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Free $(n+1)$-valued Modal Implicative Semilattices

María Cristina Canals Frau, Aldo V. Figallo and Gustavo Pelaitay

Abstract

The $(n+1)$-valued modal implicative semilattices (or MS_{n+1}-algebras) were introduced in the paper [1] by the first and second author. In this article, our main purpose is to investigate the subvariety of bounded MS_{n+1}-algebras. In particular, we describe a method to determine the structure of the bounded MS_{n+1}-algebras with a finite set of free generators.

Keywords: semilattice; modal implicative semilattice; free algebras.

Introduction

The implicative semilattices were defined by A. Monteiro [2] as algebras $\langle A, \wedge, \rightarrow, 1\rangle$ of type $(2, 2, 0)$ which satisfy the identities:

1. $x \rightarrow x = 1$,
2. $(x \rightarrow y) \wedge y = y$,
3. $x \wedge (x \rightarrow y) = x \wedge y$,
4. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

This definition is equivalent to that given by Nemitz in [3]. For more details on the theory of implicative semilattices, see [4] and [5].

Iturrioz introduced in [6] the notion of modal operators on symmetric Heyting algebras and defined the class of SHn-algebras. In [1] Canals Frau and Figallo considered some reducts of this class. In particular, they introduced the following definition.

An $(n+1)$-valued modal implicative semilattice (or MS_{n+1}-algebra) is an algebra $\langle A, \rightarrow, \wedge, \sigma_1, \ldots, \sigma_n, 1\rangle$ such that the reduct $\langle A, \rightarrow, \wedge, 1\rangle$ is an implicative semilattice and σ_1, …, σ_n are unary operations on A satisfying the following axioms:

(M1) $(\sigma_1 x \to y) \to x = x$,

(M2) $\sigma_i(x \to y) \to (\sigma_i x \to \sigma_j y) = 1$, $1 \leq i \leq j \leq n+1$,

(M3) $(\sigma_i x \to \sigma_i y) \to ((\sigma_{i+1} x \to \sigma_{i+1} y) \to \ldots ((\sigma_n x \to \sigma_n y) \to \sigma_i(x \to y)) \ldots) = 1$,

(M4) $\sigma_i(x \to \sigma_j y) = x \to \sigma_j y$, $1 \leq i, j \leq n+1$,

(M5) $\sigma_n x = (x \to \sigma_i x) \to \sigma_j x$, $1 \leq i \leq j \leq n+1$.

1 Bounded $(n+1)$-valued modal implicative semilattices

In this section we will introduce the variety of bounded MS_{n+1}-algebras.

Definition 1.1 *A bounded $(n+1)$-valued modal implicative semilattice (or MS^0_{n+1}-algebra) is an algebra $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 0, 1\rangle$ such that the reduct $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 1\rangle$ is an MS_{n+1}-algebra and it satisfies the following additional condition:*

(M6) $0 \to x = 1$.

Example 1.2 *Let be $C_{n+1} = \langle C_{n+1}, \to, \wedge, \sigma_1, \cdots, \sigma_n, 0, 1\rangle$ where $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\}$ considered as a sublattice of the real numbers, and the operations are defined by: $x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x \not\leq y \end{cases}$, $x \wedge y = \min\{x, y\}$ and $\sigma_j(\frac{k}{n}) = \begin{cases} 0 & \text{if } k+j \leq n \\ 1 & \text{if } k+j \not\leq n \end{cases}$. Then C_{n+1} is a MS^0_{n+1}-algebra.*

Remark 1.3 *The algebra C_{n+1} in Example 1.2 and its subalgebras are only simple MS^0_{n+1}-algebras.*

In what follows we will denote by $\mathcal{MS}^0_{n+1}$ the variety of MS^0_{n+1}-algebras and by $Con(A)$, $Hom(A, B)$ and $Epi(A, B)$ the set of MS^0_{n+1}-congruences, MS^0_{n+1}-homomorphisms from A into B and MS^0_{n+1}-epimorphisms from A onto B, respectively. Moreover, we will denote by $[G]$ the MS^0_{n+1}-subalgebra of A generated by G.

Definition 1.4 *Let $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 0, 1\rangle$ be a MS^0_{n+1}-algebra. $D \subseteq A$ is a modal deductive system of A if it verifies:*

(D1) $1 \in D$,

(D2) *if* $x, x \to y \in D$ *then* $y \in D$,

(D3) $x \in D$ *implies* $\sigma_1 x \in D$.

Hereafter, we will denote by $\mathcal{D}(A)$ ($\mathcal{E}(A)$) the set of all modal deductive systems (maximal modal deductive systems) of A, respectively.

Lemma 1.5 ([1]) *Let* $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 0, 1 \rangle$ *be a* MS^0_{n+1}*-algebra, then it satisfies:*

(i) $Con(A) = \{R(D) : D \in \mathcal{D}(A)\}$, *where* $R(D) = \{(x, y) \in A^2 : x \to y, y \to x \in D\}$.

(ii) *If* $h \in Hom(A, B)$ *then the set* $\ker(h) = \{x \in A : h(x) = 1\}$, *called kernel of* h, *is a modal deductive system of* A.

If $D \in \mathcal{D}(A)$ then we will denote by A/D the quotient algebra $A/R(D)$.

Definition 1.6 *Let* $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 0, 1 \rangle$ *be a* MS^0_{n+1}*-algebra.* $D \subseteq A$ *is an* $(k+1)$*-valued deductive system,* $1 \leq k \leq n$ *if* A/D *is isomorphic to* C_{k+1}.

Lemma 1.7 *Let* $\langle A, \to, \wedge, \sigma_1, \ldots, \sigma_n, 0, 1 \rangle$ *be a* MS^0_{n+1}*-algebra and* $M \in \mathcal{D}(A)$. *Then, the following conditions are equivalent:*

(i) M *is a maximal modal deductive system,*

(ii) M *is a* $(k+1)$*-valued deductive system,* $1 \leq k \leq n$.

Proof. The following conditions are equivalent to each other:

1. M is a maximal modal deductive system,
2. A/M is simple,
3. $A/M \simeq C_{k+1}$,
4. M is an $(k+1)$-valued deductive system, $1 \leq k \leq n$.

■

Theorem 1.8 ([1]) $\mathcal{MS}^0_{n+1}$ *is semisimple.*

2 Free MS^0_{n+1}-algebras

The notion of free MS^0_{n+1}-algebra is defined in the usual way as follows:

Definition 2.1 *If $m > 0$ is an arbitrary cardinal number, then we say that $\mathcal{L}_{n+1}(m)$ is the MS^0_{n+1}-algebra with m free generators if:*

(L1) *there is $G \subseteq \mathcal{L}_{n+1}(m)$ such that $[G] = \mathcal{L}_{n+1}(m)$ and $|G| = m$,*

(L2) *any mapping f from G into an arbitrary MS^0_{n+1}-algebra A can be extended to a MS^0_{n+1}-homomorphism $h : \mathcal{L}_{n+1}(m) \longrightarrow A$ such that $h_{|G} = f$.*

Since MS^0_{n+1}-algebras are equationally definable, for any cardinal number $m > 0$ there exists $\mathcal{L}_{n+1}(m)$ and it is unique up to isomorphisms. Moreover the MS^0_{n+1}-homomorphisms h of Definition 2.1 is also unique.

In what follow m is a positive integer, and $G = \{g_1, g_2, \cdots, g_m\}$ is a set of free generators of $\mathcal{L}_{n+1}(m)$.

Our next task will be to prove that the variety $\mathcal{MS}^0_{n+1}$ is locally finite.

Lemma 2.2 *Let $f \in \mathcal{F}^*_{k+1} = \{f \in (C_{k+1})^G : [f(G)] = C_{k+1}\}$. Then $\ker(h_f) \in \mathcal{E}(\mathcal{L}_{n+1}(m))$, where $h_f : \mathcal{L}_{n+1}(m) \longrightarrow C_{k+1}$ is the homomorphism extending f.*

Proof. Let $f \in \mathcal{F}^*_{k+1}$. Since $h_f(\mathcal{L}_{n+1}(m)) = [f(G)] = C_{k+1}$, we infer that h_f surjective and therefore $\mathcal{L}_{n+1}(m)/\ker(h_f) \simeq C_{k+1}$ from which we obtain that $\ker(h_f)$ is a $(k+1)$-valued modal deductive system. Then, from the above result and Lemma 1.7 we conclude that $\ker(h_f) \in \mathcal{E}(\mathcal{L}_{n+1}(m))$. ■

It is well known that all MS_{n+1}-algebras is a product subdirect of a family of chain C_{n+1} (see [1, Theorem 2.6, Theorem 2.7]), as many as maximal kernels have the algebra. We will obtain a number of maximal modal deductive systems that has finitely generated free algebra.

Proposition 2.3 $|\mathcal{E}(\mathcal{L}_{n+1}(m))| \leq |(C_{n+1})^G|$.

Proof. Let $\varphi : (C_{n+1})^G \longrightarrow \mathcal{E}(\mathcal{L}_{n+1}(m))$, defined by $\varphi(f) = \ker(h_f)$. From Lemma 2.2, φ is well defined. Let's prove that φ is surjective. Indeed, let $M \in \mathcal{E}(\mathcal{L}_{n+1}(m))$. Then, from Lemma 1.7 we have that $\mathcal{L}_{n+1}(m)/M \simeq C_{k+1}$ is a subalgebra of C_{n+1}. Hence, if α_M is the isomorphism between $\mathcal{L}_{n+1}(m)/M$ and C_{k+1} and considered $\alpha_M \circ q_{M|_G} : G \longrightarrow C_{n+1}$ it follows that $\alpha_M \circ q_{M|_G} \in (C_{n+1})^G$ and $\varphi(\alpha_M \circ q_{M|_G}) = M$. Then, φ is surjective, and therefore we have that $|\mathcal{E}(\mathcal{L}_{n+1}(m))| \leq |(C_{n+1})^G|$.

■

Proposition 2.4 *$\mathcal{L}_{n+1}(m)$ is finite.*

Proof. From Theorem 1.8 and well known results of universal algebras we have that (see [7]), $\mathcal{L}_{n+1}(m)$ is isomorphic to a subalgebra of

$$\prod_{M\in\mathcal{E}(\mathcal{L}_{n+1}(m))} \mathcal{L}_{n+1}(m)/M.$$

On the other hand, we have that $\mathcal{L}_{n+1}(m)/M \simeq C_{k+1}$ with $1 \leq k \leq n$. From this last statement and Proposition 2.3 we conclude that $\mathcal{L}_{n+1}(m)$ is finite. ■

Theorem 2.5 *$\mathcal{MS}^0_{n+1}$ is a locally finite variety.*

Proof. It is a direct consequence of Proposition 2.4 and [7]. ■

Let $\mathcal{M}_{k+1} = \{M \in \mathcal{E}(\mathcal{L}_{n+1}(m)) : \mathcal{L}_{n+1}(m)/M \simeq C_{k+1}\}$, $1 \leq k \leq n$. Then, taking into account that $\mathcal{L}_{n+1}(m)$ is finite it follows that

$$\mathcal{L}_{n+1}(m) \simeq \prod_{M\in\mathcal{M}_2} \mathcal{L}_{n+1}(m)/M \times \prod_{M\in\mathcal{M}_3} \mathcal{L}_{n+1}(m)/M \times \cdots \times \prod_{M\in\mathcal{M}_{n+1}} \mathcal{L}_{n+1}(m)/M$$

On the other hand, from the Proposition 2.3 we have that $\mathcal{M}_{k+1}$ is finite for all k, $1 \leq k \leq n$.

Hence, if we denote with $p_{k+1} = |\mathcal{M}_{k+1}|$ we conclude that:

$$\mathcal{L}_{n+1}(m) \simeq \prod_{k=1}^{n} (C_{k+1})^{p_{k+1}}.$$

To calculate the cardinal of free algebra with a finite set of free generators we only need to determine the numbers $p_{k+1} = |\mathcal{M}_{k+1}|$.

Lemma 2.6 $|\mathcal{M}_{k+1}| = |Epi(\mathcal{L}_{n+1}(m), C_{k+1})|$.

Proof. Let $\alpha_{k+1} : Epi(\mathcal{L}_{n+1}(m), C_{k+1}) \longrightarrow \mathcal{M}_{k+1}$ the function definided by the prescription $\alpha_{k+1}(h) = \ker(h)$. For each $M \in \mathcal{M}_{k+1}$ let $h = \Theta \circ q_M$, where q_M is the canonical epimorphism and Θ is the isomorphism from $\mathcal{L}_{n+1}(m)/M$ to C_{k+1}. Then, $h \in Epi(\mathcal{L}_{n+1}(m), C_{k+1})$ and $\alpha_{k+1}(h) = M$. Indeed, $\alpha_{k+1}(h) = \ker(\Theta \circ q_M)$. On the other hand the following conditions are equivalent:

1. $x \in \ker(\Theta \circ q_M)$,

2. $\Theta \circ q_M(x) = 1$,

3. $\Theta(q_M(x)) = \Theta(1)$,
4. $q_M(x) = 1$,
5. $x \in \ker(q_M)$.

Therefore, α_{k+1} is surjective. So, $|\mathcal{M}_{k+1}| \leq |Epi(\mathcal{L}_{n+1}(m), C_{k+1})|$. Moreover, if $M \in \mathcal{M}_{k+1}$ and $\alpha_{k+1}(h) = M$ then $\alpha_{k+1}^{-1}(M) = \{g \circ h : g \in Aut(C_{k+1})\}$, where $Aut(C_{k+1})$ is the set of all automorphism of C_{k+1}. Therefore,

$$|\mathcal{M}_{k+1}| = \frac{|Epi(\mathcal{L}_{n+1}(m), C_{k+1})|}{|Aut(C_{k+1})|}.$$

On the other hand, it is easy to see that the unique automorphism of C_{k+1} is the identity map. From which we conclude that $|\mathcal{M}_{k+1}| = |Epi(\mathcal{L}_{n+1}(m), C_{k+1})|$ ■

Lemma 2.7 $|Epi(\mathcal{L}_{n+1}(m), C_{k+1})| = |\mathcal{F}^*_{k+1}|$.

Proof. Let $\phi : Epi(\mathcal{L}_{n+1}(m), C_{k+1}) \longrightarrow \mathcal{F}^*_{k+1}$ defined by $\phi(h) = h\,|_G$. Since G is a free generator system of $\mathcal{L}_{n+1}(m)$ we have that ϕ is injective. On the other hand, let $f \in \mathcal{F}^*_{k+1}$, then $f : G \longrightarrow C_{k+1}$ it is that $[f(G)] = C_{k+1}$. Hence, there is a unique extension h_f of f. So, $h_f \in Epi(\mathcal{L}_{n+1}(m), C_{k+1})$ and $\phi(h_f) = f$, from which we conclude that ϕ is surjective. Then, ϕ is bijective which completes the proof. ■

From the above it follows that

$$(I)\ p_{k+1} = |\mathcal{F}^*_{k+1}|.$$

In order to determine $|\mathcal{F}^*_{k+1}|$ we note that from the definition of operations in C_{k+1}, the only subsets of this algebra that generate it are: $X_1 = \{\frac{1}{k}, \cdots, \frac{k-1}{k}\}$, $X_2 = \{0, \frac{1}{k}, \cdots, \frac{k-1}{k}\}$, $X_3 = \{\frac{1}{k}, \cdots, \frac{k-1}{k}, 1\}$ and $X_4 = \{0, \frac{1}{k}, \cdots, \frac{k-1}{k}, 1\}$.

Then, $\mathcal{F}^*_{k+1} = \bigcup\limits_{i=1}^{4} \mathcal{F}(G, X_i)$ where $\mathcal{F}(X, Y) = \{f : X \longrightarrow Y : f \text{ is surjective}\}$. Since $\mathcal{F}(G, X_i)$, $1 \leq i \leq 4$, are disjoint two to two, then we have to:

$$(\mathrm{II})\ |\mathcal{F}^*_{k+1}| = \sum_{i=1}^{4} |\mathcal{F}(G, X_i)| = |\mathcal{F}(G, X_i)| + 2|\mathcal{F}(G, X_2)| + |\mathcal{F}(G, X_4)|.$$

We observed that if $|G| < |X_i|$, then $|\mathcal{F}(G, X_i)| = 0$. Moreover, taking into account that $|G| = m$, $|X_1| = k - 1$, $|X_2| = |X_3| = k$ and $|X_4| = k + 1$ we have that

(i) $|\mathcal{F}(G,X_1)| = \sum_{j=0}^{k-2}(-1)^j \binom{k-1}{j}(k-1-j)^m = (k-1)^m - (k-1)(k-2)^m + \binom{k-1}{2}(k-3)^m - \binom{k-1}{3}(k-4)^m + \ldots + (-1)^{j-2}\binom{k-1}{j-2}(k-1-(j-2))^m + \ldots + (-1)^{k-3}\frac{(k-1)(k-2)}{2!}2^m + (-1)^{k-2}(k-1)$.

(ii) $|\mathcal{F}(G,X_2)| = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^m = k^m - \binom{k}{1}(k-1)^m + \binom{k}{2}(k-2)^m - \binom{k}{3}(k-3)^m + \ldots + (-1)^{j-1}\binom{k}{j-1}(k-(j-1))^m + \ldots + (-1)^{k-2}\binom{k}{k-2}2^m + (-1)^{k-1}\binom{k}{k-1}$.

(iii) $|\mathcal{F}(G,X_4)| = \sum_{j=0}^{k}(-1)^j\binom{k+1}{j}(k+1-j)^m = (k+1)^m - \binom{k+1}{1}k^m + \binom{k+1}{2}(k-1)^m - \binom{k+1}{3}(k-2)^m + \ldots + (-1)^{j-1}\binom{k+1}{j-1}(k-j)^m + \ldots + (-1)^{k-1}\frac{(k+1)k}{2!}2^m + (-1)^k(k+1)$.

From (i), (ii) and (iii) we have that

(III) $|\mathcal{F}(G,X_1)| + 2|\mathcal{F}(G,X_2)| + |\mathcal{F}(G,X_4)| = \Big((k+1)^m - \binom{k+1}{1}k^m + \binom{k+1}{2}(k-1)^m - \binom{k+1}{3}(k-2)^m + \ldots + (-1)^j\binom{k+1}{j}(k-(j-1))^m + \ldots + (-1)^{k-1}\frac{(k+1)k}{2!}2^m + (-1)^k(k+1)\Big) + \Big(2k^m - 2\binom{k}{1}(k-1)^m + 2\binom{k}{2}(k-2)^m - 2\binom{k}{3}(k-3)^m + \ldots + 2(-1)^{j-1}\binom{k}{j-1}(k-(j-1))^m + \ldots + 2(-1)^{k-2}\frac{k(k-1)}{2!}2^m + 2(-1)^{k-1}k\Big) + \Big((k-1)^m - (k-1)(k-2)^m + \binom{k-1}{2}(k-3)^m - \binom{k-1}{3}(k-4)^m + \ldots + (-1)^{j-2}\binom{k-1}{j-2}(k-(j-2))^m + \ldots + (-1)^{k-3}\frac{(k-1)(k-2)}{2!}2^m + (-1)^{k-2}(k-1)\Big) = (k+1)^m - \Big(\Big(\binom{k+1}{1} - 2\Big)k^m - \Big(\binom{k+1}{2} - 2\binom{k}{1} + \Big)(k-1)^m + \Big(\binom{k+1}{3} - 2\binom{k}{2} + (k-1)\Big)(k-2)^m - \Big(\binom{k+1}{4} - 2\binom{k}{3} + \binom{k-1}{2}\Big)\Big)(k-3)^m + \Big(\binom{k+1}{5} - 2\binom{k}{4} + \binom{k-1}{3}\Big)\Big)(k-4)^m + \ldots +$

$$\left((-1)^j\binom{k+1}{j}+(-1)^{j-1}2\binom{k}{j-1}+(-1)^{j-2}\binom{k-1}{j-2}\right)(k-(j-1))^m+$$
$$\ldots+\left((-1)^{k-1}\frac{(k+1)k}{2!}+(-1)^{k-2}2\frac{k(k-1)}{2!}+(-1)^{k-3}\frac{(k-1)(k-2)}{2!}\right)2^m+$$
$$\left((-1)^k\binom{k+1}{k}+(-1)^{k-1}2k+(-1)^{k-2}(k-1)\right)\Bigg).$$

We note that, $(-1)^k(k+1)+(-1)^{k-1}2k+(-1)^{k-2}(k-1)=0$. Indeed,

(i) if k is even, $(k+1)-2k+k-1=0$,

(ii) if k is odd, $-(k+1)+2k-(k-1)=0$.

Therefore, the coefficient of 1^m is 0. In an analogous way we obtain that the coefficient of 2^m is -1 if k is even and it is 1 if k if odd, this is the coefficient of 2^m is $(-1)^{k+1}$.

We will discuss the general term:

$$(-1)^j\binom{k+1}{j}+(-1)^{j-1}\,2\binom{k}{j-1}+(-1)^{j-2}\binom{k-1}{j-2}\Bigg)(k-(j-1))^m$$

If j is even, then $(-1)^j=1$, $(-1)^{j-1}=-1$ and $(-1)^{j-2}=1$ and we can write to this coefficient as $(-1)^{t+1}$ with $1\le t\le k-1$. We develop the previous term and we have that:

$$\frac{(k+1)k(k-1)(k-2)\ldots(k-(j-2))}{j(j-1)(j-2)!}-2\frac{k(k-1)(k-2)\ldots(k-(j-2))}{(j-1)(j-2)!}+$$
$$\frac{(k-1)(k-2)\ldots(k-(j-2))}{(j-2)!}=$$
$$\frac{(k-1)(k-2)\ldots(k-(j-2))\Big((k-1)k-2kj+j(j-1)\Big)}{j(j-1)(j-2)!}.$$

On the other hand,

$$\Big((k-1)k-2kj+j(j-1)\Big)=(k-(j-1))(k-j),$$

from where we can express the general term as follows:

$$\frac{(k-1)(k-2)\ldots(k-(j-2))\Big((k-(j-1))(k-j)\Big)}{j(j-1)(j-2)!}.$$

It is simple to verify that:

$$\frac{(k-1)(k-2)\dots(k-(j-2))\Big((k-(j-1))(k-j)\Big)}{j(j-1)(j-2)!} = \binom{k-1}{k-1-j}.$$

If j is odd, reasoning in an analogous way gives the same result.

In short, we have:

(IV) $|\mathcal{F}(G,X_1)| + 2|\mathcal{F}(G,X_2)| + |\mathcal{F}(G,X_4)| =$

$(k+1)^m - \sum_{j=1}^{k-1}(-1)^{j+1}\binom{k-1}{k-1-j}(k+1-j)^m.$

From (I), (II), (III) and (IV) we conclude that

$$p_{k+1} = (k+1)^m - \sum_{j=1}^{k-1}(-1)^{j+1}\binom{k-1}{k-1-j}(k+1-j)^m.$$

Theorem 2.8 *Let $\mathcal{L}_{n+1}(m)$ be a free $MS^0_{n+1}-$algebra with m free generators. Then its cardinality is given by the following formula, for $m \geq k-1$,*

$$|\mathcal{L}_{n+1}(m)| = \prod_{k=1}^{n}(k+1)^{(k+1)^m - \left(\sum_{j=1}^{k-1}(-1)^{j+1}\left(\binom{k-1}{k-1-j}\right)(k+1-j)^m\right)}$$

$$= 2^{2^m}.3^{(3^m-2^m)}.4^{(4^m-(2.3^m-2^m))}\dots.(n+1)^{(n+1)^m - \left(\left(\sum_{j=1}^{n-1}(-1)^{j+1}\left(\binom{n-1}{n-1-j}\right)(n+1-j)^m\right)\right.}.$$

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María Cristina Canals Frau
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: mcanalsfrau@gmail.com

Aldo V. Figallo
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: avfigallo@gmail.com

Gustavo Pelaitay
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: gpelaitay@gmail.com