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How May the Propositional Calculus Represent?

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Abstract

This paper is a conceptual study in the philosophy of logic. The question considered is 'How may formulae of the propositional calculus be brought into a representational relation to the world?'. Four approaches are distinguished: (1) the denotational approach, (2) the abbreviational approach, (3) the truth-conditional approach, and (4) the modelling approach. (2) and (3) are very familiar, so I do not discuss them. (1), which is now largely obsolete, led to some interesting twists and turns in early analytic philosophy which will come as news to many contemporary readers, so I discuss it in some detail. The modelling approach is, to the best of my knowledge, newly introduced here. I am not presenting it as a rival to the other approaches, but as a philosophically interesting possibility.

Keywords: philosophy of logic, propositional calculus, representation, modelling.

"I am not interested in constructing a building, so much as having a perspicuous view of the foundations of possible buildings." - Wittgenstein, *Culture and Value.*

Introduction

The formulae of formal logical systems are, among other things, supposed to be capable of being used to represent things as being certain ways. The present paper investigates the following question about the formulae of a very basic and foundational system:

(Q) How may formulae of the propositional calculus be brought into a representational relation to the world?

In answer to (Q) I want to distinguish four ways of bringing PC formulae into a representational relation to the world:

The denotational approach: Formulae are regarded as denoting (designating, referring to) propositions, sentences or truth-values.

The abbreviational approach: Formulae and connectives are regarded as abbreviating certain natural language expressions.

The truth-conditional approach: Truth-conditions are stipulated for formulae, or at least complex formulae.

The modelling approach: Formulae, together with either a valuation- or proof-theoretic apparatus, are regarded as forming an abstract structure capable of bearing (via stipulation) a representational relation to the world.

I do not consider these to be rivals. There are different ways to make PC formulae represent, and I feel no particular need to pick favourites.

There are many possible points of difference between versions of each approach. Also, the basic ideas behind each category can be combined in various ways. For example, one might regard formulae as denoting, and connectives as abbreviating - a view which I consider under the denotational heading (in accord with the descriptions above).

The abbreviational and truth-conditional approaches will probably be quite familiar to contemporary students of logic. To be sure, they raise some interesting questions. For instance, on the abbreviational approach, there is the question of which natural language expressions, and which occurrences thereof, may be regarded as fair game for abbreviation by the connectives without the calculus then leading us from truth to falsity. (This amounts to the question of which natural language expressions, and which occurrences thereof, may be regarded as truth-functional.) And on the truth-conditional approach, questions arise as to the relationship between truth-conditions and meaning. But these approaches to making PC represent, and the issues that arise on them, are well-known compared to the denotational approach, which is largely a thing of the past, and to the modelling approach, which I have not seen in the literature before. Here I want to discuss these two approaches. The denotational approach led to some interesting twists and turns in early analytic philosophy which will come as news to many contemporary readers, and I think the possibility of the modelling approach is of considerable philosophical interest.

On terminology: I use the term 'wf' for well-formed formulae and 'atom' for atomic wfs. I call the formula yielded by putting a '$\sim$' at the front of a formula F 'the tilde of F', in order to avoid the semantic connotations of the term 'negation'.

1 The denotational approach

The characteristic thing about the denotational approach is that formulae are regarded as denoting (designating, referring). The objects denoted may be propositions, sentences, truth-values, thoughts, and perhaps other things. For definiteness, let us take this description from Quine (1934) as our starting point:

> The usual sort of system treats of some manner of elements, say cardinal numbers or geometrical points, which are denoted ambiguously by variables; operative upon these elements are certain operations or relations, appropriately expressed within the language of the system. The theory of deduction, when construed as a calculus of propositions, is a system of this kind; its elements are propositions denoted by the variables '*p*', '*q*', etc., and its operations are the propositional operations of denial, alternation, material implication etc., denoted by prefixture or interfixture of the signs '$\sim$', '$\vee$', '$\supset$' etc. (Quine (1934), p. 474.)

So, atoms denote propositions on this version of the denotational approach. We shall take it that complex formulae are also to denote propositions - namely, the results of applying the operations denoted by their connectives. Note that Quine does not say exactly that connectives denote operations, but rather that operations are denoted by the prefixture or interfixture of connectives. I just take this as meaning that connectives appear as prefixtures and interfixtures, and that they denote operations. (I do not mean to make it seem as though it should be totally clear what a propositional operation is, or whether such things even exist. This matter will not be discussed.)

So far, all we have is a kind of naming system. The atoms and compounds denote. But can they be regarded as *saying* anything, or representing the world as being some way? For that, it seems like it will need to be the case that the formulae of PC can also be regarded as *sentences*. Putting these two things together, it seems like it will need to be the case that sentences - at least some sentences, i.e. those of PC - denote. Historically, the main candidate objects for denotation by sentences are propositions (variously construed) and truth-values. But what reason is there to think that sentences denote at all? People do not ordinarily speak that way; in ordinary language, sentences are said to 'refer to' such things as their constituents denote. The doctrine that sentences denote could be argued for purely on the basis of consideration of PC (though today this would seem myopic given the availability of other approaches to making the formulae of PC represent), or on a broader theoretical basis. The classic example of the latter course is Frege's (1892/2010) semantic theory, which has sentences refer to truth-values. Let us call any approach to making the formulae of PC represent which involves regarding them *both* as sentences and as denoting objects a *sentential-denotationalist* approach.

It is worth noting that for the purpose of interpreting PC, the sentential-denotationalist need not maintain that sentences in natural language refer - it may be that sentences of logical languages are peculiar in getting their meaning in a way which involves their referring.

But there is a further difficulty: even if we assume that some theoretical justification for the doctrine that (some) sentences denote can be provided, it does not follow that the complex formulae of PC *are* sentences; an expression can of course denote without being a sentence. Atoms could perhaps be taken as abbreviations of pre-existing sentences which denote, but if connectives are to denote as well (as they do on Quine's description above), a complex formula such as '$p \supset q$' will then just be a string of denoting expressions with no verb. This is not a sentence in any ordinary grammatical sense.[1]

What, then, are the options available to the would-be sentential-denotationalist?

Positing a special grammatical category. Theoretically, one option would be to argue that PC formulae belong to a special verbless category of sentences. Sentences of this form, the story would go, only contain noun-phrases, but are sentences nonetheless. If '$\sim$' refers to a function/operation called negation, and 'p' refers to the proposition that snow is white, then the formula '$\sim p$' could be rendered without symbols as 'Negation The proposition that snow is white'. If, on the other hand, atoms refer to truth-values, we would get 'Negation The True'. This latter course would apparently restrict PC to talking about truth-values (and functions/operations). However, whatever theory was previously used to motivate the view that sentences refer may have implications for this matter. For example, on a Fregean theory, the translation of '$\sim p$' as 'Negation The True' is inappropriate, since, while 'p' (i.e. 'snow is white') and 'The True' might have the same reference, they do not have the same sense.

So, one of the sentential-denotationalist's options is to adopt the view that

[1] This grammatical issue is reminiscent of the more metaphysical issue of 'the unity of the proposition', which occupied leading philosophers in the late 19th and early 20th centuries. cf. Frege (1892/1951), Bradley (1893, pp. 27-28), Russell (1903, §52), and Wittgenstein (1922, 2.03 - 2.034, 3.14 - 3.142).

a string of nouns (perhaps of a certain kind) can form a sentence of a special category. Now, this view is strictly independent of the view that sentences denote. However, adoption of the former without the latter prohibits iteration (i.e. formulae with multiple connectives): on the denotational approach, connectives must be flanked by denoting phrases, but all connective-containing phrases will be sentences, and if sentences don't denote, they won't be able to flank further connectives. Also, since atoms are to denote, then they won't be sentences: formulae with less than one connective will not get a sentential interpretation either.

Such a 'one-connective' view can be had without the theoretical cost of defending a special 'string of nouns' category of sentences. The sentential-denotationalist could depart from the denotational approach as described above by Quine and reckon connectives as verbs, instead of names of operations. '$\vee$' could be read as something like 'disjoins with', '$\supset$' as 'materially implies'. '$\sim$' could be read as a grammatically reversed 'is false' or 'is not true'. (English-speakers may use archaic grammar as a heuristic here, i.e. 'False is p'.) Just like the 'special grammatical category' strategy, this move is strictly independent of the thesis that sentences denote, but without that doctrine formulae will be limited to containing one connective. This version of one-connective-ism is less obscure than the previous, however, since it does not rest on an unusual grammatical doctrine. One plainly can make sentences of this sort, as logicians do when they claim of two sentences or propositions that one materially implies the other.

We have now examined two ways of getting a sentential version of the denotational approach - two ways of making PC more than just a naming system. The first involves recognizing a special grammatical category of sentences, while the second involves reckoning connectives as verbs. On both conceptions, to get formulae with more than one connective (and thus to retain the 'naming system' aspect) requires that sentences denote. We will now consider another way of going, due to Quine.

In his (1934), Quine suggested a different strategy for using formulae of PC to *say something*, and not just denote, while still retaining the denotational approach. His motivations were two. The first motivation was essentially the avoidance of the issues we have just been dealing with; Quine regarded the doctrine that sentences denote as *ad hoc*. Using the word 'proposition' to mean the kind of object, if such there be, which sentences denote (and thus not ruling out truth- values), he complained that 'while we are apprised of a wide array of logical properties of propositions, concerning which there is little essential disagreement, on the other hand as to the residual character of propositions we have that full latitude of choice which attends the development of gratuitous fictions' (Quine (1943), p. 473).

Quine's second motivation was quite subtle, and has to do with the possibility of reinterpreting the calculus, or considering it in abstraction from any interpretation:

> It was suggested above that in the ordinary calculus of propositions the theorems are expressions denoting certain of the elements of the system. This is an anomaly upon which mathematicians have looked askance. It is customary to consider systems in abstraction from the nature of their elements; the theorems of a system, thus viewed, become sentences telling us various properties of unidentified elements. But to abstract from the fact that the elements of the propositional calculus are propositions is to deprive the theorems themselves of their character as sentences, since in that calculus the theorems are symbols of elements of the system. The student of systems in the abstract thus comes to an impasse when he takes up the calculus of propositions. (Quine (1934), p. 475)

The point is: if the elements of the propositional calculus are denoted by formulae-as- sentences, and if sentences by their nature denote propositions and not other things, then it would seem that the elements *must* be propositions.

Quine's foremost recommendation regarding these problems is to abandon the denotational approach in favour of the abbreviational:

> Without altering the theory of deduction internally, we can so reconstrue it as to sweep away such fictive considerations; we have merely to interpret the theory as a formal grammar for the manipulation of sentences, and to abandon the view that sentences are names. (Quine (1934), pp. 473 - 474.)

But he also offers a more conservative strategy which retains denotationalism:

> But there is a way of gaining these advantages without persisting in the exclusion of the theory of deduction from the orthodox realm of systems. The theory can be reinterpreted in such a way that the signs 'p,' 'q,' etc., resume their status of variables denoting elements of the system, without return to the fiction of propositions as denotations of sentences. We can reconstrue the theory of deduction as a branch of semantic, a system whose elements are shapes, signs, specifically sentences. (Quine (1934), p. 475.)

On this view, formulae are not sentences; they merely denote sentences. To turn a wf into a sentence, the symbol '$\vdash$' ('turnstile') is introduced, 'which may be read as a predicate to the effect that the element denoted in its wake is a true (i.e. truthful, truth- telling) sentence'. This system, turnstile included, can be reinterpreted and considered in the abstract. So we no longer have any anomaly in 'the theory of systems', and nor do we need to maintain that sentences denote.

In proposing this strategy, Quine simultaneously recommends that we treat the elements as sentences, rather than, say, propositions. But the main idea could have considerable appeal to someone who is happy to speak of propositions, but wants to regard them as being *expressed*, rather than *denoted*, by sentences.

I will conclude this section with a look at an intriguing strategy for making claims using formulae of PC, employed in Russell (1903). It is likely to be of interest as a curiosity, and as further evidence of the great variety of possible strategies, but it could also be used as a reply to Quine's charge that the denotative approach makes for an anomalous system, whose elements can only be whatever sentences denote. It is an ingenious and outside-the-square idea, characteristic of the early Russell.

Sentences are taken to denote propositions. Atoms are treated as variables; not special propositional variables, but ordinary term variables. Connectives are treated as being grammatically like predicates. So expressions like '$p \supset q$' become open sentences which can be universally quantified. These quantifiers are completely unrestricted (which is what makes this a possible reply to Quine).

Theorems such as '$p \vee \sim p$', as they stand, still do not yield truths when quantified; it is not the case that, for absolutely all objects p, p disjoins with (p is false). (Remember: sentences are taken to denote.) Russell adds, to each theorem, a prefix containing, for each of its atoms p, '$(p \supset p)$', followed by a '$\supset$'. Thus '$p \vee \sim p$' becomes '$(p \supset p) \supset (p \vee \sim p)$'. Then quantifying, we get '$\forall p[(p \supset p) \supset (p \vee \sim p)]$', or translated into words:

For all p, (p materially implies p) materially implies (p disjoins with (p is false)).

Now this, according to the early Russell, is a completely general truth. The idea is that for values of p which are not propositions, e.g. Russell himself, the quantified sentence comes out vacuously true. So, the following instance is true according to Russell:

(Russell materially implies Russell) materially implies (Russell disjoins with (Russell is false)).

Russell, not being a proposition, does not materially imply himself, so the first parenthesis is false, i.e. denotes a false proposition. Thus the whole sentence, being a material implication with a false antecedent, comes out true.

This is a notable approach in its generality and fixedness: at no point is anything assigned to, or conventionally associated with, any formula.[2] Wittgenstein explicitly criticized it in a parenthetical remark in the Tractatus:

> (It is nonsense to place the hypothesis '$p \supset p$' in front of a proposition, in order to ensure that its arguments shall have the right form, if only because with a non-proposition as argument the hypothesis becomes not false but nonsensical, and because arguments of the wrong kind make the proposition itself nonsensical, so that it preserves itself from wrong arguments just as well, or as badly, as the hypothesis without sense that was appended for that purpose.) (Wittgenstein (1922, §5.5351.)

2 The modelling approach

The denotational, abbreviational and truth-conditional approaches have in common that they are supposed to make formulae represent by themselves - more specifically, as linguistic objects which refer or have truth-values. On this approach, however, formulae don't denote, abbreviate or have truth-conditions. Rather, they are reckoned as elements of an abstract structure which can be stipulated into a representational relation to the world. This can be done using either valuation- or proof-theory. I will first explain this using valuation-theory, and will then show how the same thing can be done using proof-theory.

Here is an example of a similar kind of non-linguistic representation: near where I live, there is a fire danger sign consisting of a half-dial and an arrow to indicate the level of danger. (The arrow is manually re-positioned as perceived conditions change.) The arrow by itself does not mean anything, and neither does the dial. But *that* the arrow is in such-and-such a position does mean something. Similarly, on the modelling approach to PC, a formula by itself does not mean anything, but *that* it has such-and- such a value (1 or 0, in the classical case) can mean something, i.e. have worldly consequences.

[2] For an illuminating, extended treatment of the early Russell's 'universalist' approach to logic see Korhonen (2013). You may not want to say that this strategy of Russell's brings PC formulae into a representational relation to the world, insofar as you don't think the resulting sentences represent the world as being one way rather than another, but it is clearly supposed to make *claims* (in some broad sense) using PC formulae. Also, you may think the strategy fails on its own terms insofar as sentences just don't denote.

But there is also an important difference between the fire-danger dial and PC here: the fire-danger dial is a model which we have to configure manually and directly. If we say 'Let the fire-danger dial correctly represent the current level of fire danger', this incantation will, of course, achieve nothing. We *are* able to make an analogous kind of stipulation with respect to PC conceived as an abstract structure, however. This is a sort of modelling whereby the model (if successfully configured to represent at all) will always be correct, although we may not know, and may even be wrong about, what state it is in. This should become clearer with some illustrations. I will explain the valuation-theoretic version of the approach first.

In order to see how the abstract machinery works, let us first consider a non-representational case where we simply stipulate values for atoms directly:

Let $\mathrm{v}(p) = 1$ and $\mathrm{v}(q) = 0$.

This can be read as 'Let the value of "p" be 1 and the value of "q" be 0'. (In the symbolic definition, formulae are used autonymously, i.e. to name themselves. The values, incidentally, don't have to be 1 and 0 - any two objects will do.) Given this stipulation, we can talk truth-aptly about the values of 'p' and 'q'. Values for all compounds made up of those atoms are then determined via the familiar "truth-tables". (I use scare-quotes because these tables, in this use, are not to be regarded as having to do with the truth of sentences or propositions, but simply as rules for the valuation of compound wfs, where this is regarded as non-linguistic abstract machinery. To emphasize this, I will henceforth use 'value-tables' instead.). Thus, in virtue of the above stipulation, $\mathrm{v}(\sim p) = 0$, $\mathrm{v}(q) = 1$, $\mathrm{v}(p \vee q) = 1$, $\mathrm{v}(p \wedge q) = 0$, etc. So, if we don't know which values were stipulated, but are told for example that $\mathrm{v}(\sim p) = 0$, we can "work backwards" via the value-tables and conclude that $\mathrm{v}(p) = 1$. Or if we are told that $\mathrm{v}(p \vee q) = 1$, we can work out that either $\mathrm{v}(p) = 1$, or $\mathrm{v}(q) = 1$, or both.

In order to bring this apparatus into a representational relation to the world, we stipulate the values of the atoms conditionally. For example:

Let $\mathrm{v}(p) = 1$ if snow is white, 0 otherwise.
Let $\mathrm{v}(q) = 1$ if grass is green, 0 otherwise.

With these stipulations in force, one can then assert things about the values of formulae involving 'p' and 'q', and these assertions will have implications concerning the colours of snow and grass. For example, if someone asserted that $\mathrm{v}(p) = 0$, most of us would disagree, since that assertion now carries, via the stipulation, the implication that it is not the case that snow is white. Likewise if someone asserted that $\mathrm{v}(p \wedge q) = 0$: this, via the value-tables, implies that it is not the case that v(p) = 1 and v(q) = 1, which implies that it is not the case that snow is white and grass is green. Statements about the values of formulae could in this way be used for the purpose of conveying information.

Now let us see how an analogous thing can be done using proof-theory. Any proof-theory which induces the classical consequence relation on PC formulae will do.[3] We do not proceed by associating formulae with one of two objects. Corresponding to that stipulative basis of the valuation-theoretic interpretation, we have in this case the population of a 'start-set' S according to the following convention: for each atom A we are going to use for representing, exactly one of either A or the tilde of A is to be a member of S. As before, we shall illustrate the machinery before using it representatively. In place of our stipulation that v(p) = 1 and v(q) = 0, we have:

Let $p \ni$ S and let $\sim q \ni$ S.

Given this stipulation, it becomes true to say that 'p' and '$\sim q$' follow from S (trivially) via whatever proof-theory we have chosen to work with. In symbols: S $\vdash p$ and S $\vdash \sim q$. It also becomes true to say that, e.g., S $\vdash (p \vee q)$, S $\nvdash (p \supset q)$, S $\vdash \sim(p \supset q)$, etc.[4] As before, for a representational configuration, we stipulate conditionally:

Let $p \ni$ S if snow is white, and let $\sim p \ni$ S otherwise.

Let $q \ni$ S if grass is green, and let $\sim q \ni$ S otherwise.

With these stipulations in force, we can assert that certain formulae involving 'p' and 'q' follow from the start-set, and these assertions will have implications concerning the colours of grass and snow. For example, the statement that S $\vdash p$ implies, via the stipulation, that snow is white. Furthermore, if we come to believe that, e.g., S $\vdash (p \vee q)$, we can work out via our preferred proof-theory that either S $\vdash p$ or S $\vdash q$ or both, and thus that snow is white or grass is green (or both). This working out will typically be easiest in tree systems, followed by natural-deduction systems (in virtue of their having separate rules for each connective). In an axiom system which allows assumptions, the

[3]In principle, you could use valuation-theory to induce the consequence relation, but then the apparatus of start-sets which follows above would be overkill; you could just use the valuation-theoretic version of the modelling approach explained previously.

[4]It may be wondered why we need to include tilded atoms (e.g. '$\sim q$') in S - that is, why it doesn't suffice that 'q' is not in S. This is because we need compounds such as '$\sim(p \supset q)$' to follow from S, and nothing follows from the absence of a formula (on the classical consequence relation, at least).

working out will be possible in principle but often very involved.[5] Soundness and completeness make value-tabular methods indirectly available also.

To recap: on the modelling approach, formulae have no linguistic meaning, but are reckoned as forming, together with valuation-theory, or some proof-theory plus the start-set apparatus, elements of an abstract structure which can be stipulated into a representational relation to the world.

In addition to the inherent philosophical interest of the possibility of the modelling approach, I have found the approach to be helpful in thinking about nature of formal logic,[6] and more specifically, about valuation-theory, proof-theory and the close relationship between the two. Furthermore, it seems to me that looking at things this way may help us to get clearer about whether, or to what extent, the classical consequence relation encodes the meanings of the connectives[7] - but that thought calls for a paper of its own.[8]

References

Bonnay, Denis & Westerståhl, Dag (2016). Compositionality Solves Carnap's Problem. *Erkenntnis* 81 (4):721-739.

Bradley, F. H. (1893). *Appearance and Reality: A Metaphysical Essay.* Oxford University Press.

Frege, Gottlob. (1892/1951). On concept and object. *Mind* 60 (238):168-180. Trans. Geach, P. T. & Black, Max.

Frege, Gottlob (1892/2010). On sense and reference. In Darragh Byrne & Max Kölbel (eds.), *Arguing About Language.* Routledge. pp. 36-56.

Korhonen, Anssi. (2013). *Logic as Universal Science: Russell's Early Logicism and its Philosophical Context.* Palgrave Macmillan.

Murzi, Julien & Hjortland, Ole Thomassen (2009). Inferentialism and the categoricity problem: Reply to Raatikainen. *Analysis* 69 (3):480-488.

[5] In an axiom system which does not allow assumptions, an amended strategy would be required, exploiting the correspondence between the consequence relation and '$\supset$' formulae. The start-set could in this case be thought of as a set of additional axioms.

[6] Note that, on this approach, we can alter or mangle our value tables, rules or axioms in any way we please, but this will never lead us into falsity. We might lose "expressive power", and the worst that can happen is that the system becomes completely trivial and can no longer represent anything. This is reminiscent of the remark in the *Tractatus*, 'In a certain sense, we cannot make mistakes in logic.' (§5.473).

[7] See Carnap (1947), Raatikainen (2008), Murzi & Hjortland (2009), Peregrin (2010), Bonnay & Westerståhl (2016).

[8] Thanks to Adrian Heathcote for provoking my thoughts in this area, and to N.J.J. Smith for encouragement and comments on the modelling approach.

Peregrin, Jaroslav (2010). Inferentializing Semantics. *Journal of Philosophical Logic* 39 (3):255 - 274.

Quine, W. V. (1934). Ontological remarks on the propositional calculus. *Mind* 43 (172):472-476.

Raatikainen, Panu (2008). On rules of inference and the meanings of logical constants. *Analysis* 68 (300):282-287.

Russell, Bertrand (1903). *Principles of Mathematics*. Cambridge University Press.

Wittgenstein, Ludwig (1922). *Tractatus Logico-Philosophicus*. Dover Publications.

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