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Quantified Counterfactual Temporal Alethic-Deontic Logic

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Abstract

This paper will introduce and explore a set of quantified counterfactual temporal alethic-deontic systems, that is, systems that combine counterfactual temporal alethic-deontic logic with predicate logic. I will consider three types of systems: constant, variable and constant and variable domain systems. Every system can be combined with either necessary or contingent identity. All logics are described both semantically and proof theoretically. I use a kind of possible world semantics, inspired by the so-called T × W semantics, to characterise them semantically and semantic tableaux to characterise them proof theoretically. Our models contain several different accessibility relations and a similarity relation between possible worlds, which are used in the definitions of the truth conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of these.

Keywords: Quantified modal logic; T × W logics; counterfactuals; temporal logic; deontic logic; semantic tableaux.

1 Introduction

In this paper, I will describe a set of quantified counterfactual temporal alethic-deontic systems, that is, systems that combine counterfactual temporal alethic-deontic logic with predicate logic. The paper introduces three types of systems: constant, variable and constant and variable domain systems. Every system can be combined with either necessary or contingent identity. I will use both semantic and proof theoretic techniques to describe the systems. The semantics is inspired by the so-called T × W approach and it is a kind of possible world semantics. The paper uses semantic tableaux to characterise all logics proof theoretically. Our models contain several different accessibility relations and a similarity relation between possible worlds, which are used in the definitions of the truth-conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of these.

The systems described in this essay are extensions of systems introduced in the author's earlier papers [29, 31, 32], which include information about non-quantified temporal alethic-deontic logic, quantified temporal alethic-deontic logic, and counterfactual temporal alethic-deontic logic, respectively. These articles also contain many references to the relevant literature. For more background information, see [30].[1]

As far as I know, no one has described any systems of the kind introduced in this paper. Hence, since it is not a trivial task to combine predicate logic and counterfactual temporal alethic-deontic logic, the present study is theoretically well motivated. There are also good philosophical reasons to develop logical systems that include counterfactual, temporal, alethic and deontic operators, and quantifiers, for we seem to need such systems to be able to analyse several interesting principles and arguments and make important distinctions. In our systems we can, for example, formalise several different types of normative principles and investigate their properties and implications. Consider the following examples. (1) For every x: if x is a human person, then it is not permitted that it will some time in the future be the case that x is killed [$\Pi x(Hx \to \neg P\underline{F}Kx)$], (2) For every x: it ought to be the case that if x is a human person, then it is always going to be the case that x is not killed [$\Pi xO(Hx \to \underline{G}\neg Kx)$], (3) It is (absolutely) necessary, that for every x: if x is a human person, then it is not permitted that it will some time in the future be the case that x is killed [$\mathrm{U}\Pi x(Hx \to \neg P\underline{F}Kx)$], and (4) For every x: if x were a human person, then it would be the case that it is not permitted that x will some time in the future be killed. [$\Pi x(Hx \mathrel{\Box\!\!\to} \neg P\underline{F}Kx)$]. Without a formal system, it is very difficult to get a good grip on such principles.

Now, consider the following argument.

The universalisability argument

(1) It is permitted that you will steal from Susan only if it is the case that if you were in Susan's situation and Susan were in your situation then it would be permitted that Susan will steal from you.

[1] Lewis [23] and Lewis and Langford [24] include some brief remarks about quantified modal logic. Early pioneers when it comes to combining modal and predicate logic include Barcan (Barcan-Marcus) [1, 2] and Carnap [4, 5]. Since the 50s, several philosophers and logicians have been interested in the relationships between these branches of logic, e.g. Kanger [18], Kripke [19, 20, 21, 22] and Hintikka [13, 14, 15, 16]. Other early contributions include: [3], [10], [17], [26], [27], [33]. Introductions to quantified modal logic can be found in e.g.: [6], [9], [11, 12], [17] and [28]. The literature contains few attempts to combine predicate logic with systems including both temporal and alethic concepts and even fewer attempts to combine predicate logic with systems that contain temporal, alethic and deontic concepts (but see e.g. [8] and [34]). Most of these early contributions are axiomatic.

(2) If you were in Susan's situation and Susan were in your situation, it would be obligatory that it is always going to be the case that Susan does not steal from you.

(3) For every x and y, it is absolutely necessary that: if x steals money from y, then x steals from y.

(4) It is (synchronistically) possible that: you are in Susan's situation and Susan is in your situation.

Consequently,

(5) It is obligatory that it is always going to be the case that you do not steal money from Susan.

This argument is intuitively valid. It appears to be necessary that the conclusion is true if the premises are true. However, it seems to be impossible to show this in any existing systems in the literature. In Section 6, I will establish that the conclusion is derivable from the premises in every system that includes $T-c3$ (Table 12) and that the argument therefore is valid in the class of all $C-c3$-models. $C-c3$ says that for every possible world w_j and for every moment in time t_l, if A is true in some possible world w_i at t_l, then there is a possible world w_k such that w_k is R_A accessible from w_j at t_l (see [32]). This is a good reason to be interested in the systems in this paper.

The paper is divided into six sections. Section 2 is about syntax and Section 3 about semantics. In Section 4, I describe a set of tableau rules and tableau systems. I also mention some theorems that can be proved in various systems. Section 5 includes soundness and completeness theorems. Finally, in Section 6, I show that the universalisability argument is valid in the class of all $C-c3$-models.

2 Syntax

2.1 Alphabet

Our languages will be constructed from the following alphabet: (i) A set of variables x_0, x_1, x_2, x_3 …, (ii) a set of (non-temporal, rigid) constants c_0, c_1, c_2, c_3 …, (iii) a set NT of names of times (temporal constants) t_0, t_1, t_2, t_3 …, (iv) for every natural number n, n-place predicate symbols P^0_n, P^1_n, P^2_n, P^3_n …, (v) the monadic existence predicate E, (vi) the dyadic identity predicate $=$, (vii) the primitive truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence), (viii) the alethic operators U, M, $\Box$, $\Diamond$, $\boxdot$ and $\diamonddot$, (ix) the temporal operators R (followed by a name in NT), $\underline{A}$, $\underline{S}$, $\underline{G}$, $\underline{H}$, $\underline{F}$ and $\underline{P}$, (x) the deontic operators O and P, (xi) the "possibilist" quantifiers Π, Σ, (xii) the "actualist" quantifiers $\forall$, $\exists$, (xiii) the counterfactual operators $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$, (xiv) the brackets (,).

I will use x, y and z for arbitrary variables, a, b, c for arbitrary (non-temporal) rigid constants, and t for an arbitrary temporal constant (name in NT) (possibly with primes or subscripts). I will not consider any language with non-rigid constants (descriptors) in this essay. Note that I also use s and t (with or without primes or subscripts) for arbitrary terms. I will use F_n, G_n, H_n for arbitrary n-place predicates and I will omit the subscript if it can be read off from the context.

2.2 Languages

I will consider several languages in this essay. They are all constructed from the following clauses: (i) Any (non-temporal, rigid) constant or variable is a term. (ii) If $t_1, \ldots, t_n$ are any terms and P is any n-place predicate, $Pt_1 \ldots t_n$ is an atomic formula. (iii) If t is a term, Et ("t exists") is an atomic formula. (iv) If s and t are terms, then $s = t$ ("s is identical with t") is an atomic formula. (v) If A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$. (vi) if A is a formula, then UA ("it is universally (or absolutely) necessary that A"), MA ("it is universally (or absolutely) possible that A"), $\Box A$ ("it is (historically) necessary (or settled) that A"), $\Diamond A$ ("it is (historically) possible (or open) that A"), $\boxdot A$ ("it is synchronistically (or temporally) necessary that A"), $\diamonddot A$ ("it is synchronistically (or temporally) possible that A"), $\underline{A}A$ ("It is always the case that A"), $\underline{S}A$ ("It is sometimes the case that A"), $\underline{G}A$ ("it is always going to be the case that A"), $\underline{H}A$ ("it has always been the case that A"), $\underline{F}A$ ("it will some time in the future be the case that A"), $\underline{P}A$ ("it was some time in the past the case that A"), OA ("it ought to be the case that A") and PA ("it is permitted that A") are formulas. (vii) if A is a formula and t is in NT, then RtA ("it is realized at time t that A") is a formula. (viii) If A and B are formulas, so are $(A \Box\!\!\rightarrow B)$ ("If it were the case that A, then it would be the case that B"), and $(A \Diamond\!\!\rightarrow B)$ ("If it were the case that A, then it might be the case that B"). (ix) If A is any formula and x is any variable, then ΠxA ("For every (possible) x: A") and ΣxA ("For some (possible) x: A") are formulas. (x) If A is any formula and x is any variable, then $\forall xA$ ("For every (existing) x: A") and $\exists xA$ ("For some (existing) x: A") are formulas. (xi) Nothing else is a formula.

A, B, C stand for arbitrary formulas, and Γ, Φ for sets of formulas. The concepts of bound and free variable, open and closed formula, are defined in the usual way. $(A)[t/x]$ is the formula obtained by substituting t for every free occurrence of x in A. The definition is standard. Brackets around formulas are usually dropped if the result is not ambiguous. $\mathcal{L}$ stands for a language. In constant domain systems, $\mathcal{L}$ includes the possibilist and not the actualist quantifiers; in variable domain systems, $\mathcal{L}$ includes the actualist and not the possibilist quantifiers, etc. (see Section 3). It will be obvious from the context which language '$\mathcal{L}$' denotes.

2.3 Definitions

It is possible to add the definitions introduced by [31, 32] to all our constant, variable and constant and variable systems in this essay. We can use an arbitrary tautology for $\top$ and an arbitrary contradiction for $\bot$. In Section 4.4, I will mention some theorems that include the operators $\Box\!\!\Rightarrow$ ($A \Box\!\!\Rightarrow B = (A \Diamond\!\!\rightarrow \top) \wedge (A \Box\!\!\rightarrow B)$) and $\Diamond\!\!\Rightarrow$ ($A \Diamond\!\!\Rightarrow B = \neg(A \Box\!\!\Rightarrow \neg B)$ or $(A \Box\!\!\rightarrow \bot) \vee (A \Diamond\!\!\rightarrow B)$).

3 Semantics

3.1 Constant domain semantics

Definition 1 *A (supplemented quantified counterfactual temporal alethic-deontic) constant domain model, $\mathcal{M}_s$, is a relational structure $\langle D, W, T, <, R, S, \{R_A : A \in \mathcal{L}\} \geq, v\rangle$, where D is a non-empty set of objects, W is a non-empty set of possible worlds, T is a non-empty set of times, $<$ is a binary relation on T ($< \subseteq T \times T$), R and S are two ternary accessibility relations ($R \subseteq W \times W \times T$ and $S \subseteq W \times W \times T$), $\{R_A : A \in \mathcal{L}\}$ is a set of ternary counterfactual accessibility relations, one for each sentence, A, in the language $\mathcal{L}$ ($R_A \subseteq W \times W \times T$), $\geq$ is a ternary similarity relation defined over the elements in W ($\geq \subseteq W \times W \times W$), and v is an interpretation function.*

I will usually drop the subscript if it is clear that I am talking about a supplemented model. By deleting $\geq$ from the structure, we obtain an ordinary, unsupplemented model $\mathcal{M}$.

R "corresponds" to the alethic operators $\Box$ and $\Diamond$, $<$ to the temporal operators $\underline{G}$, $\underline{F}$, $\underline{H}$ and $\underline{P}$, S to the deontic operators O and P, and R_A and $\geq$ to the counterfactual operators $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$. Informally, $\tau < \tau'$ says that the time τ is before the time τ' (or, equivalently, that τ' is later than τ), $R\omega\omega'\tau$ that the possible world ω' is alethically accessible from the possible world ω at time τ, and $S\omega\omega'\tau$ that ω' is deontically accessible from ω at τ. $R_A\omega\omega'\tau$ says that the possible world ω' is A-accessible from the possible world ω at time τ, and $\omega \geq_{\omega'} \omega''$ that the possible world ω is at least as similar to ("near" to) world ω' as is world ω'' ([32]). In a supplemented model, R_A can be defined in terms of $\geq$ (see [32] for more on this). v assigns each temporal name, t, in NT a time, $v(t)$, in T, each (non-temporal, rigid) constant, c, an element, $v(c)$, of D, and each pair comprising a world-moment pair, $\langle\omega, \tau\rangle$, and an n-place predicate, P, a subset, $v_{\omega\tau}(P)$ (the extension of P in ω at τ), of D^n. Hence, the extension of a predicate may change from world-moment pair to world-moment pair and it may be empty at a world-moment pair. In other words, $v_{\omega\tau}(P)$ is the set of n-tuples that satisfy P in the world ω at time τ (in the world-moment pair $\langle\omega,\tau\rangle$). The language of a model $\mathcal{M}$, $\mathcal{L}(\mathcal{M})$, is obtained by adding a constant k_d, such that $v(k_d) = d$, to the language for every member $d \in D$.

All constant domain systems include the "possibilist" quantifiers and no other quantifiers.

Every closed formula, A, is assigned exactly one truth value (1 = True or 0 = False), $v_{\omega\tau}(A)$, in each world ω at every time τ (in each world-moment pair $\langle\omega,\tau\rangle$). Here are the truth conditions for some sentences in our language (the truth conditions for the omitted formulas are the usual ones (see [32])).

(i)	$v_{\omega\tau}(Pa_1 \dots a_n) = 1$	iff	$\langle v(a_1),\dots,v(a_n)\rangle \in v_{\omega\tau}(P)$,
(ii)	$v_{\omega\tau}(A \wedge B) = 1$	iff	$v_{\omega\tau}(A) = 1$ and $v_{\omega\tau}(B) = 1$,
(iii)	$v_{\omega\tau}(A \mathrel{\Box\!\!\rightarrow} B) = 1$	iff	$\forall\omega' \in W$ s.t. $R_A\omega\omega'\tau$: $v_{\omega'\tau}(B) = 1$,
(iv)	$v_{\omega\tau}(A \mathrel{\Diamond\!\!\rightarrow} B) = 1$	iff	$\exists\omega' \in W$ s.t. $R_A\omega\omega'\tau$: $v_{\omega'\tau}(B) = 1$,
(v)	$v_{\omega\tau}(\Pi xA) = 1$	iff	for all $d \in D, v_{\omega\tau}(A[k_d/x]) = 1$,
(vi)	$v_{\omega\tau}(\Sigma xA) = 1$	iff	for some $d \in D, v_{\omega\tau}(A[k_d/x]) = 1$.

3.2 Variable domain semantics

Definition 2 *A (supplemented quantified counterfactual temporal alethic-deontic) variable domain model, $\mathcal{M}_s$, is a relational structure $\langle D, W, T, <, R, S, \{R_A : A \in \mathcal{L}\} \geq, v\rangle$, where D, W, T, $<$, R, S, $\{R_A : A \in \mathcal{L}\}$, $\geq$ and v are the same as in the constant domain case, except that for every world-moment pair $\langle\omega,\tau\rangle$, where $\omega \in W$ and $\tau \in T$, v maps $\langle\omega,\tau\rangle$ to a subset, $v(\omega\tau)$, of D.*

The domain of a world-moment pair, $v(\omega\tau)$ or $D_{\omega\tau}$, is the set of all things we quantify over in this world at this time. It is often reasonable to think of the domain of a world-moment pair as the class of all things that exist in this world at this time. For any n-place predicate, P, $v_{\omega\tau}(P) \subseteq D^n$ (not $D^n_{\omega\tau}$), and $v_{\omega\tau}(E)$ is $D_{\omega\tau}$. Accordingly, the extension of a predicate at a world-moment pair may change from world-moment pair to world-moment pair, it may include things that are not in the domain of this world-moment pair, and it may be empty at some world-moment pair. Even though $D_{\omega\tau}$ may be empty, D is still non-empty. The constants in our language may denote something in a world-moment pair that is not in the domain of this world-moment pair. Again, if it is clear that I am talking about a supplemented model, I will often drop the subscript. By deleting $\geq$ from the structure, we obtain an ordinary, unsupplemented model $\mathcal{M}$. In a supplemented model, R_A can be defined in terms of $\geq$, as in the constant semantics.

All variable domain systems include the existence predicate E and the "actualist" quantifiers; the possibilist quantifiers are not included in the variable systems.

The truth conditions for the "actualist" quantifiers are as follows:

(vii) $v_{\omega\tau}(\exists x A) = 1$ iff for some $d \in D_{\omega\tau}$, $v_{\omega\tau}(A[k_d/x]) = 1$,

(viii) $v_{\omega\tau}(\forall x A) = 1$ iff for all $d \in D_{\omega\tau}$, $v_{\omega\tau}(A[k_d/x]) = 1$.

The truth conditions for other sentences in our language are as in the constant domain case (Section 3.1).

3.3 Constant and variable domain semantics

The constant and variable domain semantics is the same as the variable domain semantics, except that all systems based on this kind include both the possibilist and the actualist quantifiers. A constant and variable domain model is exactly the same as a variable domain model. The difference between variable domain and constant and variable domain systems is syntactic. In a variable domain system we cannot define the possibilist quantifiers. But if we add the existence predicate, E, to a constant domain system, we can define the actualist quantifiers (see [31]).

3.4 Necessary identity semantics

I will now consider what happens when we add the identity predicate to our languages. I will investigate two kinds of semantics for the predicate: necessary and contingent. Every constant, variable, and constant and variable system can be combined either with necessary identity or with contingent identity. According to the necessary identity semantics the denotation of the identity predicate is the same at every world-moment pair in a model, i.e. $v_{\omega\tau}(=) = \{\langle d, d\rangle : d \in D\}$. This is exactly as in the quantified temporal alethic-deontic logic described by [31]. In our systems in this paper we need one more condition. I will call this the Accessibility Denotation Constraint (*ADC*) (as in Priest [28], Chapter 19). For all formulas A, and (rigid, non-temporal) constants in the language, a and b:

(*ADC*) if $v(a) = v(b)$, then $R_{A[a/x]} = R_{A[b/x]}$.

Without this constraint, the proof of the Denotation Lemma in our soundness and completeness theorems will not go through; and $Fa \mathrel{\Box\!\!\rightarrow} A$ might be true while $Fb \mathrel{\Box\!\!\rightarrow} A$ is false at a world-moment pair, even though $v(a) = v(b)$.

Let us now turn to the semantics for our contingent identity systems.

3.5 Contingent identity semantics

Definition 3 *A (supplemented quantified counterfactual temporal alethic-deontic, constant, variable or constant and variable domain) model with contingent identity, $\mathcal{M}_s$, is a relational structure $\langle D, H, W, T, <, R, S, \{R_A : A \in \mathcal{L}\} \geq, v\rangle$, where D, W, T, $<$, R, S, $\{R_A : A \in \mathcal{L}\}$, $\geq$ and v are the same as in the constant,* *variable or constant and variable domain cases, with the following exception. The elements of D are now functions from $W \times T$ to H.*

Note that D is still non-empty. However, D need not include every possible function from $W \times T$ to H. I will call the objects in H *substrata* or *manifestations*. If $d \in D$, $\omega \in W$ and $\tau \in T$, I shall say that $d(\langle \omega, \tau \rangle)$, or $|d|_{\omega\tau}$, is the manifestation or substratum of d at the world-moment pair $\langle \omega, \tau \rangle$. For every (non-temporal, rigid) constant, c, $v(c) \in D$, and for every world-moment pair, $\langle \omega, \tau \rangle$, and n-place predicate, P, $v_{\omega\tau}(P)$ is a subset of H^n, not D^n. In other words, the constants refer to objects in D, while the extensions of predicates include objects from H. The interpretation of the identity predicate, $v_{\omega\tau}(=)$, is the world-moment-invariant set $\{\langle h, h \rangle : h \in H\}$. Let $\mathcal{M}$ be a variable domain model. Then $v(\omega\tau) = D_{\omega\tau} = \{d \in D : |d|_{\omega\tau} \in v_{\omega\tau}(E)\}$. As usual, if it is clear that I am talking about a supplemented model, I will sometimes drop the subscript. By deleting $\geq$ from the structure, we obtain an ordinary, unsupplemented model $\mathcal{M}$.

The truth conditions for closed atomic formulas are as follows:

$v_{\omega\tau}(Pa_1 \ldots a_n) = 1$ iff $\langle |v(a_1)|_{\omega\tau}, \ldots, |v(a_n)|_{\omega\tau} \rangle \in v_{\omega\tau}(P)$.

For all the other sentences, the truth conditions remain the same.

3.6 Fundamental semantic concepts, conditions on frames and models, the logic of a class of models etc.

The concepts of validity, satisfiability, logical consequence etc. are essentially defined as in [29], [31] and [32]. The definitions are the same for all our semantics.

In [29], [31] and [32] various frame- and modelconditions were mentioned. All of these conditions may also be imposed on our quantified counterfactual temporal alethic-deontic models in this paper, with the exception that the conditions on the valuation function in [29] are replaced by the conditions in [31]. Due to considerations of space, I will not repeat these conditions in the present essay. As usual these conditions can be used to obtain a categorisation of the set of all models into various kinds, and these classes can then be used to define a set of logical systems. For more on this, see [29], [31] and [32]. I use the same conventions for naming systems in this essay as in [29], [31] and [32].

Without further ado, let us turn to our proof theory.

4 Proof theory

4.1 Semantic tableaux

In this section, I will develop a set of semantic tableau systems. For more information about the tableau method, see e.g. D'Agostino, Gabbay, Hähnle and Posegga [7] and Priest [28].

The concepts of semantic tableau, branch, open and closed branch etc. are essentially defined as in [29], [31] and [32].

4.2 Tableau rules

This section contains a large set of tableau rules that are used to construct a set of tableau systems. Most of these rules were introduced by [29], [31] and [32]. However, in our quantified counterfactual temporal alethic-deontic systems with necessary identity we add a rule called the Accessibility Denotation Rule (*ADR*) as in [28], Chapter 19 (see Section 4.2.16 below). For more information about these rules and a list of some derived rules, see [29], [31] and [32].

4.2.1 Propositional rules

I use the same propositional rules as in e.g. Priest [28] modified in an obvious way. I call them ($\wedge$), ($\neg\wedge$) etc.

4.2.2 Basic alethic rules (b a-rules)

U	M	$\neg U$	$\neg M$
$UA, w_i t_j$ $\downarrow$ $A, w_k t_l$ for any w_k and t_l	$MA, w_i t_j$ $\downarrow$ $A, w_k t_l$ where w_k and t_l are new	$\neg UA, w_i t_j$ $\downarrow$ $M\neg A, w_i t_j$	$\neg MA, w_i t_j$ $\downarrow$ $U\neg A, w_i t_j$
$\Box$	$\Diamond$	$\neg\Box$	$\neg\Diamond$
$\Box A, w_i t_k$ $rw_i w_j t_k$ $\downarrow$ $A, w_j t_k$	$\Diamond A, w_i t_k$ $\downarrow$ $rw_i w_j t_k$ $A, w_j t_k$ where w_j is new	$\neg\Box A, w_i t_j$ $\downarrow$ $\Diamond\neg A, w_i t_j$	$\neg\Diamond A, w_i t_j$ $\downarrow$ $\Box\neg A, w_i t_j$
$\boxdot$	$⟐$	$\neg\boxdot$	$\neg⟐$
$\boxdot A, w_i t_k$ $\downarrow$ $A, w_j t_k$	$⟐ A, w_i t_k$ $\downarrow$ $A, w_j t_k$ where w_j is new	$\neg\boxdot A, w_i t_k$ $\downarrow$ $⟐\neg A, w_i t_k$	$\neg⟐ A, w_i t_k$ $\downarrow$ $\boxdot\neg A, w_i t_k$

Table 1

4.2.3 Basic deontic rules (b d-rules)

The basic d-rules look exactly like the basic a-rules for $\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$, except that $\Box$ is replaced by O, $\Diamond$ by P, and r by s. I give them similar names.

4.2.4 Basic temporal rules (b t-rules), Id(I) and Id(II)

$\underline{A}$	$\neg\underline{A}$	$\underline{S}$	$\neg\underline{S}$
$\underline{A}A, w_it_j$ $\downarrow$ A, w_it_k for every t_k on the branch	$\neg\underline{A}A, w_it_j$ $\downarrow$ $\underline{S}\neg A, w_it_j$	$\underline{S}A, w_it_j$ $\downarrow$ A, w_it_k where t_k is new to the branch	$\neg\underline{S}A, w_it_j$ $\downarrow$ $\underline{A}\neg A, w_it_j$
$\underline{G}$	$\neg\underline{G}$	$\underline{F}$	$\neg\underline{F}$
$\underline{G}A, w_it_j$ $t_j < t_k$ $\downarrow$ A, w_it_k	$\neg\underline{G}A, w_it_j$ $\downarrow$ $\underline{F}\neg A, w_it_j$	$\underline{F}A, w_it_j$ $\downarrow$ $t_j < t_k$ A, w_it_k where t_k is new	$\neg\underline{F}A, w_it_j$ $\downarrow$ $\underline{G}\neg A, w_it_j$
$\underline{H}$	$\neg\underline{H}$	$\underline{P}$	$\neg\underline{P}$
$\underline{H}A, w_it_j$ $t_k < t_j$ $\downarrow$ A, w_it_k	$\neg\underline{H}A, w_it_j$ $\downarrow$ $\underline{P}\neg A, w_it_j$	$\underline{P}A, w_it_j$ $\downarrow$ $t_k < t_j$ A, w_it_k where t_k is new	$\neg\underline{P}A, w_it_j$ $\downarrow$ $\underline{H}\neg A, w_it_j$

Table 2

Rt	$\neg Rt$	$Id(I)$	$Id(II)$
Rt_iA, w_jt_k $\downarrow$ A, w_jt_i	$\neg Rt_iA, w_jt_k$ $\downarrow$ $Rt_i\neg A, w_jt_k$	$A(t_i)$ $t_i = t_j$ $\downarrow$ $A(t_j)$	$A(t_i)$ $t_j = t_i$ $\downarrow$ $A(t_j)$

Table 3

4.2.5 Basic counterfactual rules (b c-rules)

$\Box\!\!\rightarrow$	$\Diamond\!\!\rightarrow$	$\neg\ \Box\!\!\rightarrow$	$\neg\ \Diamond\!\!\rightarrow$
$A \Box\!\!\rightarrow B, w_it_k$ $r_Aw_iw_jt_k$ $\downarrow$ B, w_jt_k	$A \Diamond\!\!\rightarrow B, w_it_k$ $\downarrow$ $r_Aw_iw_jt_k$ B, w_jt_k where w_j is new	$\neg(A \Box\!\!\rightarrow B), w_it_k$ $\downarrow$ $A \Diamond\!\!\rightarrow \neg B, w_it_k$	$\neg(A \Diamond\!\!\rightarrow B), w_it_k$ $\downarrow$ $A \Box\!\!\rightarrow \neg B, w_it_k$

Table 4

4.2.6 CUT, CId(I), CId(II)

CUT	$CId\ (CIdI)$	$CId\ (CIdII)$
$*$ $\swarrow \searrow$ $\neg A, w_i t_k \quad A, w_i t_k$ for every A	$\alpha(w_i)$ $w_i = w_j$ $\downarrow$ $\alpha(w_j)$	$\alpha(w_i)$ $w_j = w_i$ $\downarrow$ $\alpha(w_j)$

Table 5

4.2.7 Alethic accessibility rules (a-rules)

$T - aD$	$T - aT$	$T - aB$	$T - a4$	$T - a5$
$w_i t_k$ $\downarrow$ $rw_i w_j t_k$ where w_j is new	$w_i t_j$ $\downarrow$ $rw_i w_i t_j$	$rw_i w_j t_k$ $\downarrow$ $rw_j w_i t_k$	$rw_i w_j t_l$ $rw_j w_k t_l$ $\downarrow$ $rw_i w_k t_l$	$rw_i w_j t_l$ $rw_i w_k t_l$ $\downarrow$ $rw_j w_k t_l$

Table 6

4.2.8 Temporal accessibility rules (t-rules)

$T - t4$	$T - PD$	$T - FD$
$t_i < t_j$ $t_j < t_k$ $\downarrow$ $t_i < t_k$	t_j $\downarrow$ $t_k < t_j$ where t_k is new	t_j $\downarrow$ $t_j < t_k$ where t_k is new
$T - DE$	$T - FC$	$T - PC$
$t_i < t_j$ $\downarrow$ $t_i < t_k$ $t_k < t_j$ where t_k is new	$t_i < t_j$ $t_i < t_k$ $\swarrow \downarrow \searrow$ $t_j < t_k \ \ t_j = t_k \ \ t_k < t_j$	$t_j < t_i$ $t_k < t_i$ $\swarrow \downarrow \searrow$ $t_j < t_k \ \ t_j = t_k \ \ t_k < t_j$
$T - C$	$T - UB$	$T - LB$
t_i, t_j $\swarrow \quad \downarrow \quad \searrow$ $t_i < t_j \quad t_i = t_j \quad t_j < t_i$	$t_i < t_j$ $t_i < t_k$ $\downarrow$ $t_j < t_l$ $t_k < t_l$ where t_l is new to the branch	$t_j < t_i$ $t_k < t_i$ $\downarrow$ $t_l < t_j$ $t_l < t_k$ where t_l is new to the branch

Table 7

4.2.9 Deontic accessibility rules (d-rules)

$T - dD$	$T - d4$	$T - d5$	$T - dT'$	$T - dB'$
$w_i t_k$ $\downarrow$ $sw_i w_j t_k$ where w_j is new	$sw_i w_j t_l$ $sw_j w_k t_l$ $\downarrow$ $sw_i w_k t_l$	$sw_i w_j t_l$ $sw_i w_k t_l$ $\downarrow$ $sw_j w_k t_l$	$sw_i w_j t_l$ $\downarrow$ $sw_j w_j t_l$	$sw_i w_j t_l$ $sw_j w_k t_l$ $\downarrow$ $sw_k w_j t_l$

Table 8

4.2.10 Alethic-deontic accessibility rules (ad-rules)

$T - MO$	$T - MO'$	$T - OC$	$T - OC'$	
$sw_i w_j t_k$ $\downarrow$ $rw_i w_j t_k$	$sw_i w_j t_l$ $sw_j w_k t_l$ $\downarrow$ $rw_j w_k t_l$	$w_i t_k$ $\downarrow$ $sw_i w_j t_k$ $rw_i w_j t_k$ where w_j is new	$sw_i w_j t_l$ $\downarrow$ $rw_j w_k t_l$ $sw_j w_k t_l$ where w_k is new	
$T - ad4$	$T - ad5$	$T - PMP$	$T - OMP$	$T - MOP$
$rw_i w_j t_l$ $sw_j w_k t_l$ $\downarrow$ $sw_i w_k t_l$	$rw_i w_j t_l$ $sw_i w_k t_l$ $\downarrow$ $sw_j w_k t_l$	$sw_i w_j t_m$ $rw_i w_k t_m$ $\downarrow$ $rw_j w_l t_m$ $sw_k w_l t_m$ where w_l is new	$rw_i w_j t_m$ $sw_j w_k t_m$ $\downarrow$ $sw_i w_l t_m$ $rw_l w_k t_m$ where w_l is new	$sw_i w_j t_m$ $rw_j w_k t_m$ $\downarrow$ $rw_i w_l t_m$ $sw_l w_k t_m$ where w_l is new

Table 9

4.2.11 Rules concerning R, S, $<$ and v (adt-rules)

$T - FT$	$T - BT$	$T - SP$
$A, w_i t_k$ $rw_i w_j t_k$ $\downarrow$ $A, w_j t_k$ where A is atomic	$A, w_j t_k$ $rw_i w_j t_k$ $\downarrow$ $A, w_i t_k$ where A is atomic	$rw_i w_j t_l$ $t_k < t_l$ $\downarrow$ $rw_i w_j t_k$
$T - SR$	$T - PI$	$T - WPI$
$sw_i w_j t_l$ $t_l < t_m$ $sw_j w_k t_m$ $\downarrow$ $sw_j w_k t_l$	$sw_i w_j t_l$ $t_l < t_m$ $rw_j w_k t_m$ $\downarrow$ $sw_j w_k t_m$	$sw_i w_j t_k$ $t_k < t_l$ $rw_i w_j t_l$ $\downarrow$ $sw_i w_j t_l$

Table 10

4.2.12 Counterfactual (accessibility) rules (c-rules)

$T-c0$	$T-c0'$
If D is of the form $\boxdot(A \leftrightarrow B) \rightarrow$ $((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$, $D, w_i t_l$ can be added to any open branch on which $w_i t_l$ occurs.	If D is of the form $((A \Box\!\!\rightarrow B) \wedge (B \Box\!\!\rightarrow A)) \rightarrow$ $((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$, $D, w_i t_l$ can be added to any open branch on which $w_i t_l$ occurs.

Table 11

$T-c1$	$T-c2$	$T-c3$	$T-c4$
$r_A w_i w_j t_l$ $\downarrow$ $A, w_j t_l$	$r_A w_i w_j t_l$ $B, w_j t_l$ $\downarrow$ $r_{A\wedge B} w_i w_j t_l$	$A, w_i t_l$ $\downarrow$ $r_A w_j w_k t_l$ where w_k is new	$r_A w_i w_j t_l$ $B, w_j t_l$ $r_{A\wedge B} w_i w_k t_l$ $\downarrow$ $r_A w_i w_k t_l$ $B, w_k t_l$
$T-c5$	$T-c6$	$T-c7$	
$A, w_i t_l$ $\downarrow$ $r_A w_i w_i t_l$	$A, w_i t_l$ $r_A w_i w_j t_l$ $\downarrow$ $w_i = w_j$	$r_A w_i w_j t_l$ $r_A w_i w_k t_l$ $\downarrow$ $w_j = w_k$	

Table 12

4.2.13 Possibilist quantifiers

Π	Σ	$\neg\Pi$	$\neg\Sigma$
$\Pi x A, w_i t_j$ $\downarrow$ $A[a/x], w_i t_j$ for every constant a on the branch, a new if there are no constants on the branch	$\Sigma x A, w_i t_j$ $\downarrow$ $A[c/x], w_i t_j$ where c is new to the branch	$\neg\Pi x A, w_i t_j$ $\downarrow$ $\Sigma x \neg A, w_i t_j$	$\neg\Sigma x A, w_i t_j$ $\downarrow$ $\Pi x \neg A, w_i t_j$

Table 13

4.2.14 Actualist quantifiers

$\forall$	$\exists$	$\neg\forall$	$\neg\exists$
$\forall xA, w_i t_j$ $\swarrow\searrow$ $\neg Ea, w_i t_j \quad A[a/x], w_i t_j$ for every constant a on the branch, a new if there are no constants on the branch	$\exists xA, w_i t_j$ $\downarrow$ $Ec, w_i t_j$ $A[c/x], w_i t_j$ where c is new to the branch	$\neg\forall xA, w_i t_j$ $\downarrow$ $\exists x\neg A, w_i t_j$	$\neg\exists xA, w_i t_j$ $\downarrow$ $\forall x\neg A, w_i t_j$

Table 14

4.2.15 Domain-inclusion (Barcan) rules

$T-ABF$	$T-DBF$	$T-TBF$
$Ea, w_j t_k$ $rw_i w_j t_k$ $\downarrow$ $Ea, w_i t_k$	$Ea, w_j t_k$ $sw_i w_j t_k$ $\downarrow$ $Ea, w_i t_k$	$Ea, w_i t_k$ $t_j < t_k$ $\downarrow$ $Ea, w_i t_j$
$T-ACBF$	$T-DCBF$	$T-TCBF$
$Ea, w_i t_k$ $rw_i w_j t_k$ $\downarrow$ $Ea, w_j t_k$	$Ea, w_i t_k$ $sw_i w_j t_k$ $\downarrow$ $Ea, w_j t_k$	$Ea, w_i t_j$ $t_j < t_k$ $\downarrow$ $Ea, w_i t_k$

Table 15

4.2.16 Identity rules and *ADR*

$T-R=$	$T-S=$	$T-N=$	ADR
$*$ $\downarrow$ $t = t, w_i t_j$ for every t on the branch	$s = t, w_i t_j$ $A[s/x], w_i t_j$ $\downarrow$ $A[t/x], w_i t_j$ where A is atomic	$a = b, w_i t_j$ $\downarrow$ $a = b, w_k t_l$ for any w_k and t_l	$a = b, w_0 t_0$ $r_{A[a/x]} w_i w_j t_k$ $\downarrow$ $r_{A[b/x]} w_i w_j t_k$

Table 16

ADR, the Accessibility Denotation Rule, is added to any necessary identity system. It is required by the Accessibility Denotation Condition (*ADC*) (see Section 3.4).

4.3 Some proof-theoretical concepts, tableau systems, the logic of a tableau system etc.

Basic proof-theoretical concepts such as proof, theorem, derivation, consistency, inconsistency in a system, the logic of a tableau system etc. are defined as in [29], [31] and [32].

The concepts of a constant, variable or constant and variable counterfactual temporal alethic-deontic tableau system etc. are obvious extensions of similar concepts in [29], [31] and [32]. Necessary identity systems include $T-R=$, $T-S=$, $T-N=$ and ADR; contingent identity systems include $T-R=$ and $T-S=$. Note that ADR is not needed in contingent identity systems. For more information about the classification of various tableau systems, see [29], [31] and [32].

Let us now turn to some examples of sentences that can be proved in our tableau systems.

4.4 Examples of theorems

I will consider some theorems and non-theorems that tell us something about the interaction between the quantifiers and the counterfactual operators. Many other sentences that can be proved in our systems are introduced by [29], [31] and [32].

$\Pi x(Fa \mathrel{\Box\!\!\rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Box\!\!\rightarrow} \Pi x Gx)$	$\Sigma x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Diamond\!\!\rightarrow} \Sigma x Gx)$
$\Sigma x(Fa \mathrel{\Box\!\!\rightarrow} Gx) \rightarrow (Fa \mathrel{\Box\!\!\rightarrow} \Sigma x Gx)$	$(Fa \mathrel{\Diamond\!\!\rightarrow} \Pi x Gx) \rightarrow \Pi x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx)$

Table 17

Theorem 4 *Every sentence in Table 17 is a theorem in every constant and constant and variable system in this paper. (They are not theorems in the variable systems, since these systems do not include the possibilist quantifiers.)*

$\Pi x(Fa \mathrel{\Box\!\!\Rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Box\!\!\Rightarrow} \Pi x Gx)$	$\Sigma x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Diamond\!\!\Rightarrow} \Sigma x Gx)$
$\Sigma x(Fa \mathrel{\Box\!\!\Rightarrow} Gx) \rightarrow (Fa \mathrel{\Box\!\!\Rightarrow} \Sigma x Gx)$	$(Fa \mathrel{\Diamond\!\!\Rightarrow} \Pi x Gx) \rightarrow \Pi x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx)$

Table 18

Theorem 5 *Every sentence in Table 18 is a theorem in every constant and constant and variable system in this paper that includes the following definitions:* $A \mathrel{\Box\!\!\Rightarrow} B = (A \mathrel{\Diamond\!\!\rightarrow} \top) \wedge (A \mathrel{\Box\!\!\rightarrow} B)$ *(this is an alternative analysis of "If A were the case, then B would be the case"), and* $A \mathrel{\Diamond\!\!\Rightarrow} B = \neg(A \mathrel{\Box\!\!\Rightarrow} \neg B)$ *(or* $(A \mathrel{\Box\!\!\rightarrow} \bot) \vee (A \mathrel{\Diamond\!\!\rightarrow} B)$*) (this is an alternative explication of "If A were the case, then B might be the case") (see [32], and also [25], Chapter 1).*

$\forall x(Fa \mathrel{\Box\!\!\rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Box\!\!\rightarrow} \forall xGx)$	$\exists x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Diamond\!\!\rightarrow} \exists xGx)$
$\exists x(Fa \mathrel{\Box\!\!\rightarrow} Gx) \rightarrow (Fa \mathrel{\Box\!\!\rightarrow} \exists xGx)$	$(Fa \mathrel{\Diamond\!\!\rightarrow} \forall xGx) \rightarrow \forall x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx)$
$\forall x(Fa \mathrel{\Box\!\!\Rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Box\!\!\Rightarrow} \forall xGx)$	$\exists x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx) \leftrightarrow (Fa \mathrel{\Diamond\!\!\Rightarrow} \exists xGx)$
$\exists x(Fa \mathrel{\Box\!\!\Rightarrow} Gx) \rightarrow (Fa \mathrel{\Box\!\!\Rightarrow} \exists xGx)$	$(Fa \mathrel{\Diamond\!\!\Rightarrow} \forall xGx) \rightarrow \forall x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx)$
$\forall x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx) \rightarrow (Fa \mathrel{\Diamond\!\!\rightarrow} \forall xGx)$	$(Fa \mathrel{\Box\!\!\rightarrow} \exists xGx) \rightarrow \exists x(Fa \mathrel{\Box\!\!\rightarrow} Gx)$
$\forall x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx) \rightarrow (Fa \mathrel{\Diamond\!\!\Rightarrow} \forall xGx)$	$(Fa \mathrel{\Box\!\!\Rightarrow} \exists xGx) \rightarrow \exists x(Fa \mathrel{\Box\!\!\Rightarrow} Gx)$

Table 19

Theorem 6 *The sentences in Table 19 are not provable in our weakest variable system and they are not provable in our weakest constant and variable system. (It is trivially true that they are not theorems in any constant system, since the constant systems do not include the actualist quantifiers.)*

My conjecture is that the sentences in Table 19 are unprovable in *every* system introduced in this paper.

$\Pi x(Fa \mathrel{\Diamond\!\!\rightarrow} Gx) \rightarrow (Fa \mathrel{\Diamond\!\!\rightarrow} \Pi xGx)$	$(Fa \mathrel{\Box\!\!\rightarrow} \Sigma xGx) \rightarrow \Sigma x(Fa \mathrel{\Box\!\!\rightarrow} Gx)$
$\Pi x(Fa \mathrel{\Diamond\!\!\Rightarrow} Gx) \rightarrow (Fa \mathrel{\Diamond\!\!\Rightarrow} \Pi xGx)$	$(Fa \mathrel{\Box\!\!\Rightarrow} \Sigma xGx) \rightarrow \Sigma x(Fa \mathrel{\Box\!\!\Rightarrow} Gx)$

Table 20

Theorem 7 *The sentences in Table 20 are not provable in our weakest constant system and they are not provable in our weakest constant and variable system. (It is trivially true that they are unprovable in every variable system, since the variable systems do not include the possibilist quantifiers.)*

My conjecture is that the sentences in Table 20 are unprovable in most systems introduced in this essay. However, if our logic includes $T-c7$, we can prove all the formulas in this table. So, we cannot conclude that they are not theorems in any system.

$\Pi x(Ha \Box\!\!\rightarrow (Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\rightarrow \Pi x Fx) \wedge (Ha \Box\!\!\rightarrow \Pi x Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \wedge Gx)) \leftrightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \wedge \Pi x(Ha \Box\!\!\rightarrow Gx))$
$\Sigma x(Ha \Diamond\!\!\rightarrow (Fx \vee Gx)) \leftrightarrow ((Ha \Diamond\!\!\rightarrow \Sigma x Fx) \vee (Ha \Diamond\!\!\rightarrow \Sigma x Gx))$
$(Ha \Diamond\!\!\rightarrow \Sigma x(Fx \vee Gx)) \leftrightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \vee \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$(\Pi x(Ha \Box\!\!\rightarrow Fx) \vee \Pi x(Ha \Box\!\!\rightarrow Gx)) \rightarrow (Ha \Box\!\!\rightarrow \Pi x(Fx \vee Gx))$
$(Ha \Diamond\!\!\rightarrow \Sigma x(Fx \wedge Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \wedge \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$((Ha \Box\!\!\rightarrow \Pi x Fx) \vee (Ha \Box\!\!\rightarrow \Pi x Gx)) \rightarrow \Pi x(Ha \Box\!\!\rightarrow (Fx \vee Gx))$
$\Sigma x(Ha \Diamond\!\!\rightarrow (Fx \wedge Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma x Fx) \wedge (Ha \Diamond\!\!\rightarrow \Sigma x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Pi x Fx) \rightarrow (Ha \Box\!\!\rightarrow \Pi x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Pi x Fx) \rightarrow (Ha \Diamond\!\!\rightarrow \Pi x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Sigma x Fx) \rightarrow (Ha \Box\!\!\rightarrow \Sigma x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma x Fx) \rightarrow (Ha \Diamond\!\!\rightarrow \Sigma x Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \rightarrow \Pi x(Ha \Box\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow (\Pi x(Ha \Diamond\!\!\rightarrow Fx) \rightarrow \Pi x(Ha \Diamond\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow (\Sigma x(Ha \Box\!\!\rightarrow Fx) \rightarrow \Sigma x(Ha \Box\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \rightarrow \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Pi x Fx) \leftrightarrow (Ha \Box\!\!\rightarrow \Pi x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Pi x Fx) \leftrightarrow (Ha \Diamond\!\!\rightarrow \Pi x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Sigma x Fx) \leftrightarrow (Ha \Box\!\!\rightarrow \Sigma x Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma x Fx) \leftrightarrow (Ha \Diamond\!\!\rightarrow \Sigma x Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \leftrightarrow \Pi x(Ha \Box\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow (\Pi x(Ha \Diamond\!\!\rightarrow Fx) \leftrightarrow \Pi x(Ha \Diamond\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow (\Sigma x(Ha \Box\!\!\rightarrow Fx) \leftrightarrow \Sigma x(Ha \Box\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \leftrightarrow \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$

Table 21

Theorem 8 *Every sentence in Table 21 is a theorem in every constant and constant and variable system in this paper.*

Theorem 9 *Let A be a sentence in Table 21 and let $t(A)$ be the result of substituting every occurrence of $\Box\!\!\rightarrow$ by an occurrence of $\Box\!\!\Rightarrow$ and every occurrence of $\Diamond\!\!\rightarrow$ by an occurrence of $\Diamond\!\!\Rightarrow$. E.g. if $A = \Pi x(Ha \Box\!\!\rightarrow (Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\rightarrow \Pi x Fx) \wedge (Ha \Box\!\!\rightarrow \Pi x Gx))$, then $t(A) = \Pi x(Ha \Box\!\!\Rightarrow (Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\Rightarrow \Pi x Fx) \wedge (Ha \Box\!\!\Rightarrow \Pi x Gx))$. Then $t(A)$ is a theorem in every constant and constant and variable system that includes the definitions of $\Box\!\!\Rightarrow$ and $\Diamond\!\!\Rightarrow$.*

Theorem 10 *Let A be a sentence in Table 21 and let $t(A)$ be the result of substituting every occurrence of Π by an occurrence of $\forall$ and every occurrence of Σ by an occurrence of $\exists$. E.g. if $A = \Pi x(Ha \Box\!\!\rightarrow (Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\rightarrow \Pi x Fx) \wedge (Ha \Box\!\!\rightarrow \Pi x Gx))$, then $t(A) = \forall x(Ha \Box\!\!\rightarrow (Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\rightarrow \forall x Fx) \wedge (Ha \Box\!\!\rightarrow \forall x Gx))$. Then $t(A)$ is not a theorem in our weakest variable system and $t(A)$ is not a theorem in our weakest constant and variable system.*

(It is trivially true that $t(A)$ is not at theorem in any constant system, since constant systems do not include the actualist quantifiers.)

My conjecture is that this theorem (Theorem 10) can be extended to *every* system in this paper.

$\Pi x(Ha \Box\!\!\rightarrow (Fx \wedge Gx)) \leftrightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \wedge \Pi x(Ha \Box\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \wedge Gx)) \leftrightarrow ((Ha \Box\!\!\rightarrow \Pi xFx) \wedge (Ha \Box\!\!\rightarrow \Pi xGx))$
$\Sigma x(Ha \Diamond\!\!\rightarrow (Fx \vee Gx)) \leftrightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \vee \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$(Ha \Diamond\!\!\rightarrow \Sigma x(Fx \vee Gx)) \leftrightarrow ((Ha \Diamond\!\!\rightarrow \Sigma xFx) \vee (Ha \Diamond\!\!\rightarrow \Sigma xGx))$
$(\Pi x(Ha \Box\!\!\rightarrow Fx) \vee \Pi x(Ha \Box\!\!\rightarrow Gx)) \rightarrow \Pi x(Ha \Box\!\!\rightarrow (Fx \vee Gx))$
$(Ha \Diamond\!\!\rightarrow \Sigma x(Fx \wedge Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma xFx) \wedge (Ha \Diamond\!\!\rightarrow \Sigma xGx))$
$((Ha \Box\!\!\rightarrow \Pi xFx) \vee (Ha \Box\!\!\rightarrow \Pi xGx)) \rightarrow (Ha \Box\!\!\rightarrow \Pi x(Fx \vee Gx))$
$\Sigma x(Ha \Diamond\!\!\rightarrow (Fx \wedge Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \wedge \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \rightarrow \Pi x(Ha \Box\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow (\Pi x(Ha \Diamond\!\!\rightarrow Fx) \rightarrow \Pi x(Ha \Diamond\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow (\Sigma x(Ha \Box\!\!\rightarrow Fx) \rightarrow \Sigma x(Ha \Box\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \rightarrow Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \rightarrow \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Pi xFx) \rightarrow (Ha \Box\!\!\rightarrow \Pi xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Pi xFx) \rightarrow (Ha \Diamond\!\!\rightarrow \Pi xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Sigma xFx) \rightarrow (Ha \Box\!\!\rightarrow \Sigma xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \rightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma xFx) \rightarrow (Ha \Diamond\!\!\rightarrow \Sigma xGx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow (\Pi x(Ha \Box\!\!\rightarrow Fx) \leftrightarrow \Pi x(Ha \Box\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow (\Pi x(Ha \Diamond\!\!\rightarrow Fx) \leftrightarrow \Pi x(Ha \Diamond\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow (\Sigma x(Ha \Box\!\!\rightarrow Fx) \leftrightarrow \Sigma x(Ha \Box\!\!\rightarrow Gx))$
$\Pi x(Ha \Box\!\!\rightarrow (Fx \leftrightarrow Gx)) \rightarrow (\Sigma x(Ha \Diamond\!\!\rightarrow Fx) \leftrightarrow \Sigma x(Ha \Diamond\!\!\rightarrow Gx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Pi xFx) \leftrightarrow (Ha \Box\!\!\rightarrow \Pi xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Pi xFx) \leftrightarrow (Ha \Diamond\!\!\rightarrow \Pi xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow ((Ha \Box\!\!\rightarrow \Sigma xFx) \leftrightarrow (Ha \Box\!\!\rightarrow \Sigma xGx))$
$(Ha \Box\!\!\rightarrow \Pi x(Fx \leftrightarrow Gx)) \rightarrow ((Ha \Diamond\!\!\rightarrow \Sigma xFx) \leftrightarrow (Ha \Diamond\!\!\rightarrow \Sigma xGx))$

Table 22

Theorem 11 *(i) All sentences in Table 22 are theorems in every constant and constant and variable system in this essay. (ii) Let A be a sentence in Table 22 and let $t(A)$ be the result of substituting every occurrence of Π by an occurrence of $\forall$ and every occurrence of Σ by an occurrence of $\exists$. Then $t(A)$ is a theorem in every variable and constant and variable system in this paper. (iii) Let A be a sentence in Table 22 and let $t(A)$ be the result of substituting every occurrence of $\Box\!\!\rightarrow$ by an occurrence of $\Box\!\!\Rightarrow$ and every occurrence of $\Diamond\!\!\rightarrow$ by an occurrence of $\Diamond\!\!\Rightarrow$. Then $t(A)$ is a theorem in every constant and constant and variable system that includes the definitions of $\Box\!\!\Rightarrow$ and $\Diamond\!\!\Rightarrow$. (iv) Let A be a sentence in Table 22 and let $t(A)$ be the result of substituting every occurrence* *of* Π *by an occurrence of* $\forall$, *every occurrence of* Σ *by an occurrence of* $\exists$, *every occurrence of* $\Box\!\!\rightarrow$ *by an occurrence of* $\Box\!\!\Rightarrow$ *and every occurrence of* $\Diamond\!\!\rightarrow$ *by an occurrence of* $\Diamond\!\!\Rightarrow$. *Then* $t(A)$ *is a theorem in every variable and constant and variable system the includes the definitions of* $\Box\!\!\Rightarrow$ *and* $\Diamond\!\!\Rightarrow$.

5 Soundness and completeness theorems

The concepts of soundness and completeness and the concept of a system corresponding to a class of models are defined as usual, see e.g. [29], [31] and [32].

This section proves that all systems in this paper are sound with respect to their corresponding unsupplemented class of models and that all systems not including $Tc0$ or $Tc0'$ are complete with respect to their corresponding unsupplemented class of models. The question whether the remaining systems are complete is left open. Many systems are also sound with respect to certain classes of supplemented models (see Theorem 14, part (ii)). The proofs in this section are modifications and extensions of proofs found in [29], [31] and [32]. Most parts can safely be omitted, but some steps also require some new techniques.[2]

We begin by assuming that identity is not in the language.

5.1 Locality and Denotation Lemmas

Lemma 12 ***(Locality Lemma).*** *Let* $\mathcal{M}_1 = \langle D, W, T, <, R, S, \{R_A : A \in \mathcal{L}\}, \geq, v_1\rangle$ *and* $\mathcal{M}_2 = \langle D, W, T, <, R, S, \{R_A : A \in \mathcal{L}\}, \geq, v_2\rangle$ *be two models. (We could equally well use the corresponding unsupplemented models.) The language of the two, call this* $\mathcal{L}$, *is the same. For they have the same domain. If* A *is any closed formula of* $\mathcal{L}$ *such that* v_1 *and* v_2 *agree on the denotations of all the predicates and constants in it, then for all* $\omega \in W$ *and* $\tau \in T$:
$v_{1\omega\tau}(A) = v_{2\omega\tau}(A)$.

Proof. The proofs are as in [31], except that we have to check that the lemma holds for the counterfactual connectives and $\boxdot$ and $\diamonddot$. This is easy. (IH = induction hypothesis, "for all ω'" means "for all $\omega' \in W$".)

$v_{1\omega\tau}(A \Box\!\!\rightarrow B) = 1$ iff for all ω' such that $R_A\omega\omega'\tau$, $v_{1\omega'\tau}(B) = 1$ iff for all ω' such that $R_A\omega\omega'\tau$, $v_{2\omega'\tau}(B) = 1$ [IH] iff $v_{2\omega\tau}(A \Box\!\!\rightarrow B) = 1$.

$v_{1\omega\tau}(\boxdot B) = 1$ iff for all ω': $v_{1\omega'\tau}(B) = 1$ iff for all ω': $v_{2\omega'\tau}(B) = 1$ [IH] iff $v_{2\omega\tau}(\boxdot B) = 1$.

[2]The proofs in this section are similar to ones found in [28]. However, since the counterfactual rules I use are not the same as the ones in [28], since many systems include the *CUT* rule and since our systems are embedded in a temporal dimension, there are also important differences.

The cases for $\diamond\!\!\rightarrow$ and $\diamond$ are similar. ■

Lemma 13 ***(Denotation Lemma).*** *Let $\mathcal{M} = \langle D, W, T, <, R, S, \{R_A : A \in \mathcal{L}\}, \geq, v_1\rangle$ be any model. (We could equally well use the corresponding unsupplemented model.) Let A be any formula of $\mathcal{L}(\mathcal{M})$ with at most one free variable, x, and let a and b be any two constants such that $v(a) = v(b)$. Then for any $\omega \in W$ and $\tau \in T$: $v_{\omega\tau}(A[a/x]) = v_{\omega\tau}(A[b/x])$.*

Proof. $v_{\omega\tau}((A \Box\!\!\rightarrow B)[a/x]) = 1$ iff $v_{\omega\tau}(A[a/x] \Box\!\!\rightarrow B[a/x]) = 1$ iff for all ω' such that $R_{A[a/x]}\omega\omega'\tau$, $v_{\omega'\tau}(B[a/x]) = 1$ iff for all ω' such that $R_{A[b/x]}\omega\omega'\tau$, $v_{\omega'\tau}(B[b/x]) = 1$ [IH, ADC] iff $v_{\omega\tau}(A[b/x] \Box\!\!\rightarrow B[b/x]) = 1$ iff $v_{\omega\tau}((A \Box\!\!\rightarrow B)[b/x]) = 1$.

$v_{\omega\tau}(\boxdot B[a/x]) = 1$ iff for all ω': $v_{\omega'\tau}(B[a/x]) = 1$ iff for all ω': $v_{\omega'\tau}(B[b/x]) = 1$ [IH] iff $v_{\omega\tau}(\boxdot B[b/x]) = 1$.

The cases for $\diamond\!\!\rightarrow$ and $\diamond$ are similar. The rest of the proof is as in [31]. ■

5.2 Soundness theorems

Theorem 14 ***(Soundness Theorem).*** *(i) All our constant, variable and constant and variable quantified counterfactual temporal alethic-deontic systems in this paper are (strongly) sound with respect to their corresponding unsupplemented models. (ii) Let S be a system in [32] (Theorem 7) that is (strongly) sound with respect to a certain class of supplemented models. Then the constant, variable and constant and variable quantified counterfactual temporal alethic-deontic versions of S are (strongly) sound with respect to their corresponding supplemented models.*

Proof. The proofs are similar to arguments found in [29], [31] and [32]. The details are tedious, but straightforward. ■

5.3 Completeness theorems

Theorem 15 ***(Completeness Theorem).*** *All constant, variable and constant and variable quantified counterfactual temporal alethic-deontic systems in this essay not including $Tc0$ or $Tc0'$ are (strongly) complete with respect to their corresponding class of unsupplemented models.*

Proof. The proofs are similar to ones found in [29], [31] and [32]. One important difference is that we must replace ω_i, ω_j, ... etc. with $\omega_{[i]}$, $\omega_{[j]}$, ...

etc. in many steps. But the modifications are straightforward. Furthermore, there is one new case: we have to check that the induced model satisfies the *ADC*. Suppose that $v(a) = v(b)$ in the induced model. All constants have distinct denotations, since identity is not included in our systems yet. Hence, a and b are the same constants. Accordingly, for any A, $A[a/x] = A[b/x]$. It follows that $R_{A[a/x]} = R_{A[b/x]}$, as required. ■

5.4 Soundness with necessary identity

I will now consider what happens when we add identity to our systems. I begin with necessary identity and then turn to contingent identity. The Locality and Denotation Lemmas still hold and their proofs are unaffected.

Theorem 16 ***(Soundness Theorem Necessary Identity).*** *Let S be any system in this essay (without identity). Then S + the rules for necessary identity and ADR is (strongly) sound with respect to its semantics (variable, constant or variable and constant).*

Proof. The proofs modify the arguments in [31]. The only new interesting step is that we must show that the Soundness Lemma holds for *ADR*. Assume that we have $a = b, w_0 t_0$ and $r_{A[a/x]} w_i w_j t_k$ on an open branch $\mathcal{B}$ and that we apply *ADR* and obtain $r_{A[b/x]} w_i w_j t_k$. Furthermore, suppose that f and g show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Then $a = b$ is true in $f(w_0)$ at $g(t_0)$. Hence, $v(a) = v(b)$, and $R_{A[a/x]} f(w_i) f(w_j) g(t_k)$. By the *ADC*, $R_{A[a/x]} = R_{A[b/x]}$. So, $R_{A[b/x]} f(w_i) f(w_j) g(t_k)$. Consequently, we may take $\mathcal{M}'$ to be $\mathcal{M}$. ■

5.5 Completeness with necessary identity

In [29], [31] and [32] several definitions of the concept of an induced model were introduced. Before we can prove our completeness theorem for necessary identity systems, we must first combine and modify these definitions of an induced model slightly. Instead of ω_i, ω_j, ... etc. we must often use $\omega_{[i]}$, $\omega_{[j]}$, ... etc. This modification is straightforward.

Furthermore, and more importantly, for every A, R_A is defined as follows. Say that A and A' are coidenticals if for some a and b such that $a \sim b$ (see [31]), A is of the form $B[a/x]$ and A' is of the form $B[b/x]$. Then:

$R_A \omega_{[i]} \omega_{[j]} \tau_{[k]}$ iff $r_{A'} w_i w_j t_k$ is on the branch $\mathcal{B}$ for some coidentical, A', of A.[3]

[3]This definition is similar to a definition used by Priest [28], Chapter 19. However, in our completeness proofs we can employ the present, simpler definition instead.

Note that being coidenticals is an equivalence relation. Now we can prove our completeness theorem for our necessary identity systems.

Theorem 17 ***(Completeness Theorem Necessary Identity).*** *Let S be any system in this essay (without identity) not including $Tc0$ or $Tc0'$. Then S + the rules for necessary identity and ADR is (strongly) complete with respect to its semantics (variable, constant or variable and constant).*

Proof. The proofs modify the arguments in [29], [31] and [32].

In addition, we need to check that the model satisfies the ADC.

So suppose that $v(a) = v(b)$ and $R_{A[a/x]}\omega_{[i]}\omega_{[j]}\tau_{[k]}$. Then $a = b, w_0t_0$ is on $\mathcal{B}$, and $r_{A[c/x]}w_iw_jt_k$ is on $\mathcal{B}$, where $A[c/x]$ is some coidentical of $A[a/x]$ [by the definition of an induced model]. $R_{A[b/x]}\omega_{[i]}\omega_{[j]}\tau_{[k]}$ iff $r_{A[d/x]}w_iw_jt_k$ is on $\mathcal{B}$, where $A[d/x]$ is some coidentical of $A[b/x]$ [by the definition of an induced model]. Since $a = b, w_0t_0$, $a = c, w_0t_0$ and $b = d, w_0t_0$ are on $\mathcal{B}$, so is $c = d, w_0t_0$ [by the identity rules and the fact that the branch is complete]. Hence, $r_{A[d/x]}w_iw_jt_k$ is on $\mathcal{B}$ [by ADR and the fact that the branch is complete]. It follows that $R_{A[b/x]}\omega_{[i]}\omega_{[j]}\tau_{[k]}$, as required.

In the Completeness Lemma, the step for $\Box\!\!\rightarrow$ is as follows. Suppose that $A \Box\!\!\rightarrow C, w_it_k$ is on $\mathcal{B}$, and $R_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$. Then for some coidentical, A', of A, $r_{A'}w_iw_jt_k$ is on $\mathcal{B}$. By ADR, $r_Aw_iw_jt_k$ is on $\mathcal{B}$. Accordingly, C, w_jt_k is on $\mathcal{B}$, and C is true in $\omega_{[j]}$ at $\tau_{[k]}$ by IH, as required. Suppose that $\neg(A \Box\!\!\rightarrow C), w_it_k$ is on $\mathcal{B}$. Then $(A \Diamond\!\!\rightarrow \neg C), w_it_k$ is on $\mathcal{B}$. Hence, for some w_j, $r_Aw_iw_jt_k$ and $\neg C, w_jt_k$ are on $\mathcal{B}$. Therefore, $R_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$ [by the definition of an induced model] and C is false in $\omega_{[j]}$ at $\tau_{[k]}$, the result follows by IH.

We must also show that all the different semantic constraints introduced in [29], [31] and [32] are satisfied if the corresponding rules are present. Here are some of the modified steps in this proof.

$(C - c2)$. Suppose that $R_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$ and B is true in $\omega_{[j]}$ at $\tau_{[k]}$. Then for some coidentical of A, A', $r_{A'}w_iw_jt_k$ occurs on $\mathcal{B}$ [by the definition of an induced model]. Since the tableau is complete CUT has been applied and either B, w_jt_k or $\neg B, w_jt_k$ is on $\mathcal{B}$. Suppose $\neg B, w_jt_k$ is on $\mathcal{B}$. Then B is false in $\omega_{[j]}$ at $\tau_{[k]}$ [by the completeness lemma]. But this is impossible. Hence, B, w_jt_k is on $\mathcal{B}$. By ADR, $r_Aw_iw_jt_k$ occurs on $\mathcal{B}$. Since $\mathcal{B}$ is complete, $T - c2$ has been applied and $r_{A\wedge B}w_iw_jt_k$ occurs on $\mathcal{B}$. Accordingly, $R_{A\wedge B}\omega_{[i]}\omega_{[j]}\tau_{[k]}$ as required [by the definition of an induced model].

$(C - c6)$. Suppose that A is true in $\omega_{[i]}$ at $\tau_{[k]}$ and that $R_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$. Then for some coidentical of A, A', $r_{A'}w_iw_jt_k$ occurs on $\mathcal{B}$ [by the definition of an induced model]. Since the tableau is complete CUT has been applied and either A, w_it_k or $\neg A, w_it_k$ is on $\mathcal{B}$. Suppose $\neg A, w_it_k$ is on $\mathcal{B}$. Then A is false in $\omega_{[i]}$ at $\tau_{[k]}$ [by the completeness lemma]. But this is impossible. Accordingly, A, w_it_k is on $\mathcal{B}$. By ADR, $r_Aw_iw_jt_k$ occurs on $\mathcal{B}$. Since the tableau is complete

$T - c6$ has been applied and $w_i = w_j$ is on $\mathcal{B}$. Hence, $i \approx j$. So, $[i] = [j]$. It follows that $\omega_{[i]} = \omega_{[j]}$, as required.

($C - c7$). Assume that $R_A\omega_{[i]}\omega_{[j]}\tau_{[l]}$ and $R_A\omega_{[i]}\omega_{[k]}\tau_{[l]}$. Then for some coidentical of A, A', $r_{A'}w_iw_jt_l$ occurs on $\mathcal{B}$, and for some coidentical of A, A'', $r_{A''}w_iw_kt_l$ occurs on $\mathcal{B}$ [by the definition of an induced model]. Since the tableau is complete $r_Aw_iw_jt_l$ occurs on $\mathcal{B}$ and $r_Aw_iw_kt_l$ occurs on $\mathcal{B}$ [by ADR]. Again, since the tableau is complete $T - c7$ has been applied and $w_j = w_k$ is on $\mathcal{B}$. Accordingly, $j \approx k$. Hence, $[j] = [k]$. It follows that $\omega_{[j]} = \omega_{[k]}$, as required. ■

5.6 Soundness with contingent identity

We now turn to our contingent identity systems. The Locality and Denotation Lemmas are formulated as for the necessary identity case. See [31] for a proof. The steps for the new operators are as in the proofs above.

Theorem 18 ***(Soundness Theorem Contingent Identity).*** *Let S be any system in this essay (without identity). Then S + the rules for contingent identity is (strongly) sound with respect to its semantics (variable, constant or variable and constant). (Note that ADR is not added to our contingent identity systems.)*

Proof. The proofs modify the arguments in [31]. ■

5.7 Completeness with contingent identity

Theorem 19 ***(Completeness Theorem Contingent Identity).*** *Let S be any system in this essay (without identity) not including $Tc0$ or $Tc0'$. Then S + the rules for contingent identity is (strongly) complete with respect to its semantics (variable, constant or variable and constant).*

Proof. The proof is similar to the completeness proofs for the necessary identity systems. However, now we must show that ADC is satisfied even though our contingent identity systems do not contain ADR. To do this, we use a trick described by [28], Chapter 19. When we read off information from open branches we ensure that each constant has a different denotation by taking an object (in our domain) to be a set of ordered triples <a, input, output> (rather than a set of ordered pairs <input, output>). It follows that if a and b are distinct constants, o_a (the object denoted by a) and o_b (the object denoted by b) are distinct. We can now show that ADC is satisfied as in the proof of Theorem 15 above.

Other steps in the derivation of this theorem are as in [29], [31] and [32] or above. ■

This completes the proofs of our soundness and completeness theorems in this essay.

6 Example of a valid argument

In this section, I will prove that the universalisability argument that we described in the introduction is valid in the class of all models that satisfy $C-c3$. To prove this, we first show that the conclusion in this argument is derivable from the premises in every system that includes $T-c3$. Then, we use the soundness theorem (Section 5) and conclude that the argument is valid in the class of all $C-c3$-models.

The universalisability argument can be symbolised in the following way.

(1) $P\underline{F}Sus \to ((Y \wedge A) \,\Box\!\!\to P\underline{F}Ssu)$ (It is permitted that you will steal from Susan only if it is the case that if you were in Susan's situation and Susan were in your situation then it would be permitted that Susan will steal from you).

(2) $(Y \wedge A) \,\Box\!\!\to O\underline{G}\neg Ssu$ (If you were in Susan's situation and Susan were in your situation, it would be obligatory that it is always going to be the case that Susan does not steal from you).

(3) $\Pi x \Pi y U(Txy \to Sxy)$ (For every x and y, it is absolutely necessary that: if x steals money from y, then x steals from y).

(4) $\Diamond(Y \wedge A)$ (It is (synchronistically) possible that: you are in Susan's situation and Susan is in your situation).

Consequently,

(5) $O\underline{G}\neg Tus$ (It is obligatory that it is always going to be the case that you do not steal money from Susan).

Here is our proof (MP is an abbreviation of Modus Ponens, which is a derived rule in all our systems).

$$
\begin{array}{c}
(1)\ P\underline{F}Sus \to ((Y \wedge A) \,\Box\!\!\to P\underline{F}Ssu), w_0t_0 \\
(2)\ (Y \wedge A) \,\Box\!\!\to O\underline{G}\neg Ssu, w_0t_0 \\
(3)\ \Pi x \Pi y U(Txy \to Sxy), w_0t_0 \\
(4)\ \Diamond(Y \wedge A), w_0t_0 \\
(5)\ \neg O\underline{G}\neg Tus, w_0t_0 \\
(6)\ P\neg\underline{G}\neg Tus, w_0t_0\ [5,\ \neg O] \\
(7)\ sw_0w_1t_0\ [6,\ P] \\
(8)\ \neg\underline{G}\neg Tus, w_1t_0\ [6,\ P] \\
(9)\ \underline{F}\neg\neg Tus, w_1t_0\ [8,\ \neg\underline{G}] \\
\swarrow \qquad\qquad \searrow
\end{array}
$$

(10) $\neg P\underline{F}Sus, w_0t_0$ [1, $\rightarrow$]
(12) $O\neg\underline{F}Sus, w_0t_0$ [10, $\neg P$]
(14) $\neg\underline{F}Sus, w_1t_0$ [12, 7, O]
(16) $\underline{G}\neg Sus, w_1t_0$ [14, $\neg\underline{F}$]
(18) $t_0 < t_1$ [9, $\underline{F}$]
(20) $\neg\neg Tus, w_1t_1$ [9, $\underline{F}$]
(22) $\neg Sus, w_1t_1$ [16, 18, $\underline{G}$]
(24) Tus, w_1t_1 [20, $\neg\neg$]
(26) $\Pi yU(Tuy \rightarrow Suy), w_0t_0$ [3, Π]
(28) $U(Tus \rightarrow Sus), w_0t_0$ [26, Π]
(30) $Tus \rightarrow Sus, w_1t_1$ [28, U]
(32) Sus, w_1t_1 [30, 24, MP]
(34) $*$ [22, 32]

(11) $(Y \wedge A) \Box\!\!\rightarrow P\underline{F}Ssu, w_0t_0$ [1, $\rightarrow$]
(13) $Y \wedge A, w_4t_0$ [4, $\Diamond$]
(15) $r_{Y\wedge A}w_0w_2t_0$ [13, $T-c3$]
(17) $O\underline{G}\neg Ssu, w_2t_0$ [2, 15, $\Box\!\!\rightarrow$]
(19) $P\underline{F}Ssu, w_2t_0$ [11, 15, $\Box\!\!\rightarrow$]
(21) $sw_2w_3t_0$ [19, P]
(23) $\underline{F}Ssu, w_3t_0$ [19, P]
(25) $\underline{G}\neg Ssu, w_3t_0$ [17, 21, O]
(27) $t_0 < t_1$ [23, $\underline{F}$]
(29) Ssu, w_3t_1 [23, $\underline{F}$]
(31) $\neg Ssu, w_3t_1$ [25, 27, $\underline{G}$]
(33) $*$ [29, 31]

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