what below we shall call descriptively lopsided conjunction rather than disjunctions, in view of the non-equivalence of $\forall x(\phi!(x) \to \Psi_1(x)) \wedge \Psi_2$ with $\forall x(\phi!(x) \to (\Psi_1(x) \wedge \Psi_2))$.

[13] A corresponding point concerning the analogy between $\forall$ and $\wedge$ is discussed at p. 227*f.* of [6].

[14] The use of the absorption equivalences for such 'dummying in' purposes, as well as the description of the process in these terms, can be found in Dunn [7], p. 352.

and the new formula are the following pair, which are anything but equivalent: Ψ_2 itself in the first case, and $\Psi_2 \wedge \iota x.\phi(x) = \iota x.\phi(x)$ in the second, the latter but not the former implying the existence of a unique ϕ.[15] Consider next one of the absorption equivalences: that of Ψ_2 with $(\Psi_2 \wedge Jy) \vee \Psi_2$, which, on inserting ι-terms, becomes $\big(\Psi_2 \wedge J(\iota x.\phi(x))\big) \vee \Psi_2$. Since this equivalence is sanctioned by classical logic (see note 4), there is no question of blocking it and what we have is another counterexample to Theorem 2.1. The first disjunct is equivalent to

$$\exists x\big(\phi!(x) \wedge (\Psi_2 \wedge Jx)\big) \vee \Psi_2,$$

but when we try to put the whole formula into the gap in "$\exists x(\phi!(x) \wedge \ldots)$" we are blocked exactly as in the $(*)/(**)$ example when we encountered a disjunction (namely, $G\iota x.Fx \vee \exists z(Hz)$) with a lopsided occurrence of the ι-term – having it occur, that is, in only one of the disjuncts. The other absorption law case dummied in "y" via $(\Psi_2 \vee Jy) \wedge \Psi_2$ and here the trouble sets in even earlier when an ι-term replaces "y" since we have the 'descriptively lopsided' disjunction problem immediately in the first conjunct.

At this point, the question arises as to what options there might be for repairing Forster's Theorem 2.1. Since at least one such option is most conveniently raised after we have had a look at a somewhat more orthodox modern version of a Russellian theory of descriptions, we postpone this question until the end of the following section.

Let us instead conclude the present discussion with what might be called an *a priori* argument against a combination of claims by Forster which reveals that the problem posed by disjunction for Theorem 2.1 could be detected without reference to a specific (kind of) counterexample, but simply on the basis of knowing already that $\rightarrow$ presented a problem for the inductive step. The two claims in question are (i) that $\rightarrow$ would present an obstacle to his inductive argument, and (ii) that $\vee$ presents no such obstacle, where the induction is on the number of occurrences of binary connectives and quantifiers in a formula constructed from atomic and negated atomic formulas by means of those logical devices.[16] Forster says that if the inventory of logical operators permitted is $\{\wedge, \vee, \forall, \exists\}$ one has every formula in negation normal form (i.e., negation applied only to atomic subformulas) equivalent to a formula in which a particular description has a single maximally broad scope occurrence, while

[15] At least we presume so. The instructions on p. 19 of [9], not quoted above, for how to deal with identity statements are not entirely clear. See also the discussion of $\iota x\phi = \iota x\phi$ in the middle of p. 145 of Kuhn [20]; as Kuhn reminds us, Geach [10] had pointed out that the general case of $R(\iota x\phi)(\iota x\phi)$ already raises problems for Russell – scope indicators and all.

[16] Claim (i) is made in the passage quoted from [9] in the opening paragraph of this section, as well as in note 11 above, and claim (ii) is made in the course of the argument given for Theorem 2.1: this is the inductive case for $\vee$.

this is not so if the inventory is extended to $\{\wedge, \vee, \rightarrow\ \forall, \exists\}$. Well, Forster's Theorem 2.1 does not quite put it in these terms since the derived formula is not one in which the ι-term in question appears at all – but is equivalent to one ("$\exists x(\phi!(x) \wedge \_)$") which spells out the wide scope reading, all the rest of the formula filling the blank indicated. (We return to the broad/narrow scope terminology in the following section.) Let us now show that the combination of claims (i) and (ii) does not constitute a consistent position.

The inductive hypothesis for α of the form $\beta \rightarrow \gamma$ is that each of β, γ, having fewer binary connectives or quantifiers than α, satisfies the condition. Consider the formula $\neg\beta\vee\alpha$, or rather the formula $\beta'\vee\gamma$, in which β' is a negation normal form replacement for $\neg\beta$, obtained from the latter by driving the outer negation inwards using De Morgan's Laws and the corresponding equivalence for $\rightarrow$ (i.e., the equivalence of $\neg(\delta \rightarrow \varepsilon)$ with $\delta \wedge \neg\varepsilon$), as well as the corresponding moves with quantifiers ($\neg\,\forall$ to $\exists\,\neg$, $\neg\,\exists$ to $\forall\,\neg$) until it lands one or more unnegated or negated atomic formulas – in the latter case vanishing with that preatomic occurrences of $\neg$ by appeal to the Double Negation equivalence. This process does not increase the number of binary connectives or quantifiers in $\neg\beta$, which is the same as the number of the number of such operators in β. So by the inductive hypothesis each of β' and γ has an equivalent in which the description in question has only a single maximally broad occurrence, so if the induction step for $\vee$ worked, we should have this result for $\beta' \vee \gamma$ – a formula which is equivalent to $\beta \rightarrow \gamma$, i.e., to α.

3 Forster-Inspired Russellian Explorations

Although Russell's original notation for descriptions featured a definite description appearing in singular term position, it also featured a duplicated version of that notation appearing earlier in a formula to indicate its scope. It was found – call this Fact 1 – that when the description was proper (i.e., the condition involves had a unique satisfier), the two versions were equivalent, at least when attention was restricted to contexts involving the truth-functional connectives, and – let us call this Fact 2 – that, with the same contextual restriction in force, the version in which the description had broader scope always (logically) implied, though was not necessarily implied by the version in which the description had narrower scope.[17] To avoid complications and because they were not the source of the difficulty in Section 2, we consider truth-functional ontexts only, ignoring quantificational contexts (as well as those induced by non-truth-functional connectives, touched on briefly in note 21). The double

[17]The facts described as familiar here and up to Proposition 3.2 below, can be found in Whitehead and Russell [37], pp. 65–71 and 173–186, with Fact 1 appearing (essentially), for example, as ✳14·3 on p. 185, and Smullyan [33], Kripke [18].

use of iota-expressions, both in singular term position and as scope indicators, is rather cumbersome, and several people noticed that the heart of Russell's theory could be conveyed more clearly by parsing definite descriptions as binary quantifiers. One early proponent of this approach was Prior ([27], final paragraph), who wrote "$(\iota x \,\centerdot\, Fx)Gx$", to be read: "for the unique x such that Fx, Gx". The same idea evidently occurred to others in the 1960s, Quine casually referring to it ([29] p. 327) as "Richard Sharvy's neat notation". Both Prior and Sharvy actually write an inverted iota here, and, as announced in note 3, for historical reasons we revert to that notation in the present section.[18] We follow a fairly common practice of using $\mathsf{I}x(Fx, Gx)$ for this, taking it either as an abbreviation for "$\exists x(F!x \wedge Gx)$" or else as a primitive construction so interpreted as to be equivalent to that.

Digression on Notation. Chapter 4 of Bostock [5] has (save for font choice) "$\mathsf{I}x\!:\!Fx(Gx)$" for this. Neale [22], p. 45, writes "[the x: Fx](Gx), and remarks that "this is not to propose an alternative to Russell's theory; it is just to find a more congenial way of stating it." That is right, if Russell's theory is simply Russell's account of the truth-conditions of sentences containing descriptions. There is more going on in ✳14 of [37], though. In particular there is a desire to show that on the hypothesis that the description is proper, occurrences of closed iota-terms participate in logical inferences much as they would if they were individual constants. For example, on p. 180 of [37], one reads "The above proposition shows that, provided $(℩x)(\phi x)$ exists, it has (speaking formally) all the logical properties of symbols which directly represent objects." Fact 1 above – repeated in 'headline form' below – is just one example of this. Evans [8], p. 58 has (save for choice of font and lettering) "$\mathsf{I}x[Fx; Gx]$"; Blamey [3], p. 279 has "$\mathsf{I}x[Fx, Gx]$". (The latter is treated there in a non-classical logic as true or false, respectively, depending as there is a unique F which is, or is not G, and as undefined in the absence of such an F. This all goes back to Blamey [2].) The capital "I" idea probably originated – though in the term-forming role rather than as a binary quantifier – in boldface, in Scott [31], amusing his readers with the following remark (p. 182): "Note that **I** is an *inverted* capital **I**, which the author prefers to Russell's inverted iota." Since we are discussing notation, let us recall that at the start of Section 1 above, we quoted Forster as saying that he was following Russell with "upside-down iotas" – which is actually a bit misleading: when these symbols appear on p. 17, upside down they may be but we turn out not to have Russell's notation ("℩"), in which the inverted iota is *rotated* through 180°, but something not previously encountered in polite society, in which the iota is *reflected* through 180°, so that the curved tail of the letter, now at its top, swerves disconcertingly to the right ("ɿ$x\phi(x)$")

[18] A more comprehensive list of those taking the binary quantifier route (together with detailed references) can be found in note 60 on p. 60 of Neale [22].

instead of the left ("$\imath x\phi x$"). Finally, it is apparently possible to confuse the term-forming and the quantificational uses of ι (or I), inverted or otherwise, as one sees in Horsten [15]. Here one encounters, on pp. 355 and 358, several representations in the style of the following, for 'The King of France is bald': $(\iota x)(KFx \wedge Bx)$, where what is intended is presumably $B(\iota x)(KFx)$ – here retaining Horsten's "(ιx)" with its parentheses. What is actually written is either a term ("the unique thing which is both a King of France and bald") or an attempt at a quantificational usage but with a singulary rather than binary quantifier ("for the unique thing in existence, that thing is a bald King of France"). ***End of Digression.***

Returning to the two Facts mentioned from the opening paragraph of this section, let us state them for visibility in a simple 'slogan' form:

Fact 1: Proper $\Rightarrow$ (Broad $\Leftrightarrow$ Narrow)

Fact 2: Broad $\Rightarrow$ Narrow

We can think about these issues most conveniently when the issue of broad-vs.-narrow scope arises for the definite description (regimented as a binary quantifier) and an n-ary sentence connective $\#$:

$$\begin{array}{lll} \textsc{Broad is} & \mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n)) & \text{and} \\ \textsc{Narrow is} & \#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n)). & \end{array}$$

Note, especially in view of our findings in Section 2, that while the variable v may occur free in some (or all) of the ψ_i, it need not occur free in all (or any) of them.

Example 3.1 *Taking $n = 2$ with $\#$ as $\vee$, and $\phi(v), \psi_1, \psi_2$ as (monadic) Fx, Gx, Hx, our Broad and Narrow forms are $\mathsf{I}x(Fx, Gx \vee Hx)$ and $\mathsf{I}x(Fx, Gx) \vee \mathsf{I}x(Fx, Hx)$, which would appear according to the conventions of Whitehead and Russell [37] as respectively*

$$[(\imath x)(Fx)](G(\imath x)(Fx) \vee H(\imath x)(Fx))$$

and

$$[(\imath x)(Fx)](G(\imath x)(Fx) \vee [(\imath x)(Fx)]H(\imath x)(Fx)).$$

Fleshing out Facts 1 and 2 for the general case, we have, then:

Fact 1: $\exists v(\phi!(v)) \Rightarrow \big(\mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n)) \Leftrightarrow \#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n))\big)$.

Fact 2: $\mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n)) \Rightarrow \#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n))$

Here and below the notations $\alpha \Rightarrow \beta$, $\alpha \Rightarrow (\beta \Leftrightarrow \gamma)$, $\alpha \Leftrightarrow \beta$ mean that all formulas of the forms $\alpha \to \beta$ and $\alpha \to (\beta \leftrightarrow \gamma)$, $\alpha \leftrightarrow \beta$, respectively, are true in all first-order structures (models, interpretations,...).[19]

Why is Fact 1 a fact? Without going to the trouble of setting up all the semantic apparatus needed for a rigorous proof, we can see that the first antecedent, "$\exists v(\phi!(v))$" means that if a traditional ι-term were being used, it would essentially be functioning like an individual constant picking out a single individual in any given structure – the unique satisfier of the condition $\phi(v)$, and so the Russellian scope indicators would play no disambiguating role ("names are scopeless"[20]). But that means the scope disambiguation effected by the Broad/Narrow contrast is also doing no work: the two sides are equivalent. This is why we have to disallow (as noted in Smullyan [33]) such things as modal operators or modalized truth-functional connectives as substituends for #: here the predicates do not have a single fixed extension to consider as traditional first-order structures give way to Kripke models or the like in which even proper description singling out an individual in a given structure may not surviving as the unique satisfier of the condition involved as we pass through modal vocabulary invoking other structures (associated with other possible worlds) where the predicates have different extensions.[21]

[19]We use this notation, left over from the sloganistic formulations above, simply because it is easy on the eye. Strictly we should, rather than writing (for example) "$\alpha \Rightarrow \beta \Leftrightarrow \gamma$", write instead "$\alpha \vdash \beta \leftrightarrow \gamma$", or indeed, if purifying away the appeal to specific connectives is the order of the day, write "$\alpha, \beta \vdash \gamma$" and $\alpha, \gamma \vdash \beta$". (Here $\vdash$ is the consequence relation of first-order logic supplemented with binary descriptive quantifiers.)

[20]We presume some precisification of this slogan is correct. For sources as well as qualms, see note 27 on p. 60 of Evans [8]; Evans himself discusses (p. 37) the possibility of giving scope to potentially non-denoting (but weakly rigid) names at p. 37, using a notation associated perhaps with Hughes and Londey (and Toomas Karmo). That notation can even be extended to individual variables so that the issue of scope arises for them also: see Prior [28], p. 228. (Compare Wiggins [38] p, 284: "Well, nobody proposes that *variables* should have a scope.") A nuanced version of the scopelessness thesis for genuine proper names can be found on p. 100 of Blamey [2]: "The interest of all this is that the 'more scopeless' a quantifier expression is, the more nearly it 'approximates to a singular term' for the language in question." One could refine the present discussion somewhat by making a distinction drawn (not necessarily using this terminology) by Gareth Evans in the Evans–McDowell philosophy of language seminars at Oxford in the 1970s, between *scopeless* expressions, for which the notion of scope does not arise, and *scope-indifferent* expressions, to which the notion of scope applies but different ways of fixing the expression's scope in any given sentence all yield equivalent results.

[21]Along similar lines, in Section 2.3 of [9], Forster adds (a necessity operator) $\Box$ to his underlying language and suggest that the cases for which a result like his Theorem 2.1 fails – the cases in which the definite description has narrow scope relative to the occurrence of $\Box$ – are to be subsumed as violating the restriction that "no variable free in ϕ is bound by any quantifier in Ψ", on the grounds that $\Box$-formulas are really of a form in which a free variable ranging over possible worlds is bound by an outer universal quantifier. We were rather surprised to read – [9], p. 23*f.* – that "(e)ver since Kripke it has been almost universally

Taking Fact 1 for granted, assume the antecedent of Fact 2: we have a structure verifying $\text{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n))$. Accordingly, in this structure the description is proper, so by Fact 1, $\text{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n))$ and $\#(\text{I}v(\phi(v), \psi_1), \ldots, \text{I}v(\phi(v), \psi_n))$ have the same truth-value, so our structure verifies $\#(\text{I}v(\phi(v), \psi_1), \ldots, \text{I}v(\phi(v), \psi_n))$.

While Facts 1 and 2 are familiar, less commonly canvassed is the question of which truth-functional connectives # entitle us to the converse of Fact 2, and hence to a biconditional ($\Leftrightarrow$) formulation of Fact 2 without the additional antecedent present in the formulation of Fact 1. Such a strengthening of Fact 2 is in many ways analogous to Forster's Theorem 2.1, which is also stated without a propriety condition on the descriptions involved. More will be said on the analogy in a moment. First, we need to recall the notion of an n-ary truth-function's being F-*preserving*.[22] By this is meant when each of the function's n arguments is F(alse), so is the value of the function for those arguments. (Alternatively put: $\{\text{F}\}$ is closed under the function in question.[23]) Note that for each n, exactly half of the 2^{2^n} n-ary truth-functions are F-preserving; for example when $n = 1$ the identity and constant false truth-functions are F-preserving, while the negation and constant true truth-functions are not.

Proposition 3.2 *If the truth-function associated with an n-ary connective # is* F*-preserving, then we have the converse of Fact 2 above, i.e., for all formulas $\phi, \psi_1, \ldots, \psi_n$:*

$$\#(\text{I}v(\phi(v), \psi_1), \ldots, \text{I}v(\phi(v), \psi_n)) \;\Rightarrow\; \text{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n)).$$

felt that an expression purportedly of the form $\Box\Psi$ is really of the form $(\forall\mathcal{W})(\mathcal{W} \models \Psi)$," having ourselves taken Kripke to be providing an extensional semantics for modal sentences rather than as suggesting that such sentences do not have the form they appear to have. We acknowledge, though, that the view expressed by Forster is gaining currency. The final paragraph of Wehmeier [35], for example, reads as follows: "That definition [of intensionality] derives from the familiar possible-world treatment of modal and temporal operators. While it has long been a commonplace that these operators *can be construed as* quantifiers over possible worlds or times, linguistic semantics seems to take more and more seriously the notion that they literally *are* quantifiers. But if modals and tenses are actually quantifiers, and quantifiers aren't intensional operators, as I've argued here, then intensional operators are uninteresting, merely theoretical constructs to which no feature of natural language answers." Wehmeier in unpublished work also develops a case for denying that modal contexts are extensional rather than intensional, which does not go via the contentious claim that modal operators literally are quantifiers. On such a view one could still give the Kripke semantics its due by saying that it constitutes a non-modal semantics for modal sentences.

[22] Also called 0-preserving or falsity-preserving – see for example Urquhart [34], Section 5, as well as references there given. Böhler et al. [4] use the term 0-*reproducing* for this.

[23] In what is otherwise a very helpful guide, Pelletier and Martin [23] say (p. 464), "functions closed under F" – which makes no sense at all: how could anything be 'closed under' a truth-value?

Proof. Suppose that # is interpreted by an F-preserving truth-function and we have a first-order structure verifying $\#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n))$. Then by the condition on #, at least one of the $\mathsf{I}v(\phi(v), \psi_i)$ $(i = 1, \ldots, n)$ must be true in the structure and so therefore must $\exists v(\phi!(v))$ be. Taking a as (the name of) a witness for this existential formula, and letting $\psi_i(a)$ be the result of replacing any free occurrences of v in ψ_i by a, we have that for all j $(1 \leq j \leq n)$ $\mathsf{I}v(\phi(v), \psi_j)$ is true in the structure iff $\psi_j(a)$ is. So, since # is truth-functional and $\#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n))$ is true in our structure, so is $\#(\psi_1(a), \ldots, \psi_n(a))$. But in view the truth of $\exists v(\phi!(v))$ and the choice of a, this suffices for the truth of $\mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n))$ in the structure. ■

Remark 3.3 Though this is not needed for our commentary on Forster, it is worth mentioning that we also have a converse for Proposition 3.2: if the truth-function associated with # is not F-preserving, then we can find a counterexample to the schema inset there. For example, in any structure,

$$\#\mathsf{I}v(v \neq v, v = v), \ldots, \mathsf{I}v(v \neq v, v = v),$$

is true (choosing the same formula $v = v$ for each ψ_i – and in fact it does not matter which formula is chosen here), because its immediate subformulas, the current $\mathsf{I}v(\phi(v), \psi_i)$ are all false, and our truth-function maps a sequence of n F's to T. On the other hand, the current instance of $\mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n))$ with $\phi(v)$ as $v \neq v$ and each ψ_i as $v = v$ (or anything else) is false. ■

Thus putting Fact 2 together with Proposition 3.2, we get

Corollary 3.4 *For any (primitive or derived) n-ary connective # with which is associated an* F*-preserving truth-function, and all formulas $\phi, \psi_1, \ldots, \psi_n$, we have:*

$$\#(\mathsf{I}v(\phi(v), \psi_1), \ldots, \mathsf{I}v(\phi(v), \psi_n)) \Leftrightarrow \mathsf{I}v(\phi(v), \#(\psi_1, \ldots, \psi_n)).$$

We will return to Forster's own treatment in the light of these considerations at the end of this section, after giving F-preservation some of the attention which it deserves in view of the above findings. We use $\oplus$ to represent exclusive disjunction, and we use the same notation (and terminology) for connectives as for the associated truth-functions:

Proposition 3.5 *# is an* F*-preserving connective if and only if # is definable in terms of the set of binary connectives $\{\wedge, \vee, \oplus\}$.*

Proof. *If*: It suffices to observe that $\wedge, \vee, \oplus$ are all F-preserving and that this property is preserved by composition of functions and possessed by the projection functions.

Only if: We adapt the standard proof of functional completeness for $\{\wedge, \vee, \neg\}$ (e.g., in [24]) which reads a Disjunctive Normal Form from a truth-table for the formula $\#(p_1, \ldots, p_n)$, representing each line by the corresponding conjunction of the p_i or $\neg p_i$ depending as p_i has the value T or F in that line, and then disjoining the conjunctions representing exactly the lines in which in which $\#(p_1, \ldots, p_n)$ has the value T. This disjunction is then equivalent to the desired $\#(p_1, \ldots, p_n)$, and uses only the connectives $\wedge, \vee, \neg$. From this one then proceeds to observe that either one of $\wedge$, $\vee$ can be defined in terms of the other and $\neg$, giving a further simplification. But for present purposes we retain $\wedge$ and $\vee$ and show how to get rid of $\neg$, using $\oplus$. We need the latter anyway for making the disjunction in the case that there is no line in which $\#(p_1, \ldots, p_n)$ has the value T, taking the empty disjunction as $\bot = p_1 \oplus p_1$, say. For any occurrence of $\neg p_j$ in the conjunction representing a line in which our #-compound has the value T, we now exploit the fact that # is F-preserving: this implies that some p_i (with $i \neq j$) has the value T in that line (since otherwise the value of $\#(p_1, \ldots, p_n)$ would be F), and we replace any all $\neg p_j$ with $p_i \oplus p_j$. The conjunction representing a line now has for each of its conjuncts either sentence letter or the exclusive disjunction of two sentence letters, and the (inclusive) disjunction of these conjunctions is equivalent to $\#(p_1, \ldots, p_n)$. ■

Example 3.6 *Suppose we want to define ternary # with associated truth-function mapping* $\langle \mathrm{T}, \mathrm{T}, \mathrm{T} \rangle$, $\langle \mathrm{T}, \mathrm{F}, \mathrm{T} \rangle$, $\langle \mathrm{F}, \mathrm{F}, \mathrm{T} \rangle$ *to T and all other triples to F. (Since the other triples include* $\langle \mathrm{F}, \mathrm{F}, \mathrm{F} \rangle$*, we are in F-preserving territory.) Writing* p_1, p_2, p_3 *and* p, q, r*, we have, corresponding to these three T-cases, the conjunctions* $p \wedge q \wedge r$*,* $p \wedge (p \oplus q) \wedge r$*, and* $(r \oplus p) \wedge (r \oplus q) \wedge r$*, so the desired formula representing the given truth-function is the (*$\vee$*-)disjunction of these three.*

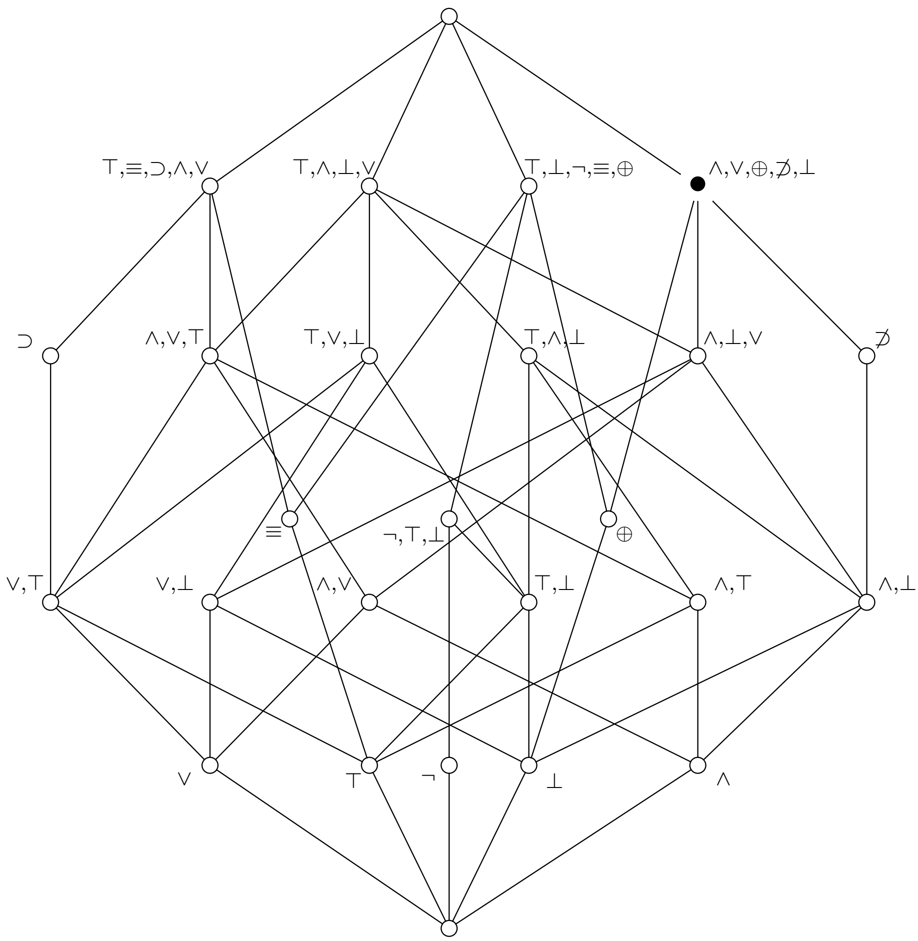

Figure 1: Hazen's Lattice of Binary Clones: See Remark 3.7(*i*).

Remarks 3.7 (*i*) The fact mentioned in the "if" direction of the proof of Proposition 3.5, that the class of F-preserving truth-functions contains the projection functions and is closed under composition of functions (the later understood in its most general sense, sometimes called superposition), is well known, the F-preserving truth-functions constituting one of Post's clones: see Pelletier and Martin [23], Chapter 3 of Part II in Lau [21], Urquhart [34]. Indeed this is one of Post's maximal clones, the addition of any truth-function not belonging to it rendering compositionally definable all truth-functions. (Here we ignore the difference – touched on below – between 'iterative systems' *à la* Post [25] and clones proper. It could well be that Prop. 3.5 itself can be found in [25], but, as is usually the case, it takes less time to prove such things for oneself from scratch than to try to extract them from Post's arcane exposition, and in any case we like the proof of it presented above. The result itself is certainly known in the literature – for example in Böhler et al. [4].) Post's lattice of clones is (countably) infinite and since the clone of F-preserving functions figures there as a co-atom, so it does also in the more manageable (finite) lattice of all – to use a term explained below – binary clones of truth-functions, a Hasse diagram for which was constructed by Allen Hazen in 1995; as usual X's being lower on the diagram (via a chain of edges) than Y means that the binary clone generated by X is properly included in that generated by Y. Hazen's accompanying notes from the period observe that the lattice is not distributive (or even modular); he called the elements of the lattice (two-variable) 'expressivity types'. We reproduce it here as Figure 1. Hazen has since pointed out to us – as he subsequently learned – that the 26 elements of this lattice were described, and noted to exhaust the number of cases, in Wernick [36] (see Wernick's note 16: the 25 there alluded to are those below the top element) in the course of his investigation of which irredundant sets of such functions were functionally complete – i.e., generated the top element of Post's and Hazen's lattices. Post's poset, at p. 101 of [25] is not quite a lattice, and its elements are not exactly clones but 'iterative systems' alluded to above. These need not contain the projections, but, partially compensating for this, are required to be closed under (loosely speaking) identification and re-ordering of variables: see §5.2 of Urquhart [34]. Similarly, a *binary clone* is a set of functions containing the (binary) projection functions to the first and second coordinate and is closed under composition. This is the definition given in Goldstern [11], p. 185, along with obvious generalized form for n-ary clones. (N.B. A binary clone of truth-functions is *not* a clone, but embodies a version of the clone idea restricted to the binary setting. However, as Goldstern remarks, we have a tight connection: C is a binary clone on a set iff C is the intersection of the set of binary functions on that set with some clone of functions on the set.) Contemporary presentations of Post make adjustments so that we have a lattice: p. 636 of Urquhart [34], p. 7 (or p. 149) of Lau [21], p. 44 of Böhler et al. [4] (perhaps the clearest diagram). In Post's lattice, so reconstructed, there are five co-atoms rather than the four of Hazen's, a discrepancy arising because two of Post's co-atoms are the set of all self-dual truth-functions and the set of linear truth-functions, but restricting attention to at most binary truth-functions, all the self-dual ones are linear, so the class of self-dual functions doesn't constitute a maximal proper subset of the class of all truth-functions. (Counterexamples to this generalization arise at the earliest stage with ternary truth-functions, such as the Majority function $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$, which are self-dual but not linear.[24]) The clone of linear truth-functions is represented in Figure 1 by the second co-atom from the right. Our recent reference to Post's lattice is to the contemporary versions of it (e.g., in [34]), rather than to the original poset on p. 101 of Post [25]. The top element in Figure 1 represents the binary clone of all at most binary truth-functions – and so it could be labelled with any basis of (up to) binary truth-functions which is functionally complete (in the weaker of the two senses distinguished in (*iii*) below), such as $\{\vee, \neg\}$, where braces are omitted in the diagram and redundant elements of the basis are included in many cases, as in Hazen's original hand-drawn diagram, to avoid arbitrarily selecting from alternative irredundant bases. (In view of the functional completeness of this set, such a basis also generates Post's top element: the clone of all truth-functions. Compare the corresponding top 1-ary clone, generated by, for example, the negation and constant false functions, which are evidently not functionally complete.) The bottom element represents the smallest binary clone of truth-functions, generated by the appropriate pair of projections (linguistically: the fragment in which we just have the sentence letters p, q). Hazen's notation is mostly preserved for the truth-functions ("$\supset$" and "$\equiv$", rather than "$\rightarrow$" and "$\leftrightarrow$", for instance, with "$\not\supset$" for the complementary function in the former case and "$\oplus$" (exclusive disjunction) in the latter (rather than Hazen's "$\not\equiv$"). The clone of principal interest in connection with our discussion – comprising the F-preserving functions – is the rightmost co-atom in the diagram, marked as a solid black node in the diagram. (The T-preserving functions make up the clone represented by the leftmost co-atom. Naturally, their lattice-meet harbours the binary clone of idempotent truth-functions.) Since *our* binary clone – or rather the similarly situated clone in Post's lattice – is the highest positioned clone not containing $\top$, linguistically it corresponds to the richest fragment of the language of classical propositional

[24]The characterization of linearity for Boolean functions given in Urquhart [34], p. 636, as those representable "in the form $x_1 \oplus \ldots \oplus x_n \oplus c$, where c is a constant (0 or 1)", is not quite right since this requires every argument of the function to be essential, while for present purposes functions with inessential variables also have to count as linear. Thus the form in question should be – and in most of the literature on Boolean functions is – written as follows (abbreviating $x \wedge y$ to xy): $c_1x_1 \oplus \ldots \oplus c_nx_n \oplus c$, where $c_1, \ldots, c_n, c \in \{0, 1\}$. Then c_i can be set to 1 for essential arguments and 0 for inessential arguments (and c itself toggles the output, fixing such things as whether it is $\oplus$ or $\equiv$ that is involved). Otherwise we don't have a clone at all, as Urquhart says on p. 639, in which passage "class" means *clone*: "when we include an n-place function in a class, then we also include all those functions that arise from it by adding irrelevant variables." "Irrelevant" is the word used in Post [25] for *inessential*, and though he does not require his iterative classes to be closed under the padding in of such variables, he allows them to contain functions with inessential argument positions; since he is treating all such classes, he makes a point (p. 55 of [25]) of including such functions in his iterative class of linear functions (or, in Post's terminology, *alternating* functions – though curiously enough, while avoiding the term *linear* altogether, on p. 106 he baptizes the class in question L_1).

logic for which the classical consequence relation is one without tautologies (also called 'purely inferential' or 'atheorematic'). If we wanted to label this node with an irredundant basis, we have several options. Following Urquhart [34], p. 638 (definition of C_3) we could choose $\{\vee, \not\supset\}$ (in Urquhart's notation, $\{\vee, \setminus\}$). Taking a lead from (*ii*) below, we could choose $\{\vee, \oplus\}$; the other economical option provided there is $\{\wedge, \oplus\}$, which chosen in Böhler et al. [4] And so on. As written, five truth-functions have been listed alongside this node in Figure 1, so without even looking to see what they are, we know *a priori* from Wernick [36] that we have a case of redundancy, since it is there shown that no binary clone of truth-functions has an irredundant basis with more than 3 elements. [36] was rather scooped by the earlier publication of Post [25], treating all clones rather than all binary clones case, and showing that each of the clones/iterative systems is finitely based.

(*ii*) As in the standard case recalled in the proof of Proposition 3.5, in which we can effect a further simplification by dropping either of $\wedge, \vee$ given the presence of $\neg$, so in the present case we can drop either of $\wedge, \vee$ thanks to the presence of $\oplus$, since we have:

$$p \wedge q \textit{ equivalent to } (p \vee q) \oplus (p \oplus q) \quad \text{and} \quad p \vee q \textit{ equivalent to } (p \wedge q) \oplus (p \oplus q).$$

(Alternatively, we could write the first definiens as $((p \vee q) \oplus p) \oplus q$, and similarly in the second case.) We could have written "$\Leftrightarrow$" in place of "equivalent to", in accordance with the convention announced after the formally regimented Facts 1 and 2 above, but were keen to avert a possible confusion: the connective $\leftrightarrow$ (with its expected logical behaviour), deployed in formulating that convention, is not available in the current fragment since it is not F-preserving. Indeed more generally – perhaps the best known consequence of using an exclusively F-preserving logical vocabulary – there are no valid formulas at all in this vocabulary, as recalled toward the end of (*i*) above. (A consequence-relation formulation in the style of note 19 would of course be available.)

(*iii*) Recalling the contrast between functional completeness and strict functional completeness, in the parlance of Grabmayer et al. [12], p. 388, or, as it is put in Humberstone [17], p. 50 (and various subsequent publications), between *weak* and *strong* functional completeness, we have the corresponding distinction between the weak/non-strict and strong/strict completeness of any basis as generating a given set of truth-functions. The proof of Proposition 3.5 establishes only the weak completeness of the basis it mentions for the class of F-preserving functions. This is evident with the move of defining $\bot = p_1 \oplus p_1$, whose left-hand side we want to represent the truth-value F (a 0-place truth-function) but whose right-hand side represents a 1-place truth-function which just constantly takes that value for any argument. The weaker kind of (relative) functional completeness is all that usually matters for logical purposes, but evidently if one wants the stronger kind one will need the nullary F, so $\bot$ will need to taken as primitive, since it qualifies as F-preserving. (Here we think of f as F-preserving provided that, if all of f's arguments are F, then f's value is F. In the case in which f is F, this conditional is secured by the truth of its consequent.) Thus the envisaged complete basis would be $\{\wedge, \vee, \oplus, \bot\}$ or, for an irredudant basis, in view of (ii) above, either $\{\vee, \oplus, \bot\}$ or $\{\wedge, \oplus, \bot\}$. ■

At the start of Section 2 we quoted Forster as observing "this inductive proof will not work for $\neg$ and $\rightarrow$" and went on to point out that his induction actually faltered on one of the cases that he had not set aside by restricting attention to the $\{\wedge, \vee, \forall, \exists\}$-formulas with negation allowed only pre-atomically: the case of $\vee$. One cannot help observing that the connectives for which the induction is here noted to falter, namely $\neg$ and $\rightarrow$, fail to be F-preserving, while the connectives for which Forster takes it that no problem arises, namely $\wedge$ and $\vee$, are F-preserving. The fact that this is mistaken is neither here nor there, when it comes to the following hypothesis: perhaps what Forster may had at the back of his mind was not the argument given in the proof of his Theorem 2.1, but instead something more along the lines of Corollary 3.4, though formulated using the linguistic resources of Forster's discussion.[25] That would turn the equivalence of Coro. 3.4 into something along the lines of:

$$\qquad\qquad \Leftrightarrow \exists v(\phi!(v) \wedge \#(\psi_1, \ldots, \psi_n)).$$

But while we can fill the gap on the left with

$$\#(\exists v(\phi!(v) \wedge \psi_1), \ldots, \exists v(\phi!(v) \wedge \psi_n)),$$

the result would not resemble Forster's Theorem 2.1 in displaying a formula from which definite descriptions are to be removed. If we follow Forster's own

[25]However, its being mistaken *is* relevant to a worry along the following lines, which the reader may now be (briefly) entertaining. In the present section we have been stressing the role of composition by F-preserving connectives in delimiting the contexts for which the kind of "Broad $\Leftrightarrow$ Narrow" result which Forster's Theorem 2.1 in effect aims at establishing. But at the end of the previous section we stressed that from the point of view of Forster's actual argument, $\vee$ and $\rightarrow$ are completely on a par, despite their differing in respect of F-preservation. What may appear puzzling here is explained, given the hypothesis mentioned in the text, by the fact that what is really going on for a result of the kind Forster wants is F-preservation, and the fact that the actual argument given does not recognise this but would work equally well for $\rightarrow$ in place of $\vee$, is itself a reflection of the unsoundness of the argument.

example, taking his $\Psi(y)$ to be the $\{\wedge,\vee\}$-formula $\#(\psi_1,\ldots,\psi_n)$ constructed from atomic and negated atomic formulas $\#(\psi_1,\ldots,\psi_n)$ in which somewhere or other the variable y appears free, then we could fill the gap on the left with

$$\#\big(\psi_1(\iota x \centerdot \phi(x)),\ldots,\psi_n(\iota x \centerdot \phi(x))\big)$$

where each $\psi_i(x)$ replaces any free y free in ψ_i with $\iota x \centerdot \phi(x)$, in then we would have a version of Forster's Theorem 2.1 simplified by not having the ι-terms on the left in the (syntactic) scope of any quantifiers. As we already know, however, this version, like the original, would not be correct, because despite metalinguistic appearances the term $\iota x \centerdot \phi(x)$ is not guaranteed to appear in the formula represented by $\psi_i(\iota x \centerdot \phi(x))$. As explained in Section 2, this is because y may not have occurred free in any given ψ_i at all. An analogously failed version of Corollary 3.4 would arise from overlooking an observation Blamey pauses to make in [2], directly following on from the passage quoted in note 20, about a quantifier expressions approximating to singular terms as they become more scope-indifferent. We give Blamey's continuation here using the above notation "$\mathsf{I}x(\phi,\psi)$" rather than his own "$\mathsf{I}x\phi\cdot\psi$" (in which "$\cdot$" is a punctuation mark rather than a connective):

> However, even within the confines of our extensional language, $\mathsf{I}x(\phi,\_)$ turns out not to be entirely scopeless: it cannot hop around a formula with total semantical freedom. For example it is easy to find ϕ,ψ, and χ such that:
>
> $$\mathsf{I}x(\phi,\psi)\vee\chi \;\not\simeq\; \mathsf{I}x(\phi,\psi\vee\chi).$$
>
> This is because the 'strong' table for $\vee$ is allowed. χ may be true making the left-hand side true, but there may be no determinately unique x such that ϕ, and this would leave the right-hand side undefined.

Here "$\simeq$" (negated in the passage quoted[26]) is Blamey's way of saying that the formulas flanking it are true in all the same models, and the reference to strong tables is to Kleene's 'strong' three-valued logic, because Blamey's account of these matters is not bivalent, as explained in the Digression (on Notation) earlier in this section. But the non-equivalence arises in the bivalent setting of our discussion for essentially the same reason that Blamey gives, dropping the word "determinately" and putting *false* for *undefined* in the final sentence. This is not in tension with Coro. 3.4 for the F-preserving $\vee$, of course, because in the inset example in the above passage from Blamey, the left-hand side has for its second disjunct χ rather than, as there, $\mathsf{I}x(\phi,\chi)$.

How might we reformulate Forster's Theorem 2.1 to make a correction analogous to that just recalled – place all the immediate subformulas of the

[26] Blamey actually writes "$\alpha \not\simeq \beta$" rather than "$\alpha \not\approx \beta$".

F-preserving compound in the context $\mathsf{I}x(\phi, \_)$ – for the binary quantifier treatment? One thing we could do is change:

> Then for any Ψ with 'y' free, $\mathcal{I}$ of $\Psi[\iota x.\phi(x)/y]$ is logically equivalent to...

by replacing "Ψ with 'y' free" to "Ψ such that 'y' is free in every subformula of Ψ", or equivalently: "...is free in every atomic subformula of Ψ". But – continuing to ignore quantificational complexity – this is rather heavy-handed, as composition by means of $\wedge$ does not require it. For a leaner approach, then, let us define what it is for a variable v to occur 'suitably free' in a formula (of the description-free language Forster calls $\mathcal{L}$) like this:

- If α is an atomic formula v is suitably free in α iff v occurs in α
- If α is the conjunction $\beta \wedge \gamma$ then v is suitably free in α iff v is suitably free in β or v is suitably free in γ.
- If α is the disjunction $\beta \vee \gamma$ then v is suitably free in α iff v is suitably free in β and v is suitably free in γ.
- If α is the formula $Qv'\beta$ for $Q \in \{\forall, \exists\}$, then v is suitably free in α iff v is suitably free in β and v and v' are distinct variables.

This differs from a similar inductive definition of occurring (merely) free in α in the appearance of "and" rather than "or" in the clause pertaining to $\vee$, called for by the observations of this and the previous section: this means that the disjunctions we are dealing with have the ι-term in each disjunct, just as the disjunctions arising in Proposition 3.2 (and Coro.3.4) have $\mathsf{I}v(\phi(v), \_)$ on each disjunct. Unsurprisingly, this difference suffices to push through the inductive case of disjunction which went wrong in the proof of Theorem 2.1 we looked at in Section 1. We are now showing that the claimed result holds provided that Ψ is as described there except that y is suitably free in Ψ, and we get to the case in which Ψ is $\Psi_1 \vee \Psi_2$, so y is suitably free in Ψ_1 and also in Ψ_2 and can proceed as Forster wanted to, with: "By induction hypothesis $\Psi_1[\iota x.\phi(x)/y]$ is equivalent to $(\exists x)(\phi(x) \wedge \Psi_1[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$ and $\Psi_2[\iota x.\phi(x)/y]$ is equivalent to $(\exists x)(\phi(x) \wedge \Psi_2[x/y] \wedge (\forall y)(\phi(y) \rightarrow y = x))$."

Of course the result is less general than the original, since y might be free in the disjunction without being suitably free in it, but then, since the original result was false, we needed to lose some generality here. Another way of 'repairing' the result might be to avoid the problem of disjunction in the same way that Forster himself avoided the problem of negation. Definition 2.1 fixed the description-free equivalents of – or, as Forster puts it, the result of applying the function $\mathcal{I}$ to – negated atomic formulas just as for atomic formulas themselves, right at the start, thus providing the basis step for the induction on the number of occurrences of $\forall, \exists, \wedge$ and $\vee$. He makes the point that since every formula is equivalent to one constructed from atomic formulas and their negations by means of these four operations, we have provided description-free equivalents for all formulas by this means. But now that we have observed disjunction is actually just as problematic as negation for an inductive argument of the kind envisaged, we might put disjunction down at the bottom layer too, and define $\mathcal{I}(\Psi[\iota x \centerdot \phi(x)/y])$ in a variant of Definition 2.1 by having the basis of the induction be disjunctions of atomic or negated atomic formulas, and this time using the number of occurrences of $\forall, \exists, \wedge$ as our measure of inductive complexity, the inductive part being that $\mathcal{I}$ commutes with these three. The justification for the restriction would be as before: every formula is equivalent to one constructed from these 'elementary disjunctions' by means of conjunction and quantification – for example, we can always take a prenex normal form version of the formula with its quantifier-free matrix in conjunctive normal form. The net effect, in Forster's way of putting things, is that $\mathcal{I}$ would no longer commute with disjunction or with negation, whereas on the treatment he (thought he) was offering, it was negation only that presented such an obstacle. On the original account, with F, G, H 1-place predicate letters, $\mathcal{I}(\neg G(\iota x \centerdot Fx))$ was not the negation of $\mathcal{I}(G(\iota x \centerdot Fx))$ and the formula $G(\iota x \centerdot Fx) \vee \neg G(\iota x \centerdot Fx)$ is not an instance of the law of excluded middle – occasioning the reference to free logic in note 4. On the modified account, a disjunction $Ha \vee G(\iota x \centerdot Fx)$ would not follow from its first disjunct, further violating our inferential expectations. In calling this the modified account, we do not intend to be taken as endorsing it over the original or over the suggestion of the previous paragraph – we are simply contrasting the proposals. The reader will by now no doubt have decided whether any of these options is more appealing than the rest, or indeed preferable indeed to the 'binary quantifier' treatment of the subject matter.

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http://jdh.hamkins.org/the-hierarchy-of-logical-expressivity/.

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References

[1] John Bacon, 'An Alternative Contextual Definition for Descriptions', *Philosophical Studies* **16** (1965), 75–76.

[2] Stephen Blamey, *Partial-Valued Logic*, Oxford University DPhil Thesis, Bodleain Library, Oxford 1980.

[3] Stephen Blamey, 'Partial Logic', Reprinted pp. 261–354 in D. M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, Second Edition, Vol. 5*, Kluwer, Dordrecht 2002. (Minor revision of pp. 1–70 in the same editors' and publisher's *Handbook of Philosophical Logic, Vol. III: Alternatives to Classical Logic*, 1986.)

[4] E. Böhler, N. Creignou, S. Reith, and H. Vollmer, 'Playing with Boolean Blocks, Part I: Post's Lattice with Applications to Complexity Theory', *ACM SIGACT News* **34** (2003) *4*, 38–52.

[5] David Bostock, *Russell's Logical Atomism*, Oxford University Press, Oxford 2012.

[6] John A. Burgess and Lloyd Humberstone, 'Natural Deduction Rules for a Logic of Vagueness', *Erkenntnis* **27** (1987), 197–229.

[7] J. Michael Dunn, 'Relevant Predication I: The Formal Theory', *Journal of Philosophical Logic* **16** (1987), 347–381.

[8] Gareth Evans, *The Varieties of Reference*, Oxford University Press, Oxford 1982.

[9] Thomas Forster, *Reasoning About Theoretical Entities*, World Scientific, New Jersey 2003.

[10] P. T. Geach, 'Russell's Theory of Descriptions', *Analysis* **10** (1950), 84–88.

[11] Martin Goldstern, 'Applications of Logic in Algebra: Examples from Clone Theory', pp. 184–194 in S. B. Cooper, H. Geuvers, A. Pillay, and J. Väänänen (eds.), *Logic Colloquium 2006*, Cambridge Universiy Press, Cambridge 2009.

[12] C. Grabmayer, J. Leo, V. van Oostrom, and A. Visser, 'On the Termination of Russell's Description Elimination Algorithm', *Review of Symbolic Logic* **4** (2011), 367–393.

[13] Norbert Gratzl, 'Hilbert and Bernays on Definite Descriptions', *Studia Philosophiae Christianae* **47** (2011), *4*, 19–29.

[14] Norbert Gratzl, 'Incomplete Symbols – Definite Descriptions Revisited', *Journal of Philosophical Logic* **44** (2015), 489–506.

[15] Leon Horsten, 'Scope and Rigidity', *Communication & Cognition* **25** (1992), 353–372.

[16] G. E. Hughes and D. G. Londey, *The Elements of Formal Logic*, Methuen, London 1965.

[17] Lloyd Humberstone, 'Zero-place Operations and Functional Completeness – and the Definition of New Connectives', *History and Philosophy of Logic* **14** (1993), 39–66.

[18] Saul Kripke, 'Russell's Notion of Scope', pp. 225–253 in Kripke, *Philosophical Troubles*, Oxford University Press, Oxford 2011. (Paper first publ. 2005.)

[19] Saul Kripke, *Reference and Existence*, Oxford University Press, Oxford 2013.

[20] Steven T. Kuhn, 'Embedded Definite Descriptions: Russellian Analysis and Semantic Puzzles', *Mind* **109** (2000), 443–454.

[21] Dietlinde Lau, *Function Algebras on Finite Sets: A Basic Course on Many-Valued Logic and Clone Theory*, Springer–Verlag, Berlin 2006.

[22] Stephen Neale, *Descriptions*, MIT Press, Cambridge, MA 1990.

[23] F. J. Pelletier and N. M. Martin, 'Post's Functional Completeness Theorem', *Notre Dame Journal of Formal Logic* **31** (1990), 462–475.

[24] F. J. Pelletier and W. D. Sharp, 'On Proving Functional Incompleteness in Symbolic Logic Classes', *Notre Dame Journal of Formal Logic* **29** (1988), 235–248.

[25] Emil L. Post, *The Two-Valued Iterative Systems of Mathematical Logic*, Princeton University Press, Princeton 1941.

[26] A. N. Prior, *Time and Modality*, Oxford University Press, Oxford 1957.

[27] A. N. Prior, 'Is the Concept of Referential Opacity Really Necessary?', *Acta Philosophica Fennica* **16** (1963), 189–198.

[28] A. N. Prior, Critical Notice of Hughes and Londey [16], *Australasian Journal of Philosophy* **44** (1966), 224–231.

[29] W. V. Quine, 'Replies', pp. 292–352 in D. Davidson and J. Hintikka (eds.), *Words and Objections*, Reidel, Dordrecht 1969.

[30] Bertrand Russell, 'On Denoting', *Mind* **14** (1905), 479–493.

[31] Dana Scott, 'Existence and Description in Formal Logic', pp. 181–200 in R. Schoenman (ed.), *Bertrand Russell: Philosopher of the Century*, Little, Brown & Co., Boston 1967.

[32] T. J. Smiley, 'The Theory of Descriptions', *Procs. British Academy* **67** (1981), 321–337.

[33] Arthur F. Smullyan, 'Modality and Description', *Journal of Symbolic Logic* **13** (1948), 31–37.

[34] Alasdair Urquhart, 'Emil Post', pp. 617–666 in *Handbook of the History of Logic, Volume 5: Logic from Russell to Church*, Amsterdam 2009.

[35] Kai Wehmeier, 'Are Quantifiers Intensional Operators?', to appear in *Inquiry.*

[36] William Wernick, 'Complete Sets of Logical Functions', *Transactions of the American Mathematical Society* **51** (1942), 117–132.

[37] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica to ∗56*, Cambridge University Press, Cambridge 1970. (Material first published 1910.)

[38] David Wiggins, 'Names, Fictional Names and "Really"', *Procs. of the Aristotelian Society, Supp. Volume* **73** (1999), 271–286.

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