

Generic Stability and Grothendieck

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For Chico Miraglia on his 70th birthday

Abstract

This is an expository paper based on a reading of [1]. We interpret Grothendieck's double limit characterization of weak relative compactness [2], in the model theoretic setting as: $\phi(x, y)$ does not have the order property in M iff and only if every complete $\phi(x, y)$ -type $p(x)$ over M is *generically stable*. We give a proof and point out the connection with [5].

Keywords: Generic stability, double limit theorem

1 Introduction

This note is a commentary on the model-theoretic interpretation of Grothendieck's double limit characterization of weak relative compactness, after having read Itai Ben-Yaacov's short paper [1] on the topic, in the model theory seminar at Notre Dame. Thanks to Gabriel Conant, Sergei Starchenko and members of the Notre Dame model theory seminar for discussions. Thanks also to Udi Hrushovski for some comments.

The Grothendieck result, Theorem 6 in [2], is that if X is a topological (Hausdorff) space, X_0 is a dense subset of X , and A is a subset of $C_b(X)$, the bounded continuous (real or complex valued) functions on X , then the following are equivalent:

- (a) the closure of A in $C_b(X)$ with respect to the weak topology on $C_b(X)$ is compact (with respect to this weak topology on $C(X)$), and
- (b) A is bounded in $C_b(X)$, and if $f_i \in A$ and $x_i \in X_0$ ($i=1,2,\dots$), then if both $\lim_i \lim_j f_i(x_j)$ and $\lim_j \lim_i f_i(x_j)$ exist then they are equal.

The classical model theory context is where M is an L -structure, $\phi(x, y)$ an L -formula, $\phi^*(y, x)$ the same formula but with y as the “variable variable”, $X = S_{\phi^*}(M)$,

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$X_0 = \{tp_{\phi^*}(b/M) : b \in M\}$ (the realized types), and A the set of (continuous) functions from X to 2 given by formulas $\phi(a, y)$ for $a \in M$. So here X is compact. Condition (b) says that $\phi(x, y)$ does not have the order property in M , namely there do NOT exist a_i, b_i in M for $i < \omega$ such that for all i, j , $M \models \phi(a_i, b_j)$ iff $i \leq j$ or for all i, j , $M \models \neg\phi(a_i, b_j)$ iff $i \leq j$. Now for condition (a): Weak compactness of a subset B of the set of continuous functions from X to 2 is equivalent to pointwise compactness of B . Hence condition (a) says that whenever $f \in 2^X$ is in the closure of $A \subseteq 2^X$ (in the pointwise convergence, equivalently Tychonoff, topology on the space 2^X of all functions from X to 2) then f is continuous, i.e. given by a ϕ^* -formula (namely a finite Boolean combination of $\phi(a, y)$'s for $a \in M$).

We will give a quick proof of this equivalence of (a) and (b) in the model-theoretic context (see Proposition 2.2 below). In fact the proof will be Grothendieck's one (proof of (d) implies (a) of Theorem 2 in [2]), which he says is based on an idea of Eberlein, but amusingly, is also essentially the proof of Proposition 3.1 from [5] where we proved that if $\phi(x, y)$ does not have the order property in M , and $a, b \in M^*$ (a saturated elementary extension of M) then $tp_{\phi}(a/M, b)$ is finitely satisfiable in M iff $tp_{\phi^*}(b/M, a)$ is finitely satisfiable in M .

As we point out in 2.3, conditions (a), (b) in the model-theoretic context imply (and are equivalent to) the statement that every $p(x) \in S_{\phi}(M)$ has an extension $p' \in S_{\phi}(M^*)$ which is both finitely satisfiable in and definable over M (where moreover the $\phi(x, y)$ -definition for p is a ϕ^* -formula over M).

In [3] we defined a *complete type* $p(x)$ over a model M to be generically stable if p has an extension to a complete type p' over M^* which is finitely satisfiable in, and definable over M . Under the assumption that T has *NIP*, we showed in [3], that generically stable complete types $p(x)$ have additional properties, such as p' being the unique nonforking extension of p . Subsequently in [6] an appropriate stronger definition of generic stability (of a complete type) was given in an arbitrary theory, in such a way that the additional properties are satisfied.

So morally, the model-theoretic meaning of the Grothendieck theorem is that the formula $\phi(x, y)$ does not have the order property in M if and only if every complete ϕ -type $p(x) \in S_{\phi}(M)$ is generically stable. And this was already implicit in [5] where we obtained generic stability of every complete type over M from “ M has no order” (i.e. no formula $\phi(x, y)$ has the order property in M). We will investigate later to what extent we can deduce the stronger notions of generic stability from not the order property in M .

Let us briefly give definitions of some of the functional analysis notions. Given a topological space X and the real Banach space $C_b(X)$ of bounded continuous functions, let $L(C_b(X), \mathbb{R})$ be the space of bounded linear functions on $C_b(X)$. The weak topology on $C_b(X)$ is the one whose basic open neighbourhoods of a point f_0 are of the form $\{f \in C_b(X) : |g_1(f - f_0)| < \epsilon, \dots, |g_r(f - f_0)| < \epsilon\}$ for some $\epsilon > 0$ and some finite

set $g_1, \dots, g_n$ from $L(C_b(X), \mathbb{R})$. This weak topology is stronger than the pointwise topology (namely where the functions g_i above are chosen to be evaluation at points $x_i \in X$). When X is compact (and so $C_b(X) = C(X)$), there is a converse: A basic fact (see Lemma D.3 of [4]) is that a bounded subset B of $C(X)$ is compact in the weak topology iff B is compact in the pointwise convergence topology. It follows for example that if B is a subset of the continuous functions from X to $\{0, 1\}$, then the closure of B in $C(X)$ (which will be contained of course in 2^X) in the weak topology on $C(X)$ is compact iff the closure of B in the space 2^X with the product topology consists of continuous functions.

We will also be using (often without mention) the result of Eberlein-Smulian that compactness in the weak topology of a subset B of $C_b(X)$ is equivalent to sequential compactness of B in the weak topology, namely every sequence from B having a convergent subsequence.

2 Theorem, proof, and discussion

Let us first fix notation. $\phi(x, y)$ is an L -formula, and M an L -structure. $S_\phi(M)$ denotes the space of complete ϕ -types over M (in variable x). $\phi^*(y, x)$ is $\phi(x, y)$ and $S_{\phi^*}(M)$ denotes the space of complete ϕ^* -types over M (in variable y). We let X denote the space $S_{\phi^*}(M)$. A ϕ -formula over M is a (finite) Boolean combination of formulas $\phi(x, b)$ for b in M . The ϕ -formulas pick out the clopen subsets of $S_\phi(M)$. Likewise for ϕ^* -formulas and $S_{\phi^*}(M)$. Let M^* be a saturated elementary extension of M and M^{**} a saturated elementary extension of M^* . Any formula $\phi(a, y)$ with $a \in M$ can be evaluated at any $q \in X$ (i.e. truth value of $\phi(a, b)$ for some/any realization b of q), and by definition the corresponding map $X \rightarrow 2$ is continuous.

Remark 2.1 *Let $f : X \rightarrow 2$ be in the closure of the set of functions $X \rightarrow 2$ given by formulas $\phi(a, y)$ for $a \in M$ (in the product topology on 2^X). Then there is $a^* \in M^{**}$ such that $tp(a^*/M^*)$ is finitely satisfiable in M (in particular M -invariant), and for $q \in X$, $f(q)$ is the value (true or false) of $\phi(a^*, b)$ for some/any $b \in M^*$ realizing q . Conversely any such a^* yields in this way a function $X \rightarrow 2$ in the closure of the set of functions given by $\phi(a, y)$ for $a \in M$.*

Modulo the discussion of weak compactness in Section 1, the equivalence of (a) and (b) below is precisely the statement of Grothendieck's theorem in the classical model-theoretic environment.

Proposition 2.2 *The following are equivalent.*

(a) *If $f \in 2^X$ is in the closure in the pointwise convergence topology (equivalently product topology) of the set of functions given by $\phi(a, y)$ for $a \in M$, then f is continuous, so*

given by a ϕ^* -formula over M .

(b) $\phi(x, y)$ does not have the order property in M .

Proof. First the “easy” direction (a) implies (b). Assume (a), and suppose (b) fails, namely ϕ does not have the order property in M witnessed without loss of generality by a_i, b_i in M for $i < \omega$ such that $M \models \phi(a_i, b_j)$ iff $i \leq j$. By (a) there is a subsequence a_{j_i} $i < \omega$ such that the functions $\phi(a_{j_i}, y)$ converge pointwise to some ϕ^* -formula $\psi(y)$ over M . This means that for every $b \in M^*$, the value of $\psi(b)$ is the eventual value of $\phi(a_{j_i}, b)$. Clearly we have to have $\models \neg\psi(b_i)$ for all i , so by compactness we can find $b \in M^*$ such that $\models \neg\psi(b)$ and $\models \phi(a_i, b)$ for all i . This is a contradiction.

Now (b) implies (a). We assume that (a) fails. It follows immediately that there is an $f \in 2^X$ which is in the closure of the set of $\phi(a, y)$ for $a \in M$, and there is $q \in S_{\phi^*}(M)$ such that for every neighbourhood U of q there is $b \in M$, $tp_{\phi^*}(b/M) \in U$, such that $f(q) \neq f(tp_{\phi^*}(b/M))$. Translating, and using Remark 2.1, this means that there are $a^* \in M^{**}$ and $b^* \in M^*$ such that

(*) for every ϕ^* -formula $\psi(y)$ over M satisfied by b^* there is $b \in M$ satisfying $\psi(y)$ such that the value of $\phi(a^*, b^*)$ is different from that of $\phi(a^*, b)$. Without loss of generality $\phi(a^*, b^*)$ is true.

We now construct inductively $a_n, b_n \in M$ for $n = 1, 2, \dots$ such that

- (i) $\models \phi(a_i, b_j)$ iff $i \leq j$,
- (ii) $\models \neg\phi(a^*, b_i)$ for all i ,
- (iii) $\models \phi(a_i, b^*)$ for all i .

Suppose a_i, b_i are constructed for $i \leq n$. As $tp_{\phi}(a^*/M^*)$ is finitely satisfiable in M , choose $a_{n+1} \in M$ such that $\models \neg\phi(a_{n+1}, b_i)$ for $i \leq n$ and $\models \phi(a_{n+1}, b^*)$. Now using (*), let $b_{n+1} \in M$ be such that $\models \neg\phi(a^*, b_{n+1})$ and $\models \phi(a_i, b_{n+1})$ for $i \leq n + 1$. So the construction can be carried out. (i) gives a contradiction to ϕ having not the order property in M . ■

The remaining material is more or less contained in [1], although we spell some things out, especially Proposition 2.3 (c), and offer some other proofs.

Proposition 2.3 *Conditions (a), (b) from Proposition 2.2. are also equivalent to each of*

- (c) *Any $p(x) \in S_{\phi}(M)$ has an extension $p'(x) \in S_{\phi}(M^*)$ which is both finitely satisfiable in and definable over M where the ϕ definition of p' is a ϕ^* -formula over M ,*
- (d) *For any sequence $(a_i)_{i < \omega}$ in M there is a ϕ^* -formula $\psi(y)$ over M and a subsequence $(a_{j_i} : i < \omega)$ of the sequence $(a_i)_i$ such that for every $b \in M^*$, the value of $\psi(b)$ equals the eventual value of $\phi(a_{j_i}, b)$.*

Proof. (a) implies (c): Given $p \in S_{\phi}(M)$ let $(a_i)_i$ be a net N in M such that $tp_{\phi}(a_i/M)$ converges to p in the space $S_{\phi}(M)$. By (a) there is a subnet N' of this net such that

the functions $\phi(a_i, y)$ converge to some ϕ^* -formula $\psi(y)$ over M . Remember this means that for all $b \in M^*$, the value of $\psi(b)$ is the “eventual on N' ” value of $\phi(a_i, b)$. So we obtain a complete ϕ -type p' over M^* as follows: for $b \in M^*$, $\phi(x, b) \in p'$ if “eventually on N' ” $\phi(a_i, b)$ iff $\psi(b)$, and $\neg\phi(x, b) \in p'$ if eventually on N' , $\neg\phi(a_i, b)$ iff $\neg\psi(b)$. We see that p' is finitely satisfiable in M , definable over M by ψ , and extends p .

(c) implies (a). Let $f \in 2^X$ be in the closure of the set of functions $\phi(a, y)$ for $a \in M$. Let a^* be as in Remark 2.1. So $tp_\phi(a^*/M^*)$ is finitely satisfiable in M . Let p be the restriction to M of this type. Then we claim that $tp_\phi(a^*/M^*)$ has to coincide with the global ϕ -type p' from (c). This is because by symmetry Proposition 2.2 also holds with ϕ^* in place of ϕ , whereby p has a unique coheir over M^* . But then it is easy to see that the ϕ -definition $\psi(y)$ of $tp_\phi(a^*/M^*)$ has to coincide with f .

(a) implies (d) is immediate because the sequence of functions $\phi(a_i, y)$ has a subsequence which converges in 2^X to a ϕ^* -formula $\psi(y)$ over M and this will do the job.

(d) implies (b): This is as in the proof of (a) implies (b) in Proposition 2.2. Namely from an example a_i, b_i in M witnessing the order property, extract a subsequence $\phi(a_i, y)$ of functions convergent to a formula and get a contradiction. ■

Remark 2.4 (Assume the equivalent conditions (a)-(d).)

(i) Given $p(x) \in S_\phi(M)$, let $\psi(y)$ be the ϕ -definition of p (and also of its global coheir p'). Then there is a sequence $(a_i : i < \omega)$ in M such that for any $b \in M$, $\psi(b)$ holds iff eventually $\phi(a_i, b)$ holds, and $\neg\psi(b)$ holds iff eventually $\neg\phi(a_i, b)$ holds.

(ii) The formula $\psi(y)$ from (i) is equivalent to a finite positive Boolean combination of formulas $\phi(a, y)$ for $a \in M$.

Proof. (i) Let M_0 be a countable elementary substructure of the reduct of M to $\phi(x, y)$ which contains the defining parameters of $\psi(y)$. Let $(a_i : i < \omega)$ be a sequence in M_0 such that $tp_\phi(a_i/M_0)$ converges to $p_0 = p|_{M_0}$. As $\phi(x, y)$ does not have the order property in M_0 , we may assume, from condition (d) in Proposition 2.3, that the formulas $\phi(a_i, y)$ converge to the defining formula $\psi(y)$ of p_0 (so also of p and p') in the space $2^{S_{\phi^*}(M_0)}$. This suffices.

(ii) This follows from (i) by compactness. Specifically, in the saturated model M^* , $\models \psi(b)$ holds iff for some n , $\models \phi(a_i, b)$ for all $i \geq n$, namely $\psi(y)$ is equivalent to a certain infinite disjunction of infinite conjunctions (of the $\phi(a_i, y)$). An easy compactness argument yields the equivalence of $\psi(y)$ with a finite subdisjunction of finite subconjunctions. ■

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