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Grasping Sets Through Ordinals: On a Weak Form of the Constructibility Axiom

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Abstract

The present paper begins with a diagnosis for the insufficiency of ZFC, especially when it comes to infinite arithmetic of sets. According to our diagnosis, the problem is that, in ZFC, sets and ordinals are only fragmentarily connected. The use of ordinals connected to sets was always one of the main points of set theory, and the usual axioms barely touched this aspect of set theory. This is related to the fact that the conceptual work upon which the axioms of ZFC are based, and also a good deal of the conceptual work on candidates for new axioms, was always very focused on the production of sets, and only indirectly related to the global connection between sets and ordinals. This diagnosis is followed by an analysis of how could we formulate a candidate for new axiom based on a very simple set-theoretic principle according to which sets and ordinals can be connected by a homomorphism from sets to ordinals which is minimal with respect to cardinality among those with the ordinals as its fixed points and such that the preimage of an infinite ordinal is equipotent to that ordinal. The formalization of this principle is called the minimal ordinal-connection axiom, which can be obtained by abstracting the coarse behaviour of the constructible rank in a definability-free way. It is shown that the basic consequences of the constructibility axiom are also consequences of the minimal ordinal-connection axiom, and, at the same time, that it is consistent with very large cardinals. A local version of the minimal ordinal-connection axiom is given in the end of the paper.

Keywords: Ordinal-Connection, Abstract Constructibility.

1 Amending the Iterative Conception of Set-Theoretic Universe

One of the most important aspects of set theory, probably its most important aspect along with its foundational role in mathematics, is its use of ordinals connected to sets. The modes of grasping sets through ordinals always rely on some *ordinal pattern* behind that plurality of sets, that is, on some assignment of ordinals to sets (or sets to ordinals), granted by the axioms, such that problems about sets can be solved using the assigned ordinals. Without a connection between sets and ordinals we could not, for example, grasp sets in a hierarchy, and we could not count them. In fact, the rank, which is a homomorphism between sets and ordinals, and the cardinality of sets, which is a well-behaved (with respect to equipotence) assignment of ordinals to sets, are basic examples of ordinal patterns granted by the usual axioms. We need to get sets and ordinals connected in some way if we want to use ordinals to understand sets. In the well established theory of sets ZFC, the recursion theorem, the axiom of foundation and the axiom of choice are basic tools for connecting sets to ordinals. That's about it, and this is not a good thing. Those tools are insufficient for a good grasping of sets through ordinals, for we cannot even know whether the power set operation is strictly increasing with respect to cardinalities on this basis. The axioms of ZFC do not even mention ordinals explicitly. We have an exceedingly important, basic concern of set theory – the connection between sets and ordinals – which is barely scratched by the well-established axioms, and this is somewhat surprising. Unsurprisingly, ZFC cannot decide basically all nontrivial questions about cardinalities. The problem we will be concerned with here is just this: How should we enhace the connection between sets and ordinals in set theory?

When it comes to account for infinite arithmetic, one of the main roles of set theory, ZFC is really weak. A diagnosis for this weakness was proposed in the above paragraph, and this point is worth expanding. ZFC is weak not because it does not fix its intended model, "the true universe of sets"– no first-order theory can do that! Rather, it is weak simply because *its axioms guarantee only a fragmentary connection between sets and ordinals in a given set-theoretic universe.* This is related to the fact that the main line of conceptual work on grounding the axioms of ZFC, the so-called iterative conception of set, was always very focused on the production of sets.[1] The global connection between sets and ordinals is, at best, a marginal concern, and the consequences of this are well-known. Once this problem is understood, it is not difficult to obtain the generalized continuum hypothesis from a simple axiom grounded on a conception of set-theoretic universe centered not only on the iterative production of sets, but also on the global connection between sets and ordinals. The amended conception of set-theoretic universe is just this: Sets, in an iterative set-theoretic universe, admit an ordinal rank, that is a membership preserving map from sets to ordinals fixing the ordinals, such that (a) the preimage of an infinite ordinal is equipotent to that ordinal and (b) the growth rate of this ordinal rank is minimal in terms of cardinality.

We will investigate an axiom system extending ZFC based on the above amen-

[1] A methodical analysis of the existential character of ZFC axioms based on the iterative conception of set can be found in [2].

ded conception of set-theoretic universe. Let us first examine a much bolder connection between sets and ordinals which is well-known: The connection given by the constructibility axiom, a connection encompassing both rank and cardinality, global well-ordering and much more. Virtually everything concerning cardinalities left undecided by ZFC is decided on this new basis. However, the constructibility axiom is not considered a solution to the above mentioned problem. First of all, it is inconsistent with very appreciated large cardinals. Moreover, it is more akin to an artificial technique for proving results about consistency strength and it is *not* grounded by the proposed amendment of the iterative conception. We are looking for an axiom which is consistent with large cardinals and is simply stated as a formal counterpart of the amended conception of set-theoretic universe giving a full, foundational connection between sets and ordinals. That is what we will attempt to accomplish in this paper, and neither the constructibility axiom nor any of its relative versions is our target. The axiom we are searching for is not to be obtained from some complex model-theoretic construction. However, discarding the constructibility axiom so fast seems like throwing the baby out with the bathwater. Let us look at an abstract, simple, definability-free axiomatization of the *coarse* structure of the **L**-hierarchy, isolating a handy connection between sets and ordinals from the undesirable definability component of constructibility.[2]

We will now anticipate how can we axiomatize, in an entirely definability-free way, the ordinal pattern given by the constructible rank when the fine behaviour of its ordinal values is covered up, and how the resulting axiom can be seen as a formalization of a very simple set-theoretic principle. Let ZF^-_ρ be the following theory: Its language has, in addition to $\in$, an unary function symbol ρ, and the axioms include all axioms of ZF^- (ZF minus foundation) along with replacement and separation axioms for formulas containing ρ. Now, consider the following axiom, the minimal ordinal-connection axiom, in ZF^-_ρ:

1. $\forall x, (\rho(x)$ *is an ordinal*$)$.

2. $\forall \alpha, (\rho(\alpha) = \alpha)$.

3. $\forall x, y, (x \in y \to \rho(x) < \rho(y))$.

4. $\forall \alpha\, \exists f; (f : \alpha \cup \omega \to \{x : \rho(x) < \alpha\}$ *is surjective*$)$.

5. For every set x, (i) if $x \in V_\omega$, then $\rho(x) = rk(x)$, and (ii) if $x \notin V_\omega$, then given a transitive set T containing x and $r : T \to T$ satisfying $1-4$ above, $\rho(x) < r(x)^+$.

[2] The definability component is usually seen as an artificial constraint on sets and a downside of the constructibility axiom. However, the rejection of $\mathbf{V} = \mathbf{L}$ on the basis that it is vaguely restrictive has been countered in [5], for example.

The set-theoretic principle behind the minimal ordinal-connection axiom can be stated as follows: It is possible to arrange a homomorphism from sets to ordinals such that (i) the ordinals are fixed, (ii) the preimage of an infinite ordinal is equipotent to that ordinal and (iii) it is minimal with respect to cardinality among the homomorphisms from sets to ordinals satisfying (i) and (ii). The axioms of ZFC give only a homomorphism with the ordinals as its fixed points and the well-ordering of sets, a small fraction of a minimal ordinal-connection. It turns out that, as we shall prove in the next sections, the most important consequences of $\mathbf{V} = \mathbf{L}$, such as the axiom of foundation, the axiom of choice, the generalized continuum hypothesis, and the statement that the standard models are unchanged, that is, if κ is inaccessible then $L_\kappa = V_\kappa$, where $L_\kappa = \{x : \rho(x) < \kappa\}$, remain valid in ZF_ρ^- with the minimal ordinal-connection axiom. Also, if θ is another function symbol satisfying $1-5$ above, then $|\theta(x)| = |\rho(x)|$, which means that this axiomatization is categorical with respect to the cardinal of $\rho(x)$. Furthermore, as far as we know, this theory is consistent with very large cardinals. We will prove this in section 5. The consistency with large cardinals follows from the fact that the natural $\mathbf{L}[A]$-rank is a minimal ordinal-connection in $\mathbf{L}[A]$, provided the J_α^A-structures are acceptable, for every α, in the sense of [11]. Indeed, the extender models $\mathbf{L}[\vec{E}]$ satisfy an appropriate acceptability condition. Therefore, the basic issue of getting sets and ordinals effectively connected without banishing large cardinals by means of a simple axiom is addressed.

2 Minimal Ordinal-Connection Axiom

Now we turn our attention to the technical work. We must show that the minimal ordinal-connection axiom has the required properties. This is mostly straightforward. We use α, β and γ as variables for ordinals, and f and g as variables for functions, and standard notation for cardinality notions: If x is a set, then $|x|$ is its cardinal, and α^+ is the least cardinal greater than the ordinal α. The basics of set theory used in this paper is very standard, and can be found in [7], for example.

Definition 2.1 *Let $\boldsymbol{K}$ be a transitive class, and r be a class function, $r : \boldsymbol{K} \to \boldsymbol{K}$. We say that r is an ordinal-connection in $\boldsymbol{K}$ iff*

- $\forall x \in \boldsymbol{K}, (r(x) \text{ is an ordinal}),$
- $\forall \alpha \in \boldsymbol{K}, (r(\alpha) = \alpha),$
- $\forall x, y \in \boldsymbol{K}, (x \in y \to r(x) < r(y)),$
- $\forall \alpha\, \exists f; (f : \alpha \cup \omega \to \{x \in \boldsymbol{K} : r(x) < \alpha\} \text{ is surjective}).$

Definition 2.1 can be read as follows: An ordinal-connection in a transitive class $\mathbf{K}$ is an assignment of ordinals to elements of $\mathbf{K}$ that fixes the ordinals, preserves membership and is such that $\{x \in \mathbf{K} : r(x) < \alpha\}$ is a set which can be obtained from $\alpha \cup \omega$ by replacing $\beta \in \alpha \cup \omega$ by $f(\beta)$, where f is an appropriate function.

If r is an ordinal-connection in a transitive *set* $\mathbf{K}$, then we say that r is a *set-like* ordinal-connection.

Remark 2.2 *If r is an ordinal-connection in $\boldsymbol{K}$ and if $\alpha \subseteq \boldsymbol{K}$, then, from the second and fourth clauses in definition 2.1, $\alpha \subseteq \{x \in \boldsymbol{K} : r(x) < \alpha\}$ and, consequently, α is equipotent to $\{x \in \boldsymbol{K} : r(x) < \alpha\}$. From the third clause, it follows that the set $\{x \in \boldsymbol{K} : r(x) < \alpha\}$ is transitive.*

Recall that ZF^-_ρ is the theory containing, in addition to $\in$, an unary function symbol ρ, and the axioms of ZF^- along with replacement and separation axioms for formulas containing ρ.

Remark 2.3 *If we were working in a theory of classes, such as NBG, the primitive language would suffice to state the existence of a (minimal) ordinal-connection in $\boldsymbol{V}$. However, working in a theory of sets there is no alternative and we must extend our language. The same thing happens when we want to state the existence of a global well-ordering, another ordinal pattern that may consistently be added to a universe of sets.*

- ***Ordinal-Connection Axiom for*** ρ: The function symbol ρ is an ordinal-connection in $\mathbf{V}$.

  Remark 2.4 *The axiom of foundation is an easy consequence of the ordinal-connection axiom for ρ in ZF^-_ρ. Indeed, since ρ preserves membership, for every nonempty set x, any element y of x with least $\rho(y)$ is $\in$-minimal.*

  Remark 2.5 *Another easy consequence of the ordinal-connection axiom for ρ in ZF^-_ρ is the inequality $\rho(x) \geq rk(x)$, where x is a set and $rk(x)$ is the* rank *of x. In fact, using $\in$-induction in the transitive class $\boldsymbol{V}$, we have that*

  $$rk(x) = sup\,\{rk(y) + 1 : y \in x\} \leq sup\,\{\rho(y) + 1 : y \in x\} \leq \rho(x).$$

- ***Minimal Ordinal-Connection Axiom for*** ρ: The function symbol ρ is an ordinal-connection in $\mathbf{V}$ such that for every set x, (i) if $x \in V_\omega$, then $\rho(x) = rk(x)$, and (ii) if $x \notin V_\omega$, then given a transitive set T containing x and a set-like ordinal-connection $r : T \to T$ in T, $\rho(x) < r(x)^+$.

It is convenient to abbreviate the expressions "ordinal-connection axiom for ρ" and "minimal ordinal-connection axiom for ρ" as $OC(\rho)$ and $MOC(\rho)$, respectively. Although it is formulated in an extended language, $MOC(\rho)$ is a weak form of the constructibility axiom in the sense that the constructible rank is a minimal ordinal-connection in $\mathbf{L}$, as it will be shown in 2.8, and it does not imply $\mathbf{V} = \mathbf{L}$. Now, we will investigate some of its consequences.

Proposition 2.6 *The axiom of choice is a theorem of* $ZF^-_\rho + OC(\rho)$.

Proof. If y is an infinite set and $\rho(y) = \alpha$, then $y \subseteq \{x \in \mathbf{V} : \rho(x) < \alpha\}$. From the last clause in definition 2.1, it follows that there is a surjective function $f : \alpha \to \{x \in \mathbf{V} : \rho(x) < \alpha\}$. Therefore, y can be well-ordered. ∎

Proposition 2.7 below shows that the axiomatization of ρ as a minimal ordinal-connection in $\mathbf{V}$ is *rigid*, that is, ρ is the *unique modulo equipotence* ordinal-connection in $\mathbf{V}$ satisfying the minimality condition.

Proposition 2.7 *For each function symbol* θ *in* $ZF^-_\rho + MOC(\rho)$ *satisfying the condition* $MOC(\theta)$,

$$|\theta(x)| = |\rho(x)|.$$

Proof. Assume that $x \notin V_\omega$. If $T = tc(\{x\}) \cup \rho(x) + 1 \cup \theta(x) + 1$, then T is transitive, closed under ρ and θ, and such that $x \in T$. Hence $\rho : T \to T$ and $\theta : T \to T$ are ordinal-connections in X. From $MOC(\rho)$ and $MOC(\theta)$, it follows that $\rho(x) < \theta(x)^+$ and $\theta(x) < \rho(x)^+$, which means that $|\theta(x)| = |\rho(x)|$. ∎

Now, we will prove the consistency of $ZF^-_\rho + MOC(\rho)$ relative to ZF^-.

Proposition 2.8 *If* ZF^- *is consistent, then so is* $ZF^-_\rho + MOC(\rho)$.

Proof. Let λ be the $\mathbf{L}$-rank in $ZF^- + \mathbf{V} = \mathbf{L}$, that is, $\lambda(x)$ is the least α such that $x \in L_{\alpha+1}$. We need only to prove $MOC(\lambda)$ in $ZF^- + \mathbf{V} = \mathbf{L}$. Of course, λ is an ordinal-connection in $\mathbf{L}(= \mathbf{V})$, and λ restricted to V_ω equals to rk restricted to V_ω. To prove $MOC(\lambda)$ in $ZF^- + \mathbf{V} = \mathbf{L}$, it only remains to verify that if $x \notin V_\omega$, T is a transitive set containing x and $r : T \to T$ is an ordinal-connection in T, then

$$\lambda(x) < r(x)^+.$$

Let α be an infinite ordinal such that $r(x) < \alpha$. Since $\{y \in T : r(y) < \alpha\}$ is transitive, it follows, from Gödel's condensation lemma for $\mathbf{L}$, that there is an ordinal β such that

$$\{y \in T : r(y) < \alpha\} \subseteq L_\beta \text{ and } |L_\beta| = |\{y \in T : r(y) < \alpha\}|.$$

Therefore, $|\beta| = |L_\beta| = |\{y \in T : r(y) < \alpha\}| = |\alpha|$, and $\lambda(x) < \beta < \alpha^+$. Taking $\alpha = r(x) + 1$, we conclude that $\lambda(x) < r(x)^+$.

■

Remark 2.9 *The proof of proposition 2.8 can be summarized as follows: The natural* $\boldsymbol{L}$*-rank is a minimal ordinal-connection in* $\boldsymbol{L}$*.*

3 GCH in $ZF^-_\rho + MOC(\rho)$

In this section we will prove the generalized continuum hypothesis, GCH, in $ZF^-_\rho + MOC(\rho)$.

Lemma 3.1 $(ZF^-_\rho + MOC(\rho))$ *If* $\alpha \geq \omega$ *and* $y \subseteq \{x \in \boldsymbol{V} : \rho(x) < \alpha\}$*, then* $\rho(y) < \alpha^+$*.*

Proof. If $y \in \{x \in \mathbf{V} : \rho(x) < \alpha\}$, then $\rho(y) < \alpha < \alpha^+$.
If $y \notin \{x \in \mathbf{V} : \rho(x) < \alpha\}$, then

$$T = \{y, \alpha\} \cup \{x \in \mathbf{V} : \rho(x) < \alpha\}$$

is a transitive set and $r : T \to T$ defined by (i) $r(y) = r(\alpha) = \alpha$ and (ii) $r(z) = \rho(z)$, for every $z \in \{x \in \mathbf{V} : \rho(x) < \alpha\}$, is an ordinal-connection in T. From $MOC(\rho)$, it follows that $\rho(y) < r(y)^+ = \alpha^+$. ■

Proposition 3.2 $(ZF^-_\rho + MOC(\rho))$ *For every* α*,* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$*.*

Proof. From lemma 3.1,

$$\wp(\{x \in \mathbf{V} : \rho(x) < \aleph_\alpha\}) \subseteq \{x \in \mathbf{V} : \rho(x) < \aleph_{\alpha+1}\},$$

and $|\wp(\{x \in \mathbf{V} : \rho(x) < \aleph_\alpha\})| \leq |\{x \in \mathbf{V} : \rho(x) < \aleph_{\alpha+1}\}|$. However,

$$|\wp(\{x \in \mathbf{V} : \rho(x) < \aleph_\alpha\})| = 2^{\aleph_\alpha} \geq \aleph_{\alpha+1} = |\{x \in \mathbf{V} : \rho(x) < \aleph_{\alpha+1}\}|.$$

Therefore, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

■

4 Standard Models of $ZF_\rho^- + MOC(\rho)$

Along with the axiom of foundation, the axiom of choice and the generalized continuum hypothesis, the statement that $V_\kappa = L_\kappa$, for κ an inaccessible cardinal, is a basic, important consequence of the constructibility axiom. For one thing, it shows that if we adopt the V_κ as the standard models of $ZF(C)$ (following the lead of Zermelo) and the L_κ as the standard models of $ZF^- + \mathbf{V} = \mathbf{L}$, then the passage to $ZF^- + \mathbf{V} = \mathbf{L}$ does not involve changing the class of standard models. This is important because the class of standard models can be seen as the subject matter of the theory, and it we want to keep the subject matter unchanged. The corresponding statement in $ZF_\rho^- + MOC(\rho)$ is the following.

Proposition 4.1 $(ZF_\rho^- + MOC(\rho))$ *If κ is an inaccessible cardinal, then*

$$\{x \in \boldsymbol{V} : \rho(x) < \kappa\} = V_\kappa.$$

Proof. Since $rk(x) \leq \rho(x)$, for every x, it follows that

$$\{x \in \mathbf{V} : \rho(x) < \kappa\} \subseteq V_\kappa.$$

For the converse inclusion, assume that $V_\kappa \setminus \{x \in \mathbf{V} : \rho(x) < \kappa\} \neq \emptyset$, and let y be a least-rank element of this set. Of course, $rk(y) \geq \omega$. From the minimality of y, we have that $tc(y) \subseteq \{x \in \mathbf{V} : \rho(x) < \kappa\}$. Since κ is inaccessible and $y \in V_k$, the set $\{\rho(x) : x \in tc(y)\}$ is strictly bounded by an infinite ordinal $\alpha < \kappa$, that is

$$tc(y) \subseteq \{x \in \mathbf{V} : \rho(x) < \alpha\}.$$

Let T be the transitive set $trcl(\{y\}) \cup \alpha + 1$, and let $r : T \to T$ be defined by (i) $r(y) = r(\alpha) = \alpha$, (ii) $r(\beta) = \beta$ for every $\beta < \alpha$ and (iii) $r(z) = \rho(z)$, for every $z \in tc(y)$.

The function r is easily seen to be an ordinal-connection in T. From $MOC(\rho)$ and the inaccessibility of κ, it follows that $\rho(y) < r(y)^+ = \alpha^+ < \kappa$, contradicting our assumption.

■

From proposition 4.1, according to $ZF_\rho^- + MOC(\rho)$, the structures

$$\langle \{x \in \mathbf{V} : \rho(x) < \kappa\}\,; \in; \rho\rangle,$$

for κ inaccessible, are models of $ZF_\rho^- + MOC(\rho)$, as they are exactly Zermelo's standard models expanded with ρ. We call these structures the *standard models* of $ZF_\rho^- + MOC(\rho)$.

5 Large Cardinals in $ZF_\rho^- + MOC(\rho)$

In this section we will finally prove that, as far as we know, $ZF_\rho^- + MOC(\rho)$ is consistent with large cardinal notions (beyond measurability). [3] This follows from the fact that the natural $\mathbf{L}[A]$-rank is a minimal ordinal-connection in $\mathbf{L}[A]$, provided the J_α^A-structures are acceptable, for every α, in the sense of [9]. Indeed, the extender models $\mathbf{L}[\vec{E}]$ satisfy an appropriate acceptability condition. [4]

Proposition 5.1 *If the J_α^A-structures are acceptable for every uncountable ordinal α, then the $\boldsymbol{L}[A]$-rank λ^A is a minimal ordinal-connection in $\boldsymbol{L}[A]$.*

Proof. All clauses of definition 2.1 are trivially satisfied, and we will be concerned only with minimality. Suppose minimality does not hold. Since the $\mathbf{L}$-rank and the usual rank are equal in V_ω, if x is a set witnessing the failure of minimality, then $x \notin V_\omega$. Let x be a least-ranked set not in V_ω for which there is a transitive set T containing x and an ordinal-connection $r : T \to T$ such that $r(x)^+ \leq \lambda^A(x)$. Notice that $r(x)^+$ is an *infinite* successor cardinal. Let α be an infinite ordinal such that $\lambda^A(x) < \alpha$ and $r(x)^+$ is a successor cardinal in J_α^A.

From the fact that $x \subseteq \{y \in T : r(y) < r(x)\}$, it follows that $|x| < r(x)^+$. Furthermore, as x is least-ranked, if $y \in x$, then $\lambda^A(y) < r(y)^+ \leq r(x)^+$, which means that $x \subseteq J_{r(x)^+}^A$. Now, from lemma 1.24 in [9], pag. 617, $x \in J_{r(x)^+}^A$. This contradicts our assumption that $r(x)^+ \leq \lambda^A(x)$, for $J_{r(x)^+}^A = L[A]_{r(x)^+}$, from the relative version of lemma 2.4 in [1], pag. 255.

■

6 The Local Minimal Ordinal-Connection Axiom

The class-free formulation of the minimal ordinal-connection axiom required an extension of the first-order language of ZF with a new unary function symbol. Now, it is natural to seek for a local version of the axiom which remains within the range of the original language. The local version that we will be concerned with can be stated as follows:

[3]This topic was discussed at *mathoverflow*, and I am very much indebted to the researchers who contributed there to the clarification of this point: http://mathoverflow.net/questions/239650/is-this-weak-form-of-v-l-inconsistent-with-large-cardinals

[4]See definition 2.4 in [10], pag. 1601, or section 6.1 in [11]. The work of Friedman and Holy in [3] and [4] is also relevant for this.

- ***Local Minimal Ordinal-Connection Axiom***: For every α there is an ordinal $\beta > \alpha$ such that there is a minimal ordinal-connection $\rho : V_\beta \to \beta$.

  Remark 6.1 *The axiom of choice easily follows from the local minimal ordinal-connection axiom in* ZF*. For every set is contained in some set of the form* $\{x \in V_\beta : \rho(x) < \alpha\}$*, and these sets can be well-ordered since* ρ *is an ordinal-connection in* V_β*.*

  Remark 6.2 *In* ZF*, if* $\rho : V_\beta \to \beta$ *is a minimal ordinal connection and* $\beta \geq \omega$*, then* β *is a fixed point of the beth function. In fact, from the definition of ordinal-connection,* V_β *is the union of the sets* $\{x \in V_\beta : \rho(x) < \alpha\}$*, for* $\alpha < \beta$*. Now, the cardinality of this union is bounded above by the cardinality of* β*. Therefore,* $\beth_\beta \leq \beta$*. Since* $\beta \leq \beth_\beta$*, we have that* β *is a fixed point of the beth function.*

Let $ZF{+}LMO$ be the theory obtained from ZF by the addition of the local minimal ordinal-connection axiom. This theory is relatively consistent with ZF, and it proves the generalized continuum hypothesis.

Proposition 6.3 *If* ZF *is consistent, then so is* $ZF + LMO$*.*

Proof. In $ZF + \mathbf{V} = \mathbf{L}$, for every α there is a $\beta > \alpha$ such that $V_\beta = L_\beta$. For if $\beta > \omega$ is any fixed point of the beth function, then $V_\beta = L_\beta$. Now, if $V_\beta = L_\beta$, then the constructible rank is a minimal ordinal-connection in V_β. ■

Now, the generalized continuum hypothesis in $ZF + LMO$:

Proposition 6.4 $(ZF + LMO)$ *For every* α*,* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$*.*

Proof. Let β be greater than $2^{\aleph_\alpha}$ and such that there is a minimal ordinal-connection ρ in V_β. As in the proof of proposition 3.2, from lemma 3.1,

$$\wp(\{x \in V_\beta : \rho(x) < \aleph_\alpha\}) \subseteq \{x \in V_\beta : \rho(x) < \aleph_{\alpha+1}\},$$

and $|\wp(\{x \in V_\beta : \rho(x) < \aleph_\alpha\})| \leq |\{x \in V_\beta : \rho(x) < \aleph_{\alpha+1}\}|$. However,

$$|\wp(\{x \in V_\beta : \rho(x) < \aleph_\alpha\})| = 2^{\aleph_\alpha} \geq \aleph_{\alpha+1} = |\{x \in V_\beta : \rho(x) < \aleph_{\alpha+1}\}|.$$

Therefore, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. ■

Lemma 6.5 $(ZF^-_\rho + MOC(\rho))$ *If* α *is an ordinal, then there is a minimal ordinal-connection* ρ_α *in* $V_\alpha \cup |V_\alpha|$*.*

Proof.

We need only to verify that if $x \in V_\alpha$, then $\rho(x) < |V_\alpha|$. Once this is proved, just take ρ_α to be the minimal ordinal-connection ρ restricted to $V_\alpha \cup |V_\alpha|$.

Suppose this is not the case, and let α be the first ordinal for which it fails, and let $x \in V_\alpha$ be a minimal set such that $\rho(x) \geq |V_\alpha|$. Of course, the rank of x must be infinite.

Since $x \in V_\alpha$, there is a $\gamma < \alpha$ such that $tc(x) \subseteq V_\gamma$. From the minimality of both α and x, if $y \in tc(x)$, then $y \in V_\gamma$ and $\rho(y) < |V_\gamma|$.

Let T be the transitive set $tc(\{x\}) \cup |V_\gamma| + 1$, and let $r : T \to T$ be defined by:

1. If $y \in tc(x)$, then $r(y) = \rho(y)$.
2. If $y \in |V_\gamma| + 1$, then $r(y) = y$.
3. $r(x) = |V_\gamma|$.

The above defined function r is an ordinal-connection in the set T. Therefore, $\rho(x) < r(x)^+ = |V_\gamma|^+ \leq |V_\alpha|$, a contradiction.

∎

We will now prove that the local minimal ordinal-connection axiom is a consequence of the global form.

Proposition 6.6 *($ZF^-_\rho + MOC(\rho)$) For every α there is an ordinal $\beta > \alpha$ such that there is a minimal ordinal-connection in V_β.*

Proof. Let β be a fixed point of the beth function which is greater than α. From the choice of β,

$$V_\beta \cup |V_\beta| = V_\beta \cup \beta = V_\beta.$$

From lemma 6.5, the restriction of the global minimal ordinal-connection ρ to V_β is a minimal ordinal-connection in V_β.

∎

7 Conclusion

Among the goals one can assign to the search for new set-theoretic axioms, there is one which is very coherent with the history of the discipline and is roughly characterized by: Endowing a set-theoretic universe with an *ordinal pattern* enriching the connection between sets and ordinals. The usefulness of highlighting this goal is twofold: It provides a point of view from which one can look at a proposed axiom, and it helps formulating new axioms with significant consequences. In this paper we were primarily concerned with a new axiom improving the ordinal pattern behind sets, at the same time avoiding some undesirable features that one such enrichment may have. The minimal ordinal-connection axiom can be approached from this perspective. An interesting connection between sets and ordinals in a set-theoretic universe cannot, of course, be an isomorphism, which is too strong and inconsistent. Also, it cannot be only a homomorphism from sets to ordinals, which is too weak and given just by the axiom of foundation.

We have shown that the minimal ordinal-connection axiom provides a bold enough connection between sets and ordinals capable of supporting large cardinals. Furthermore, the axiom is very simple, easily seen to be relatively consistent with ZF, and is based just on the possibility of a homomorphism from sets to ordinals which is minimal with respect to cardinality among those with the ordinals as its fixed points and such that the preimage of an infinite ordinal is equipotent to that ordinal. This *connection* principle is a further thought about the interrelation between sets and ordinals which is not separated from the usual practice of the discipline: Useful ordinal ranks have always been in the center of set theory, from, for example, Cantor-Bendixon analysis to the natural $\mathbf{L}[\vec{E}]$-ranks on extender models. Nevertheless, this principle seems to be a further thought, and not one which is completely grounded on the usual iterative and limitation of size conceptions.

Although our connection principle is consistent with large cardinals and does not make any definability constraint on sets, one could say that we gave no reason to suppose that sets in the *standard* set-theoretic universes must conform to it. Fair enough. The only reason that can be given is that it is based on the usual practice with $\mathbf{L}$-likeness, and if sets can be so organized, then we can understand their infinite arithmetic. This is a good reason: ZFC is very far from providing a full account for the infinite arithmetic of sets, and this is so because its axioms say nearly nothing about the connection between sets and ordinals. One could reply that this is not a weakness of ZFC, it is just that we are incapable of understanding that arithmetic. Very good, but what is the point of stipulating a kind of arithmetic with definite operations that we are incapable of understanding? In the paper [6], Jensen talks about Newtonian and Pythagorean points of view in set theory, warning that “deeply rooted differences in mathematical taste are too strong and would persist” ([6], page 401). A set-theoretic universe for which the condition stated in our axiom obtains can be properly said to be *Pythagorean*. In fact, although it seems clear that Jensen's Pythagoreanism explained in [6], page 401, is directly based on constructibility, it is reasonable to think that the essence of modern Pythagorianism is the belief in some bold connection between sets and ordinals, and *not* a definability requirement on sets. If this is so, then we can look at our axiom as a formal statement of a Pythagorean picture of set theory. It is a picture of set theory according to which we are capable of understanding the infinite arithmetic of sets.

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