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Studies on da Costa's Paraconsistent Differential Calculus: Hypermetric and Hyperfunctions

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We dedicate this paper to our friend Chico Miraglia, as a testament of our affection and respect for this exemplary scholar and tireless battler for social causes.

Abstract

In this paper we present a new proposal for the development of da Costa's paraconsistent differential calculus, from the introduction of a new concept of metric, a hypermetric. After presenting the hyperring $\mathbb{A}$ and the quasi-ring $\mathbb{A}^*$, that extend the set $\mathbb{R}$ of the real numbers, we introduce a hypermetric on $\mathbb{A}$ and generalized concepts of limit, continuity and derivability of hyperfunctions.

Keywords: Paraconsistent differential calculus, infinitesimals, infinite numbers, hyperreal numbers, hypermetric, hyperfunctions, limit of hyperfunctions, continuity, derivability.

1 Introduction

Ever since the beginning of the 18th century, the criticisms by various philosophers and mathematicians of the differential and integral calculus, created independently by Leibniz and Newton at the end of the 17th century, have been well known. The apparent intrinsic inconsistency of the concept of the infinitesimal, which was not clarified by either Leibniz or Newton, is at the center of these discussions, as seen, for instance, in George Berkeley's well-known *The Analyst* (see Berkeley (1774)).

Independently of such critiques, during the next two centuries several mathematicians contributed to the development of the calculus, in special Jean le Rond d'Alembert and Augustin-Louis Cauchy, both having proposed that the concept of the limit of a function should give the foundation of the calculus. But, from the works of Georg Cantor and Richard Dedekind, only with Karl Weierstrass' definition of the limit of a function and of his rigorous definition of real number, at the end of the 19th century that problems intrinsically concerned with the infinitesimal method were surpassed.

The recent return within mathematics to conceptual questions relative to the infinitesimals begins with the work of Abraham Robinson.

From a different perspective, Newton da Costa proposes a paraconsistent differential calculus that is able to deal with the infinitesimals and infinites, and that satisfies the so-called *Principle of l'Hôpital*:

> On demande qu'on puisse prendre indifféremment l'une pour l'autre deux quantités qui ne different entr'elles que d'une quantité infiniment petite: ou (ce qui est la même chose) qu'une quantité qui n'est augmentée ou diminuée que d'une autre quantité infiniment moindre qu'elle, puisse être considérée comme demeurant la même...[1] (l'Hôpital (1696)).

As the early formulations of the differential calculus in terms of infinitesimals offer us an example of a potentialy inconsistent theory, the paraconsistent reconstruction of the classical differential calculus reflects many of its theoretical and applied aspects and, it seems, also reflects many of the original ideas of Leibniz and Newton. It offers a new theory of mathematical analysis, that from a certain point of view conservatively extends the classical and the non-standard analyses. Besides dealing with conceptual questions relative to the infinitesimals, the paraconsistent differential calculus may also be considered as an alternative to classical mathematical analysis.

The aim of this short paper is to present a new proposal for the development of da Costa's ideas, from the introduction of a new concept of metric, the *hypermetric*.

Robinson (1961), based on a previous work of 1960, presents a new theory of mathematical analysis, non-standard analysis (see Robinson (1967) and (1996), third revised edition of the first edition of 1966).

The logic underlying non-standard analysis is a higher order (classical) logic, with a non-standard semantics (structures). In the development of his analysis, Robinson introduces extensions of the set of real numbers and of the set of natural numbers, called *sets of hyperreal numbers* and *the set of hypernatural numbers* (or *positive hyperintegers*). His analysis is based on the fact that ordered fields, which are non-standard models of the theory of real numbers, can be mathematically interpreted as non-Archimedean extensions of the field of the reals, which externally contain elements that behave as infinitesimal numbers.

Robinson and Zakon (1967) and Stroyan and Luxemburg (1976) introduce Robinson's non-standard analysis in a more comprehensible form, using set and model theory.

[1] In short, "Two finite quantities that differ by an infinitely small quantity are equal". Robinson (1967, p. 32) translates the passage from l'Hôpital as follows: "One requires that one may substitute for one another [*prendre indifféremment l'une pour l'autre*] two quantities which differ only by an infinitely small quantity: or (which is the same) that a quantity which is increased or decreased only by a quantity which is infinitely smaller than itself may be considered to have remained the same...".

Being constructed over an extension of the set of real numbers, which contains infinitesimals and infinite elements, non-standard analysis can be considered as an extension or an alternative to classical mathematical analysis.

After a pre-publication of 1996, da Costa (2000) introduces a paraconsistent differential calculus, whose underlying logic is his well-known paraconsistent predicate calculus with equality $C_1^=$, and whose underlying set theory is the paraconsistent set theory CHU_1, introduced in da Costa (1986) (see da Costa (1963) and (1974); and da Costa, Béziau and Bueno (1998)).

Based on the classical set theory ZF, da Costa (2000) introduces the *ring of the hyperreal numbers*, denoted by $\mathbb{A}$, and the *quasi-ring of the extended hyperreal numbers*, denoted by $\mathbb{A}^*$. The classical algebraic structures $\mathbb{A}$ and $\mathbb{A}^*$ are extensions of the field $\mathbb{R}$ of the standard real numbers, and the elements of $\mathbb{A}$ and $\mathbb{A}^*$, including infinitesimals and infinite numbers, are called *hyperreal numbers, generalized real numbers*, or simply *g-reals*.

From $\mathbb{A}^*$, da Costa proposes the construction of a paraconsistent differential calculus, whose language L is the language $L^=$ of the system $C_1^=$ extended to the language of CHU_1, in which the elements of $\mathbb{A}^*$ are dealt with.

Carvalho (2004) studies and improves the calculus proposed by da Costa. He presents da Costa's definitions for the basic concepts, proves some new theorems that generalize important classical results and presents some applications of these results (see also D'Ottaviano and Carvalho (2005)).

Carvalho and D'Ottaviano (201-), motivated by Robinson's non-standard analysis and by Robinson and Zakon (1967) and Stroyan and Luxembourg (1976), introduce the concept of *paraconsistent superstructure* and obtain a Transference Theorem, that "conservatively translates" the classical differential calculus into da Costa's paraconsistent calculus.

In da Costa's work, Carvalho (2004), and Carvalho and D'Ottaviano (201-), the hyperfunctions in $\mathbb{A}$ are considered as extensions of real functions in $\mathbb{R}$; and concepts of the limit and continuity of hyperfunctions are conceived relative to standard real numbers.

In this paper, we introduce a new concept of metric over the hyperring $\mathbb{A}$, named hypermetric. From this new approach, we present a general concept of *hyperfunction* and general definitions of the *limit* of hyperfunctions and of *continuity*, considering generalized hyperreal numbers.

In Section 2, we recall da Costa's definitions of the structure $\mathbb{A}$, the hyperring of the hyperreal numbers, and $\mathbb{A}^*$, the quasi-ring of the extended hyperreal numbers.

In Section 3, we introduce the concept of hypermetric over $\mathbb{A}$, which allows us to present a general definition for the concept of the limit of a hyperreal function, when the variable tends to a hyperreal number.

In Section 4, we have some final remarks.

We must here observe that the paraconsistent differential calculus introduced by da Costa (2000) is interpreted in the structure $\mathbb{A}^*$. However, in this paper, in spite of presenting some general concepts that will be useful for the improvement of da Costa's ideas, we do not properly deal with his paraconsistent differential calculus.

2 The hyperreal numbers

In this section we define the hyperring $\mathbb{A}$ of the hyperreal numbers and the quasi-ring $\mathbb{A}^*$ of the extended hyperreal numbers, which are classical algebraic structures and can be built in the classical set theory ZF, introduced and studied by da Costa (2000), and extend the set $\mathbb{R}$ of real numbers in order to incorporate, as defined elements, the infinitesimals and the infinites.

Da Costa's paraconsistent differential calculus is interpreted in the structure $\mathbb{A}^*$, and according to da Costa, under certain aspects, an infinitesimal analysis founded in $\mathbb{A}$ and $\mathbb{A}^*$ brings us back to the ideas of the pioneers of the calculus, i.e., Leibniz, Newton, the Bernoullis, l'Hôpital, etc., and recalls the *Principle of l'Hôpital*.

2.1 The ring $\mathbb{A}$ of the hyperreal numbers

Having the classical theory ZF as the underlying set theory, da Costa extends the field $\mathbb{R}$ of the real numbers to the hyperring $\mathbb{A}$ of the hyperreal numbers, which is extended to the quasi-ring $\mathbb{A}^*$ of the extended hyperreal numbers (see da Costa, Béziau and Bueno (1998)).

Let I be a fixed real interval[2] and a a fixed element of the interior of I.[3]

Definition 2.1.1: An *infinitesimal variable* is a real function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{x \to a} \varphi(x) = 0.^4$$

We denote the set of infinitesimal variables by Var. In the case when $\varphi(x) \geq 0$ in I, we write $\varphi \in \mathit{Var}_{\pm}$; when $\varphi(x) \leq 0$ in I, we write $\varphi \in \mathit{Var}_{-}$; and when $\varphi \in \mathit{Var}_{-}(\mathit{Var}_{+} \cup \mathit{Var}_{-})$ we say that $\varphi \in \mathit{Var}_{\pm}$.

[2] In particular, I can be $\mathbb{R}$.

[3] We assume the set theory ZF and the known definitions and results of the classical differential calculus.

[4] Observe that the concept of limit used here is the classical one.

Definition 2.1.2: The set of the *hyperreal numbers*, denoted by $\mathbb{A}$,[5] is defined by

$$\mathbb{A} =_{df} \{\langle r, \varphi\rangle : r \in \mathbb{R} \text{ and } \varphi \in \mathit{Var}\}.$$

Observe that every real number $r, r \in \mathbb{R}$, can be identified with the hyperreal of the form $\langle r, 0\rangle$, which is said to be a *standard real number*, via the injective homomorphism of rings $h : \mathbb{R} \to A$, such that $h(r) = \langle r, 0\rangle$.

Definition 2.1.3: An *infinitesimal* is a hyperreal number of the form $\langle 0, \varphi\rangle$, in which φ is an infinitesimal variable.

The function $\varphi(x) = 0$ can be considered an infinitesimal variable, and so the real number 0, identified with the hyperreal $\langle 0, 0\rangle$, can be considered an infinitesimal.

The set of all infinitesimals in $\mathbb{A}$ is denoted by:

$$\mathcal{J} =_{df} \{\langle 0, \varphi\rangle : \varphi \in \mathit{Var}\}.$$

For every $r \in \mathbb{R}$, the set of the hyperreals of the form $\langle r, \varphi\rangle$ is said to be a *monad* of r, denoted by $[r]$:

$$[r] =_{df} \{\langle s, \varphi\rangle \in \mathbb{A} : s = r\}$$

Therefore, according to the previous definitions, the infinitesimals are elements of $\mathbb{A}$, which in general are not real numbers, and the set of the infinitesimals is the monad of zero.

Definition 2.1.4: The *equality* or *identity* of two hyperreal numbers, denoted by $=$, is trivally defined from the equality of ordered pairs:

$$\langle r, \varphi\rangle = \langle s, \psi\rangle \text{ if, and only if, } r = s \text{ and } \varphi = \psi.$$

Definition 2.1.5: The *addition* $(+)$ and *multiplication* $(\times)$ of hyperreal numbers are defined from the usual operations of addition and multiplication of real numbers:

(i) $\langle r, \varphi\rangle + \langle s, \psi\rangle =_{df} \langle r + s, \varphi + \psi\rangle$;

(ii) $\langle r, \varphi\rangle \times \langle s, \psi\rangle =_{df} \langle rs, r\psi + \varphi s + \varphi\psi\rangle$.

Note that $\varphi + \psi, s \times \varphi, r \times \psi$ and $\varphi \times \psi \in \mathit{Var}$.

[5] We should denote Var and $\mathbb{A}$ by Var_a^I and $\mathbb{A}_a^I$, respectively. But, considering that I and a are fixed, for simplicity we use Var and $\mathbb{A}$.

According to the previous definition, for any hyperreal $\langle r, \varphi \rangle$,

$$\langle r, \varphi \rangle = \langle r, 0 \rangle + \langle 0, \varphi \rangle$$

or simply

$$\langle r, \varphi \rangle = r + \rho,$$

with ρ denoting the infinitesimal $\langle 0, \varphi \rangle$.

We observe that, in order to facilitate the notation, we use the same notation for the operations $+$ and $\times$ between hyperreal numbers and for the respective operations of addition and multiplication of real numbers.

We can now define the difference and the quotient of two hyperreals in $\mathbb{A}$.

Definition 2.1.6: The *opposite* of a hyperreal number $\langle r, \varphi \rangle$, denoted by $- \langle r, \varphi \rangle$, is defined by

$$- \langle r, \varphi \rangle =_{df} \langle -r, -\varphi \rangle .$$

Definition 2.1.7: The *difference* between two hyperreals is defined by

$$\langle r, \varphi \rangle - \langle s, \psi \rangle =_{df} \langle r, \varphi \rangle + (- \langle s, \psi \rangle) = \langle r - s, \varphi - \psi \rangle .$$

Definition 2.1.8: The elements of $\mathbb{A}$ of type $\langle r, \varphi \rangle$, with $r \neq 0$ and $\varphi(x) \neq -r$ for every x in I, are inversible, and their *inverse*, denoted by $\langle r, \varphi \rangle^{-1}$, is defined by

$$\langle r, \varphi \rangle^{-1} =_{df} \left\langle r^{-1}, \ \frac{-\varphi}{r(\varphi + r)} \right\rangle .$$

Definition 2.1.9: The *division* between two hyperreals $\langle s, \psi \rangle$ and $\langle r, \varphi \rangle$, with $r \neq 0$, is defined by

$$\langle s, \psi \rangle : \langle r, \varphi \rangle =_{df} \langle s, \psi \rangle \times \langle r, \varphi \rangle^{-1} ,$$

with $\varphi(x) \neq -r$ for every $x \in I$

The division can be extended to the case where the hyperreals are infinitesimals, $\langle 0, \varphi \rangle$ and $\langle 0, \psi \rangle$, respectively. We have to determine $k = \langle r, \eta \rangle$ such that

$$\langle 0, \varphi \rangle = \langle r, \eta \rangle \times \langle 0, \psi \rangle = \langle 0, (r + \eta)\psi \rangle ,$$

and we can write

$$\langle 0,\varphi\rangle = r\,\langle 0,\psi\rangle + \rho\,\langle 0,\psi\rangle, \text{ with } \rho = \langle 0,\eta\rangle.$$

There are three cases to consider, and when $\lim_{x\to a}\dfrac{\varphi(x)}{\psi(x)}$ and $\lim_{x\to a}\dfrac{\psi(x)}{\varphi(x)}$ do not exist, then the quotients $\langle 0,\psi\rangle : \langle 0,\varphi\rangle$ and $\langle 0,\varphi\rangle : \langle 0,\psi\rangle$ do not exist.

The proof of the following result may be seen in detail in Carvalho (2004).

Theorem 2.1.10 (da Costa (2000)): The structure $\langle \mathbb{A}, +, \times, 0, 1\rangle$ is a commutative ring with unity, where 0 and 1 are the hyperreals $\langle 0,0\rangle$ and $\langle 1,0\rangle$, respectively. ■

As in the case of the standard real numbers, the set *Var* with the trivial operations of addition and multiplication is a *commutative ring without unity*, that can be identified with the elements of type $\langle 0,\varphi\rangle \in \mathbb{A}$, that is, with $\mathcal{J}$, via the injective homomorphism of rings $h' : Var \to \mathbb{A}$, such that $h'(\varphi) = \langle 0,\varphi\rangle$. According to the homomorphism $h : \mathbb{R} \to \mathbb{A}$, previously mentioned, and h', we can consider the field $\langle \mathbb{R}, +, \times, 0, 1\rangle$ of the real numbers and *Var* as *subrings* of $\langle \mathbb{A}, +, \times, 0, 1\rangle$.

The order relation $<$ of $\mathbb{R}$ can be extended to A.

Definition 2.1.11: $\langle r,\varphi\rangle < \langle s,\psi\rangle$ if, and only if, either $r < s$, or $r = s$ and, for all $x \in I$, we have that $\varphi(x) < \psi(x)$ (that is, $(\psi - \varphi) \in Var_{\pm}$).

The order relation $<$ is non-linear (partial) in $\mathbb{A}$. In fact, two hyperreals $\langle r,\varphi\rangle$ and $\langle s,\psi\rangle$, such that $r = s$ and their infinitesimal components φ and ψ alternate positive and negative values in their respective domains ($(\psi - \varphi) \in Var_{\pm}$), are incomparable relative to $<$.

The hyperreal numbers $\langle r,\varphi\rangle$, such that $\langle r,\varphi\rangle \geq 0$, are the *positive* hyperreals; the hyperreals $\langle r,\varphi\rangle$, such that $\langle r,\varphi\rangle \leq 0$, are the *negative* hyperreals; a positive infinitesimal $\langle 0,\varphi\rangle$, where φ is never null in I, is said to be *strictly positive*; analogously we have the *strictly negative* infinitesimals.

The hyperring $\mathbb{A}$ is a *non-Archimedean* structure, for there are positive hyperreals $\langle r,\varphi\rangle$ such that there are not standard natural numbers n with $0 < 1/n < \langle r,\varphi\rangle$. Furthermore, $\mathbb{A}$ is not an integrity ring, for it has divisors of zero.

Now, let us consider the function $|\ \,| : \mathbb{A} \to \mathbb{A}$, such that

$$|\,\langle r,\varphi\rangle\,| =_{df} \langle |\,r\,|, |\,\varphi\,|\rangle\,,$$

where $|\,r\,|$ and $|\,\varphi\,|$ are the usual modules of real numbers. The image of an element of $\mathbb{A}$ by such function is also called the *module* of such element.

The following result is immediate.

Theorem 2.1.12: Given the function $| \ |: \mathbb{A} \to \mathbb{A}$, for every $\langle r, \varphi \rangle, \langle s, \psi \rangle \in \mathbb{A}$ the following conditions are valid:

i) $| \langle r, \varphi \rangle | = \langle 0, 0 \rangle$ if, and only if, $\langle r, \varphi \rangle = \langle 0, 0 \rangle$;

ii) $| \langle r, \varphi \rangle + \langle s, \psi \rangle | \ \leq \ | \langle r, \varphi \rangle | + | \langle s, \psi \rangle |$. ■

It is interesting to observe that the function $| \ |$ restricted to $\mathbb{R}$ coincides with the usual module function in $\mathbb{R}$, including the fact that $| \langle r, 0 \rangle \times \langle s \cdot 0 \rangle | = | \langle r, 0 \rangle | \times | \langle s, 0 \rangle |$.

2.2 The quasi-ring $\mathbb{A}^*$ of the extended hyperreal numbers

Da Costa's paraconsistent differential calculus is based on the extension $\mathbb{A}^*$ of $\mathbb{A}$, defined below.

As in the case of the structure $\mathbb{A}$, let I be a fixed real interval and a a fixed element of the interior of I.

Definition 2.2.1: An *infinite variable* is a function $v : I \subseteq \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to a} v(x) = \infty.$$

Definition 2.2.2: An *infinite hyperreal number*[6] is a pair of the form $\langle v, 0 \rangle$, with v an infinite variable.

Definition 2.2.3: The *set of the extended hyperreal numbers*, denoted by $\mathbb{A}^*$, is defined by:

$$\mathbb{A}^* = \{h : h \in \mathbb{A} \text{ or } h \text{ is an infinite hyperreal}\}.$$

We can extend the operations in $\mathbb{A}$ and the relation of equality of $\mathbb{A}$ to the set $\mathbb{A}^*$.

Definition 2.2.4: Two infinite hyperreal numbers $\langle u, 0 \rangle$ and $\langle v, 0 \rangle$ are equal if, and only if, the infinite variables u and v are identical, with the possible exception of the point a. That is, if and only if, $u(x) = v(x)$ for all $x \in I - \{a\}$, with a according to Definition 2.2.1.

We observe that the operation of addition between two infinite hyperreals $\langle u, 0 \rangle$ and $\langle v, 0 \rangle$ is defined only in the cases where $\lim\limits_{x \to a}(u + v) = \infty$ or $\lim\limits_{x \to a}(u + v) = r$ (r a

[6] Infinite hyperreal numbers are not hyperreal numbers.

standard real number). The operation of multiplication between an infinite hyperreal $\langle u, 0\rangle$ and an infinitesimal $\langle 0, \varphi\rangle$ is defined only in the case where $\lim_{x\to a}(u \times \varphi) = \infty$ or $\lim_{x\to a}(u \times \varphi) = 0$.

The new structure $\langle \mathbb{A}^*, +, \times, 0, 1\rangle$ preserves some of the important properties of the hyperring $\langle \mathbb{A}, +, \times, 0, 1\rangle$. But as some of the conditions of the definition of a ring are not satisfied by $\mathbb{A}^*$, as for instance the associativeness of the operations, da Costa calls it a *quasi-ring*.

As in the case of $\mathbb{A}$, the quasi-ring $\mathbb{A}^*$ is also a *non-Arquimedian* structure.

We observe that A and $\mathbb{A}^*$ are classical structures and that $\mathbb{A}^*$ may be considered as a classical model of a calculus with infinitesimals and infinites.

3 Hypermetric and limit of hyperfunctions

3.1 Hypermetric

We can introduce a special metric over the hyperring $\mathbb{A}$.

Definition 3.1.1: The function $d_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, for every $\langle r, \varphi\rangle, \langle s, \psi\rangle \in \mathbb{A}$, is defined by:

$$d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle s, \psi\rangle) =_{df} |\ \langle r, \varphi\rangle - \langle s, \psi\rangle\ | =$$
$$|\ \langle r - s, \varphi - \psi\rangle\ | = \langle |\ r - s\ |,\ |\ \varphi - \psi\ |\rangle .$$

Theorem 3.1.2: The function $d_{\mathbb{A}} : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ is such that, for every $\langle r, \varphi\rangle, \langle s, \psi\rangle, \langle t, \phi\rangle \in \mathbb{A}$:

i) $d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle s, \psi\rangle) \geq \langle 0, 0\rangle$;

ii) $d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle s, \psi\rangle) = \langle 0, 0\rangle$ if, and only if, $\langle r, \varphi\rangle = \langle s, \psi\rangle$;

iii) $d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle s, \psi\rangle) = d_{\mathbb{A}}(\langle s, \psi\rangle, \langle r, \varphi\rangle)$;

iv) $d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle s, \psi\rangle) \leq d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle t, \phi\rangle) + d_{\mathbb{A}}(\langle t, \phi\rangle, \langle s, \psi\rangle)$. ■

We call $d_{\mathbb{A}}$ a *hypermetric* over $\mathbb{A}$.

We note that $d_{\mathbb{A}}|_{\mathbb{R}\times\mathbb{R}} = d_{\mathbb{R}}$, where $d_{\mathbb{R}}$ indicates the usual metric of $\mathbb{R}$. It is also interesting to observe that, given any two elements $\langle r, \varphi\rangle$ and $\langle r, \psi\rangle$ of the monad $[r]$ of a real number $r, d_{\mathbb{A}}(\langle r, \varphi\rangle, \langle r, \psi\rangle) < \langle \varepsilon, 0\rangle$, for any real number $\langle \varepsilon, 0\rangle, \langle \varepsilon, 0\rangle > \langle 0, 0\rangle$.

3.2 Limit, Continuity and Derivative

In order to construct the paraconsistent differencial calculus, da Costa [2000] and D'Ottaviano and Carvalho [201–] consider piecewise continuous functions f, defined in $\mathbb{R}$, and extend them to *hyperreal functions* $f' : J \subseteq \mathbb{A} \to \mathbb{A}$. The concept of the limit of a hyperreal function is considered only when the free variable $\langle x, \varphi\rangle$ tends to a real number $\langle r, 0\rangle \in \mathbb{R}$; and the concept of continuity of a hyperreal function in a given element of $J \subseteq \mathbb{A}$ is defined for real numbers $\langle r, 0\rangle \in J$.

In this paper, we present a new approach. The proofs of the theorems are not developed, for they are very simple consequences of the previous definitions.

Let us consider a function $f : J \subset \mathbb{A} \to \mathbb{A}$. We call this function a *hyperfunction.*

Definition 3.2.1: We say that the *limit* of the hyperfunction $f : J \subseteq \mathbb{A} \to \mathbb{A}$, when $\langle x, \lambda\rangle$ tends to $\langle r, \varphi\rangle$, is the hyperreal number $\langle b, \psi\rangle$, what is denoted by

$$\lim_{\langle x,\lambda\rangle \to \langle r,\varphi\rangle} f(\langle x, \lambda\rangle) = \langle b, \psi\rangle ,$$

if, and only if,

$$(\forall\, \langle \epsilon, \phi\rangle > \langle 0, 0\rangle)\, (\exists\, \langle \delta, \eta\rangle > \langle 0, 0\rangle) \text{ such that}$$
$$((\langle 0, 0\rangle < |\, \langle x, \lambda\rangle - \langle r, \varphi\rangle\, | < \langle \delta, \eta\rangle) \to (|\, f(\langle x, \lambda\rangle) - \langle b, \psi\rangle\, | < \langle \epsilon, \phi\rangle))$$

Theorem 3.2.2: Let $f : \mathbb{A} \to \mathbb{A}$ and $\hat{f} = f|_{\mathbb{R}}$, such that $\hat{f}(\mathbb{R}) \subseteq \mathbb{R}$. If $\lim\limits_{\langle x,\lambda\rangle \to \langle r,0\rangle} f(\langle x, \lambda\rangle) = \langle b, 0\rangle$, then $\lim\limits_{x\to r} \hat{f}(x) = b$.

Proof: Immediate, from the above definition, noting that the $\lim\limits_{x\to r} \hat{f}(x)$ is the classical limit. ■

Theorem 3.2.3: Given the function $\hat{f} : \mathbb{R} \to \mathbb{R}$, such that $\lim\limits_{x\to r} \hat{f}(x) = b$, consider a hyperfunction $f : \mathbb{A} \to \mathbb{A}$, such that $f|_{\mathbb{R}} = \hat{f}$; for every $x \in \mathbb{R} - \{r\}$, $f([x]) \subseteq [\hat{f}(x)]$; and for $\lambda \neq 0$, $f(\langle r, \lambda\rangle) \in [b]$. Then $\lim\limits_{\langle x,\lambda\rangle \to \langle r,0\rangle} f(\langle x, \lambda\rangle) = \langle b, 0\rangle$. ■

Definition 3.2.4: We say that a hyperfunction $f : \mathbb{A} \to \mathbb{A}$ is *continuous* in $\langle r, \varphi\rangle \in \mathbb{A}$ if, and only if,

$$\lim_{\langle x,\lambda\rangle \to \langle r,\varphi\rangle} f(\langle x, \lambda\rangle) = f(\langle r, \varphi\rangle).$$

The following results show that the definition of the continuity of a hyperfunction in a hyperreal number extends the definition of continuity of real functions. They are immediate consequences of the previous theorems.

Theorem 3.2.5: Let $f : \mathbb{A} \to \mathbb{A}$ and $\hat{f} = f|_{\mathbb{R}}$, such that $\hat{f}(\mathbb{R}) \subseteq \mathbb{R}$. If f is continuous in $\langle r, 0\rangle$, then $\hat{f}$ is continuous in r. ■

Theorem 3.2.6: Given $\hat{f} : \mathbb{R} \to \mathbb{R}$, such that $\hat{f}$ is continuous in r, let us consider a hyperfunction $f : \mathbb{A} \to \mathbb{A}$ such that: $f|_{\mathbb{R}} = \hat{f}$; for every $x \in (\mathbb{R} - \{r\})$, $f([x]) \subseteq [\hat{f}(x)]$; and, for $\lambda \neq 0$, $f(\langle r, \lambda\rangle) \in [\hat{f}(r)]$. Then, f is continuous in $\langle r, 0\rangle$. ■

Following, we introduce the concept of derivative of a hyperfunction in a hyperreal number, extending the definition of derivative of real functions.

Theorem 3.2.7: Given a hyperfunction $f : \mathbb{A} \to \mathbb{A}$, we say that f is *derivable* in $\langle r, \varphi\rangle \in \mathbb{A}$ if, and only if, there exists an element $f'(\langle r, \varphi\rangle) \in \mathbb{A}$ such that

$$\lim_{\langle h,0\rangle \to \langle 0,0\rangle} \frac{f(\langle r, \varphi\rangle + \langle h, 0\rangle) - f(\langle r, \varphi\rangle)}{\langle h, 0\rangle} = f'(\langle r, \varphi\rangle).$$

3.3 The limit in $\mathbb{A}^*$

Now, if we consider hyperfunctions $f : J \subseteq \mathbb{A} \to \mathbb{A}^*$, we can generalize the concept of limit.

Definition 3.3.1:

$$\lim_{\langle x,\lambda\rangle \to \langle r,\varphi\rangle} f(\langle x, \lambda\rangle) = \langle v, 0\rangle$$

if, and only if,

$$(\forall \langle M, \rho\rangle > \langle 0, 0\rangle)\ (\exists \langle \delta, \eta\rangle > \langle 0, 0\rangle) \text{ such that}$$
$$((\langle 0, 0\rangle < |\ \langle x, \lambda\rangle - \langle r, \varphi\rangle\ | < \langle \delta, \eta\rangle) \to (f(\langle x, \lambda\rangle) > \langle M, \rho\rangle))$$

$\veebar$

$$(\forall \langle M, \rho\rangle < \langle 0, 0\rangle)\ (\exists \langle \delta, \eta\rangle > \langle 0, 0\rangle) \text{ such that}$$
$$((\langle 0, 0\rangle < |\ \langle x, \lambda\rangle - \langle r, \varphi\rangle\ | < \langle \delta, \eta\rangle) \to (f(\langle x, \lambda\rangle) < \langle M, \rho\rangle)).$$

Definition 3.3.2:

$$\lim_{\langle x,\lambda\rangle \to \langle u,0\rangle} f(\langle x, \lambda\rangle) = \langle b, \psi\rangle$$

$$\text{if, and only if, } (\forall \langle \varepsilon, \phi \rangle > \langle 0,0 \rangle)\ (\exists \langle N, \gamma \rangle > \langle 0,0 \rangle) \text{ such that}$$
$$(\forall \langle r, \varphi \rangle)\ ((\langle r, \varphi \rangle > \langle N, \gamma \rangle) \to (\mid f(\langle r, \varphi \rangle) - \langle b, \psi \rangle \mid < \langle \varepsilon, \phi \rangle))$$

$\veebar$

$$(\forall \langle \varepsilon, \phi \rangle > \langle 0,0 \rangle)\ (\exists \langle N, \gamma \rangle < \langle 0,0 \rangle) \text{ such that}$$
$$(\forall \langle r, \varphi \rangle)\ ((\langle r, \varphi \rangle < \langle N, \gamma \rangle) \to (\mid f(\langle r, \varphi \rangle) - \langle b, \psi \rangle \mid < \langle \varepsilon, \phi \rangle)).$$

For the definition of
$$\lim_{\langle x, \lambda \rangle \to \langle u, 0 \rangle} f(\langle x, \lambda \rangle) = \langle v, 0 \rangle,$$
we have four cases to consider.

4 Final remarks

Carvalho (2004), and D'Ottaviano and Carvalho (201-) have improved the paraconsistent differential calculus, proposed by da Costa, from the structure $\mathbb{A}^*$.

In this paper, we have not worked with da Costa's calculus, we have simply proposed a new approach to dealing with hyperfunctions, defined in the hyperring $\mathbb{A}$ and with values in the quasi-ring $\mathbb{A}^*$. Such approach allowed us to introduce general definitions for the concepts of the limit of a hyperfunction, continuity and derivability.

We consider that the paraconsistent differential calculus formalizes the intuitions of the piomeers of differencial calculus by giving a precise sense to infinitesimals and infinites. It seems to open a rich and useful research field, not only for logic, but also for mathematics, physics and other connected areas.

In future works, the concepts introduced in this paper may be useful for the improvement of da Costa's calculus.

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