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Identifying *Small* with *Bounded*: Unboundedness, Domination, Ideals and Their Cardinal Invariants

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Dedicated to Prof. Francisco Miraglia, on the occasion of his 70th birthday

Abstract

Cardinal invariants of the continuum – the so-called *small cardinals* – have been ubiquitous in many applications of Set Theory to Analysis and Topology over the last few decades. With this paper, we intend to provide a very intuitive framework, aiming to exhibit a number of standard examples of what are some of those cardinals trying to capture; our approach also emphasizes the more natural ways to prove inequalities between them. We handle such framework by working (in a high level of generality) with the ideal of bounded subsets of a directed pre-order. Specifically, we show that, for any fixed directed pre-order without a maximum element, the well-known cardinal invariants defined on the ideal of its bounded subsets (namely: the additivity, the uniformity, the covering number and the cofinality of the ideal) capture, in a pretty precise way, the notions of unboundedness and domination in such directed pre-order. Categorical methods to deal with both definitions and inequalities in this particular context are also discussed.

Keywords: pre-orders, unboundedness and domination, cardinal invariants defined on ideals.

1 Introduction

In what follows, the set-theoretical notation is standard, and we assume the reader is familiar with ordinals and cardinals. The cardinality of a set X is denoted by $|X|$.

The main notion of this work is that of *pre-orders*. Pre-orders, together with some cardinal invariants naturally defined on them, will be presented as providers of standard examples of what kind of information on a given structure may be captured by defining (and comparing via inequalities) certain cardinal invariants. The authors believe that pre-orders may constitute a very intuitive framework for understanding not only the definitions of several cardinal invariants but also the settling of the inequalities between them as well.

If $\mathbb{P}$ is a non-empty set and $\leqslant$ is a reflexive, transitive binary relation on $\mathbb{P}$, we will say that $\langle \mathbb{P}, \leqslant \rangle$ is a *pre-order*. A subset $B \subseteq \mathbb{P}$ is said to be *unbounded* if

$$(\forall x \in \mathbb{P})(\exists y \in B)(y \nleqslant x).$$

We define the *un*$\mathfrak{b}$*ounding number of* $\mathbb{P}$ in the following way:

$$\mathfrak{b}(\mathbb{P}) = \min\{|B| : B \subseteq \mathbb{P} \text{ is unbounded}\}.$$

Of course, $\mathfrak{b}(\mathbb{P})$ is well-defined if, and only if, $\mathbb{P}$ has no maximum element.

A subset $D \subseteq \mathbb{P}$ is said to be *dominating* if it is cofinal, that is,

$$(\forall x \in \mathbb{P})(\exists y \in D)(x \leqslant y).$$

We define the $\mathfrak{d}$*ominating number of* $\mathbb{P}$ in the following way:

$$\mathfrak{d}(\mathbb{P}) = \min\{|D| : D \subseteq \mathbb{P} \text{ is dominating}\}.$$

By reflexivity, $\mathbb{P}$ is dominating in itself, so $\mathfrak{d}$ is well-defined in any case. However, there is no interest in $\mathfrak{d}(\mathbb{P})$ in the case of $\mathbb{P}$ having a maximum element, since in this case clearly one has $\mathfrak{d}(\mathbb{P}) = 1$.

A pre-order $\mathbb{P}$ is said to be *upward directed* (or, simply, *directed*) if any finite subset of $\mathbb{P}$ has an upper bound in $\mathbb{P}$. Notice that, for an infinite directed pre-order $\mathbb{P}$ without a maximum element, $\mathfrak{b}(\mathbb{P})$ is an infinite cardinal – and so is $\mathfrak{d}(\mathbb{P})$, since obviously one has $\mathfrak{b}(\mathbb{P}) \leqslant \mathfrak{d}(\mathbb{P})$.

In Combinatorial Set Theory, one of the most important pre-orders is that of functions from ω into ω – where ω denotes the set of all natural numbers – with *eventual domination*: for any $f, g \in {}^{\omega}\omega$ we say that g *eventually dominates* f, and we denote this by $f \leqslant^* g$, if

$$\{n < \omega : g(n) < f(n)\} \text{ is a finite set.}$$

Equivalently, $f \leqslant^* g$ if, and only if, there is some $m < \omega$ such that $f(n) \leqslant g(n)$ for all $n \geqslant m$.

In the specific case of the pre-order $\langle {}^{\omega}\omega, \leqslant^* \rangle$, the cardinals given by $\mathfrak{b}(\langle {}^{\omega}\omega, \leqslant^* \rangle)$ and $\mathfrak{d}(\langle {}^{\omega}\omega, \leqslant^* \rangle)$ are referred to as, simply, $\mathfrak{b}$ and $\mathfrak{d}$ respectively. A standard diagonal argument shows that countable families of functions from ω into ω are bounded in the mod finite order, and so $\mathfrak{b}$ and $\mathfrak{d}$ are uncountable cardinals[1].

The cardinals $\mathfrak{b}$ and $\mathfrak{d}$ are examples of the so-called *cardinal invariants of the continuum*, or *small cardinals*. According to Vaughan ([15]), a small cardinal is a cardinal number which is defined as being the cardinality of a certain family which is somehow associated with the set of natural numbers. More precisely, the usual definition of a small cardinal is to consider the minimal cardinality of a subfamily of the family of all functions from ω into ω – or, instead, a subfamily of the family of all infinite subsets of ω – which does not have a property which could only fail for uncountable subfamilies. Despite their purely combinatorial definitions, such cardinals are very influential in a large number of subjects from Analysis and Topology (see, e.g., the standard references [6] (mainly for Analysis) and [9] (mainly for Topology)).

The following results are folklore. The reader will be able to prove them just by mimicking the proofs done in [9] for $\mathfrak{b}$ and $\mathfrak{d}$ themselves. We also summarize in what follows some comments already made above. For a given pre-order $\leqslant$, "$<$" stands for "$\leqslant$ and $\not\geqslant$".

Fact 1.1 *Let $\mathbb{P}$ be an infinite, directed pre-order without a maximum element. Then, the following statements hold:*

(*i*) $\mathfrak{b}(\mathbb{P})$ *is a well-defined infinite cardinal and* $\mathfrak{b}(\mathbb{P}) \leqslant \mathfrak{d}(\mathbb{P})$.

(*ii*) $(\forall x \in \mathbb{P})(\exists y \in \mathbb{P})[x < y]$.

(*iii*) $\mathfrak{b} = min\{|B| : B \subseteq \mathbb{P}$ *is unbounded and well-ordered by* $<\}$.

(*iv*) $\mathfrak{b}(\mathbb{P})$ *is regular and* $\mathfrak{b}(\mathbb{P}) \leqslant cf(\mathfrak{d}(\mathbb{P}))$. □

Let us turn to another direction. Consider a non-empty set X. A family $\mathcal{I}$ of subsets of X is said to be a *(proper) ideal of subsets of* X if the following properties are satisfied:

(*i*) $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$;

(*ii*) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;

(*iii*) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

This concept dualizes the (possibly more often present in general mathematics) notion of a *(proper) filter* – in the sense that if $\mathcal{I}$ is an ideal then the complements

[1]Countable families are, indeed, bounded in $\langle {}^{\omega}\omega, \leqslant^* \rangle$; however, the reader may easily check that, considering the pointwise defined order $\leqslant$ – that is, $f \leqslant g$ if $f(n) \leqslant g(n)$ for every $n < \omega$ –, its unbounding number is $\aleph_0$. On the other hand, it is worthwhile remarking that one has $\mathfrak{d}(\langle {}^{\omega}\omega, \leqslant^* \rangle) = \mathfrak{d}(\langle {}^{\omega}\omega, \leqslant \rangle)$; see [9], or, for a more general result, [8].

of its elements form a filter. Appealing to the intuition, ideals are usually viewed as being formed by *small* sets, while, of course, filters will be viewed as being formed by *large* sets. As any reasonable definitions of "smallness" should include all singletons, it is clear that *proper* ideals will only exist (in any interesting case) over infinite sets. Because of this, from now on we will consider that all sets X are infinite and all ideals $\mathcal{I}$ are proper.

The so-called *combinatorics of filters and ideals* has been the subject of much recent research (see, e.g., Hrušák's survey [11]). In what follows, we define the usual cardinal invariants related to ideals.

Definition 1.2 (Cardinal invariants related to ideals) *Let $\mathcal{I}$ be an ideal of subsets of an infinite set X.*

(i) add($\mathcal{I}$) (the **additivity** *of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$ – that is,*

$$add(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}.$$

(ii) cov($\mathcal{I}$) (the **covering number** *of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ which covers X – that is,*

$$cov(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}.$$

(iii) non($\mathcal{I}$) (the **uniformity** *of $\mathcal{I}$) is the smallest size of a subset of X which is not in $\mathcal{I}$ – that is,*

$$non(\mathcal{I}) = \min\{|A| : A \subseteq X \text{ and } A \notin \mathcal{I}\}.$$

(iv) cof($\mathcal{I}$) (the **cofinality** *of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ which is cofinal in $\mathcal{I}$ – that is,*

$$cof(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})[I \subseteq A]\}.$$

□

Suppose that $\mathcal{I}$ is a ideal of subsets of X which is a *σ-ideal*, meaning that it is *σ-complete* – i.e., closed under countable unions – and also suppose that $\mathcal{I}$ includes all singletons. Under these very usual requirements (which are fulfilled, e.g., by the ideal $\mathcal{M}$ of meager subsets of $\mathbb{R}$ and by the ideal $\mathcal{L}$ of the Lebesgue null subsets of $\mathbb{R}$), we have that add($\mathcal{I}$) is a regular, uncountable cardinal. It is also easy to check that, under the same mentioned requirements on a ideal $\mathcal{I}$ of subsets of an infinite set X, the following inequalities hold:

$$\begin{aligned}\aleph_1 \leqslant \operatorname{add}(\mathcal{I}) &\leqslant \min\{\operatorname{cov}(\mathcal{I}), \operatorname{non}(\mathcal{I})\}\\ &\leqslant \max\{\operatorname{cov}(\mathcal{I}), \operatorname{non}(\mathcal{I})\} \leqslant \operatorname{cof}(\mathcal{I}) \leqslant |\mathcal{I}|.\end{aligned}$$

It is easy to check that a cofinal family in $\mathcal{I}$ is also a *base* which generates the ideal. A non-empty family of non-empty sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be an *ideal base* if for every $B_1, B_2 \in \mathcal{B}$ there is some $B_3 \in \mathcal{B}$ such that $B_1 \cup B_2 \subseteq B_3$. If $\mathcal{B}$ is an ideal base, the ideal generated by $\mathcal{B}$ (which is the smallest ideal including $\mathcal{B}$) is given by $\{A \subseteq X : \exists B \in \mathcal{B}[A \subseteq B]\}$. Notice that, in particular, $\operatorname{cof}(\mathcal{I})$ is the minimal size of an ideal base of $\mathcal{I}$. For instance, as both mentioned ideals $\mathcal{M}$ and $\mathcal{L}$ have bases constituted of Borel sets, one can conclude that the cardinal invariants of such ideals have $\mathfrak{c} = 2^{\aleph_0}$ as an upper bound.

The eight cardinal invariants defined as described over the above mentioned ideals $\mathcal{M}$ and $\mathcal{L}$, together with $\aleph_1$, $\mathfrak{c}$, $\mathfrak{b}$ and $\mathfrak{d}$, constitute the well-known *Cichoń's Diagram* ([10]), displayed as follows:

$$\begin{tikzcd}
 & \operatorname{cov}(\mathcal{L}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{L}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{L}) \arrow[r] \arrow[uu] & \operatorname{add}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{non}(\mathcal{L}) \arrow[uu] &
\end{tikzcd}$$

In the diagram, arrows represent **ZFC**-provable inequalities between the corresponding cardinals. One also has $\operatorname{add}(\mathcal{M}) = \min\{\mathfrak{b}, \operatorname{cov}(\mathcal{M})\}$ and $\operatorname{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \operatorname{non}(\mathcal{M})\}$. The cardinals of Cichoń's Diagram are also considered small cardinals; notice that, in fact, one can use the Baire space of the irrationals (which is a Polish space homeomorphic to the space of functions ${}^{\omega}\omega$ endowed with the Tychonoff topology) as a substitute of $\mathbb{R}$ when it comes to compute the values of all mentioned cardinal invariants, so again we are considering cardinals which may be defined as the minimal cardinalities of certain subfamilies of the family of functions from ω into ω. We refer to the book of Bartoszinsky and Judah ([1]) for a comprehensive investigation of these cardinals and their surprising relationships with a number of subjects in both Analysis and Topology.

In the following sections, we will give an unified treatment for both contexts presented in this Introduction – the one of unboundedness and domination in pre-orders, and the other of the four cardinal invariants defined in terms of an ideal of subsets of a set. More precisely, we will show that for any fixed directed pre-order without a maximum element there is an ideal of subsets of the pre-order (indeed, a very natural ideal to be considered) whose cardinal invariants capture precisely the notions of unboundedness and domination in the directed pre-order previously fixed. Although the results of this paper seems to generalize a number of known previous results (see, e.g., the results on the σ-ideal generated by the σ-compact subsets of ${}^{\omega}\omega$ in [1], Lemma 2.2.1, or the comments on the additivity and covering number of families of thin sets ordered by inclusion in page 250 of [7]), the authors were not able to find any prior reference for them, as they stand here – and in their full generality –, in the literature. In fact, the authors believe that the results in this paper may constitute standard examples of what are some cardinal invariants trying to capture, and we hope this could lead to new insights on this subject.

2 The ideal of bounded subsets of a directed pre-order

Let $\langle \mathbb{P}, \leqslant \rangle$ be a directed pre-order without a maximum element. For any $x \in \mathbb{P}$, let

$$B_x = \{y \in \mathbb{P} : y \leqslant x\}.$$

Notice that $A \subseteq X$ is bounded in $\mathbb{P}$ if, and only if, there is some $x \in \mathbb{P}$ such that $A \subseteq B_x$.

Directedness of the pre-order ensure that $\mathcal{B} = \{B_x : x \in \mathbb{P}\}$ is an ideal base. Let $\mathcal{I}_{\mathbb{P}}$ be the ideal generated by such base. It should be clear that

$$\mathcal{I}_{\mathbb{P}} = \{A \subseteq \mathbb{P} : \ A \text{ is a bounded subset of } \mathbb{P}\}.$$

So, $\mathcal{I}_{\mathbb{P}}$ is the ideal of bounded subsets of $\mathbb{P}$ – that is, we are considering as *small* those subsets of $\mathbb{P}$ which are *bounded*. The main theorem of this paper is the following one:

Theorem 2.1 *Let $(\mathbb{P}, \leqslant)$ be an infinite, upward directed pre-order without a maximum element, and consider its ideal $\mathcal{I}_{\mathbb{P}}$ of bounded subsets. Then, the following equalities hold:*

$$add(\mathcal{I}_{\mathbb{P}}) = non(\mathcal{I}_{\mathbb{P}}) = \mathfrak{b}(\mathbb{P}) \quad and \quad cov(\mathcal{I}_{\mathbb{P}}) = cof(\mathcal{I}_{\mathbb{P}}) = \mathfrak{d}(\mathbb{P}).$$

■

In the following section, we discuss and present some of the “classical” proofs of such inequalities, say. In the last section, alternative proofs (which are based in some features of a certain category) will be presented.

3 Proof of the Main Theorem

We give here direct proofs of the equalities

$$\mathrm{add}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{b}(\mathbb{P}) \text{ and } \mathrm{cov}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{d}(\mathbb{P}).$$

The others proofs are similar (and, in fact, $\mathrm{non}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{b}(\mathbb{P})$ is clear, given the context) and left to the reader. Indeed, we will see in the following section that there is a way of proving all the four equalities just by proving two of them.

It should be also clear that the proofs below use the Axiom of Choice several times – mainly for fixing witnesses, say. Recall that $\mathcal{B} = \{B_x : x \in \mathbb{P}\}$ is a base of $\mathcal{I}_{\mathbb{P}}$.

Proof of $\mathfrak{b}(\mathbb{P}) \leqslant \mathrm{add}(\mathcal{I}_{\mathbb{P}})$: Let $\{Y_\alpha : \alpha < \mathrm{add}(\mathcal{I}_{\mathbb{P}})\}$ be a family (of minimal size) of elements of $\mathcal{I}_{\mathbb{P}}$ whose union is not in $\mathcal{I}_{\mathbb{P}}$ – that is, a minimal sized family of bounded subsets whose union is an unbounded subset. Using the Axiom of Choice, we may fix, for every $\alpha < \mathrm{add}(\mathcal{I}_{\mathbb{P}})$, a "witness of boundedness", say, which is some $y_\alpha \in \mathbb{P}$ such that $Y_\alpha \subseteq B_{y_\alpha}$. It follows that $\{y_\alpha : \alpha < \mathrm{add}(\mathcal{I}_{\mathbb{P}})\}$ is necessarily an unbounded subset of $\mathbb{P}$, since otherwise the unbounded union of the Y_α's would be included in the set B_z for some z witnessing the boundedness of $\{y_\alpha : \alpha < \mathrm{add}(\mathcal{I}_{\mathbb{P}})\}$, and this is an absurd. So, we have

$$\mathfrak{b}(\mathbb{P}) \leqslant |\{y_\alpha : \alpha < \mathrm{add}(\mathcal{I}_{\mathbb{P}})\}| \leqslant \mathrm{add}(\mathcal{I}_{\mathbb{P}}).$$

Proof of $\mathrm{add}(\mathcal{I}_{\mathbb{P}}) \leqslant \mathfrak{b}(\mathbb{P})$: Let $B \subseteq \mathbb{P}$ be an unbounded set with minimal size, that is, $|B| = \mathfrak{b}(\mathbb{P})$. Define $\mathcal{A} = \{B_x : x \in B\}$. We claim that $\bigcup \mathcal{A} \notin \mathcal{I}_{\mathbb{P}}$. Indeed, if $\bigcup \mathcal{A} \in \mathcal{I}_{\mathbb{P}}$ then for some $B_z \in \mathcal{B}$ we would have $\bigcup \mathcal{A} \subseteq B_z$. So, B_z includes B_x for every $x \in \mathcal{A}$, and therefore $x \leqslant z$ for every $x \in B$ – which is an absurd, since B is unbounded. So, we have

$$\mathrm{add}(\mathcal{I}_{\mathbb{P}}) \leqslant |B| = \mathfrak{b}(\mathbb{P}).$$

Proof of $\mathfrak{d}(\mathbb{P}) \leqslant \mathrm{cov}(\mathcal{I}_{\mathbb{P}})$: Let $\mathcal{A} = \{D_\alpha : \alpha < \kappa\}$ be a subfamily of $\mathcal{I}_{\mathbb{P}}$ which covers X and has minimal size, that is, $\mathrm{cov}(\mathcal{I}_{\mathbb{P}}) = \kappa$. For every $\alpha < \kappa$ fix $z_\alpha \in \mathbb{P}$ such that $D_\alpha \subseteq B_{z_\alpha}$. Given $z \in \mathbb{P}$, there is some $\zeta < \kappa$ such that $z \in D_\zeta$ – and therefore $z \leqslant z_\zeta$. It follows that $\{z_\alpha : \alpha < \kappa\}$ is a dominating subset of $\mathbb{P}$ and has size not larger than κ. The inequality is proved, since $\mathfrak{d}(\mathbb{P}) \leqslant |\{z_\alpha : \alpha < \kappa\}| \leqslant \kappa = \mathrm{cov}(\mathcal{I}_{\mathbb{P}})$.

Proof of $\mathrm{cov}(\mathcal{I}_{\mathbb{P}}) \leqslant \mathfrak{d}(\mathbb{P})$: Let $D \subseteq \mathbb{P}$ be a dominating subset of minimal size, that is, $|D| = \mathfrak{d}(\mathbb{P})$. Let $\mathcal{A} = \{B_x : x \in D\}$. Claim that $\bigcup \mathcal{A} = X$. Indeed, let $x \in \mathbb{P}$. As D is dominating, there is some $d \in D$ such that $x \leqslant d$, and so $x \in B_d \subseteq \bigcup \mathcal{A}$. As $\mathcal{A}$ covers X, it follows that $\mathrm{cov}(\mathcal{I}_{\mathbb{P}}) \leqslant |\mathcal{A}| \leqslant |D| = \mathfrak{d}(\mathbb{P})$.

4 Alternative proofs, using Category Theory

Peter Vojtáš ([16]) introduced a category, which he called GT (for Galois–Tukey connections), in order to express directly (via morphisms) some relations between explicit objects of Analysis. Such category is, in fact, a variant of the category $\mathrm{Dial}_2(\mathbf{Sets})$, the simplest example of the so-called *Dialectica categories*, introduced by Valeria de Paiva to provide categorical models for linear logic (see [12] and [13]); more precisely, GT is a variant of the opposite (dual) category $\mathrm{Dial}_2(\mathbf{Sets})^{op}$. As Blass did in [2], we will refer to such category as $\mathcal{PV}$ ($\mathcal{P}$ after de Paiva, $\mathcal{V}$ after Vojtáš – or, more informally, in honour of Peter and Valeria).

Objects of $\mathcal{PV}$ are triples, a generic object is $o = (A, B, E)$, where A and B are sets and $E \subseteq A \times B$ is a binary relation. A morphism from $o_2 = (A_2, B_2, E_2)$ to $o_1 = (A_1, B_1, E_1)$ is a pair of functions (φ, ψ), where $\varphi : A_1 \to A_2$ and $\psi : B_2 \to B_1$ are such that $\varphi(a)E_2b$ implies $aE_1\psi(b)$ for all $a \in A_1$ and $b \in B_2$. Morphisms induce the so-called *Galois-Tukey pre-order*, meaning that if o_1 and o_2 are objects of $\mathcal{PV}$ then

$$o_1 \leqslant_{GT} o_2 \iff \text{There is a morphism from } o_2 \text{ to } o_1.$$

Suppose that an object (A, B, E) satisfies the following condition: for every $a \in A$ there is some $b \in B$ such that aEb. In this case, one can define the *norm* (or *evaluation*) $||o|| = ||(A, B, E)||$ of the object $o = (A, B, E)$ as being the minimum cardinality of a subset $Y \subseteq B$ with the following property: for every $a \in A$ there is $b \in Y$ such that aEb. If $o = (A, B, E)$ satisfies the following (dual) condition: for every $b \in B$ there is some $a \in A$ such that $\neg aEb$, then we are able to consider the *dual object* $o^* = (B, A, E^*)$, where bE^*a means, precisely, $\neg aEb$, and then we get to the norm of the dual object, given by $||o^*|| = ||(B, A, E^*)||$ and defined accordingly.

Now, suppose that o_1 and o_2 are objects of $\mathcal{PV}$ such that all norms $||o_1||$, $||o_2||$, $||o_1^*||$ and $||o_2^*||$ are well-defined. One could ask whether are there inequalities between such cardinal numbers – and if is there some standard, fast way to prove them, if they exist. It turns out that exhibiting morphisms is an amazing way to prove inequalities between norms – indeed, every proof counts as two, as we can see in the following proposition:

Theorem 4.1 ("Folklore" ; see [4]) *Suppose o_1 and o_2 are objects such that all norms $||o_1||$, $||o_2||$, $||o_1^*||$ and $||o_2^*||$ are well-defined. Then, the following statements hold:*

1. *If $o_1 \leqslant_{GT} o_2$ then $||o_1|| \leqslant ||o_2||$.*
2. *If $o_1 \leqslant_{GT} o_2$ then $||o_2^*|| \leqslant ||o_1^*||$.*

For the sake of completeness, we present proofs of both parts of the previous well-known theorem.

Proof. For the first part: let (φ, ψ) be a morphism from the object $o_2 = (A_2, B_2, E_2)$ to the object $o_1 = (A_1, B_1, E_1)$, and let $Y_2 \subseteq B_2$ be a subset of B_2 satisfying the expected condition $(\forall a \in A_2)(\exists b \in B_2)[aE_2b]$ and with minimal size, that is, $||o_2|| = |Y_2|$. Consider $Y_1 = \psi[Y_2] \subseteq B_1$ and now we have $(\forall x \in A_1)(\exists y \in Y_1)[xE_1y]$; indeed, for a given $x \in A_1$, pick $y = \psi(b)$ for some $b \in Y_2$ such that $\varphi(x)E_2b$. It follows that Y_1 satisfies the condition required in the definition of $||o_1||$, and therefore $||o_1|| \leqslant |Y_1| \leqslant |Y_2| = ||o_2||$.

For the second part, a simple contrapositive check shows that if (φ, ψ) is a morphism from o_2 to o_1 then (ψ, φ) is a morphism from o_1^* to o_2^*. ■

The described method of norms and morphisms (which was extensively studied in the 90's by Blass, see e.g. [2],[3],[4] and [5]) fits our interests in this paper because all considered cardinal invariants in this work may be written as norms – and all proofs may be encoded by morphisms. Indeed, if $\mathbb{P}$ is a pre-order without a maximum element, then it is immediate that

$$\mathfrak{d}(\mathbb{P}) = ||(\mathbb{P}, \mathbb{P}, \leqslant)|| \text{ and } \mathfrak{b}(\mathbb{P}) = ||(\mathbb{P}, \mathbb{P}, \not\geqslant)|| = ||(\mathbb{P}, \mathbb{P}, \leqslant)^*||.$$

In particular, notice that $\mathfrak{d}(\mathbb{P})$ and $\mathfrak{b}(\mathbb{P})$ are dual to each other, meaning that they are norms of dual objects of $\mathcal{PV}$.

Let us turn to the cardinal invariants defined in terms of ideals: let X be an infinite set and let $\mathcal{I}$ be an ideal of subsets of X. Then

$$\begin{aligned} \operatorname{add}(\mathcal{I}) &= ||(\mathcal{I}, \mathcal{I}, \not\supseteq)||, \\ \operatorname{non}(\mathcal{I}) &= ||(\mathcal{I}, X, \not\ni)||, \\ \operatorname{cov}(\mathcal{I}) &= ||(X, \mathcal{I}, \in)||, \\ \operatorname{cof}(\mathcal{I}) &= ||(\mathcal{I}, \mathcal{I}, \subseteq)||. \end{aligned}$$

Indeed, the equalities for $\operatorname{cov}(\mathcal{I})$ and $\operatorname{cof}(\mathcal{I})$ are obvious and follow immediately from the definitions. For $\operatorname{add}(\mathcal{I})$, notice that, given a family $\mathcal{A} \subseteq \mathcal{I}$, $\bigcup \mathcal{A} \notin \mathcal{I}$ if, and only if, for every $I \in \mathcal{I}$ there is some $A \in \mathcal{A}$ such that $A \not\subseteq I$. For $\operatorname{non}(\mathcal{I})$, notice that, given $Y \subseteq X$, $Y \notin \mathcal{I}$ if, and only if, for every $I \in \mathcal{I}$ there is some $b \in Y$ such that $b \notin I$.

And now it is also easy to see that $\operatorname{non}(\mathcal{I})$ and $\operatorname{cov}(\mathcal{I})$ are dual to each other, and the same holds for $\operatorname{add}(\mathcal{I})$ and $\operatorname{cof}(\mathcal{I})$.

Proof of the Main Theorem, using norms and morphisms. In order to apply Theorem 4.1 to prove the Main Theorem of this paper, it suffices to exhibit morphisms in both directions between the objects $(\mathcal{I}_{\mathbb{P}}, \mathcal{I}_{\mathbb{P}}, \subseteq)$ and $(\mathbb{P}, \mathbb{P}, \leqslant)$, as well as between the objects $(\mathbb{P}, \mathcal{I}_{\mathbb{P}}, \in)$ and $(\mathbb{P}, \mathbb{P}, \leqslant)$; this would give us directly (by the first part of Theorem 4.1) the equalities $\operatorname{cof}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{d}(\mathbb{P})$ and $\operatorname{cov}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{d}(\mathbb{P})$, respectively, and then the equalities involving $\mathfrak{b}(\mathbb{P})$ are easily deduced by duality (using the second part of Theorem 4.1). Recall that $\{B_x : x \in \mathbb{P}\}$ is the base of $\mathcal{I}_\mathbb{P}$ defined and discussed in Section 2.

Proof of $(\mathcal{I}_\mathbb{P}, \mathcal{I}_\mathbb{P}, \subseteq) \leqslant (\mathbb{P}, \mathbb{P}, \leqslant)$: Let

$$\varphi : A \in \mathcal{I}_\mathbb{P} \mapsto x_A \in \mathbb{P},$$
$$\psi : y \in \mathbb{P} \mapsto B_y \in \mathcal{I}_\mathbb{P},$$

where x_A is *chosen* such that $A \subseteq B_{x_A}$. If $x_A \leqslant y$ then $A \subseteq B_{x_A} \subseteq B_y$ – and so $\varphi(A) \leqslant y$ implies $A \subseteq \psi(Y)$, as desired.

Proof of $(\mathbb{P}, \mathbb{P}, \leqslant) \leqslant (\mathcal{I}_\mathbb{P}, \mathcal{I}_\mathbb{P}, \subseteq)$: Let

$$\varphi : x \in \mathbb{P} \mapsto B_x \in \mathcal{I}_\mathbb{P},$$
$$\psi : A \in \mathcal{I}_\mathbb{P} \mapsto y_A \in \mathbb{P},$$

where y_A is *chosen* such that $A \subseteq B_{y_A}$. If $B_x \subseteq A \subseteq B_{y_A}$ then $x \leqslant y_A$ – and so $\varphi(x) \subseteq A$ implies $x \leqslant \psi(A)$, as desired.

Proof of $(\mathbb{P}, \mathcal{I}_\mathbb{P}, \in) \leqslant (\mathbb{P}, \mathbb{P}, \leqslant)$: Let

$$\varphi : x \in \mathbb{P} \mapsto x \in \mathbb{P},$$
$$\psi : y \in \mathbb{P} \mapsto B_y \in \mathcal{I}_\mathbb{P}.$$

If $x \leqslant y$ then $x \in B_y$ – and so $\varphi(x) \leqslant y$ implies $x \in \psi(y)$, as desired.

Proof of $(\mathbb{P}, \mathbb{P}, \leqslant) \leqslant (\mathbb{P}, \mathcal{I}_\mathbb{P}, \in)$: Let

$$\varphi : x \in \mathbb{P} \mapsto x \in \mathbb{P},$$
$$\psi : A \in \mathcal{I}_\mathbb{P} \mapsto y_A \in \mathbb{P}$$

where y_A is *chosen* such that $A \subseteq B_{y_A}$. If $x \in A \subseteq B_{y_A}$ then $x \leqslant y_A$ – and so $\varphi(x) \in A$ implies $x \leqslant \psi(A)$, as desired.

Notice that this procedure of morphisms gave us a sort of crystallization of all previously mentioned uses of the Axiom of Choice to fix witnesses – and the only pair of morphisms which did not depend on arbitrary choices is just a resemblance of the definition of the ideal base.

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