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Łukasiewicz logic and MV-algebras: Recent Results

Daniele Mundici

To my friend Chico Miraglia on his 70th birthday

Abstract

We present a number of recent results on MV-algebras and Łukasiewicz logic pertaining to a wider mathematical area than algebraic logic.

Keywords: Infinite-valued Łukasiewicz propositional logic, Łukasiewicz calculus, finitely presented MV-algebra, Schauder basis, projective MV-algebra, projectivity index, hopfian property, valuation, Euler characteristic, recognition problem, recursively enumerable presentation, decidable MV-algebra, Gödel incomplete MV-algebra.

1 Foreword

The aim of this paper is to highlight some categorical, geometric, topological and algorithmic properties of MV-algebras. This is intended as a tribute to Chico Miraglia and his multiform work on lattices, ordered algebraic structures, nonclassical logic, number theory, K-theory, algebraic geometry, category theory, homological algebra, in the hope of drawing his attention to an interesting class of lattice-ordered algebraic structures for nonclassical (many-valued) logic.

We refer to [22, 53] for background on MV-algebras and Łukasiewicz logic.

2 Finitely presented MV-algebras and polyhedra

Given a polyhedron P in euclidean n-space, how to *decide* in a finite number of steps whether P is homeomorphic to, say, the 4-sphere? For a precise formulation of this decision problem, P must be presented to a Turing machine $\mathcal{R}$ as a finite string of symbols. To this purpose, we first assume P to be a *rational polyhedron* in $\mathbb{R}^n$, i.e., a finite union of simplexes $S_1, \ldots, S_k \subseteq \mathbb{R}^n$ with rational vertices. Next we equip P with a triangulation whose simplexes have rational vertices. P is then presented to $\mathcal{R}$ as the list of the coordinates of the vertices of all these simplexes. Without loss of generality, [34], in the statement of our recognition problem we may replace homeomorphism by *rational PL-homeomorphism,* i.e., an invertible continuous PL-map ϕ such that every linear (in the sense of "affine linear") piece of both ϕ and its inverse has rational coefficients. In this way, the set $\mathcal{H}$ of pairs of rationally PL-homeomorphic rational polyhedra P, Q becomes recursively enumerable. However, as proved by A.A. Markov (see [64], [34] and references therein) the complementary set of $\mathcal{H}$ is not: *The rational PL-homeomorphism problem of rational polyhedra is undecidable.* The classical program of recognizing manifolds via a computable set of invariants thus fails in general.

One may refine the recognition problem by further restricting the notion of a *rational* PL-homeomorphism to that of an *integer* PL-homeomorphism. To this purpose, following [53], let us say that a continuous piecewise linear map $\zeta\colon P \to Q$ is a *$\mathbb{Z}$-map* if every linear piece of ζ has integer coefficients. Let $\mathcal{M}([0,1]^n)$ denote the MV-algebra of n-variable *McNaughton functions*, i.e., $[0,1]$-valued $\mathbb{Z}$-maps defined on $[0,1]^n$. McNaughton's theorem, [22, 9.1.5], states that $\mathcal{M}([0,1]^n)$ is the free n-generator MV-algebra. More generally, for any nonempty closed set $X \subseteq [0,1]^n$, let $\mathcal{M}(X)$ denote the MV-algebra of restrictions to X of the functions of $\mathcal{M}([0,1]^n)$. Let the functor $\mathcal{M}$ be defined as follows:

OBJECTS : For all rational polyhedra $P \subseteq [0,1]^n$, $\mathcal{M}(P) = \{f \restriction P \mid f \in \mathcal{M}([0,1]^n)\}$.

ARROWS : For all rational polyhedra $P \subseteq [0,1]^m$ and $Q \subseteq [0,1]^n$, and $\mathbb{Z}$-map $\eta\colon P \to Q$, $\mathcal{M}(\eta)$ is the function that transforms every $f \in \mathcal{M}(Q)$ into the homomorphism $f \circ \eta\colon \mathcal{M}(Q) \to \mathcal{M}(P)$. For short, $\mathcal{M}(\eta) = - \circ \eta$.

In this category, "homeomorphisms" (called $\mathbb{Z}$-homeomorphisms) are those invertible maps η from a rational polyhedron $P \subseteq \mathbb{R}^n$ onto a rational polyhedron $Q \subseteq \mathbb{R}^m$ such that both η and its inverse are $\mathbb{Z}$-maps.

Recall that a *finitely presented* MV-algebra is an isomorphic copy of the quotient of a free MV-algebra $\mathcal{M}([0,1]^n)$ by a finitely generated ideal. Note that the image of a finitely presented MV-algebra under a homomorphism is finitely presented.

The following theorem was proved in [48, 4.12], building on previous work. See, for instance, [60, 35], [46, 5.1,5.2,6.4], [14], and [53].

Theorem 2.1 *The functor $\mathcal{M}$ is a duality between rational polyhedra in euclidean space (with $\mathbb{Z}$-maps) and finitely presented MV-algebras (with homomorphisms).*

The category of rational polyhedra with $\mathbb{Z}$-maps yields a new geometry, where the isometry group in euclidean space is replaced by the affine group over the integers, [17]. In this geometry, rational polyhedra turn out to possess a wealth of new computable invariants. The proof of Theorem 2.1 also shows:

Theorem 2.2 *The isomorphism problem for finitely presented MV-algebras is decidable iff so is the $\mathbb{Z}$-homeomorphism problem for rational polyhedra.*

Owing to their dual polyhedral counterpart, finitely presented MV-algebras are endowed with a wealth of geometric extra structure, which is generally not available in general MV-algebras. This is the subject matter of the next sections in this paper.

3 Bases, valuations and Euler characteristic

We refer to [50] for the categorical equivalence Γ between MV-algebras and unital ℓ-groups. For any MV-algebra A we let $\boldsymbol{\mu}(A)$ denote its maximal spectral space, [22, 53]. The following is a key tool for the study of finitely presented MV-algebras, [53, 6.1]:

Definition 3.1 Let A be an MV-algebra. A generating set $B = \{b_1, \ldots, b_n\} \subseteq A \setminus \{0\}$ is said to be a *basis* of A if, in the corresponding unital ℓ-group of A, the unit arises as a linear combination of the b_i with integer coefficients $c_i > 0$, and for each $k = 1, 2, \ldots$ and k-element subset C of B with $\bigwedge\{b \mid b \in C\} \neq 0$ the set $\{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid \mathfrak{m} \supseteq B \setminus C\}$ is homeomorphic to a $(k-1)$-simplex.

It follows that the c_i are uniquely determined, [53, p. 70]. In [13] and [53, 6.3] the following characterization theorem is proved:

Theorem 3.2 *An MV-algebra is finitely presented iff it has a basis.*

By [53, 4.16], for every maximal ideal $\mathfrak{m} \in \mu(A)$ the quotient MV-algebra $A/\mathfrak{m}$ is uniquely isomorphic to a subalgebra J of the standard MV-algebra $[0, 1]$. Identifying $A/\mathfrak{m}$ and J, for any $a \in A$, the element $a/\mathfrak{m}$ becomes a real number. We write

$$\mathrm{supp}(a) = \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid a/\mathfrak{m} > 0\}.$$

As we have seen in the foregoing section, when A is finitely presented it is isomorphic to an MV-algebra of the form $\mathcal{M}(P)$ for some rational polyhedron in some unit cube $[0, 1]^n$. Then $\mathrm{supp}(a)$ is homeomorphic to a set of the form $P \setminus R$ for some rational polyhedron $R \subseteq P$, (see [53, §4.5, 6.2]). By definition, the *Euler characteristic* $\chi(\mathrm{supp}(a))$ is the alternating sum of the Betti numbers of $\mathrm{supp}(a)$, as given by singular homology theory. This is homotopy invariant.

In [56] one can find a proof of the following result:

Theorem 3.3 *For any finitely presented MV-algebra A let the map $\mathsf{E}\colon A \to \mathbb{Z}$ be given by $\mathsf{E}(a) = \chi(\mathrm{supp}(a))$, for all $a \in A$. Then E has the following properties:*

(i) $\mathsf{E}(0) = 0$.

(ii) (Normalization) $\mathsf{E}(b) = 1$ *for each element* b *of* A *that belongs to some basis.*

(iii) (Idempotency) *For all* $p, q \in A$, $\mathsf{E}(p \oplus q) = \mathsf{E}(p \vee q)$.

(iv) (Additivity) E *is a* valuation : *for all* $p, q \in A$, $\mathsf{E}(p \vee q) = \mathsf{E}(p) + \mathsf{E}(q) - \mathsf{E}(p \wedge q)$.

Conversely, properties (i)-(iv) *uniquely characterize* E *among all real-valued functions defined on* A.

4 Special case: Finitely generated projective MV-algebras

As a particular case of a general definition, an MV-algebra A is *projective* if whenever $\psi\colon B \to C$ is a surjective homomorphism and $\phi\colon A \to C$ is a homomorphism, there is a homomorphism $\theta\colon A \to B$ such that $\phi = \psi \circ \theta$. Finitely generated projective MV-algebras are an interesting subclass of finitely presented MV-algebras: among others, they provide an algebraic-geometric clarification of such notions as exactness and admissibility in the proof-theory of Łukasiewicz logic, [9, §4.5]. The characterization of finitely generated projective MV-algebras took several years, and today appears as an interesting chapter of algebraic topology:

Theorem 4.1 (L.M.Cabrer, D.M. [14]) *Let* A *be an* n*-generator projective MV-algebra. Then* A *is isomorphic to the MV-algebra* $\mathcal{M}(P)$ *obtained by restricting to* P *the functions of* $\mathcal{M}([0,1]^n)$*, for some contractible rational polyhedron* $P \subseteq [0,1]^n$ *that contains a vertex of the cube* $[0,1]^n$ *and has the following "strong regularity" property: For every regular triangulation* Δ *of* P *and maximal simplex* T *of* Δ*, the greatest common divisor of the denominators of the vertices of* T *is equal to* 1*.*

(L.M.Cabrer [10]) *The above conditions on* P *are also sufficient for* $\mathcal{M}(P)$ *to be isomorphic to an* n*-generator projective MV-algebra.*

The strong regularity of P is called "anchoredness" in [37], meaning that the affine hull of T contains an integer point of $\mathbb{R}^n$. Combining the foregoing theorem with Theorem 2.1 one sees that, while every finitely generated projective MV-algebra is finitely presented, the converse is not true in general. Baker and Beynon [1, 5, 6] showed that an ℓ-group G is finitely generated projective iff it is finitely presented. As shown by the foregoing theorem, the situation for unital ℓ-groups and MV-algebras is quite different.

The projectivity index.

In view of McNaughton's representation theorem for free MV-algebras, [22], a folklore result in universal algebra states that an n-generator MV-algebra A is projective iff there is an idempotent endomorphism ρ of $\mathcal{M}([0,1]^n)$ onto an isomorphic copy $R \subseteq \mathcal{M}([0,1]^n)$ of A.

Let us consider the following problem: *What is the number of idempotent endomorphisms of* $\mathcal{M}([0,1]^n)$ *onto* R*?* Note that this number is ≥ 1 iff A is projective. For every finitely generated projective MV-algebra C we define the (projectivity) *index* $\iota(C)$ as the sup of the number of idempotent endomorphisms of $\mathcal{M}([0,1]^n)$ onto C', where n is the smallest number of generators of C, and C' ranges over arbitrary isomorphic copies of C which are the image of an idempotent endomorphism of $\mathcal{M}([0,1]^n)$.

For every n-generator MV-algebra B, the construction of [53, Corollary 4.18] yields a canonical (Yosida) homeomorphism of the maximal spectral space μ_B onto a closed subset M of $[0,1]^n$. If $M = \mathrm{cl}(\mathrm{int}(M))$ then following Kuratowski, we unambiguously say that μ_B is a *closed domain* in $[0,1]^n$.

Theorem 4.2 (L.M.Cabrer, D.M. [18]) *Let A be a finitely generated projective MV-algebra. Let n be the smallest number of generators of A. Then the index of A is finite iff the maximal spectral space of A is a closed domain in* $[0,1]^n$.

The proof uses Theorem 4.1 along with the properties of the rational measure of the maximal spectral space of A, [53]. In [18] several examples are given of two-generator projective MV-algebras with arbitrarily high finite index, and with an infinite index. In particular, $\iota(\mathcal{M}([0,1]^n)) = 1$ for all $n = 1, 2, \ldots$.

5 A generalization: Hopfian MV-algebras

An algebra R is *hopfian* if every homomorphism of R onto R is an automorphism. In the category of sets, the hopfian property amounts to finiteness.

In the light of the results of the previous section, the following result, proved in [55], shows that hopfian MV-algebras are a vast generalization of finitely presented MV-algebras:

Theorem 5.1 *The following MV-algebras are hopfian:*

(i) Any finitely presented, any finitely generated projective, whence in particular, any finitely generated free MV-algebra.

(ii) Simple MV-algebras.

(iii) Finitely generated MV-algebras with only finitely prime ideals.

(iv) *For any rational point w lying in the interior of the cube $[0,1]^n$, the germinal MV-algebra $A = \mathcal{M}([0,1]^n)\,/\mathfrak{o}_w$,* [19].

(v) *Finitely generated semisimple MV-algebra A with a dense set of finite rank maximal ideals*

(vi) *Every one-generator semisimple MV-algebra*

Theorem 5.2 ([55]) *Let A be a semisimple MV-algebra with its maximal spectral space $\boldsymbol{\mu}(A)$.*

(i) If $\boldsymbol{\mu}(A)$ is an n-dimensional manifold without boundary then A is hopfian.

(ii) If $\boldsymbol{\mu}(A)$ is an n-manifold with boundary, and A has a generating set with n elements, then A is hopfian.

(iii) More generally, A is hopfian if A has a generating set of n elements and $\boldsymbol{\mu}(A)$ is homeomorphically embeddable onto a subset X of $[0,1]^n$ coinciding with the closure of its interior.

Interesting counterexamples of the hopfian property are given by the following results:

Theorem 5.3 ([55]) *The following MV-algebras are not hopfian:*

(i) *Free MV-algebras over infinitely many free generators.*

(ii) *The free product* $\mathrm{C} \amalg \mathrm{C}$.

(iii) *All countable boolean algebras.*

(iv) *Let L be an $n \times n$ matrix with integer entries and determinant equal to ± 1. Suppose L has a one-dimensional linear eigenspace E with eigenvalue $0 < \lambda < 1$, and E has a nonempty intersection with the interior of $[0,1]^n$. Then the semisimple n-generator MV-algebra $\mathcal{M}(E \cap [0,1/2]^n)$ is not hopfian.*

Corollary 5.4 ([55]) *Each of the following classes of MV-algebras contains a hopfian and a non-hopfian member:*

($\bar{\mathrm{a}}bc$) *Non-semisimple, finitely generated MV-algebras whose maximal ideals of finite rank are dense (in the maximal spectral space).*

($a\bar{\mathrm{b}}c$) *Semisimple not finitely generated MV-algebras whose maximal ideals of finite rank are dense.*

($ab\bar{\mathrm{c}}$) *Semisimple, finitely generated MV-algebras where maximals of finite rank are not dense.*

6 More general presentations of MV-algebras

A *formula* $\tau = \tau(X_1, \dots, X_n)$ of Łukasiewicz infinite-valued propositional logic $Ł_\infty$ is a string of symbols obtained from the variable symbols X_i by a finite number of applications of the Łukasiewicz connectives $\neg, \oplus, \odot$, precisely as boolean formulas are obtained from the variables and the boolean connectives $\neg, \vee, \wedge$. Following [53, 2.5] we write FORM_n for the set of formulas in the variables $X_1, \dots, X_n$. Following [53, 1.3], the map sending X_i to the ith coordinate function $x_i\colon [0,1]^n \to [0,1]$ canonically extends to a map interpreting each MV-term τ as a McNaughton function $\hat{\tau} \in \mathcal{M}([0,1]^n)$. The function $\hat{\tau}$ is said to be "associated" to τ and, τ is said to "represent" $\hat{\tau}$.

Fix a recursively enumerable set $\Phi \subseteq \mathsf{FORM}_n$. The decidability of $Ł_\infty$, [22, 4.5.3], immediately implies the recursive enumerability of the set $\overline{\Phi}$ of (syntactic) consequences of Φ.

Any set $\Theta \subseteq \mathsf{FORM}_n$ generates the ideal $\mathfrak{i}_\Theta = \{\neg\psi \mid \psi \in \overline{\Theta}\} \subseteq \mathcal{M}([0,1]^n)$ as well as the quotient MV-algebra $\mathcal{M}([0,1]^n)/\mathfrak{i}_\Theta$, called the *Lindenbaum algebra* of Θ. An MV-algebra is finitely presented iff it is the Lindenbaum algebra of some finitely axiomatizable theory.

Generalizing the classical definition for finitely axiomatizable theories and their associated finite presentations, [22, p.100], the *word problem* of Φ can be defined as follows:

INSTANCE: A formula $\psi \in \mathsf{FORM}_n$.

QUESTION: Does ψ belong to $\overline{\Phi}$?

From [8, Theorem 3.1] we have:

Theorem 6.1 *Let $\Phi \subseteq \mathsf{FORM}_n$ and $\Psi \subseteq \mathsf{FORM}_m$ be recursively enumerable sets of formulas such that $\mathcal{M}([0,1]^n)/\mathfrak{i}_\Phi$ is isomorphic to $\mathcal{M}([0,1]^n)/\mathfrak{i}_\Psi$. Then the word problem of Φ is decidable iff so is the word problem of Ψ.*

Thus, given a finitely generated MV-algebra A one may naturally say that A is decidable, undecidable, or *Gödel incomplete* (i.e., undecidable and recursively enumerable) if so is the word problem of *some* (equivalently, by Theorem 6.1 of *every*) recursively enumerable $\Theta \subseteq \mathcal{M}([0,1]^n)$ such that $A \cong \mathcal{M}([0,1]^n)/\mathfrak{i}_\Theta$. Table 1 shows that Gödel incomplete MV-algebras in the literature are more the exception than the rule.

Example: A one-generator Gödel incomplete MV-algebra. One easily constructs a Gödel incomplete *one-generator* MV-algebra A. By Table 1 (line 3), A necessarily has infinitely many maximal ideals.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Fix throughout a recursive enumeration $\nu\colon \mathbb{N} \to \mathbb{N}$ of the set of Gödel numbers of all first-order tautologies in the language of one binary relation

	Finitely generated MV-algebra A	A is not Gödel incomplete
1	chain	see [52]
2	with finitely many prime ideals	as a corollary of 1
3	one-generator, with finitely many maximal ideals	from 2 and [53, 5.13]
4	simple	see [50]
5	finitely presented	A is decidable, [53, 18.2]
6	finite	special case of 5
7	generated by an irrational (Effros-Shen), [53]	special case of 4
8	correspondent of the Behnke-Leptin AF algebra, [51]	see [51]
9	Chang algebra, [53]	A is decidable
10	finite-valued, (Grigolia) [22]	A is decidable
11	Post MV-algebra of order n, [22]	A is decidable
12	free	A is decidable, [22, 53]

Table 1: Most finitely generated MV-algebras in the literature are not Gödel incomplete. An infinitely generated example of a Gödel incomplete MV-algebra was given in [50].

symbol. Then range(ν) is a well known Gödel incomplete subset of $\mathbb{N}$. Without loss of generality, 0 and 1 do not belong to the range of ν.

For each $k \in$ range(ν) let b_k be the Schauder hat at $1/k$ whose open support $supp(b_k) = \{x \in [0,1] \mid b_k(x) > 0\}$ is the open interval with extremes $\frac{2}{k(k+1)}$ and $\frac{2}{k(k-1)}$ (see Figure 1, and refer to [53, 5.7] for details on Schauder hats.) For every $l \notin$ range(ν) let b_l be the constant function 0 over $[0,1]$.

For each $n \in \mathbb{N}$ let $B_n = b_0 \vee \cdots \vee b_n$. Since Ł_∞ is closed under the $\vee$ operation and Schauder hats are definable, ([22, 53]), there is a formula $\phi_n(X)$ such that $\hat{\phi}_n = B_n$. Let $\Phi = \{\phi_1, \phi_2, \dots\}$. The recursive enumerability of Φ follows from the recursive enumerability of the set of first-order tautologies (the latter property being a consequence of the Gödel completeness theorem) along with the recursive enumerability of range(ν). Since Ł_∞ is decidable [53, 18.3], then the deductive closure $\overline{\Phi} = \{\psi \in \mathsf{FORM}_1 \mid \psi$ is a syntactic consequence of $\Phi\}$ is recursively enumerable.

The undecidability of range(ν) entails the undecidability of $\overline{\Phi}$. Thus $\overline{\Phi}$ is a Gödel incomplete theory in one variable, and its associated Lindenbaum algebra is also Gödel incomplete.

7 MV-algebras outside algebraic logic

As mentioned above, MV-algebras are categorically equivalent to unital abelian ℓ-groups, [50]. Further, finitely presented MV-algebras are dually equivalent to rational

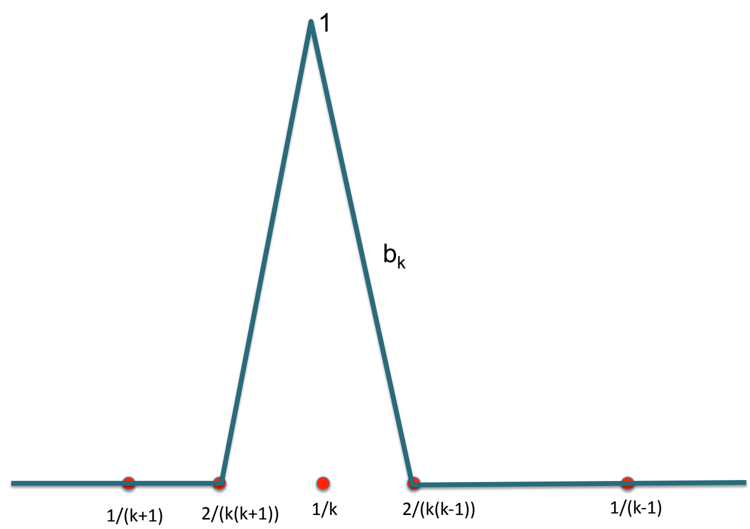

Figure 1: The Schauder hat b_k with vertex at $1/k$, where k is the Gödel number of a tautology.

polyhedra, [48]. It is also known that locally finite MV-algebras are dually equivalent to multisets, [23]. These, and other functors constructed in recent years, make MV-algebras applicable to diverse areas of mathematics. In what follows we present a selection of recent developments, well beyond the geometric algebraic and algorithmic issues that have been touched upon in this note:

— Algebraic geometry, [4]

— Categories, duality, sheafs, [20], [21], [26], [32], [33], [44], [47], [48], [49]

— Differential geometry, [7], [11], [16], [56]

— Discrete dynamical systems, [17]

— Games, [38], [39], [40], [41], [43], [45]

— Interval Algebras, [15]

— Modal logic, Belief, [30], [31], [35], [43]

— Multisets, [21], [59]

— Probability, [28], [29], [42], [63]

— Proof-theory of Łukasiewicz logic, [9], [12], [36], [37]

— Quantum structures, [27], [61], [62]

— Riesz spaces, [16], [24]

— Semantics of Łukasiewicz logic, [19], [54], [57]

— Semirings, tropical and idempotent mathematics, [2], [3], [25]

— Topology, [47], [65], [66].

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Daniele Mundici
Department of Mathematics and Computer Science "Ulisse Dini"
University of Florence
Viale Morgagni 67, 50134 Florence, Italy
E-mail: `mundici@math.unifi.it`