South American Journal of Logic
Vol. 3, n. 2, pp. 389–414, 2017
ISSN: 2446-6719

Aristotelian Relations in PDL: The Hypercube of Dynamic Oppositions

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Abstract

The aim of this paper is to study aristotelian relation in an extension of Propositional Dynamic Logic, the logic $PDL^{Q+(\neg)}$. The main result of our study is the production of a geometrical opposition structure called hypercube of Dynamic Opposition, this structure is very useful to study *negation of atomic programs* and dynamic modalities.

Keywords: PDL, negation, atomic programs, Aristotelian Relations, Hypercube of Oppositions, Modal Logics

Introduction

The motivation of this paper is to study certain oppositional structures in the fusion logic $PDL^{Q+(\neg)}$, whose fragments are Dimiter Vakarelov's *Logic of Dynamic Modalities* [20], and Propositional Dynamic Logic with negation of atomic programs of Karsten Lutz and Dirk Walter.

The main results of our studies are the following: 1) In each fragment of this fusion logic there are certain opposition structures similar to those studied in basic modal logics, only in $PDL^{(\neg)}$ there is some variation with respect to to the opposition octagon. 2) The opposition *unrelated* or *disparatae* is present in some new ways in the structure that we will study in depth. 3) It is possible to sketch a propositional negation corresponding to the negation of atomic programs, which turns out to be precisely a ngation that forms subalternations.

The relevance of our study is twofold. On the one hand, it offers results for the dynamic logician in search of new applications of the logic that we study here. On the other hand, for the oppositional theorist, our study implies a new application of the theory. Other works related to this study are those that present, on the one hand, hypercubes ([10], [11], [5]) to analyze the opposition, and applications of the theory of oppositions in PDL [8]. The first part presents a basic system PDL. The second part contains a presentation of the key concepts in opposition theory and some partial results. Part three contains the development of our study, the main results and the relations with some future prospects.

1 Basic Notions on Propositional Dynamic Logics

In this section we will briefly explain the basic elements of PDL. In the first part the main elements of dynamic languages are presented, consequently we will see how to interpret these languages. Finally, some properties and some definitions are introduced. Lets start by defining the language L_{PDL}.

Definition 1.1 ***(Alphabet)*** *The alphabet of L_{PDL} is the collection $A = \Phi \cup \Pi \cup C \cup K \cup P$, where $\Phi = \{A, B, ...\}$ is a denumerable infinite collection of propositional signs, $\Pi = \{\pi_1, \pi_2, ...\}$ is a denumerable collection of atomic program signs, $C = \{\neg, \wedge, \vee, \supset, \equiv\}$ is a collection of connectives, $K = \{;, \circ, ?, \cup, \cap, *\}$ is a collection of program constructors and $P = \{(,), \langle,\rangle, [,]\}$ is a collection of auxiliar signs.*

Auxiliary signs (round brackets) are required to produce complex formulas, but also to produce modal operators related to programs (square and angle brackets). Now we will see how to produce formulas from the alphabet.

Definition 1.2 *(π-**Grammar**) $\forall \pi \in \Pi$, $\forall \varphi \in \Phi$, $\forall k \in K$, the following production rule determines the collection of complex programs Π_{com}:*
$\pi ::= \pi_1; \pi_2 \mid \varphi? \mid \pi_1 \cup \pi_2 \mid \pi_1 \cap \pi_2 \mid \pi_1^*$

Definition 1.3 *(φ-**Grammar**) $\forall \pi \in \Pi_{com}$, $\forall \varphi \in \Phi$, the following production rule determines the language L_{PDL}[1]:*
$\varphi ::= \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \supset \psi \mid \varphi \equiv \psi \mid [\pi]\varphi \mid \langle\pi\rangle\varphi$

With these syntactical elements present we can continue with semantics and valuation conditions.

Definition 1.4 ***(Semantics)*** *PDL semantcs is defined by models of the form $\mathcal{M} = \langle L, I, \mathcal{R}, V, v\rangle$, where L is a PDL language, I is a collection of indexes, $\mathcal{R} = \{R_\pi \subseteq I \times I \mid \pi \in \Pi\}$ is a collection of relations relative to a program, $V = \{\bot, \top\}$ is a partially ordered collection of truth values, and $v : L \times I \longrightarrow V$ is a map called PDL-valuation.*

[1]When the context be clear we will omit the subscript.

Definition 1.5 *(**Valuation conditions**) The following are the conditions for logical connectives and PDL operators. Consider a semantics for PDL, the valuation mapping can be extended as follows:*

$R_{\varphi?} := \{(i,i) \in I^2 : v_i(\varphi) = \top\}$
$R_{\pi_1 \cup \pi_2} := R_{\pi_1} \cup R_{\pi_2}$
$R_{\pi_1 \cap \pi_2} := R_{\pi_1} \cap R_{\pi_2}$
$R_{\pi_1 ; \pi_2} := R_{\pi_1} \circ R_{\pi_2}$
$R_{\pi^*} := (R_\pi)^*$
$v_i(\neg\varphi) = \top$ *iff* $v_i(\varphi) = \bot$
$v_i(\varphi \wedge \psi) = inf(v_i(\varphi), v_i(\psi))$
$v_i(\varphi \vee \psi) = sup(v_i(\varphi), v_i(\psi))$
$v_i([\pi]\varphi) = inf\{v_j(\varphi) : R_\pi ij\}$
$v_i(\langle\pi\rangle\varphi) = sup\{v_j(\varphi) : R_\pi ij\}$

These are the conditions for logical connectives and operators of PDL. Due to the fact that conditional is material it can be defined in terms of conjunction or disjunction, alternatively biconditional in terms of conditional and conjunction: $(\varphi \supset \psi) =_{df} \neg(\varphi \wedge \neg\psi)$ y $(\varphi \equiv \psi) =_{df} ((\varphi \supset \psi) \wedge (\psi \supset \varphi))$. Now we will continue with the last definitions of this part.

Definition 1.6 *(**Logical consequence**) We say that a formula $\varphi \in L$ is a logical consequence of a collection of formulas Γ (and we write $\Gamma \Vdash \varphi$), if and only if $\forall\beta \in \Gamma,\ v_i(\beta) \leq v_i(\varphi)$. A Propositional Dynamic Logic, therefore, will be a pair $PDL = \langle L, \Vdash_{PDL}\rangle$*

2 Basic Notions on Oppositions Theory

2.1 Aristotelian Relations

In this paper we will use the concept of "opposition" to refer to any of the following four relations: contradiction, contrariety, subcontracting and subalternation. The usual definition of oppositional relations is the informal one, which dates back to Aristotle himself. In the known literature we can find many of them, according to the orientation and application (See for example [2], [9], [13], [15], [16], [17], [18], [19]). In [19] are given three definitions of Aristotelian opposition relations: informal, model-theoretic and abstract. Let us proceed in this order:

Definition 2.1 *(**OP1**) Let $\varphi, \psi \in L_{PDL}$ we say that:*

C) φ and ψ are contradictories*, if and only if, φ and ψ can not be neither simultaneously true, nor simultaneous false.*

CA) φ and ψ are contraries*, if and only if, φ and ψ can not be true simultaneously, but are false together.*

SC) φ and ψ are subcontraries*, if and only if, φ and ψ can not be false simultaneously, but are true together.*

SA) φ and ψ are subalterns*, if and only if, if φ is true, ψ must be true.*

This definition of oppositional relations despite being intuitive enough to understand the use of each of the concepts involved, has several shortcomings identified in [19]. A more specific definition (model-theoretic one) could be the one presented in the cited text that goes as follows.

Definition 2.2 ***(OP2)*** *Let $S = \langle L, \Vdash \rangle$ a logical system with Boolean operators $\wedge, \vee, \neg$, and a model-theoretic relation $\Vdash$, we have that $\forall \varphi, \psi \in L$:*

C) $S \Vdash \neg(\varphi \wedge \psi)$ *&* $S \Vdash (\varphi \vee \psi)$
CA) $S \Vdash \neg(\varphi \wedge \psi)$ *&* $S \nVdash (\varphi \vee \psi)$
SC) $S \nVdash \neg(\varphi \wedge \psi)$ *&* $S \Vdash (\varphi \vee \psi)$
SA) $S \Vdash \neg(\varphi \wedge \neg\psi)$ *&* $S \nVdash (\varphi \vee \neg\psi)$

These definitions have a greater degree of abstraction than the previous ones, the reference to specific logical systems is clear. Therefore, it is possible to overcome difficulties that the first definition could imply. For example, in the first characterization of the oppositions, one speaks of opposition in an unrestricted way, understanding that the concepts are applied in a *global* sense. In the second characterization we can talk about of oppositions in a *local* way in the following sense. Suppose we have two systems S_1 and S_2, a pair of formulas can be S_1-contraries and simultaneously S_2-contradictory, because S_1 and S_2 do not share specific characteristics (different semantics, different kinds of truth values, different inference rules, etc.). This is important since, it allows us to report different phenomena if we intend to work with a collection of logical systems, something that the first characterization does not allow us. Despite maintaining certain advantages, this last characterization has certain limitations. As we saw, the oppositions are satisfied only between formulas of a logical system, if we intend to account for these relationships in other types of entities, such as concepts, collections, and even relationships, the scope of the characterization is limited. In this sense, we will formulate a final characterization that can be adapted to this claim, which will not be especially useful when analyzing whether it is possible to obtain oppositions between dynamic operators.

Definition 2.3 *(**OP3**) Let* $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \bot_{\mathbb{B}} \rangle$ *a Boolean algebra,* $\forall x, y \in B$*:*

C) $x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$ *&* $x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
CA) $x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$ *&* $x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$
SC) $x \wedge_{\mathbb{B}} y \neq \bot_{\mathbb{B}}$ *&* $x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
SA) $x \wedge_{\mathbb{B}} y = x$ *&* $x \vee_{\mathbb{B}} y \neq y$

This characterization is more abstract, although the previous one is more useful when considering specific logical systems. In the following, we will use the model-theoretic one to show the main properties of the opposition structures that we present.

2.2 Basic Modal Opposition Structures and some results: Squares, Hexagons and Octagons

In this part we will present the main results in modal logics, without going into so many details and only explaining what will be required for our analysis. In the first part the modal square is presented followed by its two main extensions: the Hexagons of Sherwood-Czezowski and of Sessmat-Blnché. Finally we will talk about the Octagon of oppositions and the cube.

The basic idea behind the creation of the square of oppositions is to graphically represent the previously defined relations. Since we will analyze formulas of dynamic logics, here we present the main results in modal logics that will later be used to elaborate the Hypercube.

Figure 1 shows two ways to extend the square of oppositions following two different orientations. On the one hand, if we intend to include *singular expressions*, the alternative is to follow the technique of William of Sherwood and Tadeuz Czezowski by adding two formulas without an operator (null modalities [13, p. 175] and [6, p. 11]). The other alternative is to take a hexagon in which the idea of analyzing everything in triads is present, like Robert Blanché and Augustin Sesmat[13, p. 139]. Both extensions converge to a greater diagram, an octagon of modal formulas [13, p. 175] that is shown below and all the remaining oppositions are presented in the following corollary.

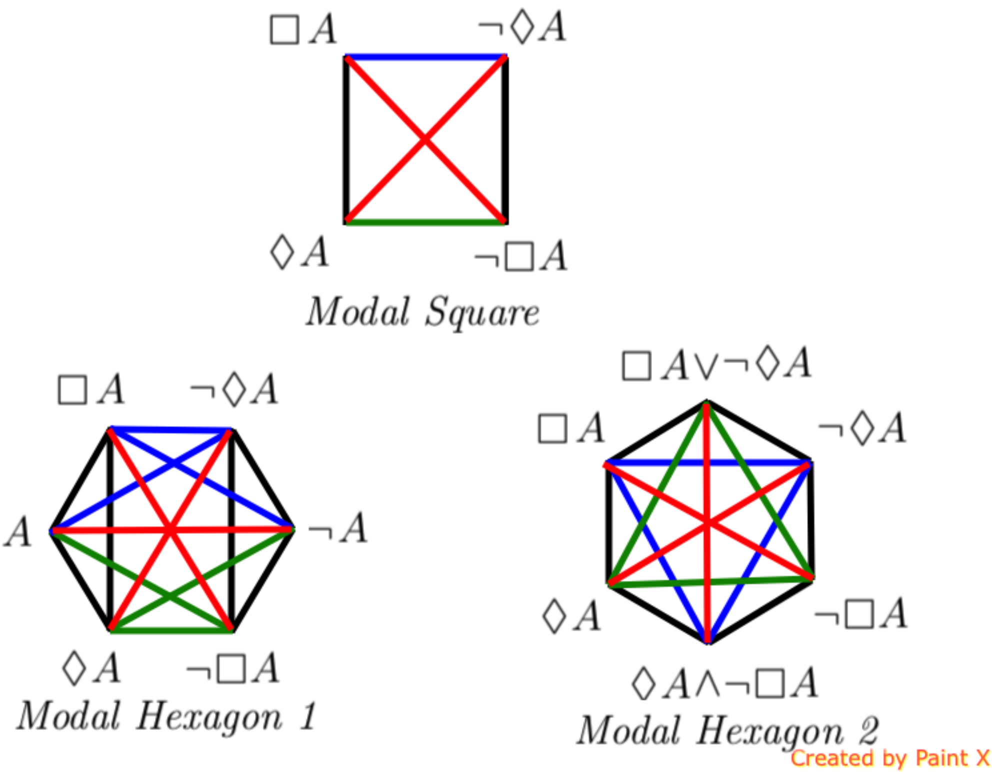


Figure 1: Modal square, Sherwood-Czezowski Hexagon (Hexagon 1) and Sesmat-Blanché Hexagon (Hexagon 2)

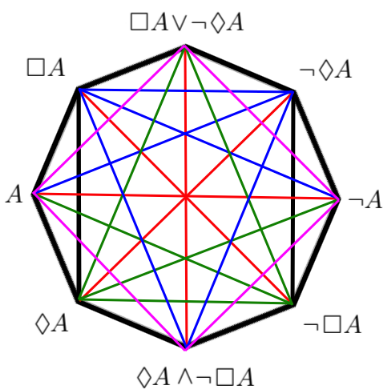
Corollary 2.4 *From Definitions 1.6 and 2.2 the following are valid:*[2]

Square relations

1. $\Vdash \neg(\Box A \wedge \neg\Box A)$ *&* $\Vdash \Box A \vee \neg\Box A$ *(Contradiction 1)*
2. $\Vdash \neg(\Diamond A \wedge \neg\Diamond A)$ *&* $\Vdash \Diamond A \vee \neg\Diamond A$ *(Contradiction 2)*
3. $\Vdash \neg(\Box A \wedge \neg\Diamond A)$ *&* $\nVdash \Box A \vee \neg\Diamond A$ *(Contrariety)*
4. $\nVdash \neg(\Diamond A \wedge \neg\Box A)$ *&* $\Vdash \Diamond A \vee \neg\Box A$ *(Subcontrariety)*
5. $\Vdash \neg(\Box A \wedge \neg\Diamond A)$ *&* $\nVdash \Box A \vee \neg\Diamond A$ *(Subalternation 1)*
6. $\Vdash \neg(\neg\Diamond A \wedge \Box A)$ *&* $\nVdash \neg\Diamond A \vee \Box A$ *(Subalternation 2)*

[2]We omit subalternation relations in hexagons because they are equivalent to contrariety relation. This fact can be seen as a kind of indication of subalternation is an opposition relation.

Hexagon 1 relations
7. $\Vdash \neg(A \wedge \neg A)$ *&* $\Vdash A \vee \neg A$ *(Contradiction 3)*
8. $\Vdash \neg(\Box A \wedge \neg A)$ *&* $\nVdash \Box A \vee \neg A$ *(Contrariety 2)*
9. $\Vdash \neg(\neg\Diamond A \wedge A)$ *&* $\nVdash \neg\Diamond A \vee A$ *(Contrariety 3)*
10. $\nVdash \neg(\Diamond A \wedge \neg A)$ *&* $\Vdash \Diamond A \vee \neg A$ *(Subcontrariety 2)*
11. $\nVdash \neg(\neg\Box A \wedge A)$ *&* $\Vdash \neg\Box A \vee A$ *(Subcontrariety 3)*
Hexagon 2 relations
12. $\Vdash \neg((\Diamond A \wedge \neg\Box A) \wedge (\Box A \vee \neg\Diamond A))$ *&* $\Vdash (\Diamond A \wedge \neg\Box A) \vee (\Box A \vee \neg\Diamond A)$ *(Contradiction 4)*
13. $\Vdash \neg((\Diamond A \wedge \neg\Box A) \wedge \Box A)$ *&* $\nVdash (\Diamond A \wedge \neg\Box A) \vee \Box A)$ *(Contrariety 4)*
14. $\Vdash \neg((\Diamond A \wedge \neg\Box A) \wedge \neg\Diamond A)$ *&* $\nVdash (\Diamond A \wedge \neg\Box A) \vee \neg\Diamond A)$ *(Contrariety 5)*
15. $\nVdash \neg((\Box A \vee \neg\Diamond A) \wedge \Diamond A)$ *&* $\Vdash (\Box A \vee \neg\Diamond A) \vee \Diamond A)$ *(Subcontrariety 4)*
16. $\nVdash \neg((\Box A \vee \neg\Diamond A) \wedge \neg\Box A)$ *&* $\Vdash (\Box A \vee \neg\Diamond A) \vee \neg\Box A)$ *(Subcontrariety 5)*
Unrelated square
17. $\nVdash \neg((\Box A \vee \neg\Diamond A) \wedge A)$ $\nVdash (\Box A \vee \neg\Diamond A) \vee A)$
18. $\nVdash \neg((\Box A \vee \neg\Diamond A) \wedge \neg A)$ $\nVdash (\Box A \vee \neg\Diamond A) \vee \neg A)$
19. $\nVdash \neg((\Diamond A \wedge \neg\Box A) \wedge A)$ $\nVdash (\Box A \vee \neg\Diamond A) \vee A)$
20. $\nVdash \neg((\Diamond A \wedge \neg\Box A) \wedge \neg A)$ $\nVdash (\Box A \vee \neg\Diamond A) \vee \neg A)$



Modal Octagon

Figure 2: Modal Octagon

This octagon (Figure 2) joins both hexagons and shows the remaining relations[3]. An interesting feature of this diagram is that there is a central square

[3]We can refer to Beziau and Moretti's work with its extensions, stellar dodecahedron and Moretti's and Smesaert's logical cuboctahedron. Both are more complex logical structures of oppositions related with this octagon.

which connects formulas without an oppositional relation. Later on we will speculate a bit about this and we will analyze more cases. This octagon can be taken as an extension of the square directly, but it is not the only alternative to extending the square to an octagon, as we can see with the work of Campos-Benítez [7] on the octagons of John Buridan and the systems of C. I. Lewis. To finalize this section we present the aforementioned diagram in Figure 3, with its respective relations[4].

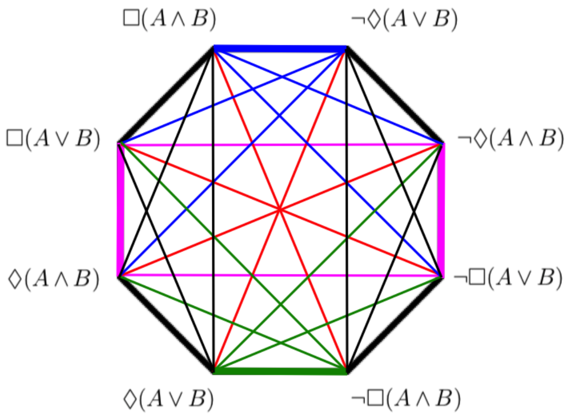


Figure 3: Modal Octagon Buridan's version

Corollary 2.5 *From Definitions 1.6 and 2.2 the following are valid:*

Contradictories
1) $\Vdash \neg(\Box(A \wedge B) \wedge \neg\Box(A \wedge B))$ & $\Vdash \Box(A \wedge B) \vee \neg\Box(A \wedge B)$
2) $\Vdash \neg(\Box(A \vee B) \wedge \neg\Box(A \vee B))$ & $\Vdash \Box(A \vee B) \vee \neg\Box(A \vee B)$
3) $\Vdash \neg(\Diamond(A \wedge B) \wedge \neg\Diamond(A \wedge B))$ & $\Vdash \Diamond(A \wedge B) \vee \neg\Diamond(A \wedge B)$
4) $\Vdash \neg(\Diamond(A \vee B) \wedge \neg\Diamond(A \vee B))$ & $\Vdash \Diamond(A \vee B) \vee \neg\Diamond(A \vee B)$
Contraries
5) $\Vdash \neg(\Box(A \wedge B) \wedge \neg\Diamond(A \vee B))$ & $\nVdash \Box(A \wedge B) \vee \neg\Diamond(A \vee B))$
6) $\Vdash \neg(\Box(A \wedge B) \wedge \neg\Diamond(A \wedge B))$ & $\nVdash \Box(A \wedge B) \vee \neg\Diamond(A \wedge B))$
7) $\Vdash \neg(\neg\Diamond(A \vee B) \wedge \Box(A \vee B))$ & $\nVdash \neg\Diamond(A \vee B) \wedge \Box(A \vee B)$
8) $\Vdash \neg(\Box(A \vee B) \wedge \neg\Diamond(A \wedge B))$ & $\nVdash \Box(A \vee B) \vee \neg\Diamond(A \wedge B))$
9) $\Vdash \neg(\Box(A \wedge B) \wedge \neg\Box(A \vee B))$ & $\nVdash \Box(A \wedge B) \vee \neg\Box(A \vee B))$
10) $\Vdash \neg(\neg\Diamond(A \vee B) \wedge \Diamond(A \vee B))$ & $\nVdash \neg\Diamond(A \vee B) \wedge \Diamond(A \vee B)$
Subcontraries
11) $\nVdash \neg(\Diamond(A \vee B) \wedge \neg\Box(A \wedge B))$ & $\Vdash \Diamond(A \vee B) \vee \neg\Box(A \wedge B)$

[4]For simplicity we present the octagon in a propositional version.

12) $\nVdash \neg(\Diamond(A \vee B) \wedge \neg\Box(A \vee B))$ & $\Vdash \Diamond(A \vee B) \vee \neg\Box(A \vee B)$
13) $\nVdash \neg(\neg\Box(A \wedge B) \wedge \Diamond(A \wedge B))$ & $\Vdash \neg\Box(A \wedge B) \vee \Diamond(A \wedge B)$
14) $\nVdash \neg(\Diamond(A \wedge B) \wedge \neg\Box(A \vee B))$ & $\Vdash \Diamond(A \wedge B) \vee \neg\Box(A \vee B)$
15) $\nVdash \neg(\neg\Box(A \wedge B) \wedge \Box(A \vee B))$ & $\Vdash \neg\Box(A \wedge B) \vee \Box(A \vee B)$
16) $\nVdash \neg(\Diamond(A \vee B) \wedge \neg\Diamond(A \wedge B))$ & $\Vdash \Diamond(A \vee B) \vee \neg\Diamond(A \wedge B)$
Unrelated
17) $\nVdash \neg(\Box(A \vee B) \wedge \neg\Diamond(A \wedge B))$ & $\nVdash \Box(A \vee B) \vee \neg\Diamond(A \wedge B)$
18) $\nVdash \neg(\Box(A \vee B) \wedge \Diamond(A \wedge B))$ & $\nVdash \Box(A \vee B) \vee \Diamond(A \wedge B)$
19) $\nVdash \neg(\neg\Diamond(A \wedge B) \wedge \neg\Box(A \vee B))$ & $\nVdash \neg\Diamond(A \wedge B) \vee \neg\Box(A \vee B)$
20) $\nVdash \neg(\neg\Box(A \vee B) \wedge \Diamond(A \wedge B))$ & $\nVdash \neg\Box(A \vee B) \vee \Diamond(A \wedge B)$

It is possible to use a cube as a visual resource to represent the same relations of the octagon, although it is still debated whether such a structure exists as a direct extension of the square [3]. In this work we will use cubes and octagons, independently of the considerations related to the debate between the existence or non-existence of the cube of oppositions, keeping the idea that when we manipulate cubes we are manipulating octagons in three-dimensional form. The above is due to something very simple. Our analysis in this paper ends with an opposition structure of 16 vertices, if such vertices are ordered in a plane generating a hexadecagon, the lines that represent the relations will saturate the structure and can create noise. We consider it useful to use the third axis to debug some of this noise. In the end, what remains is to decide between which structure best represents the intentions in each case (something that is outside our objectives), a hexadecagon or an hypercube; but, if we do not commit ourselves to the debate, we can use both.

3 The Hypercube of Dynamic Oppositions

In this section we will present two extensions of the basic system defined above. Two new operators are presented that give rise to two extended *PDL* logics, in which opposition octagons with specific characteristics can be produced. As we will see, an octagon is an instance of Buridan's Octagon, while the other is neither modal nor Buridan one. In this last part we will explore the characteristics of both, and the function of the second with respect to the general structure produced in this part, *i.e.* the Hypercube.

3.1 Extending *PDL* with $[\Pi]^{\forall}$ and $[\overline{\pi}]$

We will start with Dimiter Vararelov's work "Dynamic Modalities", in which certain operators are presented that are susceptible of comparison with modalities analyzed by Buridan [7]. Vakarelov's orientation is different from ours, but what we will say is consequence of his definitions. We will continue with an extension of PDL called *Propositional Dynamic Logic with Negation of Atomic Programs* (henceforth $PDL^{(\neg)}$), which is due to Karsten Lutz and Dirk Walter. One of the questions that the authors leave aside, in our view, is in what sense the operator generated by a negated atomic program can be taken as a negation of formulas? In other words, if $[\overline{\pi}]$ is a modal operator, in what sense can it be used as a formula negation? In both logics we will analyze the opposition relations and compare the results obtained, ending with the union of both extensions.

3.1.1 Vakarelov's Dynamic Modalities: The meaning of $[\Pi]^{\forall}$ and $[\Pi]^{\exists}$

Dimiter Vakarelov presents a logic called *Logic of Dyamic Modalities* (in the following LDM), with some operators with the following intuitive meaning [20, p. 387]:

1. $\Box^{\forall}$ always necessary, necessary in all situations,
2. $\Box^{\exists}$ sometimes necessary, necessary in some situations,
3. $\Diamond^{\forall}$ always possibly, possibly in all situations, and
4. $\Diamond^{\exists}$ sometimes possibly, possibly in some situations.

He offers two formal interpretations for these operators, the relevant to us is the one that uses models of dynamic logic. Let us begin extending the collection of signs of PDL with the quantifiers $\forall$ and $\exists$, to generate the four new operators:

1. $[\Pi]^{\forall}$, always after all program from Π,
2. $[\Pi]^{\exists}$, always after some program from Π,
3. $\langle\Pi\rangle^{\forall}$, sometimes after all program from Π,
4. $\langle\Pi\rangle^{\exists}$, sometimes after some program from Π.

Semantics is the same as in 1.4. Our interpretation of operators is determined by the following definition:

Definition 3.1 *(PDL^Q-**valuation conditions**) The following are the conditions for logical connectives and PDL^Q operators. Consider a semantics for PDL, the valuation mapping can be extended as follows:*

$v_i(\neg\varphi) = \top$ *iff* $v_i(\varphi) = \bot$
$v_i(\varphi \wedge \psi) = inf(v_i(\varphi), v_i(\psi))$
$v_i(\varphi \vee \psi) = sup(v_i(\varphi), v_i(\psi))$
$v_i([\pi]\varphi) = inf\{v_j(\varphi) : R_\pi ij\}$
$v_i(\langle\pi\rangle\varphi) = sup\{v_j(\varphi) : R_\pi ij\}$
$v_i([\Pi]^\forall\varphi) = inf\{inf\{v_j(\varphi) : R_\pi ij\} : \pi \in \Pi\}$
$v_i([\Pi]^\exists\varphi) = sup\{inf\{v_j(\varphi) : R_\pi ij\} : \pi \in \Pi\}$
$v_i(\langle\Pi\rangle^\forall\varphi) = inf\{sup\{v_j(\varphi) : R_\pi ij\} : \pi \in \Pi\}$
$v_i(\langle\Pi\rangle^\exists\varphi) = sup\{sup\{v_j(\varphi) : R_\pi ij\} : \pi \in \Pi\}$

There are several properties that we can highlight that Vakarelov mentions in his work. First of all Vakarelov [20, 389] highlights the fact that the modalities $[\Pi]^\forall$ and $[\Pi]^\exists$ can be taken as primitives and define the others as duals as follows:
$\langle\Pi\rangle^\exists A =_{def} \neg[\Pi]^\forall\neg A$
$\langle\Pi\rangle^\forall A =_{def} \neg[\Pi]^\exists\neg A$

On the other hand, because $[\Pi]^\forall$ is a normal modality [20, Lemma 1.1], when proposing its axiomatic system, consider the following formulas:
$[\Pi]^\forall(A \supset B) \supset ([\Pi]^\forall A \supset [\Pi]^\forall B)$ (K Axiom)
$[\Pi]^\forall(A \supset B) \supset ([\Pi]^\exists A \supset [\Pi]^\exists B)$ (Mono $[\Pi]^\exists$)
$[\Pi]^\forall A \supset [\Pi]^\exists A$ (Cond)

In addition he adds necessitation rule and modus ponens due to the same fact. In addition to the monotonicity rule, Vakarelov includes four alternative monotonicity rules for each operator: $(A \supset B) \Vdash ([\Pi]^\forall A \supset [\Pi]^\forall B)$
$(A \supset B) \Vdash ([\Pi]^\exists A \supset [\Pi]^\exists B)$
$(A \supset B) \Vdash (\langle\Pi\rangle^\forall A \supset [\Pi]^\forall B)$
$(A \supset B) \Vdash (\langle\Pi\rangle^\exists A \supset [\Pi]^\exists B)$

Finally, we can highlight that the modalities $[\Pi]^\exists$ and $\langle\Pi\rangle^\forall$ are not normal modalities, therefore with these modalities *modus ponens* and K axiom are not valid. Finally the following two are taken as theorems:
$[\Pi]^\exists\top$
$[\Pi]^\forall A \wedge [\Pi]^\exists B \supset [\Pi]^\exists(A \wedge B)$

3.1.2 Lutz and Walter's negation of atomic programs: The meaning of $[\overline{\pi}]$

In *PDL* with negation of atomic programs Karsten Lutz and Dirk Walter present the logic $PDL^{(\neg)}$, motivated by the logic of negation of programs $PDL^{\neg}$. The latter, as they report in their article, has the main disadvantage of being undecidable. In the first part of his article they present three examples of the use of this logic: the use of the negation of programs to express the intersection, the use of negation to express the universal modality $\square^{U}$[5], and the use of program negation to express the window operator $\boxminus_a$ to express *sufficency* rather than necessity. Taking advantage of the second issue, we will analyze the operators with negation of programs in an oppositional context. The proposal of Lutz and Walter is to locate a decidable fragment of $PDL^{\neg}$ that satisfies the three mentioned characteristics, this is how they present $PDL^{(\neg)}$ as the indicated logic.

The first ingredient we require to present $PDL^{(\neg)}$ is the sign of negation of atomic programs, which we add to the $\mathcal{K}$ collection of program constructors. This sign allows to expand the π-Grammar with a clause to obtain only negation of atomic programs[6]. Finally the semantics is the same as in Definition 1.4 and 1.5 but adding the following condition for the negation of programs:

$R_{\overline{\pi}} := I^2 \backslash R_\pi$

Definitions of logical consequence and validity are the same as in Definition 1.6. The aspect that we wish to highlight is that the operators of Vakarelov can be defined in this logic as shown below:

$([\pi]A \wedge [\overline{\pi}]A) =_{def} [\Pi]^{\forall}A$
$([\pi]A \vee [\overline{\pi}]A) =_{def} [\Pi]^{\exists}A$
$(\langle\pi\rangle A \wedge \langle\overline{\pi}\rangle A) =_{def} \langle\Pi\rangle^{\forall}A$
$(\langle\pi\rangle A \vee \langle\overline{\pi}\rangle A) =_{def} \langle\Pi\rangle^{\exists}A$

The link with these formulas and Buridan's octagon is evident, in the following section are analyzed two cubes of opposition to finalize with the analysis of the Hypercube.

3.2 Opposition in $PDL^{Q+(\neg)}$: From squares to cubes

We will begin by briefly describing a logic that can be presented as the *fusion* of PDL^{Q} and $PDL^{(\neg)}$. The definitions 1.1 - 1.4 are adapted including the

[5]In our case corresponds to the operator $\square^{\forall}$ of Vakarelov.
[6]Lutz and Walter defines this in Definition 1

operators of dynamic modalities and negation of atomic programs. The only definition that suffers the most relevant modification is the following:

Definition 3.2 *($PDL^{Q+(\neg)}$**-valuation conditions**) The following are the conditions for logical connectives and PDL^{Q} and $PDL^{(\neg)}$ operators. Consider a semantics for PDL, the valuation mapping can be extended as follows:*

$R_{\varphi?} := \{(i,i) \in I^2 : v_i(\varphi) = \top\}$
$R_{\pi_1 \cup \pi_2} := R_{\pi_1} \cup R_{\pi_2}$
$R_{\pi_1 \cap \pi_2} := R_{\pi_1} \cap R_{\pi_2}$
$R_{\pi_1 ; \pi_2} := R_{\pi_1} \circ R_{\pi_2}$
$R_{\overline{\pi}} : = I^2 \backslash R_{\pi}$
$R_{\pi^*} := (R_{\pi})^*$
$v_i(\neg\varphi) = \top$ *iff* $v_i(\varphi) = \bot$
$v_i(\varphi \wedge \psi) = inf(v_i(\varphi), v_i(\psi))$
$v_i(\varphi \vee \psi) = sup(v_i(\varphi), v_i(\psi))$
$v_i([\pi]\varphi) = inf\{v_j(\varphi) : R_{\pi}ij\}$
$v_i(\langle\pi\rangle\varphi) = sup\{v_j(\varphi) : R_{\pi}ij\}$
$v_i([\Pi]^{\forall}\varphi) = inf\{inf\{v_j(\varphi) : R_{\pi}ij\} : \pi \in \Pi\}$
$v_i([\Pi]^{\exists}\varphi) = sup\{inf\{v_j(\varphi) : R_{\pi}ij\} : \pi \in \Pi\}$
$v_i(\langle\Pi\rangle^{\forall}\varphi) = inf\{sup\{v_j(\varphi) : R_{\pi}ij\} : \pi \in \Pi\}$
$v_i(\langle\Pi\rangle^{\exists}\varphi) = sup\{sup\{v_j(\varphi) : R_{\pi}ij\} : \pi \in \Pi\}$

Definition 1.6 remains unaltered, therefore $PDL^{\neg+Q} = \langle L_{PDL}, \Vdash\rangle$.

3.2.1 V-Cube of dynamic modalities

In this part we present the opposition relations in Vakarelov's logic of dynamic modalities. Returning to the results presented in section 2.2, we can start with some assumptions. First, considering that both PDL^{Q} and $PDL^{(\neg)}$ are a kind of modal logics, a plausible assumption is that in both the opposition relations are met and therefore structures opposition are just as in that section 2.2. In that sense, let's start with the square and the hexagons in each structure. Second, considering that there are only two normal modalities in PDL^{Q}, it is possible that there are variations between the octagons outlined above and the octagon presented in this part. Finally, as visual appeal we present octagons in its three dimensional form as cubes of opposition, assuming that simply is an alternative presentation.

We will start with the square of oppositions. Because there are four operators, the first difficulty is determining how to present the square, in case there is such a square. We can consider several alternatives if we take theorems in section 3.1.1, the axiom that interests us is the following:

$$\Vdash [\Pi]^{\forall} A \supset [\Pi]^{\exists} A$$

Due to the fact that conditional is material this formula can be rewritten as subalternation relation as follows:

$$\Vdash \neg([\Pi]^{\forall} A \wedge \neg[\Pi]^{\exists} A) \ \& \ \nVdash [\Pi] A \vee \neg[\Pi]^{\exists} A$$

From this relation, by *Equipollence Rule* [21, p. 498], we can construct the square diagram from the remaining relations. If we consider the remaining operators, we obtain five squares of oppositions, as shown in Figure 4:

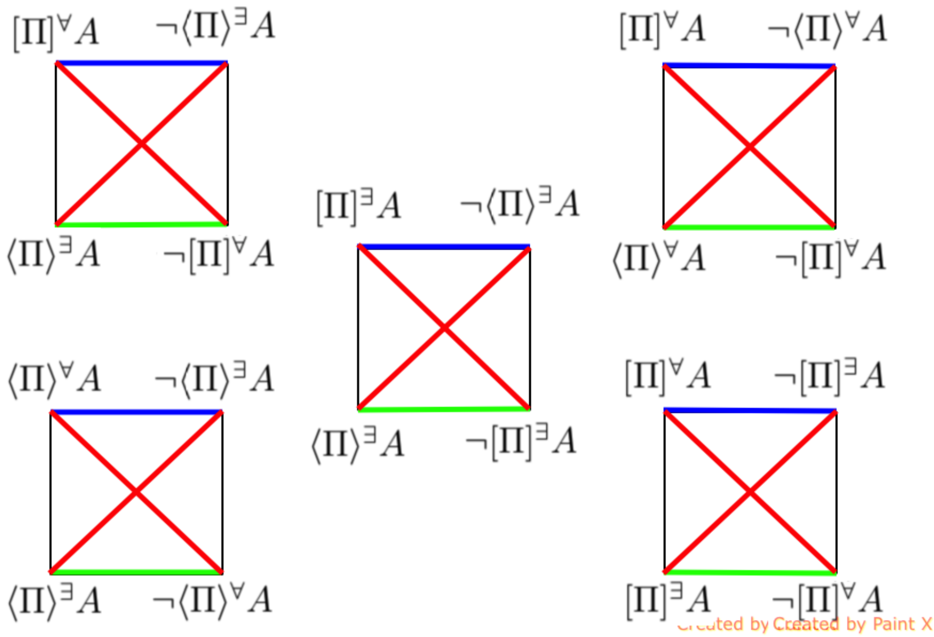


Figure 4: Five squares of opposition in $PDL^{[}Q]$

These five squares can be classified with reference to the operators used in each figure. On the one hand we have a classic square (Fig. 4.1), this structure contains only normal modalities [20, Lemma 1.1]. Consequently we have two squares that satisfy "modal uniformity" *i. e.*, they contain only modal operators of a single type. In this case we have a square of necessity (Fig. 4, 4) and one of possibility (Fig. 4, 3). The remaining squares satisfy something that we will call "quantificational uniformity" *i. e.*, all modal operators contain one type of quantifiers. In that sense, we have a universal square (Fig. 4, 2) and an existential square (Fig. 4, 5). Classical square does not satisfy uniformity neither with respect to the modality nor with respect to the quantification, therefore, we consider that it is "quantificational and modal hybrid". This is not the only square that satisfies this property. The last square that serves to connect these five, is in addition to hybrid, the square of the "unrelated".

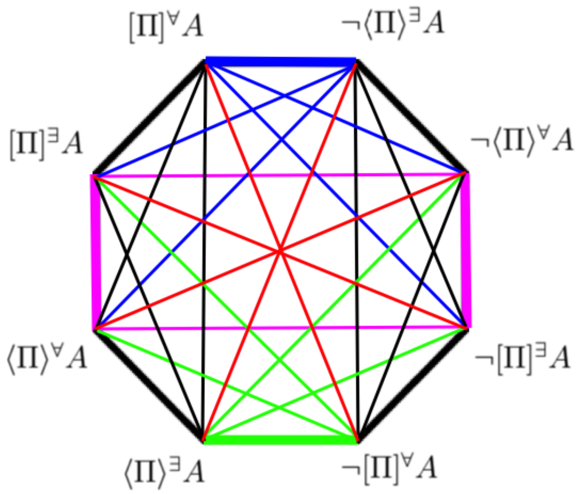


Figure 5: Octagon of oppositions Buridan style PDL^Q

In Figure 5 we can see the octagon formed by the union of the six squares, while in Fig. 6 the same formulas are shown in another three-dimensional configuration, a cube of oppositions. The octagon is analogous to Buridan's octagons, and satisfies the same conditions as these. As in Buridan's octagon, in this octagon, the unrelated square serves as a link between all contradictory formulas. As in the Corollary 2.5, the same relations are satisfied in this octagon, but in this case it is necessary to make some important modifications that are presented in the following Corollary:

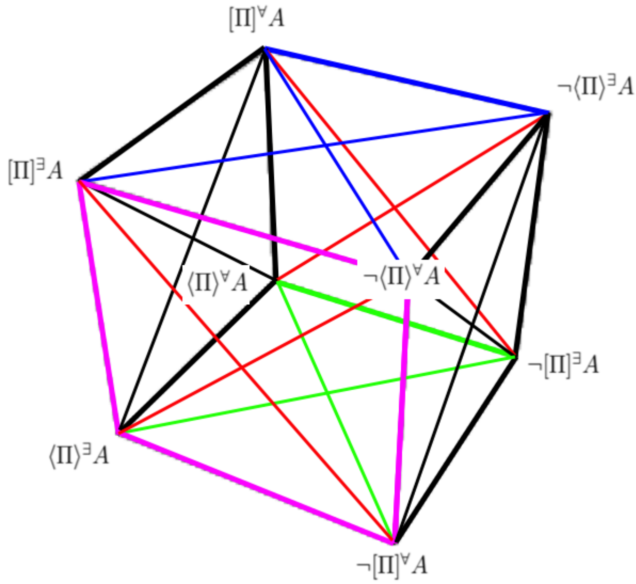


Figure 6: Cube of oppositions in PDL^{Q}

Corollary 3.3 *From Definitions 1.6 and 2.2 the following are valid:*

Contradictories
1) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg[\Pi]^{\forall}A)$ & $\Vdash [\Pi]^{\forall}A \vee \neg[\Pi]^{\forall}A$
2) $\Vdash \neg([\Pi]^{\exists}A \wedge \neg[\Pi]^{\exists}A)$ & $\Vdash [\Pi]^{\exists}A \vee \neg[\Pi]^{\exists}A$
3) $\Vdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg\langle\Pi\rangle^{\forall}A)$ & $\Vdash \langle\Pi\rangle^{\forall}A \vee \neg\langle\Pi\rangle^{\forall}A$
4) $\Vdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg\langle\Pi\rangle^{\exists}A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg\langle\Pi\rangle^{\exists}A$
Contraries
5) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg\langle\Pi\rangle^{\exists}A)$ & $\nVdash [\Pi]^{\forall}A \vee \langle\Pi\rangle^{\exists}A$
6) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg\langle\Pi\rangle^{\forall}A)$ & $\nVdash [\Pi]^{\forall}A \vee \langle\Pi\rangle^{\forall}A$
7) $\Vdash \neg(\neg\langle\Pi\rangle^{\exists}A \wedge [\Pi]^{\exists}A)$ & $\nVdash \neg\langle\Pi\rangle^{\exists}A \wedge [\Pi]^{\exists}A$
8) $\Vdash \neg([\Pi]^{\exists}A \wedge \neg\langle\Pi\rangle^{\forall}A)$ & $\nVdash [\Pi]^{\exists}A \vee \neg\langle\Pi\rangle^{\forall}A$
9) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg[\Pi]^{\exists}A)$ & $\nVdash [\Pi]^{\forall}A \vee \neg[\Pi]^{\exists}A$
10) $\Vdash \neg(\neg\langle\Pi\rangle^{\forall}A) \wedge \langle\Pi\rangle^{\exists}A$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \wedge \langle\Pi\rangle^{\exists}A$
Subcontraries
11) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg[\Pi]^{\forall}A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg[\Pi]^{\forall}A$
12) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg[\Pi]^{\exists}A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg[\Pi]^{\exists}A$
13) $\nVdash \neg(\neg[\Pi]^{\forall}A \wedge \langle\Pi\rangle^{\forall}A)$ & $\Vdash \neg[\Pi]^{\forall}A \vee \langle\Pi\rangle^{\exists}A$
14) $\nVdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg[\Pi]^{\exists}A)$ & $\Vdash \langle\Pi\rangle^{\forall}A \vee \neg[\Pi]^{\exists}A$
15) $\nVdash \neg(\neg[\Pi]^{\forall}A \wedge [\Pi]^{\exists}A)$ & $\Vdash \neg[\Pi]^{\forall}A \vee [\Pi]^{\exists}A$

16) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg\langle\Pi\rangle^{\forall}A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg\langle\Pi\rangle^{\forall}A$
Unrelated
17) $\nVdash \neg([\Pi]^{\exists}A \wedge \neg\langle\Pi\rangle^{\forall}A)$ & $\nVdash [\Pi]^{\exists}A \vee \neg\langle\Pi\rangle^{\forall}A$
18) $\nVdash \neg([\Pi]^{\exists}A \wedge \langle\Pi\rangle^{\forall}A)$ & $\nVdash [\Pi]^{\exists}A \vee \langle\Pi\rangle^{\forall}A$
19) $\nVdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge \neg[\Pi]^{\exists}A)$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \vee \neg[\Pi]^{\exists}A$
20) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge \langle\Pi\rangle^{\forall}A)$ & $\nVdash \neg[\Pi]^{\exists}A \vee \langle\Pi\rangle^{\forall}A$

3.2.2 LW-Cube of negation of atomic programs

To conclude this section we present some opposition structures in $PDL^{(\neg)}$. The basic dynamic square is only an interpretation of the basic modal square with the language of $PDL^{(\neg)}$. In that case the only possible squares are the aforementioned and the square of program negations, both presented in Fig. 7.

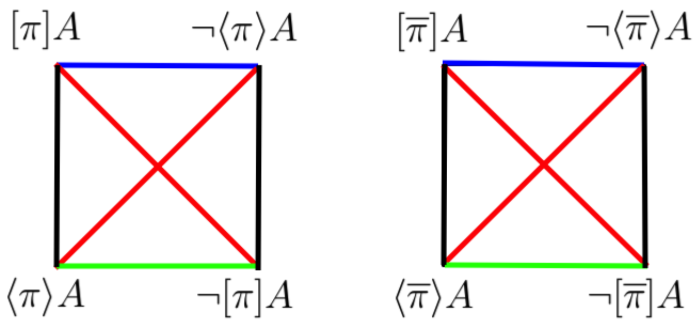


Figure 7: Squares of opposition in $PDL^{(\neg)}$

Because they are the only opposition squares the only way to produce an octagon is from a link between the formulas $[\pi]A$ and $[\overline{\pi}]A$. As we saw in 3.1.2 the dynamic modalities can be defined with this language. The problem is that there is no opposition relationship between the formulas, but it is possible to produce an octagon with both as shown in Fig. 8 and Fig. 9. The key is found in the function that meets the square of the *unrelated* formulas in the octagon. The main characteristic of this square is to link contradictory formulas by means of the composition of two unrelated relations, that is, if the pairs φ, ψ and ψ, ρ are unrelated, then, the pair φ, ρ is a pair of contradictory formulas. The same is preserved for each pair of formulas of the unrelated square. Something similar happens in this case, the main difference is that the composition of two unrelated relations can produce a pair of formulas of any of the four types of oppositions. In this sense, the unrelated relationship serves as a link between formulas with complementary modalities. This property becomes more important when joining both cubes.

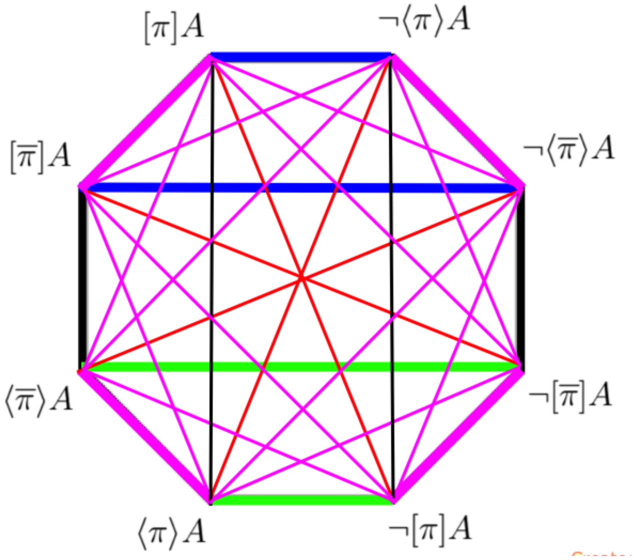


Figure 8: Octagon of opposition $PDL^{(\neg)}$

In the previous case the composition of unrelated only works between formulas of the unrelated square, in this case any formula of the octagon can be taken and the composition of two Unrelated produces an oppositional relation. In that sense, unrelated continues to fulfill the function of linking between the opposite formulas, as well as in Buridan's octagon. The main difference is the lack of proportion between the relationships and the predominance of the unrelated. Finally we present the list of relations satisfied in this structure.

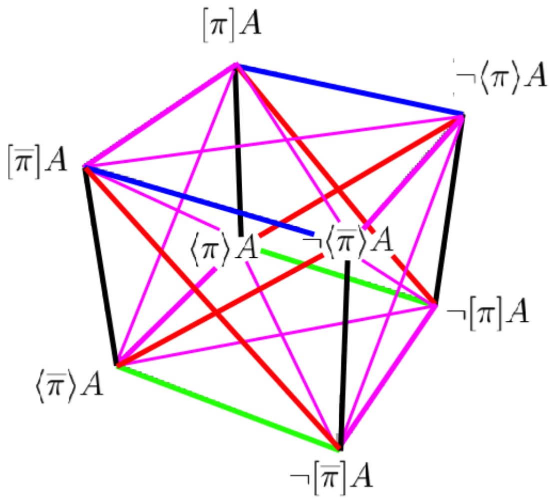


Figure 9: Cube of opposition $PDL^{(\neg)}$

Corollary 3.4 *From Definitions 1.6 and 2.2 the following are valid:*

Contradictories
1) $\Vdash \neg([\pi]A \wedge \neg[\pi]A)$ & $\Vdash [\pi]A \vee \neg[\pi]A$
2) $\Vdash \neg([\overline{\pi}]A \wedge \neg[\overline{\pi}]A)$ & $\Vdash [\overline{\pi}]A \vee \neg[\overline{\pi}]A$
3) $\Vdash \neg(\neg\langle\pi\rangle A \wedge \langle\pi\rangle A)$ & $\Vdash \neg\langle\pi\rangle A \vee \langle\pi\rangle A$
4) $\Vdash \neg(\neg\langle\overline{\pi}\rangle A \wedge \langle\overline{\pi}\rangle A)$ & $\Vdash \neg\langle\overline{\pi}\rangle A \vee \langle\overline{\pi}\rangle A$
Contraries
5) $\Vdash \neg([\pi]A \wedge \neg\langle\pi\rangle A)$ & $\nVdash [\pi]A \vee \neg\langle\pi\rangle A$
6) $\Vdash \neg([\overline{\pi}]A \wedge \neg\langle\overline{\pi}\rangle A)$ & $\nVdash [\overline{\pi}]A \vee \neg\langle\overline{\pi}\rangle A$
Subcontraries
7) $\nVdash \neg(\langle\pi\rangle A \wedge \neg[\pi]A)$ & $\Vdash \langle\pi\rangle A \vee \neg[\pi]A$
8) $\nVdash \neg(\langle\overline{\pi}\rangle A \wedge \neg[\overline{\pi}]A)$ & $\Vdash \langle\overline{\pi}\rangle A \vee \neg[\pi]A$
Unrelated
9) $\nVdash \neg([\pi]A \wedge [\overline{\pi}]A)$ & $\nVdash [\pi]A \vee [\overline{\pi}]A$
10) $\nVdash \neg([\pi]A \wedge \langle\overline{\pi}\rangle A)$ & $\nVdash [\pi]A \vee \langle\overline{\pi}\rangle A$
11) $\nVdash \neg([\pi]A \wedge \neg[\overline{\pi}]A)$ & $\nVdash [\pi]A \vee \neg[\overline{\pi}]A$
12) $\nVdash \neg([\pi]A \wedge \neg\langle\overline{\pi}\rangle A)$ & $\nVdash [\pi]A \vee \neg\langle\overline{\pi}\rangle A$
13) $\nVdash \neg(\neg\langle\pi\rangle A \wedge [\overline{\pi}]A)$ & $\nVdash \neg\langle\pi\rangle A \vee [\overline{\pi}]A$
14) $\nVdash \neg(\neg\langle\pi\rangle A \wedge \langle\overline{\pi}\rangle A)$ & $\neg \nVdash \langle\pi\rangle A \vee \langle\overline{\pi}\rangle A$
15) $\nVdash \neg(\neg\langle\pi\rangle A \wedge \neg[\overline{\pi}]A)$ & $\nVdash \neg\langle\pi\rangle A \vee \neg[\overline{\pi}]A$
16) $\nVdash \neg(\neg\langle\pi\rangle A \wedge \neg\langle\overline{\pi}\rangle A)$ & $\nVdash \neg\langle\pi\rangle A \vee \neg\langle\overline{\pi}\rangle A$
17) $\nVdash \neg(\langle\pi\rangle A \wedge [\overline{\pi}]A)$ & $\nVdash \langle\pi\rangle A \vee [\overline{\pi}]A$
18) $\nVdash \neg(\langle\pi\rangle A \wedge \langle\overline{\pi}\rangle A)$ & $\nVdash \langle\pi\rangle A \vee \langle\overline{\pi}\rangle A$

19) $\nVdash \neg(\langle\pi\rangle A \wedge \neg[\overline{\pi}]A)$ & $\nVdash \langle\pi\rangle A \vee \neg[\overline{\pi}]A$
20) $\nVdash \neg(\langle\pi\rangle A \wedge \neg\langle\overline{\pi}\rangle A)$ & $\nVdash \langle\pi\rangle A \vee \neg\langle\overline{\pi}\rangle A$
21) $\nVdash \neg(\neg[\pi]A \wedge [\overline{\pi}]A)$ & $\nVdash \neg[\pi]A \vee [\overline{\pi}]A$
22) $\nVdash \neg(\neg[\pi]A \wedge \langle\overline{\pi}\rangle A)$ & $\nVdash \neg[\pi]A \vee \langle\overline{\pi}\rangle A$
23) $\nVdash \neg(\neg[\pi]A \wedge \neg[\overline{\pi}]A)$ & $\nVdash \neg[\pi]A \vee \neg[\overline{\pi}]A$
34) $\nVdash \neg(\neg[\pi]A \wedge \neg\langle\overline{\pi}\rangle A)$ & $\nVdash \neg[\pi]A \vee \neg\langle\overline{\pi}\rangle A$

To finish the work now let's see how both cubes can join to build a Hypercube in which the $PDL^{(\neg)}$ cube satisfies a function similar to the square of unrelated formulas.

3.3 Hypercube of Dynamic Oppositions: V+LW

Figure 10 presents a structure resulting from the union of the cubes presented previously. This structure of oppositions can be presented as a hexadecagon, as in Figure 11. We only present some results and some reflexions about this union of structures in the final remarks. In specific, we will talk about what type of opposition produces the operator $[\overline{\pi}]$. The list of relations is extended with the relation presented in Corollary 3.5, in which is evident the predominance of contrariety and subcontrariety.

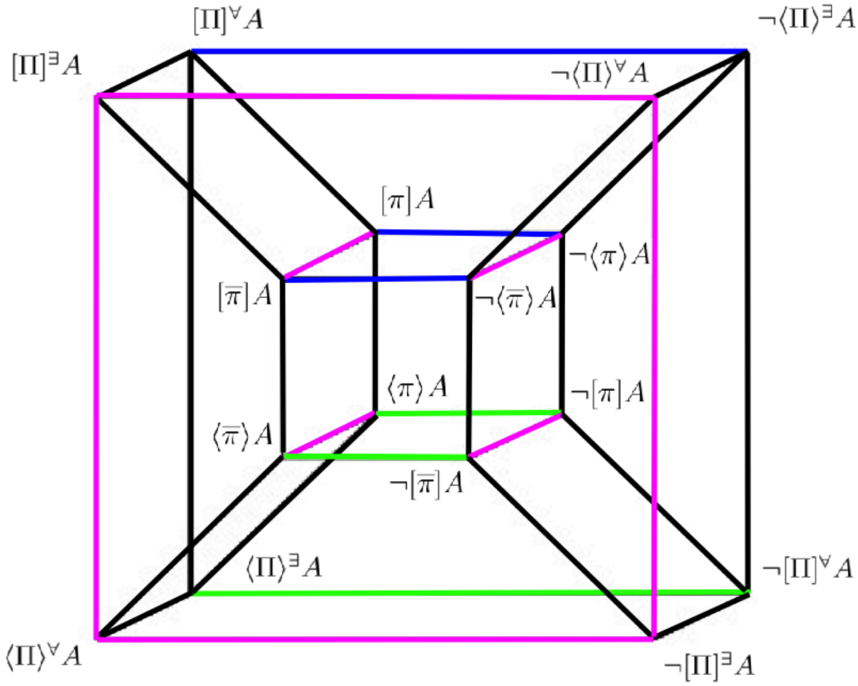


Figure 10: Hypercube of opposition in $PDL^{(\neg)+Q}$

Corollary 3.5 *All the relations from Corollary 3.3 and 3.4 are valid, and from Definitions 1.6 and 2.2 the following are valid:*
Contraries
1) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg[\pi_1]A)$ & $\nVdash [\Pi]^{\forall}A \vee \neg[\pi_1]A$
2) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg[\overline{\pi_1}]A)$ & $\nVdash [\Pi]^{\forall}A \vee \neg[\overline{\pi_1}]A$
3) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg\langle\pi_1\rangle A)$ & $\nVdash [\Pi]^{\forall}A \vee \neg\langle\pi_1\rangle A$
4) $\Vdash \neg([\Pi]^{\forall}A \wedge \neg\langle\overline{\pi_1}\rangle A)$ & $\nVdash [\Pi]^{\forall}A \vee \neg\langle\overline{\pi_1}\rangle A$
5) $\Vdash \neg([\Pi]^{\exists}A \wedge \neg\langle\pi_1\rangle A)$ & $\nVdash [\Pi]^{\exists}A \vee \neg\langle\pi_1\rangle A$
6) $\Vdash \neg([\Pi]^{\exists}A \wedge \neg\langle\overline{\pi_1}\rangle A)$ & $\nVdash [\Pi]^{\exists}A \vee \neg\langle\overline{\pi_1}\rangle A$
7) $\Vdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg\langle\pi_1\rangle A)$ & $\nVdash \langle\Pi\rangle^{\forall}A \vee \neg\langle\pi_1\rangle A$
8) $\Vdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg\langle\overline{\pi_1}\rangle A)$ & $\nVdash \langle\Pi\rangle^{\forall}A \vee \neg\langle\overline{\pi_1}\rangle A$
9) $\Vdash \neg(\neg\langle\Pi\rangle^{\exists}A \wedge \langle\pi_1\rangle A)$ & $\nVdash \neg\langle\Pi\rangle^{\exists}A \vee \langle\pi_1\rangle A$
10) $\Vdash \neg(\neg\langle\Pi\rangle^{\exists}A \wedge \langle\overline{\pi_1}\rangle A)$ & $\nVdash \neg\langle\Pi\rangle^{\exists}A \vee \langle\overline{\pi_1}\rangle A$
11) $\Vdash \neg(\neg\langle\Pi\rangle^{\exists}A \wedge [\pi_1]A)$ & $\nVdash \neg\langle\Pi\rangle^{\exists}A \vee [\pi_1]A$
12) $\Vdash \neg(\neg\langle\Pi\rangle^{\exists}A \wedge [\overline{\pi_1}]A)$ & $\nVdash \neg\langle\Pi\rangle^{\exists}A \vee [\overline{\pi_1}]A$
13) $\Vdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge [\pi_1]A)$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \vee [\pi_1]A$
14) $\Vdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge [\overline{\pi_1}]A)$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \vee [\overline{\pi_1}]A$
15) $\Vdash \neg(\neg[\Pi]^{\exists}A \wedge [\pi_1]A)$ & $\nVdash \neg[\Pi]^{\exists}A \vee [\pi_1]A$

16) $\Vdash \neg(\neg[\Pi]^{\exists}A \wedge [\overline{\pi_1}]A)$ & $\nVdash \neg[\Pi]^{\exists}A \vee [\overline{\pi_1}]A$

Subcontraries

17) $\nVdash \neg([\Pi]^{\exists}A \wedge \neg[\pi_1]A)$ & $\Vdash [\Pi]^{\exists}A \vee \neg[\pi_1]A$

18) $\nVdash \neg([\Pi]^{\exists}A \wedge \neg[\overline{\pi_1}]A)$ & $\Vdash [\Pi]^{\exists}A \vee \neg[\overline{\pi_1}]A$

19) $\nVdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg[\pi_1]A)$ & $\Vdash \langle\Pi\rangle^{\forall}A \vee \neg[\pi_1]A$

20) $\nVdash \neg(\langle\Pi\rangle^{\forall}A \wedge \neg[\overline{\pi_1}]A)$ & $\Vdash \langle\Pi\rangle^{\forall}A \vee \neg[\overline{\pi_1}]A$

21) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg\langle\pi_1\rangle A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg\langle\pi_1\rangle A$

22) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg\langle\overline{\pi_1}\rangle A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg\langle\overline{\pi_1}\rangle A$

23) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg[\pi_1]A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg[\pi_1]A$

24) $\nVdash \neg(\langle\Pi\rangle^{\exists}A \wedge \neg[\overline{\pi_1}]A)$ & $\Vdash \langle\Pi\rangle^{\exists}A \vee \neg[\overline{\pi_1}]A$

25) $\nVdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge \langle\pi_1\rangle A)$ & $\Vdash \neg\langle\Pi\rangle^{\forall}A \vee \langle\pi_1\rangle A$

26) $\nVdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge \langle\overline{\pi_1}\rangle A)$ & $\Vdash \neg\langle\Pi\rangle^{\forall}A \vee \langle\overline{\pi_1}\rangle A$

27) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge \langle\pi_1\rangle A)$ & $\Vdash \neg[\Pi]^{\exists}A \vee \langle\pi_1\rangle A$

28) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge \langle\overline{\pi_1}\rangle A)$ & $\Vdash \neg[\Pi]^{\forall}A \vee \langle\overline{\pi_1}\rangle A$

29) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge [\pi_1]A)$ & $\Vdash \neg[\Pi]^{\exists}A \vee [\pi_1]A$

30) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge [\overline{\pi_1}]A)$ & $\Vdash \neg[\Pi]^{\forall}A \vee [\overline{\pi_1}]A$

31) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge \langle\pi_1\rangle A)$ & $\Vdash \neg[\Pi]^{\exists}A \vee [\pi_1]A$

32) $\nVdash \neg(\neg[\Pi]^{\exists}A \wedge \langle\overline{\pi_1}\rangle A)$ & $\Vdash \neg[\Pi]^{\forall}A \vee \langle\overline{\pi_1}\rangle A$

Unrelated

33) $\nVdash \neg([\Pi]^{\exists}A \wedge \langle\pi_1\rangle A)$ & $\nVdash [\Pi]^{\exists}A \vee \langle\pi_1\rangle A$

34) $\nVdash \neg([\Pi]^{\exists}A \wedge \langle\overline{\pi_1}\rangle A)$ & $\nVdash [\Pi]^{\exists}A \vee \langle\overline{\pi_1}\rangle A$

35) $\nVdash \neg(\langle\Pi\rangle^{\forall}A \wedge [\pi_1]A)$ & $\nVdash \langle\Pi\rangle^{\forall}A \vee [\pi_1]A$

36) $\nVdash \neg(\langle\Pi\rangle^{\forall}A \wedge [\overline{\pi_1}]A)$ & $\nVdash \langle\Pi\rangle^{\forall}A \vee [\overline{\pi_1}]A$

37) $\nVdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge \neg[\pi_1]A)$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \vee \neg[\pi_1]A$

38) $\nVdash \neg(\neg\langle\Pi\rangle^{\forall}A \wedge \neg[\overline{\pi_1}]A)$ & $\nVdash \neg\langle\Pi\rangle^{\forall}A \vee \neg[\overline{\pi_1}]A$

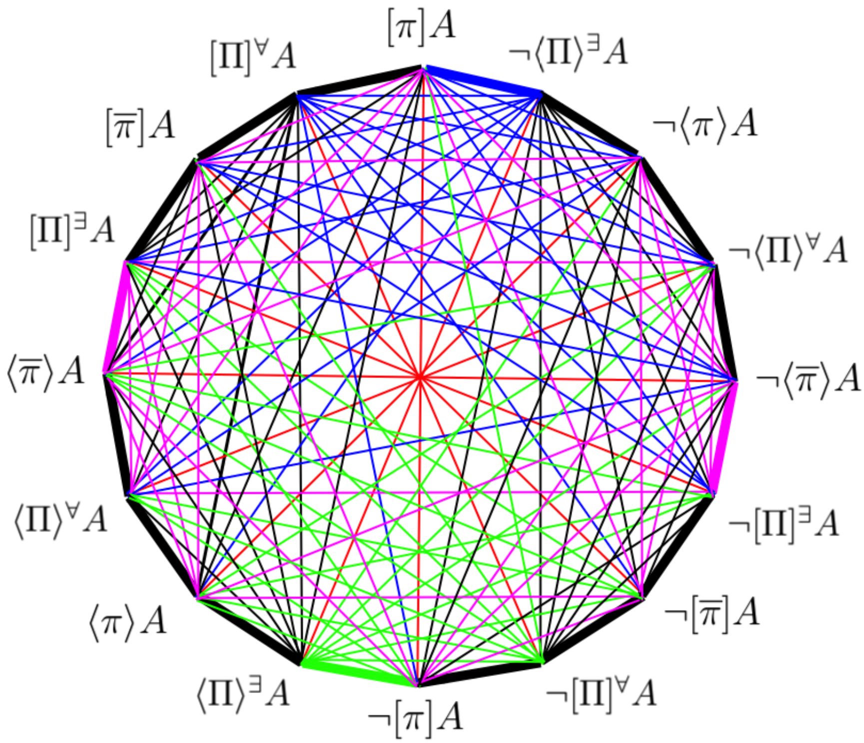


Figure 11: Hexadecagon of opposition in $PDL^{(\neg)+Q}$ (2D Hypercube)

4 Final remarks

The final question related with the concept of negation and its relations with opposition is what kind of negation could be $[\overline{\pi}]$ operator? To solve this problem consider the following hypothesis:

(Forming operator hypothesis) Let be $\varphi \in L_{PDL}$, its *program negation* produces a formula $[\overline{\pi}]\varphi$ with some opposition relation, in case that $[\overline{\pi}]$ can be an opposition forming operator.

What remains is to find what is the relation between $[\overline{\pi}]\varphi$ and φ. This relation is absent in the Hypercube, but there is no problem to finding it. The possible candidates are subalternation and *superalternation*, if they are, this means that the strong operator of program negation is a subaltern operator and the weak operator of program negation is a superalternation operator. We

consider that this is so, since the following formulas are valid in $PDL^{Q+(\neg)}$[7]:
$\Vdash ([\overline{\pi}]A \supset A)$
$\Vdash (A \supset \langle\overline{\pi}\rangle A)$

Due to the fact that conditional is material, we may conclude that between $[\overline{\pi}]A$ and subalternation relation holds and between A and $\langle\overline{\pi}\rangle A$ superalternation relation holds, therefore, negation of atomic programs is a subalternation/superalternation forming opposition.

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[7] An interesting aspect (that we leave for another paper) is to investigate the properties of $[\overline{\pi}]$ and its dual, as negations.

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