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Model Theory in Sheaves

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In the honor of Chico Miraglia's 70th birthday

Abstract

In this work, we will be interested in the results in model theory which can be proven inside **sheaf models** over a frame. We know that these sheaf models generalize classical models, Kripke models, topological models and Beth models. The results in model theory with sheaf models are intuitionistic results. We will explain how to define sheaf models and give an overview over the model theoretic results obtained so far in the context explained.

Keywords: Models in sheaves, intuitionistic models, sheaf theory, frames, complete Ω-sets.

Introduction

The point of investigation is a generalization of intuitionistic models as Kripke, Beth, topological and algebraic models, for these models see for example [15, 35, 38]. We will not work inside a topos theoretic model but in sheaf models over a fixed frame or if necessary in sheaves over a (fixed) topological space. Working in a special topological space has some advantages which we will see in the following. Topological properties of the space considered can help proving theorems in model theory, see sections 2 and 5. Working in sheaf models over a fixed frame Ω, we are able to use the rather algebraic definition of complete Ω-set. On the other side, we have clearly the disadvantage that these sheaf models could be strictly generalized: The sheaf models over topological spaces are generalized by sheaf models over a frame and the latter are generalized by the topos theoretic models, i.e., models in a general topos, cf. [28].

This article is to be intended as a survey presenting results obtained so far in models of sheaves from Chico Miraglia and the author. Most of the results presented here are due to Chico Miraglia. The author had the honor and pleasure to work and to learn with Chico in the second half of the 1990's. In occasion of Chico Miraglia's 70th birthday in september 2016, the author gave a talk about these results at University of São Paulo. We resume all these results in this article. The author hopes that this article helps for a further understanding of intuitionistic model theory. It seems that today this branch of mathematical logic has only a few researchers, nevertheless the author thinks that the research subject is very interesting and helps to understand intuitionistic logic. We generalize model theory for intuitionistic logic and in a first approach we *do not* use topos theory. Also, we consider *classical* logic in the meta theory. Chico Miraglia always said to me as I learned with him: "Only if we have a clear sight of intuitionistic results in this way we can think to start generalization of these obtained results, for example, with an intuitionistic meta theory or in topos theory." Clearly, whenever possible, we always use in the meta theory constructive arguments in the sense of Brouwer's constructive reasoning, cf [3, 19, 38]. In section 4 we are able to prove the generalization of the method of constants for sheaf models with a constructive meta theory. A very good overview of the origin and philosophy of Brouwer's intuitionistic and constructive reasoning is given in [1], which can be accessed online.

The seminal paper of model theory in sheaves over a frame Ω, named as *complete* Ω*-set* by the authors, is to the best of the authors knowledge the 1979 paper of Fourman and Scott, cf. [16]. We think that this paper gives a kind of new foundation and origin of the research in model theory using sheaf models over a frame Ω. Clearly, before and after Fourman and Scott, there were research in model theory using sheaf models over a topological space, see for example the results of [8, 9, 11, 12, 14, 26, 33, 39]. The idea of Fourman and Scott was to introduce sheaf models using Ω*-sets, finitely complete and complete* Ω*-sets*, where Ω is a frame. This approach seemed to be very interesting because we do not use the definitions of presheaves and sheaves known and used so far: presheaf as a functor and sheaf as functor with the additional property of the existence of gluings for compatible sections. The value of a formula interpreted in a sheaf model so far was given by sub-(pre-)sheaf of the interpreting (pre-)sheaf. After Fourman and Scott's work, we were be able to see sheaf in a kind of algebraic way and the value of a formula is given by a characteristic function, and this characteristic function is in general easier to treat as the sub-(pre-)sheaves. Also we are working a little bit more general as in sheaves over a topological space[1] using these complete Ω-sets.

The paper is divided in the following way. In the next section, we recall some definitions and basic facts important for understanding the model theoretic results obtained in the other sections. We define frame, Ω-set, complete Ω-set, presheaf over a frame. We recall the notion of characteristic maps, important for interpretation of the formulas. We give the ideas of sheafification and the interpretation of terms and formulas. In the third section, we present two model existence theorems, cf. [4, 6] and the Łoś theorem for a restricted language $L^\sharp$, cf. [31] – a language without the universal

[1]About the history of sheaf theory see [18].

quantifier. In section three, we sketch the downward Löwenheim Skolem Theorem in presheaf structures over a certain frame, having countable determined meets and joins, cf. [30]. Section four, gives a generalizing method of diagrams in the sense of A. Robinson. We have basically two versions, one in our usual (total) language L and the other in an extended language $L^{\square}$, cf. [5]. In this section, we are able to argue constructively in the meta theory. The last section treats an intuitionistic version of the Omitting Types Theorem, there a restriction to the language $L^{\sharp}$ and sheaf models over topological spaces[2] is made, cf. [6].

1 Previous Definitions and Results about Sheaf Theory

We do not assume the reader is familiar with the basic notions of Ω-sets, presheaves and sheaves over Ω, where Ω is a frame. We try to give all preliminaries to a good understanding of the following results. For further information the reader can take a look in the following books or articles: [4, 16, 28, 29, 30, 38]. In all that follows, notation and terminology will be that of [4] and [29]. Almost all definitions and results of this section are due to [29]. We hope to make the article independent, by giving these definitions and results in the following.

We begin with the following standard notation.

Notation 1.1 *We shall adhere to standard notation for lattice operations. Thus, we write $\leq$, $\wedge$, $\bigwedge$, $\vee$, $\bigvee$, for the partial order, binary and arbitrary meets and joins, respectively. By $\top$ and $\bot$ we indicate the largest (top) and least (bottom) elements of a lattice.*

The following remark is fundamental for the following.

Remark 1.2 a) Recall that a **frame**[3] is a complete lattice, Ω, such that for all $S \subseteq \Omega$ and $a \in \Omega$:

$[\wedge, \bigvee]$ $$a \wedge \bigvee S \;=\; \bigvee_{s\in S} a \wedge s.$$

In a frame Ω we always have implication, i.e., for all p, $q \in \Omega$, **implication and negation** are defined by

$$p \to q \;:=\; \bigvee \{x \in L : x \wedge p \leq q\} \quad \text{and} \quad \neg p \;:=\; p \to \bot.$$

Also the adjunction between $\wedge$ and $\to$ is easily verified in a frame :

[ad] $$\forall\, x \in \Omega,\;\; x \wedge p \leq q \;\text{ iff }\; x \leq p \to q.$$

In the following the symbol Ω will always stand for a frame.

[2]Sometimes called simply *topological sheaves*.

[3]For some more information about frames the reader is asked to consult [22, 34].

b) If Ω, Λ are frames, a map $\Omega \xrightarrow{f} \Lambda$ is a **frame morphism** if it preserves finite meets and arbitrary joins. So, we can consider the category of frames.
c) If X is a topological space, its topology, $\Omega(X)$, is a classical example of a frame, under the inclusion partial order. For $\{U, V\} \cup \{U_i : i \in I\} \subseteq \Omega(X)$, we have, where *int* is the interior operation and $(\cdot)^c$ is the set theoretic complementation:
$\bigwedge_{i\in I} U_i := int \bigcap_{i\in I} U_i$; $\bigvee_{i\in I} U_i := \bigcup_{i\in I} U_i$; $U \to V := int\,(U^c \cup V)$; $\neg U := int\, U^c$.
Let $\overline{A}$ be the topological closure of $A \subseteq X$. Then, for $U \in \Omega(X)$ we have, $\neg\neg U = int\, \overline{U}$. ■

We next introduce the important notion of complete Ω-set, which is equivalent to the notion of sheaf.

Definition 1.3 *a) A Ω-set, A, consists of a non-empty set $|A|$ (its domain), together with a map,*
$[\![\cdot = \cdot]\!]_A : A \times A \longrightarrow \Omega$, *the* **equality of *A***, *verifying, for all a, b, $c \in |A|$*
[eq 1] : $[\![a = b]\!]_A = [\![b = a]\!]_A$;   [eq 2] : $[\![a = b]\!]_A \wedge [\![b = c]\!]_A \leq [\![a = c]\!]_A$.
Whenever A is clear from context, we drop its mention from the notation. The elements of A are called **sections of *A***. *For $x \in |A|$, $Ex := [\![x = x]\!]_A$ is the* **extent** *or* **domain** *of x. It is straightforward that for all x, $y \in |A|$,*
[E] $$[\![x = y]\!]_A \leq Ex \wedge Ey.$$
We say that A is **separated** *or* **extensional** *if for all x, $y \in |A|$,*
[ext] $$Ex = Ey = [\![x = y]\!]_A \quad \Rightarrow \quad x = y.$$
b) For $p \in \Omega$, $A(p) = \{x \in |A| : Ex = p\}$ is **the set of sections of extent *p***. *In particular, $A(\top)$ is the set of* **global sections** *of A.*
c) Let A be an Ω-set; a subset $S \subseteq |A|$ is **compatible** *if for all $s, s' \in S$, $Es \wedge Es' = [\![s = s']\!]$. If S is a compatible set of sections in A, a* **gluing** *of S is a section $t \in |A|$ such that*
[glu 1] : $Et = \bigvee_{s\in S} Es$;   [glu 2] : For all $s \in S$, $Es = [\![s = t]\!]$.
It can be shown that if A is separated, gluings are unique (whenever they exist).
d) An Ω-set D is **complete** *or a* **sheaf** *if all compatible sets of sections of D have a* unique gluing in D.
e) (Morphisms) *If A, B are Ω-sets, a* **morphism**, $A \xrightarrow{f} B$, *consists of a map, $f : |A| \longrightarrow |B|$, such that for all x, $y \in |A|$*
[mor 1] : $E_B fx = E_A x$;   [mor 2] : $[\![x = y]\!]_A \leq [\![fx = fy]\!]_B$.
Ω-sets, separated Ω-sets and sheaves over Ω, with the notion of morphism defined above are categories, written **Ω-set**, **$\epsilon\Omega$-set** *and* **Sh(Ω)**, *respectively.*
f) Whenever clear from context, we omit the mention of the Ω-set from the notation.

Example 1.4 *It is easy to see that given a topological space X and considering $\mathcal{C}(X;\mathbb{R}) := \{f \mid f : X \to \mathbb{R}$ is continuous $\}$ is a complete separated $\Omega(X)$-set.* ■

As a special case of Lemma 25.21, cf. [29] we have the next

Lema 1.5 *For morphism in* **Ω-set**, $A \xrightarrow{f} B$, *with A separated, the following are equivalent:*
(1) *f is an injection of $|A|$ into $|B|$;* (2) *For all $x, y \in |A|$,* $[\![x = y]\!]_A = [\![fx = fy]\!]_B$. ∎

The next definition gives the notion of presheaf introduced by Fourman and Scott, cf. [16].

Definition 1.6 *a) A* **presheaf over Ω, $\boldsymbol{A}$,** *is a set,* $|A|$ (its domain), *together with two maps*

$$\cdot_{|\cdot} : |A| \times \Omega \longrightarrow |A| \quad \text{(restriction)} \quad \textit{and} \quad E : |A| \longrightarrow \Omega \quad \text{(extent)},$$

satisfying the following axioms, for all $x \in |A|$ and $p, q \in \Omega$:
[psh 1] : $E(x_{|p}) = Ex \wedge p$; [psh 2] : $x_{|Ex} = x$; [psh 3] : $x_{|(p \wedge q)} = (x_{|p})_{|q}$.
b) A presheaf A is **separated** *if for all $x, y \in |A|$ and $D \subseteq \Omega$*

$$\forall\, p \in D,\ x_{|p} = y_{|p} \text{ and } Ex = Ey = \bigvee D \quad \Rightarrow \quad x = y.$$

c) If A, B are presheaves over Ω, a **presheaf morphism**, $A \xrightarrow{f} B$, *is a* (set-theoretic) *map, $f : |A| \longrightarrow |B|$, such that for all $x \in |A|$ and $p \in \Omega$,*
[pmor 1] : $E_B f(x) = E_A x$; [pmor 2] : $f(x_{|p}) = f(x)_{|p}$.
d) (Restriction of a presheaf) *If A is a presheaf over Ω and $p \in \Omega$, the* **restriction** *of A to p, $A_{|p}$, is the presheaf whose domain is $\bigcup_{q \leq p} A(q)$ and whose restriction map is that induced by A. Thus, the elements of $|A_{|p}|$ are the sections of A of extent $\leq p$.*

Write **pSh(Ω)** *for the category of presheaves over Ω.*

In case of topological presheaves A, we can introduce the stalk of A in a given point y, which is important in theorem 5.2. For this, see also the localization of a Ω-set, cf. 2.5. We explain in the next definition how to do this.

Definition 1.7 *Let Y be a topological space, A a topological presheaf over Y, i.e., a presheaf over $\Omega(Y)$ and $y \in Y$ a point.*
a) We say that ν_y is the set of open neighborhoods of y in Y, $\nu_y := \{u \in \Omega(Y) : y \in u\}$.
b) Let $A^y := \{a \in |A| : y \in Ea\} = \{a \in |A| : Ea \in \nu_y\}$ *be the set of sections of A whose extent (or domain) is a neighborhood of y.*
c) Define a binary relation θ_y in A^y by $a\ \theta_y\ b$ *iff* $y \in [\![a = b]\!]_A$.
d) The equivalence class of $a \in A^y$ by θ_y is indicated by a_y and called **germ of $\boldsymbol{a}$ in $\boldsymbol{y}$**. *For $\vec{a} = \langle a_1, \ldots, a_n \rangle \in (A^y)^n$, write $\vec{a}_y := \langle a_{1y}, \ldots, a_{ny} \rangle$, the sequence of corresponding germs.*
e) The set of equivalence classes of A^y by θ_y is the **stalk of $\boldsymbol{A}$ in the point $\boldsymbol{y}$**, *written A_y.*[4]

[4]The stalk A_y is therefore the direct limit of $A(p)$, $p \in \nu_y$.

Clearly, every presheaf can be associated a Ω-set, which is described in the next remark.

Remark 1.8 *If A is a presheaf over Ω and x, $y \in |A|$, define*
$$[\![x = y]\!]_A = \bigvee \{p \wedge Ex \wedge Ey : x_{|p} = y_{|p}\}.$$
Then, $[\![\cdot = \cdot]\!]_A$ is an equality on $|A|$, with which it becomes an Ω-set, the **Ω-set associated to the presheaf A**. *Whenever A is clear from context, we omit its mention from the notation.* ∎

The relations between restriction, extent and the equality associated with a presheaf over Ω are described in the next result:

Proposition 1.9 *Let A be a presheaf over Ω, let x, $y \in |A|$ and let p, $q \in \Omega$.*
a) (i) $Ex = [\![x = x]\!]$;
(ii) $[\![x_{|p} = y_{|q}]\!] = p \wedge q \wedge [\![x = y]\!]$;
(iii) $x_{|[\![x = y]\!]} = y_{|[\![x = y]\!]}$;
(iv) *A is an separated as a presheaf iff A is separated as an Ω-set.*
b) Let P, Q be presheaves over Ω and let $f : |P| \longrightarrow |Q|$ be a map and consider the following conditions:
(i) *f is a presheaf morphism;* (ii) *f is a Ω-set morphism.*
Then, $(i) \Rightarrow (ii)$ and these conditions are equivalent if Q is extensional.
c) Let P, Q be extensional presheaves over Ω. For a presheaf morphism, $f : P \longrightarrow Q$, the following are equivalent:
(i) *f is an injection of $|P|$ into $|Q|$;* (ii) *For all x, $y \in |P|$, $[\![x = y]\!]_A = [\![fx = fy]\!]_B$.*

Proof: Immediate from proof of Theorem 26.8, [29]. ∎

Every Ω-set can be transformed without loss of information into a **separated** Ω-set, which is the content of the next remark.

Remark 1.10 a) The process of *extensionalization*, i.e., making a presheaf separated, of a presheaf of first-order structures is described in Theorem 23.21, [29]. The method is the analogue using equivalence relation for inducing a partial order in a preorder.
b) The assumption that presheaves and Ω-sets be separated, simplifies many arguments. From the point of view of Model Theory of first-order structures in the category **pSh(Ω)**, nothing is lost by considering only *separated* presheaves. Therefore, we may assume that all **Ω**-sets and presheaves over **Ω** are **separated**. ∎

The next definition introduce the product of Ω-sets, which is essential for the interpretation of our formulas in the following. It is the same as in [29].

Definition 1.11 *If A_i, $i \in I$, are Ω-sets, their product, $\prod_{i\in I} A_i$, is the Ω-set given by:*
(i) $|\prod_{i\in I} A_i| \;=\; \{\langle x_i\rangle \in \prod_{i\in I} |A_i| : \forall\, i,\, j \in I,\, Ex_i = Ex_j\}$;
(ii) $[\![\langle x_i\rangle = \langle y_i\rangle]\!] \;=\; \bigwedge_{i\in I} [\![x_i = y_i]\!]$.
Note the distinction between $|\prod A_i|$ and $\prod |A_i|$; the restriction to $|\prod A_i|$ of the canonical projections of $\prod |A_i|$ onto its coordinates yield Ω-set morphisms from $\prod A_i$ to A_i, written π_i. The same construction works in the categories **pSh(Ω)** *and* **Sh(Ω)**.

We will use the following

Notation 1.12 *If A is a Ω-set, $n \geq 1$ is an integer and $\vec{a} = \langle a_1, \ldots, a_n\rangle$, $\vec{c} = \langle c_1, \ldots, c_n\rangle \in |A|^n$, define*
* $E\vec{a} \;:=\; \bigwedge_{i=1}^n Ea_i$; * $[\![\vec{a} = \vec{c}]\!] \;:=\; \bigwedge_{i=1}^n [\![a_i = c_i]\!]$;
* *If A is a presheaf and $p \in \Omega$, $\vec{a}_{|p} := \langle a_{1|p}, \ldots, a_{n|p}\rangle$, where $\cdot_{|\cdot}$ is the restriction of A.* ■

The notion of density and dense elements in Ω-sets is used in the model theoretic results, see the sections 3 and 4.

Definition 1.13 *Let A be a Ω-set.*
a) A subset $D \subseteq |A|$ is **dense in *A*** *if for all $a \in |A|$, $Ea \;=\; \bigvee_{d\in D} [\![a = d]\!]$.*
b) The **density** *of A, $d(A)$, is the least cardinal κ such that A has a dense subset of cardinal κ. We say that A is* **separable** *if $d(A) \leq \aleph_0$.*
c) For p, $q \in \Omega$, we say that q **is dense in** *p iff $q \leq p \leq \neg\neg q$. In case q is dense in $\top$ (i.e., $\neg\neg\, q = \top$), we say that q is dense in Ω.*

The notion of *density* seems to be the reasonable analogue of set-theoretic cardinality in the sheaf-theoretic context, as illustrated by the Löwenheim-Skolem results in [30], see also section 3. If A is a presheaf, then D is dense in A iff every section of A locally coincides with one in D, or, more precisely, for all $a \in |A|$, there is $\{\langle d_i, p_i\rangle \in D \times \Omega : i \in I\}$ such that $Ea = \bigvee_{i\in I} p_i$ and $\forall i \in I$, $a_{|p_i} = d_{i|p_i}$, where $\cdot_{|\cdot}$ is the restriction map of A (see definition 1.6 and [29]). We have the next Lemma whose proof can be found in [29], more precisely Lemma 25.33 and Theorem 37.8 in [29]:

Lema 1.14 *Let A, B be Ω-sets and let D be a dense set of sections in A.*
a) The relation of being dense is transitive.
b) If f, $g : A \longrightarrow B$ are Ω-set morphisms, then $f_{|D} = g_{|D} \;\Rightarrow\; f = g$.
c) A morphism $f : A \longrightarrow B$ is epic iff $f(A)$ is dense in B.
d) For all $\vec{a} \in |A|^n$, $E\vec{a} \;=\; \bigvee \{[\![\vec{a} = \vec{d}]\!] \in \Omega : \vec{d} \in D^n\}$. ■

In [29], chapter 37, and in [4], chapter 1, there is an extensive account of characteristic maps that are useful, among other things, to define the interpretation of formulas in a first-order structure in **Ω-set**. We shall here collect the basics on this topic, referring the reader to [4, 29] for details and proofs. The vector notation in 1.12 will be of constant use.

Definition 1.15 (Characteristic Maps) *Let A be a Ω-set and let $n \geq 0$ be an integer. A **n-characteristic map** on A is a map $h : |A|^n \longrightarrow \Omega$, such that for all $\vec{x}, \vec{y} \in |A|^n$:*

$$[\text{ch } 1] : \; h(\vec{x}) \;\; \leq \;\; E\vec{x}; \qquad\qquad [\text{ch } 2] : \; [\![\vec{x} = \vec{y}]\!] \wedge h(\vec{x}) \;\; \leq \;\; h(\vec{y}).$$

*Write $\mathfrak{K}_n A$ for the set of n-characteristic maps on A. If $h \in \mathfrak{K}_n A$, its **extent** is defined by*

$$Eh \;\; := \;\; \bigvee \{h(\vec{x}) \in \Omega : \vec{x} \in |A|^n\}.$$

We have the following important remark from [29]:

Remark 1.16 Let A be a Ω-set and $h \in \mathfrak{K}_n A$.
a) Condition [ch 2] is equivalent to $\;[\text{ch } 2'] : [\![\vec{x} = \vec{y}]\!] \wedge h(\vec{x}) \;\; = \;\; [\![\vec{x} = \vec{y}]\!] \wedge h(\vec{y})$.
b) $\mathfrak{K}_n A$ has a natural partial order, given by $h \leq k$ iff $\forall\, \vec{x} \in |A|^n, \;\; h(\vec{x}) \leq k(\vec{x})$. With this partial order
∗ In $\mathfrak{K}_n A$, $\perp$ is the constant $\perp$-valued function, while $\top$ is the map $\vec{x} \longmapsto E\vec{x}$;
∗ $\mathfrak{K}_n A$ is a frame, with meets and joins computed pointwise, while implication and negation are given by

$$[h \rightarrow k](\vec{x}) \;=\; E\vec{x} \wedge (h(\vec{x}) \rightarrow k(\vec{x})) \quad \text{and} \quad [\neg\, h](\vec{x}) \;=\; E\vec{x} \wedge \neg\, h(\vec{x}),$$

where $\rightarrow$ and $\neg$ in the right-hand side of these equations are the operations in Ω. ■

Further properties of n-characteristic maps are contained in the Proposition 1.17 below, and are consequences of Theorem 37.8 and Proposition 37.12, in [29]:

Proposition 1.17 *Let A be a Ω-set, let D be a dense set of sections in A and let $h \in \mathfrak{K}_n A$.*
a) For all $\vec{x} \in |A|^n$, $\;h(\vec{x}) \;=\; \bigvee_{\vec{\alpha} \in D^n} [\![\vec{\alpha} = \vec{x}]\!] \wedge h(\vec{\alpha})$.
b) If $k \in \mathfrak{K}_n A$, then $\;h_{|D^n} \;=\; k_{|D^n} \;\Rightarrow\; h = k$.
c) If A is a presheaf over Ω, then for all $\vec{a} \in |A|^n$ and $p \in \Omega$ we have $\;h\left(\vec{a}_{|p}\right) \;=\; p \wedge h(\vec{a})$. In particular,

$$h\left(\vec{a}_{|E\vec{a}}\right) \;=\; h\left(\vec{a}_{|h(\vec{a})}\right) \;=\; h(\vec{a}).$$

d) Let $D^n \xrightarrow{k_0} \Omega$ be a map such that for $\vec{x}, \vec{y} \in D^n$, $\;k_0(\vec{x}) \leq E\vec{x}\;$ and $\;k_0(\vec{x}) \wedge [\![\vec{x} = \vec{y}]\!] \leq \; k_0(\vec{y})$. Then, there is a unique $k \in \mathfrak{K}_n A$ such that $k_{|D^n} = k_0$.
e) If B is a sheaf over Ω and $f_0 : D \longrightarrow |B|$ is map such that for all $x, y \in D$, $Ef_0(x) = Ex$ and $[\![x = y]\!] \leq [\![f_0 x = f_0 y]\!]$, there is a unique Ω-set morphism, $f : A \longrightarrow B$, satisfying $f_{|D} = f_0$. ■

Using Scott's idea of a *singleton* (see [16]), every Ω-set A has a completion (or sheafification); for a proof of the following result see theorems 27.9, 37.8 and Corollary 37.9 in [29].

Theorem 1.18 *If A is a Ω-set, there is a sheaf cA over Ω and a morphism, $A \xrightarrow{c} cA$, such that*
a) For every a, $b \in |A|$, $[\![a = b]\!] = [\![ca = cb]\!]$.
b) The image of c is dense in cA.
c) If B is a sheaf over Ω and $f : A \longrightarrow B$ is a morphism, there is a unique morphism, $cf : cA \longrightarrow B$, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "c"] \arrow[dr, "f"'] & & cA \arrow[dl, "cf"] \\ & B & \end{tikzcd}$$

d) Let $n \geq 1$ be an integer. Each characteristic map $h \in \mathfrak{K}_n A$, has a unique extension to a characteristic map, $h^c \in \mathfrak{K}_n cA$, and for all $\vec{x} \in |cA|^n$,
$(*)$ $$h^c(\vec{x}) = \bigvee_{\vec{a} \in |A|^n} [\![\vec{x} = c\vec{a}]\!]_{cA} \wedge h(\vec{a}),$$
where $c\vec{a} = \langle ca_1, \ldots, ca_n \rangle$. In particular, for all $\vec{a} \in |A|^n$, $h^c(c\vec{a}) = h(\vec{a})$. ■

When dealing with first-order languages one frequently encounters operations that are not unary. In general, the completion of a finite product of Ω-sets *is not* the product of the completion of the coordinates.The situation for presheaves is smoother, for in this case the completion functor is finitely complete[5]. If A is a Ω-set and $n \geq 2$ is an integer, recall (1.11) that $\langle a_1, \ldots, a_n \rangle \in |A|^n$ is in A^n iff all a_i have the same extent. If A is a presheaf over Ω and $D \subseteq |A|$ is dense in A, the set $D^n_* := \{\vec{d}_{|E\vec{d}} : \vec{d} \in D^n\}$, *is a dense subset of A^n* (see Proposition 26.22 in [29]).[6]

In the following, we will start with model theory in sheaves, and therefore we have to make use of a first order language L, which we introduce in the next remark. For the method of constants, cf. 4 we modify – without loss of generality, cf. [4, 6] – our notion of the constant symbols.

Remark 1.19 *We consider the following first-order language with equality, L, where*

[5] We say that a Ω-set A is *finitely complete* iff every finite $S \subseteq_f |A|$ compatible has a gluing, cf. 1.3 (c).

[6] If A is not a presheaf, there is no canonical way to lift a dense subset of A to a dense subset of A^n (in general, D^n *is not* a subset of A^n). We need restriction!

• *$rel(n, L)$ is the set of n-ary relation symbols in L,*
• *$op(m, L)$ is the set of m-ary function symbols in L, and*
• *C is the set of constant symbols, or if necessary when a frame Ω is fixed, C is a* presheaf[7] *(of constant symbols) over Ω.*

Clearly, we need the interpretation of our language introduced in 1.19. We use characteristic functions to interprete relation symbols and the equality of the Ω-set related to the presheaf for interpreting logical equality. By the discussion above we can use almost always presheaves over a frame Ω instead of sheaves, cf. 1.18.

Definition 1.20 *Let Ω be a frame, L a first-order language, with equality and a presheaf of constant symbols, C. An* **interpretation of L in $pSh(\Omega)$** *consists of a presheaf A over Ω, together with the following data*
[rel] : *For every relation symbol $R \in rel(n, L)$,* **distinct from $=$**, *a characteristic function*

$$[\![R(\cdot, \dots, \cdot)]\!]_A : |A|^n \longrightarrow \Omega, \quad \vec{a} \mapsto [\![R(\vec{a})]\!]_A;$$

[=] : *The equality symbol is interpreted by the equality of A;*
[fun] : *For every function symbol $\omega \in op(n, L)$, a presheaf morphism*

$$\omega^A : A^n \longrightarrow A, \quad \vec{a} \ \mapsto \ \omega^A(\vec{a});$$

[Con] : *A presheaf morphism*[8] *from C to A, $\cdot^A : C \longrightarrow A, \quad c \mapsto c^A$.*
A presheaf A over Ω, together with an interpretation of L in A is a **L-structure in pSh(Ω)** *or a* **presheaf of L-structures over Ω**.

Remark that there exist many sheaves *without* global sections, consider for example the recovering space $\mathcal{S}^n := (\mathbb{R}^n, \pi, \mathbb{R}^n/\mathbb{Z}^n)$, where π is the canonical projection, of a n-torus; for more examples we refer the reader to [4, 29]. So the admittance of a presheaf of constant symbols is in this sense very useful. We define free and bound variables, terms of L, formulas of L and (free) substitution of a term in a formula as in the classical case. We must define the extension of a term and a formula, because by considering partial existence of constant symbols, terms and formulas can exist locally. See the next definition.

Definition 1.21 *Let L be a first order language with a presheaf of constants C over Ω, τ be a term of L and φ be a formula of L. Then we define*

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[7]Initially in the theory of sheaf models, a set C of constant symbols is used and every symbol is interpreted as global section. But if we want to generalize the method of constants, see section 4, we have to consider the partial existence of the constants, to create the diagram of a theory. In [4] we explain that we do not loss generality considering a presheaf of constants. So in the following we can use simply a set C of constant symbols, or if necessary a presheaf over a frame of constant symbols.

[8]In the case, considering a set C of constant symbols, this set C can be seen as a presheaf C over the frame Ω with only global sections. The interpretation of a symbol $c \in C$ is then given by a global section $c^A \in A(\top)$. See also [16].

$$\begin{cases} E\tau \ := \ \bigwedge \{E_C c \ : \ c \ occurs \ in \ \tau\} \\ E\varphi \ := \ \bigwedge \{E_C c \ : \ c \ occurs \ in \ \varphi\}, \end{cases}$$

the **extent** *of τ and φ, respectively.*

Let $p \in \Omega$, then $\tau_{|p}$ and $\varphi_{|p}$ indicate the term and the formula obtained by substituting every occurrence of a symbol $c \in C$ in τ or φ, respectively, by the symbol $c_{|p}$; it is easy to see, that $\tau_{|p}$ is a term and $\varphi_{|p}$ is a formula of L, with the same complexity as the originals. We call them **restriction** *of τ to p and* **restriction** *of φ to p, respectively. It is also clear, that in case of τ and φ without constant symbols, the restrictions are identical to the original formulas.*

The next Lemma will give some simple but helpful properties.

Lema 1.22 *Let $\tau(x_1, \ldots, x_m)$, $\tau_1, \ldots, \tau_m$ be terms of L, $\omega \in op(m, L)$, $R \in rel(m, L)$ and φ, ψ formulas of L. Let C be the presheaf of constants of L and $p \in \Omega$. Then*
a) $E\varphi_{|p} = \bigwedge \{p \wedge Ec : c \in |C| \text{ and } c \text{ occurs in } \varphi\}$.
b) If there are no occurrences of elements of $|C|$ in φ and in τ, then $E\tau = E\varphi = \top$.
c) $ER(\tau_1, \ldots, \tau_m) = E\omega(\tau_1, \ldots, \tau_m) = \bigwedge_{i=1}^m E\tau_i$.
d) $E(\varphi \diamond \psi) = E\varphi \wedge E\psi$, where $\diamond$ represents a binary connective of L.
e) $E\neg\varphi = E\,\forall v\varphi = E\,\exists v\varphi = E\varphi$.
f) If $\varphi = \varphi(x_1, \ldots, x_m)$ and χ is the result obtained by substituting τ_i for x_i in φ, $1 \leq i \leq m$, then $E\chi = E\varphi \wedge \bigwedge_{i=1}^n E\tau_i$. ■

We have to talk about L-terms and their interpretation in a given presheaf A over Ω.

Definition 1.23 *Let A be a presheaf of L-structures. Every term $\tau(v_1, \ldots, v_n)$ in L induces a morphism of presheaves*

$$\tau^A : A^n_{|E\tau} \longrightarrow A,$$

called the **interpretation of $\boldsymbol{\tau}$ in $\boldsymbol{A}$**, *defined, for $\vec{a} \in |A_{|E\tau}|^n$, by induction on complexity as follows:*
1. *If τ is a variable v_i, then $E\tau = \top$ and τ^A is the i-th projection, that is, $v_i^A(\vec{a}) = a_i$.*
2. *If τ is a constant $c \in |C|$, then for every $p \in \Omega$, $c^A(p) = c^A{}_{|(p \wedge Ec)}$.*
3. *If $\tau_1(\vec{v}), \ldots, \tau_m(\vec{v})$ are terms in L, $\omega \in op(m, L)$ and $\tau(\vec{v}) := \omega(\tau_1(\vec{v}), \ldots, \tau_m(\vec{v}))$, then*

$$\tau^A(\vec{a}) = \omega^A(\tau_1^A(\vec{a}), \ldots, \tau_m^A(\vec{a})).$$

With these definitions in hands, we are able to define the interpretation of any first order formula φ by complexity in an arbitrary presheaf over Ω. We will do this in the next definition.

Definition 1.24 *Let A be a presheaf of L-structures over Ω. By induction on complexity, we associate to each formula $\varphi(v_1, \ldots, v_n)$ in L, a characteristic function*

$$[\![\varphi(\cdot, \ldots, \cdot)]\!]_A : |A|^n \longrightarrow \Omega,$$

called the **interpretation of φ in A**, *defined for $a_1, \ldots, a_n \in |A|$ as follows:*

1. [atom] : If $\tau_1(\vec{v}), \ldots, \tau_m(\vec{v})$ are terms and $R \in rel(m, L)$, then

$[\![(\tau_1(\vec{v}) = \tau_2(\vec{v}))(\vec{a})]\!]_A := [\![\tau_1^A(\vec{a}_{|_p}) = \tau_2^A(\vec{a}_{|_p})]\!]_A$, *where $p = E\tau_1 \wedge E\tau_2$;*

$[\![R(\tau_1, \ldots, \tau_m)(\vec{a})]\!]_A := [\![R(\tau_1^A(\vec{a}_{|_q}), \ldots, (\tau_m^A(\vec{a}_{|_q}))]\!]_A$, *where $q = \bigwedge_{i=1}^m E\tau_i$.*

2. [con] : If φ, ψ are formulas with $p = E\varphi$ and $q = E\varphi \wedge E\psi$, then

$[\![\neg\,\varphi(\vec{a})]\!]_A := p \wedge E\vec{a} \wedge \neg\,[\![\varphi(\vec{a}_{|_p})]\!]_A$;

$[\![(\varphi \diamond \psi)(\vec{a})]\!]_A := q \wedge E\vec{a} \wedge \Big([\![\psi(\vec{a}_{|_q})]\!]_A \diamond [\![\chi(\vec{a}_{|_q})]\!]_A\Big)$, *where* $\diamond \in \{\wedge, \vee, \rightarrow\}$.

3. [$\exists$] : $[\![\exists x\varphi(x, \vec{a})]\!]_A := \bigvee_{t\in|A|} [\![\varphi(t; \vec{a})]\!]_A$.

4. [$\forall$] : $[\![\forall x\varphi(x, \vec{a})]\!]_A := E\varphi \wedge E\vec{a} \wedge \bigwedge_{t\in|A|} \big(Et \rightarrow [\![\varphi(t; \vec{a})]\!]_A\big)$.

Next, we define the *intuitionistic* forcing relation, which we will use in the following sections.

Definition 1.25 *Let A be a presheaf of L-structures over Ω.*
a) Let $\varphi(v_1, \ldots, v_n)$ be a L-formula and $a_1, \ldots, a_n \in |A|$. We say that **A forces φ** *at $a_1, \ldots, a_n$ and write $A \Vdash \varphi[a_1, \ldots, a_n]$ iff $[\![\varphi(a_1, \ldots, a_n)]\!]_A = E\varphi \wedge E\vec{a}$.*
b) Let σ be a L-**sentence**. *We say that* **A is a model of σ**, *written $A \Vdash \sigma$, in case $[\![\sigma]\!]_A = E\sigma$.*
c) Let Σ be a set of L-sentences, $\Lambda(v_1, \ldots, v_n)$ a set of L-formulas in the variables $v_1, \ldots, v_n$ and $a_1, \ldots, a_n \in |A|$. Then the expressions

$$A \Vdash \Sigma \quad \text{and} \quad A \Vdash \Lambda[a_1, \ldots, a_n]$$

mean that $A \Vdash \sigma$ and $A \Vdash \varphi[a_1, \ldots, a_n]$, for all $\sigma \in \Sigma$ and $\varphi \in \Lambda$, respectively.

For a better understanding, we recall in the next definition the Gödel translation, which will be used in section 4.

Definition 1.26 *Let L be a first-order language with equality, as above. The* **Gödel translation** *of a L-formula, φ, written φ^g, is defined by induction on complexity as follows:*

- *If φ is atomic, then φ^g is $\neg\neg\,\varphi$;*
- *If $\diamond \in \{\wedge, \rightarrow\}$, then $(\varphi \diamond \psi)^g = \varphi^g \diamond \psi^g$. Moreover, $(\neg\,\varphi)^g = \neg\ \varphi^g$;*
- *$(\varphi \vee \psi)^g = \neg\neg\,(\varphi^g \vee \psi^g)$;*
- *$(\exists x\,\varphi)^g = \neg\neg\,\exists x\ \varphi^g$; $(\forall x\,\varphi)^g = \forall x\ \varphi^g$.*

Also in section 4 we can prove a generalization of the diagram lemma extending our language by adding a family of new connectives, which were considered first by Caicedo and Sette in the paper [9]. We introduce a new family $\{\Box_q\}_{q\in\Omega}$ for a fixed frame Ω, in the next remark.

Remark 1.27 *We fix some frame Ω and introduce a new language $L^{\Box}$ adding the symbols $\Box_q$, $q \in \Omega$, as unary connectives, to L.*
∗ For every $p \in \Omega$, if φ is a formula, then $\Box_p\varphi$ is a formula in $L^{\Box}$.
Clearly, the notion of $L^{\Box}$-term is the same as for L. The notions of $L^{\Box}$-structure A in $\mathbf{pSh(\Omega)}$ *and the interpretation of $L^{\Box}$-term in A, are the same as earlier. We easily show that*

$\boldsymbol{A}$ **is a $\boldsymbol{L^{\Box}}$-structure in $\mathbf{pSh(\Omega)}$** *iff* $\boldsymbol{A}$ **is a $\boldsymbol{L}$-structure in $\mathbf{pSh(\Omega)}$**.

Concluding, we have to add the following conditions for the extent and the value of a $L^{\Box}$-formula:
∗ If $\varphi(v_1, \ldots, v_n)$ is a L-formula with $E\varphi = q$ and $p \in \Omega$, we set $E(\Box_p\varphi) := E\varphi = q$;
∗ If $\varphi(v_1, \ldots, v_n)$ is a $L^{\Box}$-formula with $q = E\varphi$ and $\vec{a} \in |A|^n$, then

$$[\Box_p] \qquad [\![\Box_p\varphi(\vec{a})]\!]_A \;=\; E\vec{a}_{\,|q} \wedge ([\![\varphi(\vec{a}_{|q})]\!]_A \leftrightarrow p).$$

∗ We remark that for a dense element $d \in D$, the formula $\Box_d\varphi(\vec{a})$ behaves in some sense classically, see also section 4, theorem 4.10. ∎

We argue that these preliminaries are sufficient for the presentation of our model theoretic results in (pre-)sheaves of L-structures over a frame Ω. The reader interested in the subject is recalled to take a look in [4, 16, 29] or every other bibliography treating first order sheaf models. In the next section we begin with the important question of completeness for these sheaf models which we answer positively.

2 Model Existence and Łoś Theorem

In this section, the question of completeness and the existence of models in (pre-)sheaves over a frame Ω are treated. In [38], there is done an approach to construct a model in Ω-sets for the first order language L. There, only global sections are considered for the proof of completeness. We can do a little better and prove the following completeness theorem in pre-sheaves of L-structures.

The proof can be seen in [4]. The idea of the proof is simple, we construct a term model seen as Ω-set and adapt the ideas of the original algebraic proof using the Lindenbaum-Tarski algebra, cf. [36]. More explicitly, we consider the set of L-terms and L-formulas and introduce the usual equivalence relation in terms indentifying two terms iff they are equal in the theory; two formulas are identified iff they are logically equivalent in the theory. Then, we are able to construct the Lindenbaum-Tarski algebra $A(\mathfrak{T})$ of the theory $\mathfrak{T}$ which we show to be a frame. For example, see the construction in [36] for the proof of completeness of the intuitionistic first order logic. The set of the names of the L-terms is now easily shown to be a separated $A(\mathfrak{T})$-set with equality defined as equivalence class of the equality in the Lindenbaum-Tarski algebra. The interpretation of the formulas is also defined simply as equivalence class of the formula. So, we are done and have our desired term model which proves the next theorem.

Theorem 2.1 (Brunner & Miraglia, 2000, [4]) *Let L be a first order language with equality, φ a L-formula and $\mathfrak{T}$ an intuitionistic theory in L . Are equivalent*

(*i*) $\mathfrak{T} \vdash_I \varphi$*; and*

(*ii*) *For all frame Ω and every L-structure A in* $\mathbf{pSh(\Omega)}$*, $\mathfrak{T} \Vdash \varphi$, i.e.,*

$$A \Vdash \mathfrak{T} \quad \Longrightarrow \quad A \Vdash \varphi. \qquad \blacksquare$$

If we consider a fragment of the language L, $L^\sharp$ consisting of the formulas (intuitionistically equivalent to those) constructed from the atomic ones using the propositional connectives and the existential quantifier (a language without the universal quantifier) and denote by $\forall L^\sharp$ the fragment of L composed of the formulas (intuitionistically equivalent to those) of the type $\forall \vec{x} \varphi(\vec{x}; \vec{y})$, where φ is in $L^\sharp$, then we have a different model existence theorem, given in 2.2. Observe that in particular, a sentence in $\forall L^\sharp$ is (intuitionistically equivalent to) the universal closure of a formula in $L^\sharp$.

The next theorem is only valid for $\forall L^\sharp$-formulas, and the proof is more complicated as the proof of 2.1. All the details can be joined in [4, 5]. Roughly speaking, we will construct for the set Σ of L-sentences, the space $X(\Sigma, L)$ consisting of all prime theories extending Σ. We topologize this space by the base $V_\Sigma := \{V_\varphi | \ \varphi$ is a L-sentence $\}$,[9] where $V_\varphi := \{P \in X(\Sigma, L) | \ \varphi \in P\}$. We are able to show that the space $X(\Sigma, L)$ with this topology is a spectral space[10]. The analogue of the Boolean Prime Ideal Theorem is proved and so we are able to prove the completeness theorem 2.2 using the Henkin idea of witnessing the existential formulas of L by new constants. The construction of the complete Ω-set or sheaf is made in the same way of theorem 2.1 adapting ideas from Rasiowa and Sikorski explained above.

Theorem 2.2 (Brunner & Miraglia, 2004, [4, 5]) *Let L be a first order language with equality and let $\Sigma \subseteq \forall L^\sharp$ be a consistent set of sentences. Then, there exist an expansion L' of L by constants, a spectral space Z and a L'-structure A in $Sh(Z)$ such that*

(1) *A is a model of Σ;*

(2) *The set $\{c^A \in A(\top) : c \in \mathfrak{N}\}$ is dense in A;*

(3) *For every formula $\varphi(x_1, \ldots, x_n)$ in L and $a_1, \ldots, a_n \in |A|$,*

$$[\![\varphi(a_1, \ldots, a_n)]\!]_A = \bigvee_{\vec{c} \in \mathfrak{N}^n} \bigcap_{j=1}^n [\![a_j = c_j^A]\!]_A \cap [\![\varphi(c_1^A, \ldots, c_n^A)]\!]_A,$$

where $\mathfrak{N}$ is the set of constants in $L' \setminus L$. $\blacksquare$

As immediate consequence we have the completeness theorem for the language $\forall L^\sharp$ and with the existence of a spectral space Z a model as a sheaf over this spectral space Z. Clearly, correctness theorem is straightforward to prove, and so we did not comment anything so far.

[9] This topology is a Zariski style topology.

[10] A spectral space is compact T0 and sober, with a base of compact opens. Furthermore, it is a Baire space. In section 5 we will use this fact to obtain stalks omitting a certain type.

Corollary 2.3 *Let L be a first order language with equality, Σ a set of sentences in $\forall L^\sharp$ and φ a sentence of L. The following are equivalent :*

(1) $\Sigma \vdash_I \varphi$;

(2) *For every spectral space Z and every L-structure A in $Sh(Z)$,*

$$A \Vdash \Sigma \quad \Rightarrow \quad A \Vdash \varphi. \qquad \blacksquare$$

The next principle, the maximum principle, is first shown for intuitionistic logic, by Chico Miraglia in [31]. The notable difference between classical and intuitionistic logic, is the fact that an existential formula has a witness, which is *only* dense. This result has been expected for intuitionistic logic and in section 4 we have a similar result for the diagram, where density appears. Clearly, for showing that an existential formula has a witness, gluing is important and necessary, so the result only holds in sheaves over a frame. A proof can be accompanied in [30, 31]. We omit the details.

Theorem 2.4 (Maximum Principle, Miraglia, 1990, [31]) *Let $\exists x\varphi(x;\vec{x})$ be a L-formula and A an L-structure in* $\mathbf{Sh(\Omega)}$. *Then* $\forall \vec{a} \in |A|^n$, $\exists b \in |A|$ *such that*

$$[\![\varphi(b;\vec{a})]\!]_A \leq [\![\exists x\varphi(x;\vec{a})]\!]_A \leq \neg\neg[\![\varphi(b;\vec{a})]\!]_A. \qquad \blacksquare$$

For proving of Łoś' Theorem we need this Maximum Principle and use a kind of quotient Ω/θ_F-set, where F is a filter in Ω and θ_F is the congruence generated by F. We assume the reader is familiar with the following constructions, some of them use ideas from universal algebra, for details look [2, 7, 17], the other are basically straightforward constructions, which can be consulted in [31].

Remark 2.5 (Localization of an Ω-set) *Let Ω be a frame, $F \subseteq \Omega$ a filter and θ_F the congruence generated by the filter F, i.e.:*

$$\forall p, q \in \Omega, \quad p\theta_F q \quad \textit{iff} \quad (p \leftrightarrow q) \in F$$

Then we have the following:

(a) The quotient Ω/θ_F defined in the known way, cf. [7].

(b) Let now A be a (separated) Ω-set, then we construct naturally the following Ω/θ_F-set A_F:

$|A_F| := |A|/\theta_F = \{[a]_F |\ a \in |A|\}$, *where we have* $[a]_F := \{b \in |A| |\quad (Ea \vee Eb \to [\![a = b]\!]_A) \in F\}$, *and*

$$[\![[a]_F = [b]_F]\!]_{A_F} := [\![a = b]\!]_A/\theta_F, \quad \textit{for } [a]_F, [b]_F \in A_F$$

We say that A_F is the **localization of A in the filter** *F. We denote the set of the global sections in A_F by $A_F^*(= A_F(\top)) = \{[a]_F \in A_F |\ Ea \in F\}$.*

(c) For presheaves we can do localization in a similar way, we define restriction:

$$\cdot_{|\cdot} : |A|_F \times \Omega/F \longrightarrow |A|_F\ , \quad [x]_{F_{|[p]_F}} := [x_{|p}]_F,$$

and we have the localization presheaf.[11]

[11]This method cannot be generalized for sheaves, i.e., the localization A_F can be **no** sheaf anymore.

(d) For an example of localization of a presheaf, consider the recovering space $(E;\pi;X)$, with $\pi : E \to X$ a local homeomorphism. Fix a point $x \in X$ and consider the neighborhood filter F of x. Then the $A_F = A_x = \pi^{-1}(x)$ is the classical stalk of A at x, cf. 1.7.

(e) Localization can be seen as a functor, sometimes said to be the localization in F, cf. [31].

(f) Having a presheaf A of L-structures over Ω, by localization, we can make A_F a presheaf of L-structures over Ω/θ_F in a natural way commuting the equivalence classes. We omit the details and appoint the reader to [31]. ■

We have a generalization of Łoś Theorem cf. 2.6, for presheaves satisfying the maximum principle 2.4[12]. The proof of 2.6 is made by induction in the L-formulas, being the atomic case easy and the connective case almost immediate (using calculus in frames and the fact that F is a prime filter in the frame Ω). The maximum principle 2.4 is used in the step for the existential quantifier in the direction from right to the left. The direction from left to right follows by the induction hypothesis and the filter properties. The universal quantifier is *not* going through. We are able to show the direction from left to right, but we are not able to show the direction from right to left. So we can formulate the Łoś Theorem only for the language $L^\sharp$.

Theorem 2.6 (Łoś Theorem, Miraglia, 1990, [31]) *Let A be an L-structure in* $\mathbf{pSh(\Omega)}$ *satisfying the maximum principle, $U \subseteq \Omega$ an ultrafilter and $\varphi(\vec{x})$ a $L^\sharp$-formula. Let $a_1, \ldots, a_n \in |A|$ be sections in A such that $E\vec{a} \in U$, then*

$$A^*_U \models \varphi([\vec{a}]_U) \qquad \textit{iff} \qquad [\![\varphi(\vec{a})]\!]_A \in U$$ ■

We have so the following open question: Exists a Łoś theorem in $\mathbf{pSh(\Omega)}$ satisfying the maximum principle for the *whole* language L, i.e. including the universal quantifier?

3 Downward Löwenheim Skolem Theorem

We know that classically there are two versions of the Löwenhein Skolem theorems, the downward version and the upward version. Intuitionistically, we can show the downward theorem and this downward Löwenheim Skolem Theorem is proved in frames which have countable caracter, cf. 3.1, considering our (pre-)sheaf models introduced earlier. We remind the reader that the proof can be studied in [30]. For the results, the following definition about some kind of frames is essential.

Definition 3.1 *Let Ω be a frame. Then*

[12] Clearly the generalization is valid also for sheaves!

(a) We say that Ω has **countable determined meets**, **cdm**, *iff for all $\{p_i\}_{i\in I} \subseteq \Omega$ there is a countable $J \subseteq I$ such that $\bigwedge_{i\in I} p_i = \bigwedge_{j\in J} p_j$.*
(b) We say that Ω has **countable determined joins**, **cdj**, *iff for all $\{p_i\}_{i\in I} \subseteq \Omega$ there is a countable $J \subseteq I$ such that $\bigvee_{i\in I} p_i = \bigvee_{j\in J} p_j$.*
(c) We say that Ω has **countable caracter**, **cc**, *iff it has cdm and cdj.*
(d) We say that Ω has **countable chain condition**, **ccc**, *iff for all $\{p_i\}_{i\in I} \subseteq \Omega \setminus \{\perp\}$, if $p_i \wedge p_j = \perp$ for $i \neq j$, then I is countable.*

It is possible to connect for a topological space conditions of the space X and conditions of the frame of open sets $\Omega(X)$ in the following way:

Remark 3.2 *Let X be a topological space and $\Omega(X)$ the frame of the open sets of X. Then*
(i) $\Omega(X)$ has ccc iff the space X has ccc,
(ii) $\Omega(X)$ has cdm iff the space X is hereditarily separable, and
(iii) $\Omega(X)$ has cdj iff the space X is hereditarily Lindelöf. ■

The proof of the next fundamental theorem is made basically in three steps. In the first step we will construct by induction in n a chain $\{D_n\}_{n\in\mathbb{N}}$ of countable subsets of the support $|A|$ – considering restriction, values of the function symbols in the structure A and values of the quantified formulas in A, using the fact that Ω has *cc*. For details see [30]. In the second step, we prove some properties of this chain $\{D_n\}_{n\in\mathbb{N}}$: the set $D := \bigcup_{n\in\mathbb{N}} D_n$ is countable with $card(D_n) \leq max\{card(S); card(L)\}$, for each n. We take now D^* the subsheaf with support D and name it B. By construction, we have then trivially that D is dense in B, and also that B is separable with subset $S \subseteq |B|$. In the third and last step of the proof, we show that B is an elementary substructure[13] of A by induction in the complexity of the L-formulas. For the quantifier steps, we use the properties of the sets D_n constructed in the first step. So we have the following

Theorem 3.3 (Downward Löwenheim-Skolem, Miraglia 1988, [30]) *Let Ω be a frame with cc, L a countable first order language with equality and A a L-structure in* $\mathbf{pSh(\Omega)}$*. For all countable $S \subseteq |A|$, there is a separable elementary substructure B of A such that $S \subseteq |B|$.* ■

The following corollary is easy to handle. For the sheaf case, we get an elementary substructure B that is a presheaf containg D. By sheafification we get the sheaf cB which of course is a separable substructure of A, cf. 1.18.

Corollary 3.4 (Downward Löwenheim-Skolem Theorem in $\mathbf{Sh(\Omega)}$, [30]) *Let Ω be a frame with cc, L countable and A a L-structure in* $\mathbf{Sh(\Omega)}$*. For all separable $S \subseteq |A|$ there exists a separable B in A such that B is a L-structure in* $\mathbf{Sh(\Omega)}$*, $S \subseteq B$ and B is an elementary substructure of A.* ■

[13]For the definition of elementary substructure see section 4 definiton 4.1(5).

4 Method of Constants

In this section, we generalize results considering positive diagram, diagram and elementary diagram, in an intuitionistic context. Clearly, we work with (intuitionistic) sheaf models introduced earlier. Classically, the notion of diagram was first introduced to the best of the authors knowledge by A. Robinson. We start with a model A and add to our language L new constant symbols $\underline{a}$, for each $a \in |A|$. In the new language L_A, we form the diagram of this model, which contains all valid atomic and negated atomic sentences in A. The classical theorem of diagram postulates that knowing the diagram, we know (up to isomorphism) the model A. There are some variations of this theorem for the positive and elementary diagrams, cf. [10, 21]. We are able to generalize all of them.

In our context of sheaf models, we are doing basically the same as classically, we add to our language *names* of the sections $a \in |A|$, where A is a sheaf of L-structures. Because of the fact that in sheaves (and presheaves) we have always partial sections – sometimes there are no global sections at all ! –, we have to modify the original definition of the set of constant symbols and we admit *partial* constant symbols. More clearly, we admit a (pre-)sheaf C over some given frame Ω of constant symbols, cf. 1.19. Then we can form our new language, L_A – adding the names for $a \in |A|$ – and prove the diagram theorems, cf. 4.4, 4.5, 4.7 and 4.10. We start with the following definition, recalling the various notions of L-morphisms.

Definition 4.1 *Let L be a first-order language with equality, as above. Let A, B be L-structures in* $\mathbf{pSh(\Omega)}$ *and let $f : A \longrightarrow B$ be a presheaf morphism. For $a_1, \ldots, a_n \in |A|$, set*

$$f(\vec{a}) := \langle f(a_1 \restriction E\vec{a}), \ldots, f(a_n \restriction E\vec{a})\rangle = \langle f(a_1) \restriction E\vec{a}, \ldots, f(a_n) \restriction E\vec{a}\rangle.^{14}$$

(1) *f is a* ***L-morphism*** *iff for every $n \geq 1$ and $a_1, \ldots, a_n \in |A|$*

a) $\forall R \in rel(n, L), \quad [\![R(\vec{a})]\!]_A \leq [\![R(f(\vec{a})]\!]_B$;

b) $\forall\ \omega \in op(n, L), \quad (f \circ \omega^A)(\vec{a} \restriction E\vec{a}) = \omega^B(f(\vec{a}))$;

c) $\forall\ c \in |C|, \quad f(c^A) = c^B$.

(2) *f is a* ***weak L-monomorphism*** *iff f is a L-morphism and for all $n \geq 1$, all $R \in rel(n, L)$ (equality included) and each $\vec{a} \in |A|^n$,*

$$\neg\neg\, [\![R(\vec{a})]\!]_A = \neg\neg\, [\![R(f(\vec{a}))]\!]_B.$$

(3) *f is a* ***L-monomorphism***[15] *iff f is a L-morphism and for all $n \geq 1$, all $R \in rel(n, L)$ (= included) and each $\vec{a} \in |A|^n$,* $[\![R(\vec{a})]\!]_A = [\![R(f(\vec{a}))]\!]_B$.

(4) *f is a* ***weak elementary L-monomorphism*** *iff f is a L-morphism and for every L-formula $\varphi(v_1, \ldots, v_n)$ and $\vec{a} \in |A|^n$,*

$$[\![\varphi^g(\vec{a})]\!]_A = [\![\varphi^g(f(\vec{a}))]\!]_B.$$

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[14] Note that $Ef(\vec{a}) = E\vec{a}$.

[15] Alternatively we call an L-monomorphism also an ***L*-embedding** or if f is the inclusion, ***L*-substructure**.

(5) *f is an **elementary L-monomorphism**[16] iff for every L-formula $\varphi(v_1, \dots, v_n)$ and $\vec{a} \in |A|^n$,*

$$[\![\varphi(\vec{a})]\!]_A \;=\; [\![\varphi(f(\vec{a}))]\!]_B.$$

In the next definiton, we introduce our new language L_A. For simplicity, we use the classical notation.

Definition 4.2 *A presheaf of L-structures, A, becomes a presheaf of L_A-structures by the morphism[17] $\underline{a} \mapsto a$, i.e., for every $a \in |A|$, the interpretation of $\underline{a}$ in A is a. We write*

$$\widetilde{A} = (A, a)_{a \in |A|}$$

to indicate the expansion of A to a presheaf of L_A-structures.
More generally, if B is a presheaf of L-structures and $f : A \longrightarrow B$ a morphism, $\widetilde{B} = (B,\, fa)_{a \in |A|}$ indicates the expansion of B to a presheaf of L_A-structures, interpreting $\underline{a}$ by fa.

The definitions of the (positive) diagram are the same as in the classical theory.

Definition 4.3 *Let A be a L-structure in* $\mathbf{pSh(\Omega)}$.
a) The **positive diagram of *A***, *Δ_A^+, is the set of all atomic L_A-sentences forced by $\widetilde{A}$.*[18]
b) The **diagram of *A***, *Δ_A, is the set of all atomic and negated atomic L_A-sentences forced by $\widetilde{A}$.*

For the proof of the next result we refer the reader to [6]. Obviously, the result is a generalization of the classical result in context of sheaf models over a frame.

Theorem 4.4 (Brunner & Miraglia 2000, [6]) *Let A and B be L-structures in* $\mathbf{pSh(\Omega)}$ *and let $f : |A| \longrightarrow |B|$ be a function preserving extents, i.e., $Efa = Ea$, for each $a \in |A|$. The following are equivalent:*
(1) *f is a L-morphism.*
(2) *For every atomic formula $\varphi(v_1, \dots, v_n)$ and $\vec{a} \in |A|^n$, $[\![\varphi(\vec{a})]\!]_A \;\le\; [\![\varphi(f(\vec{a}))]\!]_B$.*
(3) *$\widetilde{B} = (B, fa)_{a \in |A|} \Vdash \Delta_A^+$.* ■

Let us formulate and prove the intuitionistic theorem about diagrams. The result claims the existence of a *weak* L-monomorphism; treating with intuitionistic logic, the notion "weak" could be expected and is the best possible, as example 4.12. of [6] shows – using ultrasheaves, cf. [13, 14].

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[16] Alternatively we call an elementary L-monomorphism also an **elementary *L*-embedding** or if f is the inclusion, **elementary *L*-substructure**.
[17] This is a (pre-)sheaf morphism!
[18] See 1.25.

Theorem 4.5 (Brunner & Miraglia 2000, [6]) *Let A and B be L-structures in* **pSh(Ω)** *and let $f : |A| \longrightarrow |B|$ be a function preserving extents, i.e., $Efa = Ea$, for each $a \in |A|$. The following are equivalent:*

(1) *There is a weak L-monomorphism $f : A \longrightarrow B$.*

(2) *There is an expansion $\widetilde{B}$ of B to a L_A-structure such that $\widetilde{B} \Vdash \Delta_A$.*

Proof: The following proof is from [6].

(1) ⇒ (2). Suppose that $f : A \longrightarrow B$ is a weak L-monomorphism. By 4.4, $\widetilde{B} \Vdash \Delta_A^+$. If $A \Vdash \neg\,\varphi(\underline{a}_1, \ldots, \underline{a}_n)$, with φ atomic in L, then

$$E\varphi \wedge E\vec{a} = [\![\neg\,\varphi(\vec{a})]\!]_A = E\vec{a}_{\,|E\varphi} \wedge \neg\,[\![\varphi(\vec{a}_{\,|E\varphi})]\!]_A.$$

Since f is a weak L-monomorphism, we obtain

$$\neg\neg\,[\![\varphi(\vec{a})]\!]_A = \neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B,$$

and so $\neg\,[\![\varphi(\vec{a})]\!]_A = \neg\,[\![\varphi(f(\vec{a}))]\!]_B$. The preceding equalities yield

$$E\varphi \wedge Ef(\vec{a}) = E\varphi \wedge E\vec{a} = Ef(\vec{a}_{|E\varphi}) \wedge \neg\,[\![\varphi(f(\vec{a}_{\,|E\varphi}))]\!]_B = [\![\neg\,\varphi(f(\vec{a}))]\!]_B,$$

and so $\widetilde{B} \Vdash \neg\,\varphi[\underline{a}_1, \ldots, \underline{a}_n]$, as needed.

(2) ⇒ (1). Let $\widetilde{B}$ be a L_A-structure, such that $\widetilde{B} \Vdash \Delta_A$. Define $f : |A| \longrightarrow |B|$ by $f(a) := \underline{a}^B$.

It is straightforward that f is a morphism of presheaves, while lemma 4.4 shows that f is a L-morphism. If $\varphi(v_1, \ldots, v_n)$ is an atomic formula in L and $\vec{a} \in |A|^n$, then the double negation of (2) in the statement of 4.4 gives

$$\neg\neg\,[\![\varphi(\vec{a})]\!]_A \;\leq\; \neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B.$$

Without loss of generality, assume $E\vec{a} \leq E\varphi$ and set $\vec{c} := \vec{a}_{|\neg[\![\varphi(\vec{a})_A]\!]}$. By lemma 4.4, we have $E\vec{c} = E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A$ and $\neg\,\varphi(\underline{c}_1, \ldots, \underline{c}_n) \in \Delta_A$, and so $\widetilde{B} \Vdash \neg\,\varphi[\underline{c}_1, \ldots, \underline{c}_n]$, i.e., $E\vec{c} = Ef(\vec{c}) = [\![\neg\,\varphi(f(\vec{c}))]\!]_B$. The definition of $\vec{c}$ and the fact that $[\![\cdot]\!]_B$ is a characteristic map yield

$$[\![\neg\,\varphi(f(\vec{c}))]\!]_B = E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A \wedge [\![\neg\,\varphi(f(\vec{a}))]\!]_B = E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A \wedge \neg\,[\![\varphi(f(\vec{a}))]\!]_B.$$

The above equations imply

$$(*) \qquad E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A \;=\; E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A \wedge \neg\,[\![\varphi(f(\vec{a}))]\!]_B.$$

Taking the meet with $\neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B$ in both of the sides of $(*)$, entails

$$E\vec{a} \wedge \neg\,[\![\varphi(\vec{a})]\!]_A \wedge \neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B = \bot,$$

and so $E\vec{a} \wedge \neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B \leq \neg\neg\,[\![\varphi(\vec{a})]\!]_A$. Since $[\![\varphi(f(\vec{a}))]\!]_B \leq E\vec{a}$, we obtain

$$\neg\neg\,[\![\varphi(f(\vec{a}))]\!]_B \leq \neg\neg\,[\![\varphi(\vec{a})]\!]_A,$$

as desired. ∎

We define the elementary diagram and the Gödel theory – using Gödel translation, cf. 1.26.

Definition 4.6 *Let A be a L-structure in* **pSh(Ω)**.

a) The **theory of $\widetilde{A}$** *or the* **elementary diagram of A** *is the set*

$$\mathfrak{Th}(\widetilde{A}) \;:=\; \{\sigma : \sigma \text{ is a } L_A\text{-sentence and } \widetilde{A} \Vdash \sigma\}.$$

b) The **Gödel theory of $\widetilde{A}$** *or the* **elementary Gödel diagram of A** *is the set*

$$\mathfrak{Th}^g(\widetilde{A}) \ := \ \Delta_A \cup \{\sigma^g : \sigma \text{ is a sentence of } L_A \text{ and } \widetilde{A} \Vdash \sigma^g\},$$

where σ^g is the Gödel translation.

With the above definition in mind, we prove our first version of intuitionistic elementary diagram. Details can be joined in [4, 6].

Theorem 4.7 (Brunner & Miraglia 2000, [6]) *Let A and B be L-structures in* $\mathbf{pSh}(\boldsymbol{\Omega})$. *The following conditions are equivalent:*

(1) *There exists a weak elementary L-monomorphism $f : A \longrightarrow B$.*

(2) *There is an expansion $\widetilde{B}$ of B to a L_A-structure such that $\widetilde{B} \Vdash \mathfrak{Th}^g(\widetilde{A})$.* ■

For Boolean-valued structures, we have the following result showing that theorem 4.7 is indeed the generalization of Robinson's classical theorems for (elementary) diagrams. The proof of the next corollary is immediate from the preceding results.

Corollary 4.8 *Let $\mathfrak{B}$ be a complete Boolean algebra and A, B L-structures in* $\boldsymbol{pSh}(\boldsymbol{\mathfrak{B}})$.[19]

a) The following conditions are equivalent:

(1) *There is a L-embedding from A into B;*

(2) *There is an expansion $\widetilde{B}$ from B to a L_A-structure such that $\widetilde{B} \Vdash \Delta_A$.*

b) The following conditions are equivalent:

(1) *There is an elementary L-embedding from A into B;*

(2) *There is an expansion $\widetilde{B}$ from B to a L_A-structure such that $\widetilde{B} \Vdash \mathfrak{Th}(\widetilde{A})$.* ■

We can do a little different, considering the new connective $\Box_p, p \in \Omega$, cf. 1.27, and we introduce the following definition, remembering A. Robinson. In the next theorem 4.10, we can prove that the new connective about dense elements makes in some sense the logic classical, and we prove diagram theorems in a *classical* sense. For the proof we refer to [6].

Definition 4.9 *Let A be a L-structure in* $\mathbf{pSh}(\boldsymbol{\Omega})$ *and let D be the set of dense elements in Ω.*

a) The **Robinson diagram of A** *is the following collection of $L_A^{\Box}$-sentences:*

$$\Delta_A^R \ := \ \Delta_A \cup \{\Box_d\varphi : d \in D, \varphi \text{ is atomic in } L_A \text{ and } \widetilde{A} \Vdash \Box_d\varphi\}.$$

b) The **Robinson theory of $\widetilde{A}$** *or* **elementary Robinson diagram of A**, *is the following collection of $L_A^{\Box}$-sentences:*

$$\mathfrak{Th}^R(\widetilde{A}) := \{\Box_d\varphi : d \in D, \varphi \text{ is a sentence in } L_A \text{ and } \widetilde{A} \Vdash \Box_d\varphi\}.$$

[19]Note that the structures A and B are so called Boolean valued structures.

Theorem 4.10 (Brunner & Miraglia 2000, [6]) *Let A and B be L-structures in* $\mathbf{pSh(\Omega)}$ *and D be the set of dense elements in Ω.*
a) The following conditions are equivalent:
(1) *There is a L-embedding, $f : A \longrightarrow B$.*
(2) *There is an expansion $\widetilde{B}$ of B to a L_A-structure such that $\widetilde{B} \Vdash \Delta^R_A$.*
b) The following conditions are equivalent:
(1) *There is an elementary L-embedding, $f : A \longrightarrow B$.*
(2) *There is an expansion $\widetilde{B}$ of B to a L_A-structure such that $\widetilde{B} \Vdash \mathfrak{Th}^R(\widetilde{A})$.* ■

5 Omitting Types Theorem

In this section, we formulate an intuitionistic version of the Omitting Types Theorem in the context of topological sheaves. For the proof, it is essential that spectral spaces have Baire's property.[20] This property will guarantee that the stalks omitting a certain type are dense. We have to restrict our language to the earlier mentioned language $L^\sharp$ – this is the language *without* the universal quantifier. We also use our spectral space constructed earlier in section 2 and theorem 2.2 for the proof. For more details, we refer the reader to [4, 5].

The next definition is the same as in classical logic.

Definition 5.1 *Let L be a first order language with equality, Σ a consistent set of sentences in L and $\Gamma = \Gamma(x_1, \dots, x_n)$ a set of formulas in L in (at most) the free variables $x_1, \dots, x_n$.*
a) A formula $\varphi(x_1, \dots, x_n)$ is **consistent with Σ** *iff $\Sigma \cup \{\exists\varphi\}$[21] is consistent.*
b) If the formulas of Γ are consistent with Σ, we say that **Σ locally omits Γ** *if for every formula $\varphi(x_1, \dots, x_n)$ of L, which is consistent with Σ, there is $\gamma \in \Gamma$ such that $\Sigma \cup \{\exists(\varphi \wedge \neg\gamma)\}$ is consistent.*
c) For a L-structure A in $pSh(\Omega)$, ***A*** **omits Γ** *iff for every $\vec{a} \in |A|^n$, $E\vec{a} = \bigvee_{\gamma\in\Gamma} [\![\neg\gamma(\vec{a})]\!]_A$.*

For our restricted language $L^\sharp$ we can prove the following result.

Theorem 5.2 (Brunner & Miraglia 2004, [5]) *Let L be a countable language with equality and Σ a consistent set of $\forall L^\sharp$-sentences. Let $\Gamma(x_1, \dots, x_n)$ be a set of $L^\sharp$-formulas, which are consistent with Σ. If Σ locally omits Γ, then there is a spectral space Z and a separable L structure A in $Sh(Z)$ such that*
a) A is a model of Σ in ***Sh(Z)*** *omitting Γ;*
b) The set $\mathfrak{O} = \{z \in Z : A_z$[22] is a classical model of Σ omitting $\Gamma\}$ is dense in Z. ■

[20]For the proof of this fact see for example [4, 31]
[21]$\exists\varphi$ denotes the existential closure of the formula φ.
[22]A_z is the stalk of A in the point z. See definition 1.7

The reader may have noticed that the above theorem 5.2 is clearly an *intuitionistic* version. But we were not able to work in the whole language L and not with structures of sheaves over an arbitrary frame. We have to use the notion of topological sheaf, because properties of the topological spectral space Z are needed to obtain the omitting structures of the type. We argue nevertheless that a more general intuitionistic version is possible, considering the whole language and frames with some kind of Baire's property.

Remark 5.3 *(a) There is a version of Model completeness for Boolean-valued Structures, made by Macintyre in the 70's, cf. [26].*
(b) Fraïssés Theorem[23] *using* **back and forth** *was done by Caicedo and Sette in the language $L^{\Box}$ and in the context of topological sheaves, [9]. They created and used the new connective $\Box_p, p \in \Omega$.*
(c) In 1976, H. Volger published a work [39], generalizing the Feferman-Vaught theorem for Boolean-valued structures. Is there an intuitionistic version?

Further Remark and Work 5.4 *(i) Finally, we remark that all stated results are intuitionistic results. We were working in sheaves over a frame, whenever possible. So we obtain the results without considering in most cases points at all. Sometimes it was necessary to exclude the universal quantifier from the language. The last theorem 5.2, theorem 2.2 and theorem 2.6 are valid only in the langauge $L^{\sharp}$ and some of them used points of the given topological space.*

(ii) In [34] the authors stated that points in topology are not necessary in almost all important cases. So it seems that all theorems using topological sheaves can be proved in sheaves over an arbitrary frame, this is in pointless topology.

(iii) Clearly, there is a lot of work to do in the model theory of sheaves. We give only a few questions:
1. Prove a Completeness Theorem for the language $L^{\Box}$. Perhaps, a Pavelka-style theorem can be stated and proved.
2. Prove Łoś Theorem for the whole language L, or for the language $L^{\Box}$.
3. Prove an intuitionistic Feferman-Vaught Theorem for $L^{\Box}$ or an intuitionistic Feferman-Vaught Theorem for L.
4. Prove an intuitionistic Omitting Types Theorem for the whole language L considering sheaves over a frame.
5. Define types in models of sheaves $A \in \boldsymbol{Sh(\Omega)}$. Formulate and prove an intuitionistic Omitting Types Theorem. Use language L, or $L^{\Box}$.
6. Make a categorial approach for model theory, in topos.

[23]This theorem can be generalized in the context of sheaves of arbitrary frames – the proof made by Caicedo and Sette is working. For details see [4].

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