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The von Neumann-Regular Hull of (Preordered) Rings and Quadratic Forms

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In honor of Chico Miraglia on his 70th birthday

Abstract

The class of (commutative, unitary) von Neumann-regular rings (vN-rings) has been studied under algebraic and model-theoretic aspects. It is closed under several constructions and it can be characterized as the class of rings isomorphic to the ring of global sections of a sheaf of rings over a Boolean space such that the stalks are fields– from a broader logical perspective they are "fields". In this work we build, by sheaf-theoretic methods, a vN-Hull for every commutative unitary ring, giving a left adjoint to the inclusion of categories $vNRings \hookrightarrow Rings$. This result is immediately extended to categories of preordered rings and we present some applications to abstract codifications of the algebraic theory of quadratic forms over rings with 2^{-1} (ATQF), that turns out to be an alternative approach to the first-principle axiomatic approach of "well-behaved" quadratic form theory of pre-ordered rings, introduced and developed in [**DM9**]. For instance we address two subjects in the theory of Special Groups ([**DM1**]). **(I)** We determine interesting classes of rings relative to ATQF ([**DM7**], [**DM9**]): we show that the class of rings whose induced (proto)special group morphism into the special group of its vN-hull is a pure embedding is an elementary class in the language of rings that can be axiomatized by sets of Horn sentences or by ∀∃-sentences. **(II)** We determine a class of reduced special groups (rsg) of interest for a variant of the representation problem in SG-theory (see for instance [**DM8**]): we show that the class of reduced special groups that can be purely embedded into a special group of a preordered vN-ring is an elementary class in the language of special groups which can be axiomatized by sets of Horn sentences. Moreover, every rsg in the class satisfies the K-theoretic property called [SMC] ([**DM6**]).

Keywords: von Neumann regular rings, Special Groups, preordered rings, quadratic forms

Introduction

In this work, “ring” will always mean unitary commutative ring and “regular ring” means von Neumann regular ring (vN-regular ring or simply vN-ring), i.e., for each a in the ring, there is x such that $a^2x = a$ and $(x) = (e)$.

The theory of quadratic forms over rings more general than fields is addressed in the setting of the theory of *Special Groups* in [**DM7**], [**DM8**], [**DM9**]; see also [**DP1**], [**Mar1**], [**Mar2**], [**Mar3**] for other abstract encodings of the algebraic theory of quadratic forms. Among all rings, the vN-rings form a particularly well-behaved class, for which it is easier to access the information on quadratic forms than for a general ring. The reason is that vN-regular rings are, in a precise sense, the rings which are closest to fields. From a broader logical and set-theoretical perspective, vN-regular rings are “fields” ([**Smi**]).

It is known, that vN-regular rings form a reflective subcategory of all rings, i.e. for every ring A there is a vN-regular hull $R(A)$, i.e. a *universal* vN-regular ring $R(A)$ and a homomorphism $\eta_A\colon A \to R(A)$ which is initial among maps to vN-regular rings.

This suggests the strategy to transfer results about quadratic forms over vN-regular rings back to arbitrary rings along this map. Indeed, we show in Thm. 41 below that for any ring one can detect Witt-equivalence of quadratic forms still after passage to the vN-regular hull. We further, in Thm. 45, determine elementary classes of rings and pre-ordered rings whose quadratic form theory is satisfactorily encoded in their vN-hulls, namely those rings for which the the morphism to their vN-hull induces an elementary equivalence of the associated proto-special groups. This turns out to be an alternative approach to the first-principle axiomatic approach of “well-behaved” quadratic form theory of pre-ordered rings, introduced and developed in [**DM9**]. Finally, we address the representation problem for special groups by showing in Thm. 48 that there is an elementary class of special groups with a good representation theory by vN-regular rings.

As our approach is based on the initial map $\eta_A\colon A \to R(A)$ to a vN-regular ring, we spend the first four sections discussing this construction in detail. In the classical papers [**Car**], [**LS**] the model-theory of the elementary class of vN-regular rings is analyzed. A key point in these works is the simple fact that a reduced ring is (canonically) embedded in a vN-regular ring through the “diagonal” homomorphism: $\delta_A : A \rightarrowtail \prod_{p\in Spec(A)} k_p(A)$, where $k_p(A)$ is the field of fractions of A/p (or equivalently, the field A_p/pA_p, where A_p is the localization of A at the prime ideal p).

There is a refinement of the map above: we prove that for an *arbitrary* ring the “diagonal arrow” $A \longrightarrow \prod_{p\in Spec(A)} A/p \rightarrowtail \prod_{p\in Spec(A)} k_p(A)$ factors through the universal vN-regular ring $R(A) \subseteq \prod_{p\in Spec(A)} k_p(A)$. We show that $R(A)$ arises as the global sections of a sheaf of fields canonically associated to A.

These last results are not new — we summarize the history towards the end of section 1.

It is especially the sheaf construction, which is useful. Sheaves over boolean spaces already feature prominently in the literature on special groups and this construction connects well with our desired applications, e.g. it is used in the proof of Thm. 41. Therefore we chose to dedicate parts of the article to a self-contained proof, avoiding the topos-theoretic machinery of other existing proofs.

Overview of the article:

We will start in section 1 by reviewing the concept of vN-regular ring, giving a quick proof that there is a reflection, summarizing the history of the result and showing that one "cannot get any closer to fields" via a reflection functor. Since the sheaf theoretic point of view on the reflection functor provides additional insight, in sections 2 and 3 we give a self-contained and elementary proof that the vN-hull arises in this way. In section 4 we present some simple examples. We then proceed in section 5 with the observation that the result is easily extended to categories of pre-ordered rings. Finally, in section 6 we give the model-theoretic applications of this to the theory of *Special Groups* ([**DM1**]) which were mentioned above.

1 vN-regular rings

We start by giving a selection of equivalent characterizations of vN-regular rings, that we will make use freely in the sequel.

Proposition 1 *Let A be a ring. Then the following are equivalent:*

(i) A is vN-regular, i.e. $\forall a \in A\ \exists x \in A:\ a = a^2x$.

(ii) Every principal ideal of A is generated by an idempotent element, i.e. $\forall a \in A\ \exists e \in A \exists y, z \in A:\ e^2 = e, ey = a, az = e$.

(iii) $\forall a \in A\ \exists b \in A:\ a = a^2b, b = b^2a$.

Moreover, when A is vN-regular, then A is reduced (i.e., $Nil(A) = \{0\}$) and for each $a \in A$, the idempotent element $e \in A$ satisfying (ii) and the element b satisfying (iii) are uniquely determined.

Proof. (iii) $\Rightarrow$ (i): is obvious.

(i) $\Rightarrow$ (ii) Let $a, x \in A$ such that $a = a^2x$ and define $e := ax$, then: $e^2 = a^2x^2 = ax = e$ and $ea = axa = a$.

(ii) $\Rightarrow$ (i) Let $a, e, x, z \in A$ such that $e^2 = e, ey = a, az = e$ and define $x := z^2y$, then $a^2x = a^2z^2y = e^2y = ey = a$.

(i) $\Rightarrow$ (iii) Let $a, x \in A$ such that $a = a^2x$. There can be many x satisfying this role, but there is a "minimal" one: the element ax is idempotent and we can project any chosen x down with this idempotent, obtaining $b := ax^2$. Then: $aba = aab^2a = (ax)(ax)a = axa = a$ and $bab = (ax^2)a(ax^2) = (ax)^3x = (ax)x = b$.

Now suppose that A is vN-regular and let $a \in A$ such that $a^n = 0$, for some $n \in \mathbb{N}\backslash\{0\}$. Then let e be an idempotent such that $ey = a, az = e$, for some $y, z \in A$. Then $e = e^n = a^n z^n = 0z^n = 0$ an then $a = ey = 0y = 0$, showing that $Nil(A) = \{0\}$.

Let $e, e' \in A$ idempotents in an arbitrary ring satisfying $(e) = (e')$. Select $r, r' \in A$ such that $er' = e'$ and $e'r = e$. Then $e' = er' = er'e = e'e = e're' = e'r = e$. Thus, if an ideal a ring is generated by a single idempotent, this idempotent is uniquely determined.

Let A be a vN-regular ring. Select a member $a \in A$ and consider $b, b' \in A$ such that $a^2b' = a = a^2b, b = b^2a, b' = b'^2a$. Then $(b - b')^2a^2 = (b - b')(ba^2 - b'a^2) = (b - b')(a - a) = (b - b')0 = 0$. Since A is reduced, we have $(b - b')a = 0$, therefore $bb' = b^2a - b'^2a = (b^2 - b'^2)a = (b + b')(b - b')a = (b + b')0 = 0$.

■

Remark 2 On vN-regular rings:

(i) Other equivalent descriptions of a vN-regular ring A:

- A is reduced and has Krull dimension 0 (i.e., prime ideals are maximal).
- Every A-module is flat.

(ii) Fields and boolean rings are natural examples of regular rings. A domain or a local ring is vN-regular iff it is a field. If A is vN-regular ring, then $Spec(A)$ is a boolean space.

(iii) The subclass of vN-regular rings is closed under isomorphisms, products and coequalizers (=homomorphic images) in the category of all rings.

□

Remark 3 vN-regular rings and Logic: We will denote $L_{Rings} = \{+, \cdot, -, 0, 1\}$, the first-order language adequate for description of rings.

(i) The subclass of vN-regular rings is closed under and pure subrings[1].

(ii) If the "diagonal" homomorphism $\delta_A : A \longrightarrow \prod_{p \in Spec(A)} k_p(A)$ is a L_{Ring}-pure embedding, then A is a vN-regular ring. It follows from Proposition 3.2.(d) in [**DM7**], that the converse statement also holds.

[1]Recall that, if L a language and M, M' be L-structures, an L-homomorphism $j : M \longrightarrow M'$ is called L-pure embedding if it (preserves and) reflects the satisfaction of existential positive L-formulas with parameters in M.

(iii) Since a ring A is vN-regular iff for each $a \in A$ there exists a *unique* $b \in A$ such that $a = a^2b, b = b^2a$. then the class of vN-regular rings is axiomatizable by a (finite) set of *limit sentences* in the language L_{Rings}, i.e. sentences of the form $\forall \bar{x}(\phi(\bar{x}) \rightarrow \exists! \bar{y}\theta(\bar{x}, \bar{y}))$[2].

(iv) Since vN-regular rings can be presented as structures for a signature $L^* := L_{Rings} \cup \{(\ )^*\}$, with an additional unary operation symbol which, for any x forms describes its "quasi-inverse" x^*, where $x = x^2x^*, x^* = (x^*)^2x$. This makes it clear that vN-regular rings form a variety (= equational class) in this expanded language. Thus: the subclass of vN-regular rings is closed under homomorphic images, substructures and products in the class of all L^*-structures; there exists a free vN-regular ring over any set.

□

In the sequel, we present some closure properties of *RegRings* as a subclass of L_{Rings}-structures.

Proposition 4 *The inclusion functor RegRings* $\hookrightarrow$ *Rings creates filtered colimits, i.e., RegRings⊆Rings is closed under directed/filtered colimits.*

Proof. A filtered colimit of vN-regular rings, taken in *Rings*, is a vN-regular ring again. Indeed, filtered colimits in *Rings* are formed by taking the colimit of the underlying sets and defining the sum, resp. product, of two elements a, b by mapping them both into a common ring occurring in the diagram and taking the sum, resp. product, there. Thus for an element a in the colimit, there is a ring A_i in the diagram containing an element $a_i \in A_i$ which is mapped to a by the canonical map to the colimit. Since A_i is vN-regular, we have $x_i \in A_i$ with $a_i x_i a_i = a_i$, and the image of x_i in the colimit will satisfy the corresponding relation with a. ■

Proposition 5 *RegRings⊆Rings is closed under localizations.*

Proof. Let A be a vN-regular ring and $S \subseteq A$ be a multiplicative submonoid of A. We will show that $A[S]^{-1}$ is a vN-regular ring:

Case (1) $S = S_a\langle a \rangle = \{a^k : k \in \mathbb{N}\}$, for some $a \in A$.

Let e be the unique idempotent element such that $(a) = (e)$. Then, by Remark 15, $D_a = D_e$ and $\sigma(A)^a_e : A_a \cong \longrightarrow A_e$. By Fact 14, $A_e \cong A/(1-e) \cong A \cdot e$. Since *RegRings⊆Rings* is closed under homomorphic images, then $A_a = A[S_a]^{-1}$ is a vN-regular ring.

[2]Note that any equation $\forall \bar{x}(t(\bar{x}) = s(\bar{x}))$ is logically equivalent to the limit sentence $\forall \bar{x}(\top \rightarrow \exists! y(t(\bar{x}) = y \wedge y = s(\bar{x})))$.

Case (2) $S = \langle a_1, \cdots, a_k \rangle$, for some $\{a_1, \cdots a_k\} \subseteq_f inA$.
Since $A[S]^{-1} \cong A[S_a]^{-1}$, where $a = \prod_{i=1}^{n} a_i$, then $A[S]^{-1}$ is a vN-regular ring, by the case (1).

Case (3) For general S.
Write $S = \bigcup_{X \subseteq_{fin} S} \langle X \rangle$: this is a directed reunion. Then $A[S]^{-1} \cong \underrightarrow{lim}_{X \subseteq_{fin} S} A[\langle X \rangle]^{-1}$.
Since $RegRings \subseteq Rings$ is closed under directed/filtered colimits, then $A[S]^{-1}$ is a vN-regular ring, by the case (2).

■

Proposition 6 *The limit in Rings of a diagram of vN-regular rings is vN-regular. In particular RegRings is a complete category and the inclusion functor RegRings* $\hookrightarrow$ *Rings preserves all limits.*

Proof. It is clear from the definition that the class *RegRings* of vN-regular rings is closed under arbitrary products in the class *Rings* of all rings. Thus it suffices to show that it is closed under equalizers. So let A, B be vN-regular rings and $f, g \colon A \to B$ ring homomorphisms. Their equalizer in the category *Rings* is given by the set $E := \{a \in A : f(a) = g(a)\}$, endowed with the restricted ring operations from A.

To see that E is vN-regular, we need to show that for $a \in E$, the (unique) element $b \in A$ satisfying $a^2 b = a$ and $b^2 a = b$ also belongs to E.

First we note that the idempotent element ab belongs to E. Indeed, we have $f(ab) = f(a)f(b) = g(a)f(b) = g(a^2 b)f(b) = g(a)g(ab)f(b) = f(a)g(ab)f(b) = f(ab)g(ab)$. Exchanging f and g in this chain of equations, we also get $g(ab) = f(ab)g(ab)$. Altogether we obtain $g(ab) = f(ab)$, and hence $ab \in E$.

Now we use this, as well as the fact that we also have the equation $b = ab^2$, and conclude $f(b) = f(ab^2) = f(b)f(ab) = f(b)g(ab) = f(b)g(a)g(b) = f(b)f(a)g(b) = f(ab)g(b) = g(ab)g(b) = g(ab^2) = g(b)$. ■

Clearly, in many ways fields are the best behaved kind of commutative ring. Often problems in general commutative algebra are treated by reducing them to the case of fields, for example when using local-global principles in module theory. It is thus natural to ask whether an arbitrary commutative ring A admits some kind of universal approximation by a field, i.e. a map $A \to F$, with F a field, which is initial among maps to fields.

The answer is negative, because if every ring would admit such a map, then we would obtain a reflection functor *Rings* $\to$ *Fields*, left adjoint to the inclusion *Fields* $\hookrightarrow$ *Rings*. But then *Fields* would be a cocomplete (and complete) category and the inclusion functor would preserve limits. This contradicts the fact that the product of two fields is a ring with non-trivial idempotents, thus it not a field.

However, it turns out that this problem with limits is the *only* obstruction to the existence of a universal approximation. If we take the closure of the subcategory of fields under limits in *Rings*, then we obtain the category *RegRings* of vN-regular rings (see Prop. 8 below), and this category does admit a reflection functor. We give a quick, but non-constructive, proof that the inclusion functor $RegRings \hookrightarrow Rings$ has a left adjoint.

Proposition 7 *The inclusion functor $i : RegRings \hookrightarrow Rings$ has a left adjoint.*

Proof. Since *RegRings*, *Rings* are categories of models of first order theories given by *limit sentences* then, by [**AR**, Thm 5.9], they are locally finitely presentable categories. By the previous propositions 4 and 6, the inclusion functor $i : RegRings \hookrightarrow Rings$ is a finitely accessible (= preserves filtered colimits), limit preserving functor between locally presentable categories. Thus, by the adjoint functor theorem for locally presentable categories, [**AR**, Thm 1.66], the functor i has a left adjoint. ∎

The above proof of Prop. 7 provides no information about how to compute the left adjoint. For this, more constructive proofs are preferable.

The first proof of Prop. 7 in the literature is that of Olivier, [**Oli**, Prop. 5]. He proceeds by formally adjoining a quasi-inverse to every element a of a ring A (where quasi-inverse means the x such that $a^2x = a$, $x^2a = x$). It is easy to see that the outcome is the universal vN-regular ring associated to A, but hard to compute this outcome, as it is given in terms of generators and relations.

Next, Popescu and Vraciu [**PV**, Sect. 3] noted that the reflection functor can be described as follows. To a ring A one associates the diagonal map $\delta_A : A \rightarrowtail \prod_{p \in Spec(A)} k_p(A)$, where $k_p(A)$ is the field of fractions of A/p (or equivalently, the field A_p/pA_p, where A_p is the localization of A at the prime ideal p). Then one takes the smallest zero-dimensional subring of $\prod_{p \in Spec(A)} k_p(A)$ containing the image of δ_A (which exists, because zero-dimensional subrings of a ring are closed under intersection by [**Gil**, Thm 2.1]).

It follows from Theorem 10.3 in [**Pie**] that the class of vN-regular rings coincides with the closure under isomorphisms of the class of the rings of global sections of sheaves of rings over a boolean space such that the stalks are fields. Finally, Coste [**Cos**, Prop. 4.5.5 and Sect. 5.2.2] gave a description of the functor as the global sections of a sheaf of fields on the boolean space arising by endowing the space $Spec(A)$ with the constructible topology. The étale space of this sheaf is given by $\coprod_{p \in Spec(A)} k_p(A)$ with an appropriate topology, which links this construction to the previous one. This space with its sheaf of fields is called the "field spectrum" and had previously been considered by Johnstone [**Joh**, Prop. 5.6]. It arises from a general topos theoretic machinery which is able to gives a universal approximation of a ring by a field — with the downside that the field lives in a different topos than the the topos of set, where the original ring A is situated.

It is only this last construction, as the global sections of a sheaf of fields, that allows to prove that our solution of taking vN-regular rings for approximating a ring by something close to a field is the best possible:

Proposition 8 *The category RegRings is the smallest reflective subcategory of Rings containing all fields.*

Proof. Clearly all fields are vN-regular, and by Prop. 6 so are limits of fields, so *RegRings* contains all limits of fields. On the other hand the ring of global sections of a sheaf can be expressed as a limit of a diagram of products and ultraproducts of the stalks by [**Ken**, Lemma 2.5]. All these occurring (ultra)products are vN-regular as well and hence so is their limit by Prop. 6. ∎

In the following two sections we will give an elementary and self-contained proof that the reflection functor arises as the global sections, without any use of general topos theory.

2 The candidate

It is well known that any ring is isomorphic to the ring of global sections of the structure sheaf of its affine scheme. It is a sheaf of rings over the *spectral space* $Spec(A)$, the prime spectrum of A, with stalks that are (isomorphic to) the local rings A_p, the localizations of the ring in the prime ideals $p \in Spec(A)$. This *suggests* that if we get a sheaf of rings over the *booleanization space* of the spectral space $Spec(A)$ (i.e. $Spec^{const}(A) := Spec(A)$ with the *constructive topology*) and with stalk on the prime ideal p the residue field $k_p(A) := A_p/p.A_p$ of the local ring A_p, then the ring of the global sections of this sheaf should be "the closest" vN-regular ring to A.

In this section we will see that **if** the ring A has a vN-regular hull $\eta_A : A \longrightarrow R(A)$ (i.e., η_A is a ring homomorphism from A to a regular ring $R(A)$ with the universal property: for any regular ring V and any ring homomorphism $f : A \longrightarrow V$ there is a unique ring homomorphism $\widetilde{f} : R(A) \longrightarrow V$ such that $\widetilde{f} \circ \eta_A = f$), **then** the spectral sheaf of $R(A)$ *must be* such that the topological space $Spec(R(A))$ is homeomorphic to the booleanization of $Spec(A)$ and the stalks $R(A)_{p'}$, $p' \in Spec(R(A))$, are isomorphic to the residue fields of the local ring A_p: $k_p(A)$, $p \in Spec(A)$.

Notation 9

(i) If p is a proper prime ideal of the ring A, write $\pi_p^A : A \twoheadrightarrow A/p$ for the quotient homomorphism and $\alpha_p^A : A \longrightarrow k_p(A)$ for the composition $A \twoheadrightarrow A/p \rightarrowtail k_p(A)$. We will consider the "diagonal" homomorphism $\delta_A := (\alpha_p^A)_{p \in Spec(A)} : A \longrightarrow \prod_{p \in Spec(A)} k_p(A)$.

(ii) If $f : A \longrightarrow B$ is a ring homomorphism and $q \in Spec(B)$, we will denote $\bar{f}_q : A/f^\star(q) \rightarrowtail B/q$, the "quotient" monomorphism between the associated domains, $\tilde{f}_q : A_{f^\star(q)} \longrightarrow B_q$, the "canonical" local homomorphism between the associated local rings and $\widehat{f}_q : k_{f^\star(q)}(A) \rightarrowtail k_q(B)$, the "canonical" monomorphism between the associated fields[3]. We have $\widehat{f}_q \circ \alpha^A_{f^\star(q)} = \alpha^B_q \circ f$.

We will write $\widehat{f} : \prod_{p \in Spec(A)} k_p(A) \longrightarrow \prod_{q \in Spec(B)} k_q(B)$ for the unique homomorphism such that $proj^B_q \circ \widehat{f} = \widehat{f}_q \circ proj^A_{f^\star(q)}$, for each $q \in Spec(B)$ (note that $\widehat{f} \circ \delta_A = \delta_B \circ f$). □

Remark 10 On spectral spaces: (see [**Hoc**], [**DST**])

- A topological space is *spectral* if: (i) it is *sober*, i.e. each closed irreducible subset has a unique generic point; (ii) it has a basis that is closed under finite intersections (and contains the space) and whose elements are compact-open subsets. A boolean space is a compact Hausdorff space that is 0-dimensional (or equivalently, it has a basis whose elements are clopens). The boolean spaces are the spectral T_1-spaces. If A is a ring, then $Spec(A)$ is a spectral space. For each spectral space S and field K, there is a K-algebra A, such that S is homeomorphic to $Spec(A)$.

- A map between spectral spaces is called *spectral map* if the inverse image of a compact open subset of the codomain is a compact open subset of the domain. In particular, spectral maps are continuous; a map between boolean spaces is spectral iff it is continuous. If $f : A \longrightarrow B$ is a ring homomorphism, then $f^\star : Spec(B) \longrightarrow Spec(A) : q \mapsto f^{-1}[q]$ is a spectral map.

- Every spectral space S has a "canonically" associated boolean space: S^{const} is the set S with the *constructive topology*, constructed as follows. Take any base $\{D_i : i \in I\}$ of S whose elements are compact open, that contains $\varnothing$, S and that is closed under finite intersections. Then $\{D_{i_0} \cap (S \backslash D_{i_1}) \cap \ldots \cap (S \backslash D_{i_n}) : i_0, i_1, \ldots, i_n \in I\}$ is a basis of S^{const} whose elements are clopen subsets (that contains $\varnothing, S$ and is closed under finite intersections). Clearly, the identity function $id : S^{const} \longrightarrow S$ is spectral. This construction satisfies a universal property: it is the *booleanization* of the spectral space S, i.e. given T a boolean topological space and $F : T \longrightarrow S$ a spectral map, there is a unique continuous/spectral map between the boolean spaces $\widehat{F} : T \longrightarrow S^{const}$ such that $id \circ \widehat{F} = F$. As continuous/spectral bijections between boolean spaces are homeomorphisms, a spectral map $G : U \longrightarrow S$ from a boolean space U is a/the booleanization of S iff it is a bijection.

□

[3]Recall that $k_q(B) := Res(B_q) = B_q/q.B_q \cong Frac(B/q) = (B/q)[B/q\backslash\{0\}]^{-1}$, and that $B_q \twoheadrightarrow k_q(B)$ is a *local* homomorphism.

Proposition 11 *Suppose that the ring A has a vN-regular hull $\eta_A : A \to R(A)$* [4]. *Then:*

(i) $(\eta_A)^\star : Spec(R(A)) \longrightarrow Spec(A)$ *is a surjective spectral function.*

(ii) For any $q \in Spec(R(A))$, $\widehat{\eta_A}_q : k_{\eta_A^\star(q)}(A) \overset{\cong}{\longrightarrow} k_q(R(A))$.

(iii) $(\eta_A)^\star : Spec(R(A)) \overset{\approx}{\longrightarrow} Spec^{const}(A)$ *is a homeomorphism where* $Spec^{const}(A)$ *is the (boolean) topological space on the set of prime ideals of A with the* constructive topology.

(iv) (iv) $ker(\eta_A) = nil(A) = \bigcap Spec(A) = \{a \in A : \exists n \in N,\ a^n = 0\}$. *Thus η_A is injective iff A is reduced.*

Proof.

(i) Take $p \in Spec(A)$ and consider the canonical homomorphism α_p from A to the *regular ring* (field) $k_p(A)$. By the universal property of $\eta_A : A \longrightarrow R(A)$ we have a unique "extension" $\widetilde{\alpha}_p^A : R(A) \longrightarrow k_p(A)$ with $\widetilde{\alpha}_p^A \circ \eta_A = \alpha_p^A$. Now take $\gamma_A(p) := (\widetilde{\alpha}_p)^\star(\{0\}) \in Spec(R(A))$. In this way we have $(\eta_A)^\star(\gamma_A(p)) = (\eta_A)^\star((\widetilde{\alpha}_p)^\star(\{0\})) = (\widetilde{\alpha}_p \circ \eta_A)^\star(\{0\}) = (\alpha_p)^\star(\{0\}) = p$, thus $(\eta_A)^\star \circ \gamma_A = id_{Spec(A)}$ and $(\eta_A)^\star$ is surjective.

(ii) Let $q \in Spec(R(A))$, by definition of $\widehat{\eta}_A$ we have (see diagram below) $\alpha_q^{R(A)} \circ \eta_A = \widehat{(\eta_A)}_q \circ \alpha^A_{\eta^\star(q)}$, so we have the field (mono)morphism $\widehat{\eta_A}_q : k_{\eta_A^\star(q)}(A) \rightarrowtail k_q(R(A))$. Let us prove that it is surjective: consider the extension of $\alpha^A_{\eta_A^\star(q)}$ to $R(A) : \widetilde{\alpha}^A_{\eta_A^\star(q)} : R(A) \longrightarrow k_{\eta_A^\star(q)}(A)$, then $\widehat{\eta_A}_q \circ \widetilde{\alpha}^A_{\eta_A^\star(q)} \circ \eta_A = \widehat{\eta_A}_q \circ \alpha^A_{\eta_A^\star(q)} = \alpha_q^{R(A)} \circ \eta_A$, thus $\widehat{\eta_A}_q \circ \widetilde{\alpha}^A_{\eta_A^\star(q)} = \alpha_q^{R(A)}$, by the universal property of η_A. But $q \in Spec(R(A))$ is a *maximal* ideal so $R(A)/q \cong k_q(R(A))$ and $\alpha_q^{R(A)} : R(A) \twoheadrightarrow R(A)/q \overset{\cong}{\longrightarrow} k_q(R(A))$ is *surjective*, therefore $\widehat{\eta_A}_q$ is surjective too, since $\widehat{\eta_A}_q \circ \widetilde{\alpha}^A_{\eta_A^\star(q)} = \alpha_q^{R(A)}$.

(iii) By Remark 10, to prove that $(\eta_A)^\star : Spec(R(A)) \overset{\cong}{\longrightarrow} Spec^{const}(A)$ is a homeomorphism it is necessary and sufficient to prove that the spectral map $(\eta_A)^\star : Spec(R(A)) \longrightarrow Spec(A)$ is a *bijection* from the boolean space $Spec(R(A))$ to the spectral space $Spec(A)$. Keeping the notation in the proof of item (i), we will show that γ_A is the inverse map of $\eta_A^\star$. By the proof of (i), it is enough to prove that $\gamma_A \circ (\eta_A)^\star = id_{Spec(R(A))}$. Let $q \in Spec(R(A))$, then $\gamma_A(\eta_A^\star(q)) = ker(\widetilde{\alpha}_{\eta_A^\star(q)}) = ker(\alpha_q^{R(A)}) = q$, since $\widehat{\eta_A}_q \circ \widetilde{\alpha}^A_{\eta_A^\star(q)} = \alpha_q^{R(A)}$ and $\widehat{\eta_A}_q$ is injective.

(iv) We will see that the result follows from the fact that $Spec(R(A))$ is homeomorphic to the booleanization of $Spec(A)$. Take any ring B and consider the "diagonal" homomorphism

$$\delta_B := (\alpha_p^B)_{p \in Spec(B)} : B \longrightarrow \prod_{p \in Spec(B)} k_p(B).$$

[4] For instance, if A is already regular then $\eta_A : A \cong R(A)$

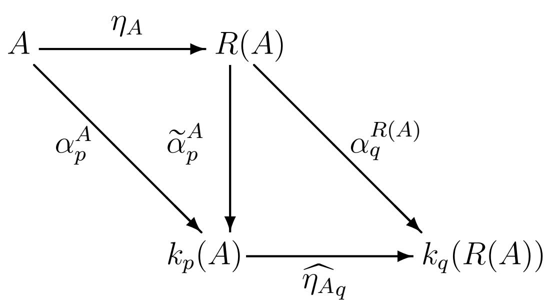
From the equality of morphisms

$$(B \overset{(\alpha_p^B)_{p\in Spec(B)}}{\longrightarrow} \prod_{p\in Spec(B)} k_p(B)) = (B \overset{(\pi_p^B)_{p\in Spec(B)}}{\longrightarrow} \prod_{p\in Spec(B)} B/p \rightarrowtail \prod_{p\in Spec(B)} k_p(B)))$$

we have $ker(\delta_B) = \bigcap Spec(B) = nil(B)$.

In particular, for $B = R(A)$, we have $ker(\delta_{R(A)}) = nil(R(A)) = \{0\}$ and therefore obtain that $\delta_{R(A)} : R(A) \longrightarrow \prod_{q\in Spec(R(A))} k_q(R(A))$ is *injective*.

As $\eta_A^\star : Spec(R(A)) \longrightarrow Spec(A)$ is bijective we get that the arrow $\widehat{\eta_A} : \prod_{p\in Spec(A)} k_p(A) \longrightarrow \prod_{q\in Spec(R(A))} k_q(R(A))$ *is isomorphic to* the homomorphism $(\widehat{\eta_{A}}_q)_{q\in Spec(R(A))} : \prod_{q\in Spec(R(A))} k_{\eta_A^\star(q)}(A) \longrightarrow \prod_{q\in Spec(R(A))} k_q(R(A))$. By the previous items, $(\widehat{\eta_{A}}_q)_{q\in Spec(R(A))}$ is an isomorphism, thus $\widehat{\eta_A}$ is an isomorphism too. As $\widehat{\eta_A} \circ \delta_A = \delta_{R(A)} \circ \eta_A$ and $\widehat{\eta_A}, \delta_{R(A)}$ are injective, we have $ker(\eta_A) = ker(\eta_A \circ \delta_{R(A)}) = ker(\widehat{\eta_A} \circ \delta_A) = ker(\delta_A) = nil(A)$.



where $p := \eta_A^\star(q)$

■

Proposition 12 *Suppose that we have a functor $R : Rings \longrightarrow RegRings$ and a natural transformation $(\eta_A)_{A\in obj(Rings)}$ $\eta_A : A \longrightarrow R(A)$.*
(i) Suppose that the following condition is satisfied.

(E) *For each vN-regular ring V the arrow $\eta_V : V \longrightarrow R(V))$ is a* section *(i.e. it has a left inverse).*

Then every homomorphism $f : A \longrightarrow V$ to a vN-regular ring V factors through η_A.
(ii) Suppose that the following conditions are satisfied.

(U) *$\eta_A^\star : Spec(R(A)) \longrightarrow Spec(A)$ is a bijection ($Spec(R(A))$ is homeomorphic to the booleanization of $Spec(A)$).*

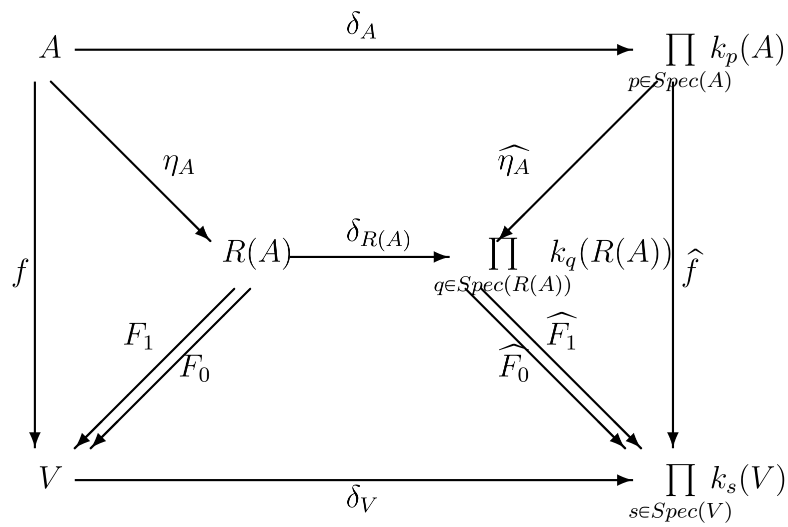
(U') *the stalk of the spectral sheaf of $R(A)$ at a prime ideal p "in A" is isomorphic to $k_p(A)$, more precisely, $\widehat{\eta_{A}}_q : k_{\eta_A^\star(q)}(A) \longrightarrow k_q(R(A))$ is an isomorphism, $q \in Spec(R(A))$.*

Then a homomorphism $f : A \longrightarrow V$ to a vN-regular ring V admits at most one factorization through η_A.

Thus if all of **(E)**, **(U)** *and* **(U')** *are satisfied, then the map* η_A *is initial among maps to vN-regular rings, i.e. a homomorphism* $f : A \longrightarrow V$ *to a vN-regular ring* V *factors uniquely through* η_A.

Proof. (i) Let V be a vN-regular ring and $f : A \longrightarrow V$ a homomorphism. As η is a natural transformation we have $R(f) \circ \eta_A = \eta_V \circ f$. By hypothesis **(E)** there is a homomorphism $r_V : R(V) \longrightarrow V$ such that $r_V \circ \eta_V = id_V$. Now *define* $F : R(A) \longrightarrow V$ as the composition : $F := r_V \circ R(f)$: clearly F is an "extension" of f ($F \circ \eta_A = f$).

(ii) Let V be a vN-regular ring and $F_0, F_1 : R(A) \longrightarrow V$ be homomorphisms such that $F_0 \circ \eta_A = f = F_1 \circ \eta_A$. Then we get $\widehat{F_0} \circ \widehat{\eta_A} = \widehat{f} = \widehat{F_1} \circ \widehat{\eta_A}$ (to see that, just compose these homomorphisms with $proj_s^V$, $s \in Spec(V)$). From the hypotheses **(U)** and **(U')** we obtain that the arrow $\widehat{\eta_A} : \prod_{p \in Spec(A)} k_p(A) \longrightarrow \prod_{q \in Spec(B)} k_q(B)$ is an *isomorphism*, and therefore $\widehat{F_0} = \widehat{F_1}$. It follows that $\delta_V \circ F_0 = \widehat{F_0} \circ \delta_{R(A)} = \widehat{F_1} \circ \delta_{R(A)} = \delta_V \circ F_1$. As V is reduced, we get $ker(\delta_V) = Nil(V) = \{0\}$, thus δ_V is injective and, by the commutativity of the bottom trapezoid below, we can conclude that $F_0 = F_1$, proving the uniqueness of extensions.



■

3 Building sheaves

In the remainder of this section we will show that the inclusion functor $RegRings \hookrightarrow Rings$ has a left adjoint through the following steps:
(I) Firstly we build a (mono)presheaf of rings over a basis of the constructive topology of $Spec(A)$ with stalks $k_p(A)$, $p \in Spec(A)$: this construction turns out to be canonical, i.e. describes a functor $A \mapsto P(A)$;

(II) Then we take the *associated sheaf* $S(A)$ of the presheaf basis $P(A)$: this has the same underlying space of $P(A)$ and isomorphic stalks over each point of the space;
(III) Next, we consider the ring $R(A)$ of global sections of the sheaf $S(A)$ and the *canonical* A-algebra arrow $\eta_A : A \longrightarrow \Gamma(S(A))$;
(IV) Finally we prove that, for each ring A, $R(A)$ is vN-regular ring and that $\eta_A : A \longrightarrow R(A)$, satisfies the conditions in Proposition 12, thus has the universal property wanted.

Let us fix now a ring A.

13 On the prime spectrum of rings:

- For each $a, b \in A$, consider the sets $D_a := \{p \in Spec(A) : a \notin p\}$, $Z_b := \{p \in Spec(A) : b \in p\} = Spec(A) \backslash D_a$. Then $D_1 = Z_0 = Spec(A)$, $D_0 = Z_1 = \emptyset$, $D_a \cap D_{a'} = D_{a \cdot a'}$ and the set $\{D_a \subseteq Spec(A) : a \in A\}$ is a basis of a topology in $Spec(A)$ whose elements are compact-open subsets, that contains $\emptyset, Spec(A)$ and that it is closed under finite intersections. Moreover with this topology $Spec(A)$ is a *spectral* space. If $f : A \longrightarrow A'$ is a ring homomorphism, then $(f^\star)^{-1}[D_a] = D'_{f(a)}$, thus $f^\star : Spec(A') \longrightarrow Spec(A)$ is a spectral map.

- Now, for each $a \in A$ and $\bar{b} = \{b_1, \ldots, b_n\} \subseteq_{fin} A$, we consider the subset $U_{a,\bar{b}} := D_a \cap Z_{b_1} \cap \ldots \cap Z_{b_n} \subseteq Spec(A)$. Then the set $\beta(A) := \{U_{a,\bar{b}} \subseteq Spec(A) : a \in A, \bar{b} = \{b_1, \ldots, b_n\} \subseteq_{fin} A\}$ is a basis of the *boolean* space $Spec^{const}(A)$ whose elements are clopen subsets, that contains $\emptyset, Spec(A)$ and is closed under finite intersections. $Spec^{const}(A)$ is the booleanization of the spectral space $Spec(A)$ (see Remark 10). If $f : A \longrightarrow A'$ is a ring homomorphism, then $(f^\star)^{-1}[U_{a,\bar{b}}] = U'_{f(a), f(\bar{b})}$, thus $f^\star : Spec^{const}(A') \longrightarrow Spec^{const}(A)$ is a spectral/continuous map.

□

We register the following:

Fact 14

- *(i) Let A be a ring. Denote $B(A) := \{e \in A : e^2 = e\}$. Then:*
  $*$ $(B(A), \wedge, \vee, *, \leqslant, 0, 1)$ *is a boolean algebra, where* $e \wedge e' := e.e'$, $e \vee e' := e + e' - e.e'$, $e^* := 1 - e$, $e \leqslant e' \Leftrightarrow e = e.e'$, *top* $:= 1$, *bottom* $:= 0$.
  $*$ *Concerning principal ideals:* $(e) \subseteq (e') \Leftrightarrow e \leqslant e'$; $(e) = (e') \Leftrightarrow e = e'$; $(e) + (e') = (e \vee e')$; $(e) \cap (e') = (e).(e') = (e \wedge e')$.
  $*$ *The mapping $e \mapsto D_e$ determines an* injective *boolean algebra homomorphism* $j_A : B(A) \rightarrowtail Clopen(Spec(A))$. *If A is a vN-regular ring, then* $j_A : B(A) \xrightarrow{\cong} Clopen(Spec(A))$.
  $*$ *For each $e \in B(A)$, consider the canonical A-algebra morphisms* $q : A \twoheadrightarrow$

$A/(1-e)$, $f : A \longrightarrow A_e$ and $m : A \twoheadrightarrow A.e$ (note that $A.e := \{x.e : x \in A\}$ is a ring with unity e). Then there are unique isomorphisms of A-algebras: $A_e \cong A/(1-e) \cong A.e$.

- *(ii) Let $f : A \longrightarrow A'$ be a ring homomorphism, then $f_\restriction : B(A) \longrightarrow B(A')$ is a (well-defined) boolean algebra homomorphism. Moreover, this defines a functor $\mathcal{B} : Rings \longrightarrow BA$, where BA is the category of boolean algebra and its homomorphisms.*

□

Let us now recall some definitions and facts about the *affine scheme functor* from the category of rings to the category of *locally ringed spaces*, *Locringed*. Recall first that:
$*$ the objects of *Locringed* are locally ringed spaces, i.e. pairs (X, F) where X is a topological space and $F : (Open(X), \subseteq)^{op} \longrightarrow Rings$ is a sheaf whose stalks, $F_x := \underset{U \in Open(X): x \in U}{\underrightarrow{lim}} F(U)$ (denote $\phi_{U,x} : F(U) \longrightarrow F_x$ the cocone arrow), are local rings (denote m_x the unique maximal ideal in F_x).
$*$ the morphisms in *Locringed* are certain morphisms of sheaves (over variable base spaces), i.e. pairs $(h, \tau) : (X, F) \longrightarrow (X', F')$ given by a continuous function $h : X' \longrightarrow X$ and a natural transformation $\tau : F' \longrightarrow h_\bullet(F)$, that induces local homomorphism on stalks. It follows from the definition of identities and composition in *Locringed* that (h, τ) is an isomorphism iff h is a homeomorphism and τ is a natural isomorphism.

Remark 15 On the affine scheme functor: ([**EGA**])

(i) For each $a, c \in A$, $D_a \subseteq D_c$ iff $Z_a \supseteq Z_c$ iff $\sqrt{(a)} \subseteq \sqrt{(c)}$.

(ii) If $D_a \subseteq D_c$, then there is a unique ring homomorphism $\sigma(A)^c_a : A_c \longrightarrow A_a$ such that $(A \overset{\sigma(A)_c}{\longrightarrow} A_c \overset{\sigma(A)^c_a}{\longrightarrow} A_a) = (A \overset{\sigma(A)_a}{\longrightarrow} A_a)$ where A_b denotes the ring of fractions of A w.r.t. the multiplicative set $\{b^n : n \in \mathbb{N}\}$ and $\sigma(A)_b : A \longrightarrow A_b$, $x \mapsto x/1$ is the canonical arrow.

(iii) $\sigma(A)^a_a = id_{A_a}$; if $D_a \subseteq D_c \subseteq D_e$, then $\sigma(A)^e_a = \sigma(A)^c_a \circ \sigma(A)^e_c$; if $D_a = D_c$, then $\sigma(A)^c_a$, $\sigma(A)^a_c$ is a pair of inverse isomorphisms.

(iv) The *structure sheaf* of the affine scheme of the ring A, $\Sigma_A : (Open(Spec(A)), \subseteq)^{op} \longrightarrow Rings$ is the (essentially) unique sheaf of rings such that for each $a \in A$, $\Sigma_A(D_a) \cong A_a$. We denote $\Sigma_A(V) \overset{\sigma(A)^V_U}{\longrightarrow} \Sigma_A(U)$, the morphism induced by the inclusion $U \subseteq V$, $U, V \in Open(Spec(A))$. The canonical arrow $s(A) : A \longrightarrow \Sigma_A(Spec(A))$ is naturally identified with the canonical homomorphism $A \longrightarrow A_1$, thus it is a ring isomorphism. The stalk of Σ_A over the point $p \in Spec(A)$ is isomorphic to the local ring A_p of fractions of A w.r.t. the multiplicative set $A \backslash p$. Thus Σ_A is a locally ringed space.

(v) If $f : A \longrightarrow A'$ is a ring homomorphism, then there is an induced morphism of ringed spaces $(f^\star, \widehat{f}) : \Sigma_A \longrightarrow \Sigma_{A'}$ where, $f^\star : Spec(A') \longrightarrow Spec(A)$ is the induced spectral map and $\widehat{f} : \Sigma_A \longrightarrow (f^\star)_\bullet(\Sigma_{A'})$ is the natural transformation given on basic opens D_a and $D'_{f(a)} = (f^\star)^{-1}[D_a]$ by the induced homomorphism $A_a \longrightarrow A'_{f(a)}$, $x/a^n \mapsto f(x)/f(a)^n$. For each $p' \in Spec(A')$, $A_{f^\star(p')} \longrightarrow A'_{p'}$ is a local homomorphism. Thus $(f^\star, \widehat{f}) : \Sigma_A \longrightarrow \Sigma_{A'}$ is a morphism of locally ringed space. Moreover, the map $(A \xrightarrow{f} A') \mapsto \Sigma_A \xrightarrow{(f^\star, \widehat{f})} \Sigma_{A'}$ defines a functor $\Sigma : Rings \longrightarrow Locringed^{op}$.

(vi) If $\Gamma : Locringed^{op} \longrightarrow Rings$, denotes the global sections functor, i.e. $((X, F) \xrightarrow{(h,\tau)} (X', F')) \overset{\Gamma}{\mapsto} (F(X) \xrightarrow{\tau_X} F'(X') = F'(h^{-1}[X])$, then:

$*$ the family of canonical (iso)morphisms $s(A) : A \longrightarrow \Gamma(\Sigma_A)$, $A \in obj(Rings)$, determines a natural transformation $s : Id_{Rings} \longrightarrow \Gamma \circ \Sigma$ (and it is a natural isomorphism);

$*$ there is another family of (canonical) arrows $e_{(X,F)} = (h_F, \tau_F) : \Sigma_{\Gamma(X,F)} \longrightarrow (X, F)$, $(X, F) \in obj(Locringed)$, that determines a natural transformation

$$e : \Sigma \circ \Gamma \longrightarrow Id_{Locringed}$$

- $h_F : X \longrightarrow Spec(F(X))$, $x \mapsto \phi_{X,x}^{-1}[m_x]$; h_F is continuous since, for each $a \in F(X)$, $h_F^{-1}[D_a] = \{x \in X : \phi_{X,x}(a)$ is invertible in $F_x\} = W_a :=$ the largest open U such that $a_U := F_U^X(a) \in F(U)$ is invertible.
- $\tau_F : \Sigma_{F(X)} \longrightarrow (h_F)_\bullet(F)$ is the natural transformation that, for each $a \in F(X)$, $(\tau_F)_{D_a} : \Sigma_{F(X)}(D_a) \longrightarrow F(h_F^{-1}[D_a])$ that is naturally identified with the homomorphism $F(X)_a \longrightarrow F(W_a)$, obtained from the arrow $F_{W_a}^X : F(X) \longrightarrow F(W_a)$ by the universal property of the ring of fractions $F(X) \longrightarrow F(X)_a$; τ_F is uniquely determined by the condition above since $\Sigma_{F(X)}$ and $(h_F)_\bullet(F)$ are sheaves on $Spec(F(X))$ and $\{D_a : a \in F(X)\}$ is a basis of the space $Spec(F(X))$.

(vii) It is straightforward to check that the pair natural transformations (s, e) satisfies both triangular equations above ([**Mac**]):

$*$ $(\Sigma_A \xrightarrow{\Sigma(s(A))} \Sigma(\Gamma(\Sigma_A)) \xrightarrow{(e_{\Sigma_A})} \Sigma_A) = (\Sigma_A \xrightarrow{Id_{\Sigma_A}} \Sigma_A)$, $A \in obj(Rings)$;

$* \ (\Gamma(X,F) \stackrel{s(F(X))}{\longrightarrow} \Gamma(\Sigma_{F(X)}) \stackrel{\Gamma(e_{(X,F)})}{\longrightarrow} \Gamma(X,F)) = (\Gamma(X,F) \stackrel{Id_{F(X)}}{\longrightarrow} \Gamma(X,F))$, $(X,F) \in obj(Locringed)$.

(viii) ([**Mac**]) In this way, the affine scheme functor Σ is the left adjoint of the global sections functor Γ with s being the unit and e being the counit of this adjunction. Thus $s(A)$ has the universal property between all ring homomorphisms $A \longrightarrow \Gamma(X,F)$, for each locally ringed space (X,F). Moreover, since the unit of the adjunction $s : Id_{Rings} \longrightarrow \Gamma \circ \Sigma$ is a natural isomorphism, Σ is a *full and faithful* functor from the category of rings to the category of locally ringed spaces.

□

We will now recall some results from Commutative Algebra and develop some preparations.

16 Let A be a ring.

$*$ The *Prime Ideal Theorem*: let $S \subseteq A$ be a multiplicative submonoid and $I \subseteq A$ an ideal. Then $S \cap I = \emptyset$ iff $\exists p \in Spec(A)$ s.t. $I \subseteq P, S \cap P = \emptyset$.

$*$ Let $J \subseteq A$ be an ideal and denote $q_J^A : A \twoheadrightarrow A/J$ the canonical homomorphism onto que quotient ring. Then $(q_J^A)^\star : Spec(A/J) \stackrel{\approx}{\longrightarrow} Z_J \underset{closed}{\hookrightarrow} Spec(A)$. Recall that $Z_J = Z_{\sqrt{J}}$.

$*$ Let $a \in A$ and denote $\sigma_a^A : A \longrightarrow A_a$ the canonical homomorphism into the ring of fractions of $\{a^k : k \in \mathbb{N}\}$. Then $(\sigma_a^A)^\star : Spec(A_a) \stackrel{\approx}{\longrightarrow} D_a \underset{open}{\hookrightarrow} Spec(A)$.

$*$ Let $a \in A$ and let $I \subseteq A$ be a radical ideal ($I = \sqrt{I}$).
Denote $A_{a,I} := (A/I)_{a/I}$ the a/I-fractions ring of the quotient ring A/I.
Denote $t_{a,I}^A := \sigma_{a/I}^{A/I} \circ q_I^A : A \longrightarrow A_{a,I}$.
Then $ker(t_{a,I}^A) = \{x \in A : \exists n \in \mathbb{N}, x.a^n \in I\}$.
Denote $U_{a,I} := D_a \cap Z_I \subseteq Spec(A)$.
Then $(t_{a,I}^A)^\star : Spec(A_{a,I}) \stackrel{\approx}{\longrightarrow} U_{a,I} \hookrightarrow Spec(A)$. □

Lemma 17 *Let A be a ring.*

(a) Consider $a, a' \in A$ and $I, I' \subseteq A$ be radical ideals of A. Then the following are equivalent:

(i) $U_{a,I} \subseteq U_{a',I'}$;

(ii) Both the conditions $(ii)_Z$ and $(ii)_D$ below hold:

$(ii)_Z \quad U_{a,I} \cap D_{I'} = \emptyset \qquad (ii)_D \quad U_{a,I} \cap Z_{a'} = \emptyset$

(iii) Both the conditions $(iii)_Z$ and $(iii)_D$ below hold:

$(iii)_Z \quad I' \subseteq ker(t_{a,I}^A) \qquad (iii)_D \quad t_{a,I}^A(a') \in Unit(A_{a,I})$

(iv) There is an A-algebra homomorphism $h(A)_{a,I}^{a',I'} : A_{a',I'} \longrightarrow A_{a,I}$ i.e. $h(A)_{a,I}^{a',I'}$ is a ring homomorphism such that $h(A)_{a,I}^{a',I'} \circ t_{a',I'}^A = t_{a,I}^A$.

(b) Suppose that the equivalent conditions in (a) above hold. Then the A-algebra homomorphism $h_{a,\bar{b}}^{c,\bar{d}}$ *is* <u>*unique*</u>*. Moreover,* $\exists k \in \mathbb{N}\ \exists \lambda \in A\ \lambda.a' - a^k \in I$ *and* $\frac{(x/I')}{(a'^n/I')} \in A_{a',I'}$ $\overset{h_{a,I}^{a',I'}}{\mapsto}$ $\frac{(x\lambda^n/I)}{(a^{kn}/I)} \in A_{a,I}$.

(c) $h(A)_{a,I}^{a,I} = id_{A_{a,I}}$;
if $U_{a,I} \subseteq U_{a',I'} \subseteq U_{a'',I''}$*, then* $h(A)_{a,I}^{a',I'} \circ h(A)_{a',I'}^{a'',I''} = h(A)_{a,I}^{a'',I''}$;
if $U_{a,I} = U_{a',I'}$*, then* $h(A)_{a,I}^{a',I'}$ *and* $h(A)_{a',I'}^{a,I}$ *are a inverse pair of A-algebra isomorphisms.*

□

Proof. Item (c) follows directly from items (a) and (b).

(a) The equivalence between (i) *and* (ii) is clear.

$(ii)_Z \Leftrightarrow (iii)_Z$:

$U_{a,I} \cap D_{I'} \neq \emptyset$ iff $\exists p \in D_a \cap Z_I \cap D_{I'}$ iff $\exists p \in Spec(A) \exists x' \in I', I \subseteq p,$ *and* $a, x' \notin p$ iff $\exists p \in Spec(A) \exists x' \in I' \forall m, k \in \mathbb{N}, I \subseteq p, (a^m.x')^k \notin p$ iff $\exists x' \in I' \forall m \in \mathbb{N}, a^m.x' \notin I$ iff $I' \subsetneq ker(t_{a,I}^A)$

$(ii)_D \Leftrightarrow (iii)_D$:

$U_{a,I} \cap Z_{a'} \neq \emptyset$ iff $\exists p \in D_a \cap Z_I \cap Z_{a'}$ iff $\exists p \in Spec(A), \{a^k : k \in \mathbb{N}\} \cap p = \emptyset, (a') + I \subseteq p$ iff $\{a^k : k \in \mathbb{N}\} \cap (a') + I = \emptyset$ iff $\forall k \in \mathbb{N} \forall \lambda \in A, a^k - \lambda.a' \notin I$ iff[5] $\forall k, l \in \mathbb{N} \forall \lambda \in A, a^l.(a^k - \lambda.a') \notin I$ iff $t_{a,I}(a') \notin Unit(A_{a,I})$

$(iii) \Rightarrow (iv)$: by $(iii)_Z$, there is a unique homomorphism $\bar{t}_{a,I}^A : A/I' \longrightarrow A_{a,I}$ such that $\bar{t}_{a,I}^A \circ q_{I'}^A = t_{a,I}^A$ thus, by $(iii)_D$, $\bar{t}_{a,I}^A(a'/I') = t_{a,I}^A(a') \in Unit(A_{a,I})$, and then there is a unique homomorphism $h(A)_{a,I}^{a',I'} : A_{a',I'} \longrightarrow A_{a,I}$ such that $h(A)_{a,I}^{a',I'} \circ \sigma_{a'/I'}^{A/I'} = \bar{t}_{a,I}^A$. Composing with $q_{I'}^A : A \twoheadrightarrow A/I'$, we get $h(A)_{a,I}^{a',I'} \circ t_{a',I'}^A = t_{a,I}^A$.

$(iv) \Rightarrow (iii)$: Let $h(A)_{a,I}^{a',I'} : A_{a',I'} \longrightarrow A_{a,I}$ be a ring homomorphism such that $h(A)_{a,I}^{a',I'} \circ t_{a',I'}^A = t_{a,I}^A$. Define $f_{a,I}^{a'} := h(A)_{a,I}^{a',I'} \circ \sigma_{a'/I'}^{A/I'}$. Then $f_{a,I}^{a'}(a'/I') \in Unit(A_{a,I})$. Then $f_{a,I}^{a'} \circ q_{I'}^A = t_{a,I}^A$ and then $I' \subseteq ker(t_{a,I}^A)$, establishing $(iii)_Z$. Since $q_{I'}^A : A \twoheadrightarrow A/I'$ is surjective, $f_{a,I}^{a'} = \bar{t}_{a,I}^A$ and $h(A)_{a,I}^{a',I'}$ is the unique ring homomorphism such that $h(A)_{a,I}^{a',I'} \circ \sigma_{a'/I'}^{A/I'} = f_{a,I}^{a'}$. Thus ${t^A}_{a,I}(a') = \bar{t}_{a,I}^A(a'/I') = f_{a,I}^{a'}(a'/I') \in Unit(A_{a,I})$, establishing $(iii)_D$.

(b) The uniqueness of $h(A)_{a,I}^{a',I'}$ was established in the course of the proof of equivalence $(iii) \Leftrightarrow (iv)$ above. We leave it to the reader to check the correctness of the concrete description of $h_{a,I}^{a',I'}$ when acting on elements. ■

18 Let A be a ring. For each $a \in A, \bar{b} \subseteq_{fin} A$ consider:

[5] Since $U_{a,I} \neq \emptyset$.

(i) the quotient ring $A/I_{\bar{b}}$, where $I_{\bar{b}} := \sqrt{\sum_{i=1}^{i=k}(b_i)} \subseteq A$;

(ii) the fraction ring $A_{a,\bar{b}} := A_{a,I_{\bar{b}}} = (A/I_{\bar{b}})_{a/I_{\bar{b}}}$;

(iii) the canonical homomorphism $t^A_{a,\bar{b}} := t^A_{a,I_{\bar{b}}} : A \longrightarrow (A/I_{\bar{b}})_{a/I_{\bar{b}}}$ i.e. the composition of projection on the quotient $q^A_{\bar{b}} : A \twoheadrightarrow A/I_{\bar{b}}$ with the universal homomorphism of $a/I_{\bar{b}}$-fractions ring $\sigma_{A/I_{\bar{b}}} : A/I_{\bar{b}} \longrightarrow (A/I_{\bar{b}})_{a/I_{\bar{b}}}$.

(iv) if $a' \in A, \bar{b}' \subseteq_{fin} A$ is such that $U_{a,\bar{b}} \subseteq U_{a',\bar{b}'}$, then $h(A)^{a',\bar{b}'}_{a,\bar{b}} := h(A)^{a',\bar{b}'}_{a,I_{\bar{b}}} : A_{a',\bar{b}'} \longrightarrow A_{a,\bar{b}}$ is the unique ring homomorphism such that $h(A)^{a',\bar{b}'}_{a,\bar{b}} \circ t^A_{a',\bar{b}'} = t^A_{a,\bar{b}}$. Besides, $\exists k \in \mathbb{N}$ $\exists \lambda \in A$ $\lambda a' - a^k \in I_{\bar{b}}$ and $\frac{(x/I_{\bar{b}'})}{(a'^n/I_{\bar{b}'})} \in A_{a',\bar{b}'} \overset{h^{a',\bar{b}'}_{a,\bar{b}}}{\mapsto} \frac{(x\lambda^n/I_{\bar{b}})}{(a^{kn}/I_{\bar{b}})} \in A_{a,\bar{b}}$.

Note that:

(a) Since $Nil(A) \subseteq I_{\bar{b}}$, then $t^A_{a,\bar{b}}$ factors through $q_0 : A \twoheadrightarrow A/Nil(A)$ and $A_{a,\bar{b}}$ is an $A/Nil(A)$-algebra.

(b) $U_{a,\bar{b}} = Spec(A) \Leftrightarrow a \in Unit(A)$ and $\forall i < n, b_i \in Nil(A) \Rightarrow$ the canonical homomorphism $A/Nil(A) \longrightarrow A_{a,\bar{b}}$ is an isomorphism.

(c) $U_{a,\bar{b}} = \varnothing \Leftrightarrow \bigcap_{i<n} Z_{b_i} \subseteq Z_a \Leftrightarrow \exists k \in \mathbb{N}, a^k \in I_{\bar{b}} \Leftrightarrow A_{a,\bar{b}} = \{0\}$.

□

Construction 19 For each ring A we build a presheaf basis of A-algebras[6]. Keep the notation from Lemma 17. The work is done in two steps:

(i) We build a (contravariant) functor $P'(A) : (\beta(A), \subseteq)^{op} \longrightarrow A - alg$ (recall from paragraph 13 that $\beta(A)$ was a basis for the constructible topology). This is a "large" presheaf base of A-algebras:

For each $U \in \beta(A)$ consider the ring $P'(A)(U) := \prod\{A_{a,\bar{b}} : U = U_{a,\bar{b}}, a \in A, \bar{b} \subseteq_{fin} A\}$ and the ring homomorphism $t'(A)_U := (t^A_{a,\bar{b}})_{U_{a,\bar{b}}=U} : A \longrightarrow P'(A)(U)$. Let $U, V \in \beta(A)$ with $U \subseteq V$ and consider the ring homomorphism $h'(A)^V_U : P'(A)(V) \longrightarrow P'(A)(U)$, the unique ring homomorphism such that for each $a, \bar{b}, c, \bar{d}$ with $U = U_{a,\bar{b}}, V = U_{c,\bar{d}}$, $proj_{U_{(a,\bar{b})}} \circ h'(A)^V_U = h(A)^{c,\bar{d}}_{a,\bar{b}} \circ proj_{U_{(c,\bar{d})}}$; $h'(A)^V_U$ is an A-algebra homomorphism, i.e. $h'(A)^V_U \circ t'(A)_V = t'(A)_U$: to see that just take composition of these arrows with the projections $proj_{U_{(a,\bar{b})}} : P'(A)(U) \longrightarrow A_{a,\bar{b}}$ such that $U = U_{a,\bar{b}}$ and use Lemma 17.(a).(iv) . $P'(A)$ is indeed a (contravariant) functor, i.e. if $U, V, W \in \beta(A)$ is such that $U \subseteq V \subseteq W$ then $h'(A)^V_U \circ h'(A)^W_V = h'(A)^W_U$ and $h'(A)^U_U = id_{P'(A)(U)}$: this follows from Lemma 17.(c), taking compositions with appropriate projections.

(ii) We build a subfunctor $P(A) : (\beta(A), \subseteq)^{op} \longrightarrow A - alg$ of $P'(A)$. This will be a "good" presheaf base of A-algebras:

[6] In fact, by 18.(a), we will obtain a presheaf basis of $A/Nil(A)$-algebras.

For each $U \in \beta(A)$ consider the subring $P(A)(U) \overset{i(A)_U}{\hookrightarrow} P'(A)(U)$ given by $P(A)(U) := \{\vec{x} \in P'(A)(U) : h'(A)^{a',\bar{b}'}_{a,\bar{b}}(proj_{U_{(a',\bar{b}')}}(\vec{x})) = proj_{U_{(a,\bar{b})}}(\vec{x})$, for each $a, \bar{b}, a', \bar{b}'$ with $U = U_{a,\bar{b}} = U_{a',\bar{b}'}\}$. By Lemma 17.(a).(iv), the ring homomorphism $t'(A)_U : A \longrightarrow P'(A)(U)$ factors (uniquely) as $t'(A)_U = i(A)_U \circ t(A)_U$ with $t(A)_U : A \longrightarrow P(A)(U)$ is a ring homomorphism. Let $U, V \in \beta(A)$ with $U \subseteq V$, then the ring homomorphism $P(A)(V) \overset{i(A)_V}{\hookrightarrow} P'(A)(V) \overset{h'(A)^V_U}{\longrightarrow} P'(A)(U)$ factors (uniquely) as $P(A)(V) \overset{h(A)^V_U}{\longrightarrow} P(A)(U) \overset{i(A)_U}{\hookrightarrow} P'(A)(U)$ and $h(A)^V_U : P(A)(V) \longrightarrow P(A)(U)$ is a ring homomorphism; $h(A)^V_U$ is an A-algebra homomorphism, i.e. $h(A)^V_U \circ t(A)_V = t(A)_U$. To see that, just take composition of these arrows with the inclusion $i(A)_U : P(A)(U) \hookrightarrow P'(A)(U)$. It is easy to see that $P(A)$ is a contravariant functor and the family of inclusions $(i(A)_U : P(A)(U) \hookrightarrow P'(A)(U))_{U \in \beta(A)}$ gives a natural transformation $i(A) : P(A) \hookrightarrow P'(A)$.

□

Proposition 20 *Let A be a ring.*

(i) *For each $U \in \beta(A)$, $P(A)(U) = \{0\}$ iff $U = \varnothing$. $A/Nil(A) \overset{\cong}{\longrightarrow} P(A)(Spec(A))$.*

(ii) *For each $p \in Spec^{const}(A)$, we have that the stalk of $P(A)$ at p is*

$$P(A)_p := \varinjlim_{U \in \beta(A): p \in U} P(A)(U) \overset{\cong}{\longrightarrow} k_p(A).$$

(iii) *In general, $P(A)$ is not a sheaf basis, but it always is a monopresheaf basis.*

(iv) *If A is a vN-regular ring, then $Spec^{const}(A) = Spec(A)$ and $P(A)$ is a sheaf that is naturally isomorphic to Σ_A, the usual* structure sheaf of the affine scheme *associated to A.*

Proof.

(i) Since $P(A)(U_{a,\bar{b}}) \cong A_{a,\bar{b}}$, by 18.(c) we have that $P(A)(U_{a,\bar{b}}) = \{0\}$ iff $U_{a,\bar{b}} = \varnothing$.

Since $P(A)(Spec(A)) \cong A_{a,\bar{b}}$ whenever $U_{a,\bar{b}} = Spec(A)$, we have $A/Nil(A) \overset{\cong}{\longrightarrow} A_{a,\bar{b}} \cong P(A)(Spec(A))$ (see 18.(b)).

(ii) Let $p \in Spec(A)$. Recall the canonical isomorphism

$$k_p(A) := Res(A_p) = A_p/p.A_p \cong Frac(A/p) = A/p[A/p\backslash\{0\}]^{-1}.$$

(1) Let $a, b_1, \cdots, b_k \in A$ such that $p \in U_{a,\bar{b}} = D_a \cap Z_{I_{\bar{b}}}$. Then $I_{\bar{b}} \subseteq p$ and we have a unique A-algebra epimorphism $q^A_{I_{\bar{b}},p} : A/I_{\bar{b}} \twoheadrightarrow A/p$ such that $q^A_{I_{\bar{b}},p} \circ q^A_{I_{\bar{b}}} = q^A_p$. Since $a \notin p$, $a/p \in (A/p\backslash\{0\})$ and the canonical homomorphism $j^A_p : A/p \rightarrowtail A/p[A/p\backslash\{0\}]^{-1}$ is such that $j^A_p(q^A_p(a)) \in Unit(Frac(A/p))$. Define $f^A_{I_{\bar{b}},p} := j^A_p \circ q^A_{I_{\bar{b}},p} : A/I_{\bar{b}} \longrightarrow Frac(A/p)$. Then

clearly $f^A_{I_{\bar{b}},p}(a/I_{\bar{b}}) \in Unit(Frac(A/p))$. By the universal property of $\sigma^{A/I_{\bar{b}}}_{a/I_{\bar{b}}} : A/I_{\bar{b}} \longrightarrow A_{a,\bar{b}}$, we obtain a unique homomorphism $h^{a,\bar{b}}_p : A_{a,\bar{b}} \longrightarrow Frac(A/p)$ such that $h^{a,\bar{b}}_p \circ \sigma^{A/I_{\bar{b}}} = j^A_p \circ q^A_{I_{\bar{b}},p}$. Since $q^A_{I_{\bar{b}}}$ is surjective, we have:

(2) $h^{a,\bar{b}}_p : A_{a,\bar{b}} \longrightarrow Frac(A/p)$ is the unique homomorphism such that $h^{a,\bar{b}}_p \circ t^A_{a,\bar{b}} = j^A_p \circ q^A_p$. Moreover $\frac{x/I_{\bar{b}}}{a^n/I_{\bar{b}}} \in A_{a,\bar{b}} \overset{h^{a,\bar{b}}_p}{\mapsto} \frac{x/p}{a^n/p} \in Frac(A/p)$.

(3) If $c, d_1, \cdots, d_l \in A$ are such that $U_{a,\bar{b}} \subseteq U_{c,\bar{d}}$, then $h^{a,\bar{b}}_p \circ h(A)^{c,\bar{d}}_{a,\bar{b}} = h^{c,\bar{d}}_p$: this follows from (2) since, by 18.(iv), $h(A)^{c,\bar{d}}_{a,\bar{b}} : A_{c,\bar{d}} \longrightarrow A_{a,\bar{b}}$ is the unique homomorphism such that $h(A)^{c,\bar{d}}_{a,\bar{b}} \circ t^A_{c,\bar{d}} = t^A_{a,\bar{b}}$.

(4) It follows from (3) and Construction 19 that, for each $V \in \beta(A)$ such that $p \in V$, we have a unique (well defined) homomorphism $h^V_p : P(A)(V) \longrightarrow Frac(A/p)$ such that for each $c, d_1, \cdots, d_l \in A$ such that $V = U_{a,\bar{b}}$, $h^V_p = h^{c,\bar{d}}_p \circ proj_{c,\bar{d}}$. Moreover, if $U \in \beta(A)$ is such that $p \in U \subseteq V$, then $h^U_p \circ h(A)^V_U = h^V_p$.

(5) Denote $\beta_p(A) := \{U \in \beta(A) : p \in U\}$. By (4) we have a co-cone $(h^V_p)_{V \in \beta_p(A)}$ over the diagram $(P(A)(V) \overset{h(A)^V_U}{\longrightarrow} P(A)(U))_{U \subseteq V, U, V \in \beta_p(A)}$. Thus we have a unique homomorphism

$$h(A)_p : \underset{U \in \beta_p(A)}{\underrightarrow{lim}} \; P(A)(U) \longrightarrow Frac(A/p)$$

such that for each $U \in \beta_p(A)$, $h(A)_p \circ \phi_{U,p} = h^U_p : P(A)(U) \longrightarrow Frac(A/p)$, where

$$\phi_{V,p} : P(A)(V) \longrightarrow \underset{U \in \beta_p(A)}{\underrightarrow{lim}} \; P(A)(U), \qquad z \mapsto [(z, V)]$$

is the canonical arrow. We will show that $h(A)_p$ is an isomorphism.

(6) $h(A)_p$ is surjective:
Let $\frac{x/p}{c/p} \in A/p[A/p \backslash \{0\}]^{-1}$ and select representatives $x \in A$, $c \in A \backslash p$. Let $d_1 = 0$ and consider $V = U_{c,\bar{d}} = D_c \cap Z_0 = D_c$, then $V \in \beta_p(A)$ (and $I_{\bar{d}} = Nil(A) \subseteq p$). By (2), $\frac{x/I_{\bar{d}}}{c/I_{\bar{d}}} \in A_{c,\bar{d}} \overset{h^{c,\bar{d}}_p}{\mapsto} \frac{x/p}{c/p} \in Frac(A/p)$. Consider $z := (h^{c,\bar{d}}_{a,\bar{b}}(\frac{x/I_{\bar{d}}}{c/I_{\bar{d}}}))_{U_{a,\bar{b}}=V} \in P(A)(V)$. Then $h(A)_p([(z, V)]) = h^V_p(z) = h^{c,\bar{d}}_p(\frac{x/I_{\bar{d}}}{c/I_{\bar{d}}}) = \frac{x/p}{c/p} \in Frac(A/p)$, showing that $h(A)_p$ is surjective.

(7) $h(A)_p$ is injective:
Let $[(z, V)] \in ker(h(A)_p)$ for some $V \in \beta_p(A)$ and $z \in P(A)(V)$. Consider $c, d_1, \cdots, d_l$ such that $V = U_{c,\bar{d}}$ and let $\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}} \in A_{c,\bar{d}}$ such that $proj_{c,\bar{d}}(z) = \frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}$. Since $[(z, V)] \in ker(h(A)_p)$, we have $\frac{x/p}{c^n/p} = h^{c,\bar{d}}_p(\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}) = h^V_p(z) = h(A)_p([(z, V)] = 0 \in Frac(A/p)$. Thus $x \in p$ Let $a = c$ and $\bar{b} = \{d_1, \cdots, d_l\} \cup \{x\}$, then $p \in U_{a,\bar{b}} \subseteq U_{c,\bar{d}}$ and $h^{c,\bar{d}}a, \bar{b}(\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}) = \frac{(x\lambda^n/I_{\bar{b}})}{(a^{kn}/I_{\bar{b}})} \in A_{a,\bar{b}}$, for some $k \in \mathbb{N}$ and some $\lambda \in A$ such that $\lambda.a - a^k \in I_{\bar{b}}$. Let $V' := U_{a,\bar{b}}$

and $z' := (h^{a,\bar{b}}_{a',\bar{b}'}(\frac{(x\lambda^n/I_{\bar{b}})}{(a^{kn}/I_{\bar{b}})}))_{U_{a',\bar{b}'}=V'}$. Then $V', V \in \beta_p(A), V' \subseteq V$ and $h^V_{V'}(z) = z'$, thus $[(z', V')] = [(z, V)] \in \varinjlim_{U \in \beta_p(A)} P(A)(U)$. We will show that $z' = 0 \in P(A)(V')$, thus $[(z, V)] = 0 \in \varinjlim_{U \in \beta_p(A)} P(A)(U)$ and $ker(h(A)_p) = \{0\}$, as desired. To ensure that $z' = 0 \in P(A)(V')$, it is enough to prove that $\frac{(x\lambda^n/I_{\bar{b}})}{(a^{kn}/I_{\bar{b}})} = 0 \in A_{a,\bar{b}}$, i.e., that $\exists l \in \mathbb{N}, a^l x \lambda^n \in I_{\bar{b}}$: Since $x \in \bar{b}$, we have $x \in I_{\bar{b}}$ and we can choose any $l \in \mathbb{N}$.

(iii) Consider the following three facts:
- any sheaf of rings over a boolean space whose stalks are fields has a vN-regular ring as its global section ring (see Theorem 10.3 in [**Pie**], alternatively this follows from the facts that fields are vN-regular and that vN-regular rings are closed under limits by Prop. 6, since the ring of global sections can be expressed as a limit of a diagram of ultraproducts of the stalks [**Ken**, Lemma 2.5]);
- $P(A)$ is a presheaf basis over the boolean space $Spec^{const}(A)$ and its stalks are fields (item (ii) above);
- a sheaf basis (as opposed to a presheaf basis) of rings and its associated sheaf over any given space have isomorphic global section ring and isomorphic stalks (see Remark 23 below).

Thus, to see that, in general, $P(A)$ is not a sheaf basis of rings, it is enough remark that its global section ring is not a vN-regular ring in general, since $P(A)(Spec(A)) \cong A/Nil(A)$ (item (i) above). For this just take for A any domain that is not a field (see Remark 2).

Now we will show that $P(A) : (\beta(A), \subseteq)^{op} \longrightarrow A - alg$ is a *mono*presheaf basis. We have to prove that for any $U \in \beta(A)$ and $\{U_i : i \in I\} \subseteq \beta(A)$ such that $U = \bigcup_{i \in I} U_i$, the A-algebra homomorphism $(h(A)^U_{U_i})_{i \in I} : P(A)(U) \longrightarrow \prod_{i \in I} P(A)(U_i)$ is injective. Clearly it is enough to prove that for any $c \in A, \bar{d} \subseteq_{fin} A$ and any $a_i \in A, \bar{b_i} \subseteq_{fin} A$, $i \in I$ such that $U_{c,\bar{d}} = \bigcup_{i \in I} U_{a_i,\bar{b_i}}$, the A-algebra homomorphism $(h(A)^{c,\bar{d}}_{a_i,\bar{b_i}})_{i \in I} : A_{c,\bar{d}} \longrightarrow \prod_{i \in I} A_{a_i,\bar{b_i}}$ is injective. Suppose that there is $\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}} \neq 0 \in A_{c,\bar{d}}$ such that $h(A)^{c,\bar{d}}_{a_i,\bar{b_i}}(\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}) = 0 \in A_{a_i,\bar{b_i}}$, $\forall i \in I$. From this we will derive a contradiction in the following four steps:

(1) $\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}} \neq 0 \in A_{c,\bar{d}}$ iff $\forall k \in \mathbb{N}, c^k \cdot x \notin I_{\bar{d}}$ iff $\exists p \in Z_{I_{\bar{d}}}, c^k \cdot x \notin p$

(2) $U_{a_i,\bar{b_i}} \subseteq U_{c,\bar{d}} \Rightarrow \exists k_i \in \mathbb{N} \exists \lambda_i \in A, \lambda_i \cdot c - a_i^{k_i} \in I_{\bar{b_i}}$ and $h(A)^{c,\bar{d}}_{a_i,\bar{b_i}}(\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}) = \frac{x.\lambda_i^n/I_{\bar{b_i}}}{a_i^{k_i n}/I_{\bar{b_i}}}$.

(3) $h(A)^{c,\bar{d}}_{a_i,\bar{b_i}}(\frac{x/I_{\bar{d}}}{c^n/I_{\bar{d}}}) = 0 \in A_{a_i,\bar{b_i}}$ iff $\exists l_i \in \mathbb{N}, (a_i^{k_i n})^{l_i} \cdot x \cdot \lambda_i^n \in I_{\bar{b_i}}$ iff $\forall q \in Spec(A), (q \in Z_{I_{\bar{b_i}}} \Rightarrow \exists l_i \in \mathbb{N}, (a_i^{k_i n})^{l_i} \cdot x \cdot \lambda_i^n \in q$

(4) According to (1), we can choose $p \in Spec(A)$ such that $p \in Z_{I_{\bar{d}}}, c^k.x \notin p$. Then $p \in Z_{I_{\bar{d}}} \cap D_c = U_{c,\bar{d}} = \bigcup_{i \in I} U_{a_i,\bar{b_i}}$, thus there is $i \in I$ such that $p \in Z_{\bar{b_i}} \cap D_{a_i}$. Applying (2), $\exists k_i \in \mathbb{N} \exists \lambda_i \in A, \lambda_i.c - a_i^{k_i} \in p$. Applying (3), $\exists l_i \in \mathbb{N}, (a_i^{k_i n})^{l_i}.x.\lambda_i^n \in p$. Since $p \in D_{a_i}$

and $x \notin p$, we must have $\lambda_i \in p$ and, since $\lambda_i.c - a_i^{k_i} \in p$, we conclude that $a_i \in p$, contradicting $p \in Z_{\bar{b}_i} \cap D_{a_i}$. This finishes the proof of item (iii)

(iv) Let A be a vN-regular ring:

(1) By Fact 14.(i), for each $x \in A$ there is a *unique* idempotent $e_x \in B(A)$ such that $(x) = (e_x)$.

(2) Therefore every ideal $I \subseteq A$ in A is radical ($\sqrt{I} = I$): for each $x \in A$ we have $x \in I$ iff for $e_x \in I$ iff $\exists n \in N$ ($e_x^n \in I$) iff $\exists n \in N$ ($x^n \in I$) iff $x \in \sqrt{I}$.

(3) By the result (2) above and Remark 15.(i), it follows that $D_a = D_c$ iff $(a) = (c)$.

(4) Let $a, b_0, \ldots, b_{n-1} \in A$ and denote $e, f_0, \ldots, f_{n-1} \subseteq B(A)$ their (uniquely given) corresponding idempotents (so $(a) = (e)$ and $(b_i) = (f_i)$, $i < n$) and write $f := \bigvee_{i<n} f_i$. Then:
$U_{a,\bar{b}} := D_a \cap \bigcap_{i<n} Z_{b_i} = D_e \cap \bigcap i < n Z_{f_i} = D_e \cap \bigcap_{i<n} D_{1-f_i} = D_{e.(\prod_{i<n} 1-f_i)} = D_{e \wedge (1-\bigvee_{i<n} f_i)} = D_{e \wedge f*} = D_{e \wedge f*} \cap Z_0$;

(5) By results (3), (4) above and by Fact 14.(i), for each $U \in \beta(A)$ there is a *unique* $g^U \in B(A)$ such that $U = D_{g^U} = D_{g^U} \cap Z_0$ (6) For each $U \in \beta(A)$, by result (5) above, $P(A)(U) \cong (A/\sqrt{(0)})_{g^U/\sqrt{(0)}} = (A/(0))_{g^U/(0)} \cong A_{g^U} \cong \Sigma_A(U)$. It can be proved that all those isomorphisms are compatible so they give an isomorphism $\gamma(A)_U : P_A(U) \xrightarrow{\cong} \Sigma_A(U)$.

(7) By result (4) and Fact 14.(i), $U_{a,\bar{b}} \subseteq U_{a',\bar{b}'}$ iff $D_{e \wedge f*} \subseteq D_{e' \wedge f'*}$ iff $e \wedge f^* \leqslant e' \wedge f'^*$.

(8) By results (6), (7) above, if $U, U' \in \beta(A)$, $U \subseteq U'$, then:

$$(P_A(U') \overset{h(A)_U^{U'}}{\longrightarrow} P_A(U) \overset{\gamma(A)_U}{\longrightarrow} \Sigma_A(U)) = (P_A(U') \overset{\gamma(A)_{U'}}{\longrightarrow} \Sigma_A(U') \overset{\sigma(A)_U^{U'}}{\longrightarrow} \Sigma_A(U)).$$

(9) By results (6), (8) above: $\gamma(A) : P(A) \xrightarrow{\cong} \Sigma_A$.

■

Denote *Boofield* the full subcategory of *Locringed* whose objects are the sheaves of rings over boolean spaces whose stalks are fields. In Theorem 10.3 in [**Pie**] is shown that the ring of global sections of each sheaf in *Boofield* is a vN-regular ring. On the other hand, Proposition 5.6 in [**DM7**], provides an(other) explicit proof that the affine scheme of a vN-regular ring is a sheaf in *Boofield*. A natural question suggested by item (iv) of Proposition 20 above is ask if, in general, any sheaf in *Boofield* is $Boofield-isomorphic$ to the affine sheaf of the ring of global sections. The answer is the content of the following:

Proposition 21 **(i)** *Let X be a boolean space and $F : (Open(X), \subseteq)^{op} \longrightarrow Ring$ be a sheaf of rings over X such that for each $x \in X$, $F_x := \underset{x \in U \in Open(X)}{\underrightarrow{lim}} F(U)$ is a field. Then $e_F := (h_F, \tau_F) : \Sigma_{\Gamma(F)} \longrightarrow F$ (see Remark 15.(vi)) is a $Boofield-$*

isomorphism. In more details:
(a) $\Gamma(F) := F(X)$ *is a vN-regular ring ([**Pie**], Theorem 10.3), thus* $\Sigma_{F(X)} \in obj(Boofield)$ *([**DM7**], Proposition 5.6);*
(b) $h_F : X \longrightarrow Spec(F(X))$ *is a homeomorphism of boolean spaces;*
(c) $\tau_F : \Sigma_{F(X)} \longrightarrow (h_F)_\bullet(F)$ *is a natural isomorphism of sheaves over* $Spec(F(X))$.

(ii) *The adjunction* $(\Sigma, \Gamma, s, e) : Rings \rightleftarrows Locringed$*, described Remark 15, restricts to an adjunction equivalence* $(\Sigma, \Gamma, s, e) : Regrings \rightleftarrows Boofield$*; in particular, the categories* $Regrings, Boofield$ *are equivalent.*

Proof. Item **(ii)** is a direct consequence of item **(i)**, since we already have that for *every* ring A, $s(A) : A \xrightarrow{\cong} \Gamma(\Sigma_A)$ (see items (iv) and (viii) in Remark 15).

(i) (a) We just present an alternative proof that $F(X)$ is a vN-regular ring. By Proposition 3.2.(d) in [**DM7**], for each $C \in Clopen(X)$, the mapping $(\phi_{C,x})_{x\in C} : F(C) \longrightarrow \prod_{x\in C} F_x$ reflects the satisfiability of geometrical formulas, it is pure in particular. As fields are vN-regular rings, products of vN-regular rings are vN-regular rings and pure subrings of vN-regular rings are vN-regular rings (see Remark 3), then it follows that $F(C)$ is a vN-regular ring; in particular, $\Gamma(F) = F(X)$ is a vN-regular ring.

(b) Since $F(X)$ is a vN-regular ring, $Spec(F(X))$ is a boolean space. By Stone duality, to show that $h_F : X \longrightarrow Spec(F(X))$ is a homeomorphism of boolean spaces, is equivalent to show that $(h_F)^{-1} : Clopen(Spec(F(X))) \longrightarrow Clopen(X)$ is a boolean algebra isomorphism. Since $F(X)$ is a vN-regular ring, the mapping $j_{F(X)} : B(F(X)) \longrightarrow Clopen(Spec(F(X)))$, given by $e \mapsto D_e$, is a boolean algebra isomorphism (see Fact 14.(i)). Thus, it is enough to provide an inverse map to the BA-homomorphism $(h_F)^{-1} \circ j_{F(X)} : B(F(X)) \longrightarrow Clopen(X)$, $e \mapsto W_e =$ the largest open subset U of X such that $F^X_U(e)$ is invertible in $F(U)$; note that $W_{1-e} = X \backslash W_e$, thus W_e is a clopen.

Let $C \in Clopen(X)$ and denote $C' := X \backslash C$; thus $\{C, C'\}$ is a disjoint (cl)open cover of X. Since F is a sheaf of rings, we have $F(\emptyset) = \{0\}$ and $(F^X_C, F^X_{C'}) : F(X) \xrightarrow{\cong} F(C) \times F(C')$. Denote by $e(C), e(C') \in F(X)$ the elements that under this isomorphism correspond to, respectively, $(1_C, 0_{C'})$ and $(0_C, 1_{C'})$. Clearly $e(C), e(C') \in B(F(X))$, $e(C) \cdot e(C') = 0$ and $e(C) + e(C') = 1$.

We will show that the mapping $E_X : Clopen(X) \longrightarrow B(F(X))$, $C \mapsto e(C)$, is the inverse of $(h_F)^{-1} \circ j_{F(X)}$.

$*$ $E_X \circ (h_F)^{-1} \circ j_{F(X)} = id_{B(F(X))}$:

Let $e \in B(F(X))$ and denote $e' := 1 - e \in B(F(X))$. We must show that $F^X_{W_e}(e) = 1_{W_e}$ and $F^X_{W_{e'}}(e) = 0_{W_{e'}}$. But $F^X_{W_e}(e)$ is invertible and idempotent in $F(W_e)$, thus $F^X_{W_e}(e) = 1_{W_e}$; likewise $1_{W_{e'}} = F^X_{W_{e'}}(e') = F^X_{W_{e'}}(1-e)1_{W_{e'}} - F^X_{W_{e'}}(e)$, thus $F^X_{W_{e'}}(e) = 0_{W_{e'}}$.

$*$ $(h_F)^{-1} \circ j_{F(X)} \circ E_X = id_{Clopen(X)}$:

Let $C \in Clopen(X)$ and denote $C' := X \backslash C \in Clopen(X)$. We must show that $C = W_{e(C)}$. By the definitions, $F^X_C(e(C)) = 1_C$ is invertible in $F(C)$, thus $C \subseteq W_{e(C)}$;

likewise $C' \subseteq W_{e(C')} = W_{1-e(C)} = X \backslash W_{e(C)}$. In this way $\{C, C'\}$ and $\{W_{e(C)}, W_{e(C')}\}$ are both disjoint (cl)open covers of X the former one refining the latter: thus $C = W_{e(C)}$.

(c) Since $\tau_F : \Sigma_{F(X)} \longrightarrow (h_F)_\bullet(F)$ is a natural transformation of sheaves over $Spec(F(X))$, to prove that τ_F is a natural *isomorphism*, it is enough to show that $(\tau_F)_U : \Sigma_{F(X)}(U) \longrightarrow F(h_F^{-1}[U])$ is a ring isomorphism, for each U in some open basis of X that is closed under finite intersections. In Remark 15.(vi), we saw that, for each $a \in F(X)$, $(\tau_F)_{D_a} : \Sigma_{F(X)}(D_a) \longrightarrow F(h_F^{-1}[D_a])$ is naturally identified with the homomorphism $F(X)_a \longrightarrow F(W_a)$, obtained from the arrow $F^X_{W_a} : F(X) \longrightarrow F(W_a)$ by the universal property of the ring of fractions $F(X) \longrightarrow F(X)_a$.

Since $F(X)$ is a vN-regular ring, $(a) = (e)$ for a unique $e \in B(F(X))$. Therefore $D_a = D_e$, $W_a = W_e$ and $F(X)_a \overset{\sigma^a_{1-e}}{\cong} F(X)_e$. Thus we only need consider the homomorphisms $F(X)_e \longrightarrow F(W_e)$, obtained from the arrows $F^X_{W_e} : F(X) \longrightarrow F(W_e)$ by the universal property of the ring of fractions $F(X) \longrightarrow F(X)_e$, $e \in B(F(X))$.

Since $\{W_e, W_{e'}\}$ is a disjoint (cl)open cover of X and F is a sheaf, we have an isomorphism $(F^X_{W_e}, F^X_{W_{e'}}) : F(X) \overset{\cong}{\longrightarrow} F(W_e) \times F(W'_e)$. Thus $F^X_{W_e} : F(X) \longrightarrow F(W_e)$ can be identified with the canonical epimorphism $F(X) \twoheadrightarrow F(X) \cdot e$. Let $e' := 1 - e$. By Fact 14.(i), we have the canonical inverse isomorphisms of $F(X)$-algebras $F(X)_e \cong F(X)/(e') \cong F(X) \cdot e$. Thus $F^X_{W_e} : F(X) \longrightarrow F(W_e)$ can be identified with the canonical (epi)morphism of fractions $F(X) \twoheadrightarrow F(X)_e$, then $F(X)_e \longrightarrow F(W_e)$ is an isomorphism, finishing the proof. ■

The association $A \mapsto P(A)$ extends to a functor from the category of rings to the category of presheaves (or presheaf bases) over variable spaces. Keeping the notations in 17, 19, this is content of the following:

Fact 22 *Let $f : A \longrightarrow A'$ a ring homomorphism.*

(i) $f^\star : Spec^{const}(A') \longrightarrow Spec^{const}(A)$ is a spectral map such that $(f^\star)^{-1} : \beta(A) \longrightarrow \beta(A') : U_{a,\bar{b}} \mapsto U'_{f(a),f(\bar{b})}$. If $f = id(A)$ then $(f^\star)^{-1} = id(\beta(A))$. If $f' : A' \longrightarrow A''$ is a ring homomorphism then $((f' \circ f)^\star)^{-1} = (f'^\star)^{-1} \circ (f^\star)^{-1}$.

(ii) Consider $a \in A$, $\bar{b} \subseteq_{fin} A$. There is a unique ring homomorphism $f_{a,\bar{b}} : A_{a,\bar{b}} \longrightarrow A'_{f(a),f(\bar{b})}$ such that $f_{a,\bar{b}} \circ t^A_{a,\bar{b}} = t^{A'}_{f(a),f(\bar{b})} \circ f$. Moreover:

- *if $c \in A$, $\bar{d} \subseteq_{fin} A$ is such that $U_{a,\bar{b}} \subseteq U_{c,\bar{d}}$, then $f_{a,\bar{b}} \circ h(A)^{c,\bar{d}}_{a,\bar{b}} = h(A')^{f(c),f(\bar{d})}_{f(a),f(\bar{b})} \circ f_{c,\bar{d}}$;*
- *if $f = id_A$, then $f_{a,\bar{b}} = id_{A_{a,\bar{b}}}$;*
- *if $f' : A' \longrightarrow A''$ is a ring homomorphism, then $(f' \circ f)_{a,\bar{b}} = f'_{f(a),f(\bar{b})} \circ f_{a,\bar{b}}$.*

(iii) There is a canonical natural transformation $\check{f} : P(A) \longrightarrow (f^\star)_\bullet P(A')$ of contravariant functors over $(\beta(A), \subseteq)$, where $(f^\star)_\bullet P(A')$ is the direct image presheaf base *of $P(A')$ under $f^\star$. Moreover,*

$$(A \overset{f}{\longrightarrow} A') \;\mapsto\; (P(A) \overset{(f^\star, \check{f})}{\longrightarrow} P(A'))$$

is a functorial association. More explicitly:
For $U \in \beta(A)$ *there is a unique ring homomorphism* $\check{f}_U : P(A)(U) \longrightarrow P(A')((f^\star)^{-1}[U])$ *such that* $\check{f}_U \circ t(A)_U = t(A')_{(f^\star)^{-1}[U]} \circ f$ *and with the following properties:*
$*$ *if* $V \in \beta(A)$ *is such that* $U \subseteq V$*, then* $\check{f}_U \circ h(A)_U^V = h(A')_{(f^\star)^{-1}[U]}^{(f^\star)^{-1}[V]} \circ \check{f}_V$*;*
$*$ *if* $f = id_A$*, then* $\check{f}_U = id_{P(A)(U)}$ *;*
$*$ *if* $f' : A' \longrightarrow A''$ *is a ring homomorphism then* $\widecheck{(f' \circ f)}_U = \check{f'}_{(f^\star)^{-1}[U]} \circ \check{f}_U$. □

Remark 23 On the associated sheaf of a presheaf basis: ([**EGA**])
Let $B \subseteq Open(X)$ be a basis of the topology. For each $U \in Open(X)$ define $B(U) := \{D \in B : D \subseteq U\}$; if $U \subseteq U'$ are open sets note that $B(U) \subseteq B(U')$.

Consider a "presheaf basis" F of rings defined on the basis B (i.e., a contravariant functor $F : (B, \subseteq)^{op} \longrightarrow Rings$) and a "sheaf basis" G of rings defined on the basis B (i.e., G is a presheaf basis and satisfies the condition that any compatible family of sections has a unique gluing). Then:

(i) As the stalks of the presheaf basis F on B are determined, there is a sheaf basis $\tilde{F} : (B, \subseteq)^{op} \longrightarrow Rings$ and a natural transformation $\sigma_F : F \longrightarrow \tilde{F}$, satisfying the universal property that characterizes the construction up to unique isomorphism.

(ii) If $U \subseteq U'$ are open sets, define $\widehat{G}(U) := \underset{D \in B(U)}{\underleftarrow{lim}} G(D)$ and $\widehat{G}(U \hookrightarrow U') : \underset{D' \in B(U')}{\underleftarrow{lim}} G(D') \longrightarrow \underset{D \subseteq U}{\underleftarrow{lim}} G(D)$ is the projection homomorphism.

(iii) $\widehat{G}$ is a sheaf on $open(X)$. If $D \in B$, then $\widehat{G}(D) \cong G(D)$ (through the canonical projection): This characterizes the construction $G \mapsto \widehat{G}$ up to unique isomorphism.

(iv) The mapping $F \mapsto S(F) := \widehat{\tilde{F}}$ satisfies a universal property that characterizes the construction up to unique isomorphism. Moreover, F, $\tilde{F}$ and $\widehat{\tilde{F}}$ have (canonically) isomorphic stalks.

□

As the *direct image* of a sheaf under a continuous function is a sheaf (over the codomain space), the following result is a direct consequence of the universal property of the associated sheaf in Remark 23 and Fact 22.(iii).

Fact 24

(i) *Let A be a ring. Denote by $S(A) : (Open(Spec^{const}(A)), \subseteq)^{op} \longrightarrow A - alg$ the* associated sheaf *of the presheaf base $P(A) : (\beta(A), \subseteq)^{op} \longrightarrow A - alg$. For each $U \in Open(Spec^{const}(A))$, denote $\theta(A)_U : A \longrightarrow S(A)(U)$ the ring homomorphism that encodes the A-algebra structure on the ring $S(A)(U)$; denote $\eta_A := \theta(A)_{Spec^{const}(A)} : A \longrightarrow S(A)(Spec^{const}(A))$.*

(ii) *Let $f : A \longrightarrow A'$ be a ring homomorphism and still denote by $S(A)$ and $S(A')$ the sheaves with codomain in the category $Rings$. There is a canonical natural transformation $\check{f} : S(A) \longrightarrow (f^\star)_\bullet S(A')$ of contravariant functors over $(Open(Spec^{const}(A)), \subseteq)$, where $(f^\star)_\bullet S(A')$ is the* direct image (pre)sheaf *of $S(A')$ under $f^\star$.*

Moreover, $(A \xrightarrow{f} A') \mapsto (S(A) \xrightarrow{(f^\star, \check{f})} S(A'))$ is a functorial association. More explicitly:

Consider $U \in Open(Spec^{const}(A))$, then there is a unique ring homomorphism $\check{f}_U : S(A)(U) \longrightarrow S(A')((f^\star)^{-1}[U])$ such that $\check{f}_U \circ \theta(A)_U = \theta(A')_{(f^\star)^{-1}[U]} \circ f$. Moreover:
$*$ *if $V \in \beta(A)$ is such that $U \subseteq V$, then $\check{f}_U \circ s(A)^V_U = s(A')^{(f^\star)^{-1}[V]}_{(f^\star)^{-1}[U]} \circ \check{f}_V$;*
$*$ *if $f = id_A$, then $\check{f}_U = id_{P(A)(U)}$;*
$*$ *if $f' : A' \longrightarrow A''$ is a ring homomorphism, then $\widecheck{(f' \circ f)}_U = \check{f'}_{(f^\star)^{-1}[U]} \circ \check{f}_U$.*

□

Consider now the following:

Construction 25

- Let A be a ring. Define $R(A) := \Gamma(S(A)) = S(A)(Spec^{const}(A))$: Note that $R(A) \in obj(RegRings)$ because it is given by the global sections of a sheaf of rings over a boolean space whose stalks are fields.

- Let $f : A \longrightarrow A'$ be a ring homomorphism. Define

$$R(f) := \check{f}_{Spec^{const}(A)} : S(A)(Spec^{const}(A)) \longrightarrow S(A')(Spec^{const}(A')).$$

  Note that $f^{\star -1}[Spec^{const}(A)] = Spec^{const}(A')$.

- By Fact 24.(ii) above, these mappings determine a functor $R : Rings \longrightarrow RegRings$.

- For each ring A, consider the ring homomorphism $\eta_A : A \longrightarrow R(A)$ described in Fact 24.(i) above. Then, by Fact 24.(ii), $(\eta_A)_{A \in obj(Rings)}$ defines a the natural transformation $\eta : Id_{Rings} \longrightarrow i \circ R$, where i is the inclusion functor $i : RegRings \hookrightarrow Rings$.

□

Finally we are ready to state and prove the following:

Theorem 26 *The inclusion functor $i : RegRings \hookrightarrow Rings$ has a left adjoint given by the functor $R : Rings \longrightarrow RegRings$ and the natural transformation $\eta = (\eta_A)_{A \in obj(Rings)}$ is the unit of this adjunction.*

Proof. We will check that the conditions **(E)**, **(U)**, **(U')** in Proposition 12 are satisfied.

(E) Let V be a vN-regular ring. By Proposition 20.(iv) we have $Spec^{const}(V) = Spec(V)$ and $\Sigma_V \xrightarrow{\cong} P(V)$ thus, in particular, $P(V) \xrightarrow{\cong} S(V)$ and $V \xrightarrow{\cong} \Sigma(V)(Spec(V)) \xrightarrow{\cong} P(V)(Spec(V))$ (see Remark 15). Keeping track of the former isomorphisms, we can conclude that $\eta_V : V \longrightarrow S(V)(Spec^{const}(V))$ is an isomorphism, establishing (E).

By Proposition 21, $h_{S(A)} : Spec^{const}(A) \xrightarrow{\approx} Spec(R(A))$ and, for each $U \in Open(Spec(R(A)))$, we have $\tau_{S(A)}(U) : \Sigma_{R(A)}(U) \xrightarrow{\cong} S(A)(h^{-1}_{S(A)}[U])$.

(U) A diagram chase shows that

$$(Spec^{const}(A) \xrightarrow{h_{S(A)}} Spec(cR(A)) \xrightarrow{\eta_A^\star} Spec(A)) = (Spec^{const}(A) \xrightarrow{id} Spec(A))$$

Thus, since $h_{S(A)}$ is a homeomorphism, we conclude that $(\eta_A)^\star : Spec(R(A)) \xrightarrow{\approx} Spec(A)$ is a spectral bijection, establishing (U).

(U') By Proposition 20.(ii) and Remark 23.(iv), for each $p \in Spec^{const}(A)$, $P(A)_p \cong k_p(A) \cong S(A)_p$. As $h_{S(A)}(p) = ker(R(A) \twoheadrightarrow S(A)_p)$, $(\tau_{S(A)})_p : (\Sigma_{R(A)})_{h_{S(A)}(p)} \xrightarrow{\cong} S(A)_p$ and $(\Sigma_{R(A)})_{h_{S(A)}(p)} \cong \Sigma_{R(A)}/h_{S(A)}(p) \cong k_{h_{S(A)}(p)}(R(A))$ (since $R(A)$ is vN-regular), we get an isomorphism $k_p(A) \xrightarrow{\cong} k_{h_{S(A)}(p)}(R(A))$.

Keeping track of the former isomorphisms and by the above proof of (U), we can conclude that, for each $q \in Spec(R(A))$, $\widehat{\eta_A}_q : k_{\eta_A^\star(q)}(A) \xrightarrow{\cong} k_q(R(A))$, establishing (U').
■

Corollary 27 *The functor R preserves all colimits. In particular it preserves:*

1. *directed inductive limits;*
2. *coproducts (= tensor products in Rings);*
3. *coequalizers/quotients.*

Proof. Since it is a left adjoint, R preserves all colimits. We explain the meaning of the preservation of quotients. Consider the induced homomorphism $R(q_I)$:

$R(A) \twoheadrightarrow R(A/I)$: it is surjective since the coequalizers in *Rings* and *RegRings* coincide with the surjective homomorphisms. Then $\bar{I} := ker(R(q_I)) \subseteq R(A)$ is such that $\overline{R(q_I)} : R(A)/\bar{I} \xrightarrow{\cong} R(A/I)$ and, since $R(A) \rightarrowtail \prod_{p \in Spec(A)} k_p(A)$, $\bar{I}$ can be identified with $R(A) \cap \{\vec{x} = (x_p)_{p \in Spec(A)} \in \prod_{p \in Spec(A)} k_p(A) : \forall p \supseteq I, x_p = 0 \in k_p(A)\}$. Note in particular, that if $I \subseteq Nil(A)$, then $R(q_I) : R(A) \xrightarrow{\cong} R(A/I)$, thus $\bar{I} = \{0\}$. ■

The following results are specific to the functor R, i.e., they are not general consequences of it being a left adjoint functor.

Proposition 28 *R preserves localizations. More precisely, given a ring A and a multiplicative submonoid $S \subseteq A$, denote $S' := \eta_A[S] \subseteq R(A)$ the corresponding multiplicative submonoid and let $\eta_A^S : A[S]^{-1} \longrightarrow R(A)[S']^{-1}$ be the induced arrow, i.e., η_A^S is the unique homomorphism such that $\eta_A^S \circ \sigma(A)_S = \sigma(R(A))_{S'} \circ \eta_A$. Then $\eta_{A[S]^{-1}} : A[S]^{-1} \longrightarrow R(A[S]^{-1})$ thus it is isomorphic to the arrow η_A^S, through the obvious pair of inverse (iso)morphisms $R(A[S]^{-1}) \rightleftarrows R(A)[S']^{-1}$.*

Proof. First of all, note that $R(A)[S']^{-1}$ is a vN-regular ring, cf. Prop. 5. For each vN-regular ring V, the bijection $(- \circ \eta_A) : RegRings(R(A), V) \xrightarrow{\cong} Rings(A, V)$ restricts to the bijection

$(- \circ \eta_A)_\restriction : \{H \in RegRings(R(A), V) : H[S'] \subseteq Unit(V)\}$

. $\xrightarrow{\cong} \{h \in Rings(A, V) : h[S] \subseteq Unit(V)\}$.

Composing the last bijection with the bijections below, obtained from the universal property of localizations,

$$(- \circ \sigma(A)_S)^{-1} : \{h \in Rings(A, V) : h[S] \subseteq Unit(V)\} \xrightarrow{\cong} Rings(A[S]^{-1}, V),$$

$$(- \circ \sigma(R(A))_{S'}) : Rings(R(A)[S']^{-1}, V) \xrightarrow{\cong} \{H \in Rings(R(A), V) : H[S'] \subseteq Unit(V)\},$$

we obtain, since *RegRings* is a full subcategory of *Rings*, the bijection

$$(- \circ \eta_A^S) : RegRings(R(A)[S']^{-1}, V) \xrightarrow{\cong} Rings(A[S]^{-1}, V).$$

Summing up, the arrow $\eta_A^S : A[S]^{-1} \longrightarrow R(A)[S']^{-1}$ satisfies the universal property of vN-regular hull, thus it is isomorphic to the arrow $\eta_{A[S]^{-1}} : A[S]^{-1} \longrightarrow R(A[S]^{-1})$. ■

Proposition 29 *R preserves* finite *products. More precisely, let I be a finite set and $\{A_i : i \in I\}$ any family of rings. Denote $\pi_j : \prod_{i \in I} A_i \twoheadrightarrow A_j$ the projection homomorphism, $j \in I$. Then $\prod_{i \in I} \eta_{A_i} : \prod_{i \in I} A_i \longrightarrow \prod_{i \in I} R(A_i)$ satisfies the universal property of vN-regular hull, thus it is isomorphic to the arrow $\eta_{\prod_{i \in I} A_i} : \prod_{i \in I} A_i \longrightarrow R(\prod_{i \in I} A_i)$, through the obvious pair of inverse (iso)morphisms $R(\prod_{i \in I} A_i) \rightleftarrows \prod_{i \in I} R(A_i)$.*

Proof. (Sketch) First of all, note that $\prod_{i\in I} R(A_i)$ is a vN-regular ring. If $I = \varnothing$, then $\prod_{i\in I} A_i$, $\prod_{i\in I} R(A_i)$ and $R(\prod_{i\in I} A_i)$ are isomorphic to the trivial ring $\{0\}$, thus the result holds in this case.

By induction, we only need to see that $(R(\pi_1), R(\pi_2)) : R(A_1 \times A_2) \xrightarrow{\cong} R(A_1) \times R(A_2)$. But we have:

- $[\pi_1^\star, \pi_2^\star] : Spec(A_1) \sqcup Spec(A_2) \xrightarrow{\approx} Spec(A_1 \times A_2)$, $(p_i, i) \mapsto \pi_i^{-1}[p_i]$, $i = 1, 2$.
- $[\pi_1^\star, \pi_2^\star] : Spec^{const}(A_1) \sqcup Spec^{const}(A_2) \xrightarrow{\approx} Spec^{const}(A_1 \times A_2)$.
- Let $a := (a_1, a_2) \in A_1 \times A_2$ and $\bar{b} := \{(b_{1,1}, b_{2,1}), \cdots, (b_{1,n}, b_{2,n})\} \subseteq_{fin} A_1 \times A_2$, then $(A_1 \times A_2)_{a,\bar{b}} \cong A_{1_{a_1,\bar{b}_1}} \times A_{2_{a_2,\bar{b}_2}}$.
- $\sqcup_{p\in Spec(A_1\times A_2)} k_p(A_1 \times A_2) \approx (\sqcup_{p_1\in Spec(A_1)} k_{p_1}(A_1)) \sqcup (\sqcup_{p_2 \in Spec(A_1)} k_{p_2}(A_2))$.
- Let $U \in Open(Spec(A_1 \times A_2))$, then $U \approx U_1 \sqcup U_2$, where $U_i = (\pi_i^\star)^{-1}[U] \in Open(Spec(A_i))$. Then any sheaf S satisfies $S(A)(U) \cong S(A_1)(U_1) \times S(A_2)(U_2)$.

Now the construction of R as the global sections of of a sheaf allows to conclude $R(A_1 \times A_2) = S(A_1 \times A_2)(Spec(A_1 \times A_2)) \cong (S(A_1)(Spec(A_1)) \times S(A_2)(Spec(A_2))) = R(A_1) \times R(A_2)$.

■

Remark 30 By a combination of Theorem 26, Proposition 21.(ii) and Remark 15.(viii), in some sense, any ring viewed as an object of *Locringed* has a "nearest field"[7]. In more details, for each affine scheme $\mathcal{A}$ (i.e $\mathcal{A} \cong \Sigma_A$), there is an affine scheme $\mathcal{V}_A$ in $Boofield$ and there is a *Locringed*-morphism $(h_A, \tau_A) : \mathcal{A} \longrightarrow \mathcal{V}_A$, such that for each (X, F) in $Boofield$, and for each *Locringed*-morphism $(h, \tau) : \mathcal{A} \longrightarrow (X, F)$, then there is a unique *Locringed*-morphism $(\tilde{h}, \tilde{\tau}) : \mathcal{V}_A \longrightarrow (X, F)$ such that $(\tilde{h}, \tilde{\tau}) \circ (h_A, \tau_A) = (h, \tau)$. □

4 Some calculations and remarks

In Proposition 11.(iv) we saw that, given a ring A, then for each $a \in A$, $a \in Nil(A) \Leftrightarrow \eta_A(a) \in Nil(R(A)) = \{0\}$. It is natural to ask if the canonical homomorphism $\eta_A : A \longrightarrow R(A)$ reflects other ring-theoretic properties.

31 If V is a vN-regular ring then the diagonal ring homomorphism $\delta_V : V \longrightarrow \prod_{s\in Spec(V)} k_s(V)$ is injective (because $ker(\delta_V) = Nil(V) = \{0\}$), but much more holds: as V is isomorphic to the ring of global sections of its spectral sheaf and the space $Spec(V)$ is *boolean*, it follows from Proposition 3.2.(d) in [**DM7**] that $\delta_V : V \rightarrowtail \prod_{s\in Spec(V)} k_s(V)$ is an L_{ring}*-pure embedding*, i.e., if we consider the language $L_{ring} = \{+, -, 0, \cdot, 1\}$ and we take any existential positive L_{ring}-formula, say $\phi(x_1, \ldots, x_n)$, then

[7]Clearly, the inclusion functor $Fields \hookrightarrow Rings$ does not have a left adjoint. On other hand, each ring A admits a essentially unique family of "nearest fields": $\{\alpha_p^A : A \longrightarrow k_p(A)\}$, establishing that $Fields \hookrightarrow Rings$ has a *multi* left adjoint.

for each $b_1, \dots b_n \in V$, $\phi[b_1, \dots, b_n]$ is true in V iff $\phi[\delta_V(b_1), \dots, \delta_V(b_n)]$ is true in $\prod_{s \in Spec(V)} k_s(V)$. □

32 Let us write $B^\bullet$ for the set of *invertible* elements of a ring B. For a given ring A, we have that for each $a \in A$, $a \in A^\bullet \Leftrightarrow \eta_A(a) \in R(A)^\bullet$. Indeed, it follows from some well known results in Commutative Algebra and the calculations in Section 2, that

$a \in A$ is invertible iff $\forall p \in Spec(A)\ a/p \neq 0/p \in A/p$
iff $\forall p \in Spec(A) \alpha_p^A(a) \in k_p(A)$ is invertible
iff $\delta_A(a) \in \prod_{p \in Spec(A)} k_p(A)$ is invertible
iff $\widehat{\eta_A} \circ \delta_A(a) \in \prod_{q \in Spec(R(A))} k_q(R(A))$ is invertible
iff $\delta_{R(A)} \circ \eta_A(a) \in \prod_{q \in Spec(R(A))} k_q(R(A))$ is invertible
iff $\eta_A(a) \in R(A)$ is invertible (since $\delta_{R(A)}$ is a pure embedding) □

33 The sheaf theoretic description of the vN-regular hull, and the fact that $\delta_{R(A)} : R(A) \rightarrowtail \prod_{q \in Spec(R(A))} k_q(R(A))$ is a pure embedding, give us a guide to "calculate" some vN-regular hulls. For instance:

(i) Since $R(A)$ is the ring of global sections of the sheafification of a known (mono)presheaf, its elements are obtained by gluing of sections of this presheaf, and the canonical map $\eta_A : A \to R(A)$ is the one that maps elements of A to the compatible families they represent (and which appear in more general form in the process of gluing). Potentially, this more concrete description of the vN-hull can provide information easier than just by the use of the universal property.

(ii) Let A be a ring. Then the following are equivalent:
(a) $Spec(A)$ is a finite spectral space.
(b) $Spec^{const}(A)$ is a finite and *discrete* space.
(c) $R(A)$ is isomorphic to a finite product of fields.
We analyze only the non-obvious implication: (b) $\Rightarrow$ (c). Assuming $Spec^{const}(A)$ is a finite and discrete space, then $S(A)(\{p\}) \cong S(A)_p \cong k_p(A)$; moreover $S(A)(Spec^{const}(A)) = S(A)(\bigcup_{p \in Spec(A)} \{p\}) \cong \prod_{p \in Spec(A)} S(A)(\{p\})$, since $S(A)$ is a sheaf. Thus $R(A) = S(A)(Spec^{const}(A)) \cong \prod_{p \in Spec(A)} k_p(A)$ and $\eta_A : A \longrightarrow R(A)$ can be identified with the diagonal homomorphism $\delta_A : A \longrightarrow \prod_{p \in Spec(A)} k_p(A)$.
In particular:
• $R(A)$ is a trivial ring iff $Spec(A) = \emptyset$ iff A is a trivial ring.
• $R(A)$ is a field iff $Spec(A) = \{m\}$ iff A is a zero-dimensional local ring; moreover, in this case, $R(A) \cong Res(A) = A/m$.
• If A is a finite ring, then $Spec(A)$ is a finite space and $k_p(A) \cong A/p$. Then $R(A) = S(A)(Spec^{const}(A)) \cong \prod_{p \in Spec(A)} A/p$ is a finite ring (a finite product of finite fields) and $\eta_A : A \longrightarrow R(A)$ can be identified with the diagonal homomorphism $(q_p^A)_{p \in Spec(A)} : A \longrightarrow \prod_{p \in Spec(A)} A/p$.

(iii) If $F[x]$ the ring of polynomials in one variable over the field F, $R(F[x])$ is a pure subring of $F(x) \times \prod\{$ simple algebraic extensions of $F\}$ and $F[x]$ is in the diagonal. In particular, when F is an algebraically closed field, then the (boolean) space of 1-types of F is homeomorphic to $Spec^{const}(F[x])$ and $R(F[x]) \subseteq F(x) \times F^F$ is a pure subring containing $F[x]$.

(iv) $R(\mathbb{Z})$ is a pure subring of $\mathbb{Q} \times \prod_{p \in \mathbb{N}, p\ prime} \mathbb{Z}/p\mathbb{Z}$. Consider $n \in \mathbb{N}\backslash\{0,1\}$, say $n = p_1^{e_0}. \cdots .p_k^{e_k}$, with $p_i > 0$ distinct primes and $e_i > 0$, $i \leqslant n$; if $A = \mathbb{Z}/n\mathbb{Z}$ then the canonical arrow $\eta_A : A \to R(A)$ can be identified with the projection onto the quotient $q_{n,n'} : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n'\mathbb{Z}$, where $n' = p_1 \cdot \ldots \cdot p_k$: as $\delta_{\mathbb{Z}/n\mathbb{Z}} = (q_{n,p_i})_{i \leqslant k} : \mathbb{Z}/n\mathbb{Z} \longrightarrow \prod_{i \leqslant k} \mathbb{Z}/p_i\mathbb{Z}$ we get $Nil(A) = ker(\delta_A) = n'\mathbb{Z}/n\mathbb{Z} \subseteq \mathbb{Z}/n\mathbb{Z}$, and because $R(A) \cong R(A/Nil(A))$, we just have to see that $\delta_{\mathbb{Z}/n'\mathbb{Z}} = (q_{n',p_i})_{i \leqslant k} : \mathbb{Z}/n'\mathbb{Z} \longrightarrow \prod_{i \leqslant k} \mathbb{Z}/p_i\mathbb{Z}$ is an *isomorphism*, this follows from the injectivity of $\delta_{\mathbb{Z}/n'\mathbb{Z}}$ and a counting argument or by Chinese Remainder Theorem.

(v) If A is a ring with Krull dimension $= 0$, then $Jac(A) = Nil(A)$ and $A/Nil(A)$ is vN-regular, since it is reduced and zero dimensional. It is straightforward to check that to the quotient map $q_{Nil(A)} : A \twoheadrightarrow A/Nil(A)$ satisfies the universal property of vN-hull. Thus we can conclude that $\eta_A : A \to R(A)$ is isomorphic to $q_{Nil(A)} : A \twoheadrightarrow A/Nil(A)$.

□

5 The vN-Hull in categories of preordered rings and applications

In this section, we extend the construction of the vN-Hull to categories of preordered rings.

Remark 34 On preordered rings: ([**Lam2**], [**Mar2**])

(i) A *preorder* in a ring A is a subset $T \subseteq A$ such that $A^2 = \{a^2 : a \in A\} \subseteq T, T + T \subseteq T, T.T \subseteq T$. The intersection of any set of preorders in A is a preorder in A. It follows that the set $po(A) = \{T \subseteq A$: T is a preorder in $A\}$, ordered by inclusion, is a *complete lattice*, in which $\sum A^2$ and A are the *extremal* preorders in A. A preorder $T \subseteq A$ is *proper* if $T \neq A$. If $2 \in A^\bullet$ ($\Leftrightarrow 2 \in R(A)^\bullet$) then $T \subseteq A$ is proper iff $-1 \notin T$. If $T \subseteq A$ is a preorder, then $T \cap -T$ is an ideal in A. An *order* in A is a (proper) preorder $P \subseteq A$ such that $P \cup -P = A$ and $P \cap -P \in Spec(A)$. If $-1 \notin T \subseteq A$ is a (proper) preorder then a maximal preorder P such that $-1 \in P$ and $T \subseteq P$ is a (maximal) order in A. *Not* every order is maximal under inclusion.

(ii) The set $Sper(A) = \{P\subseteq A\colon P \text{ is an order in } A\}$ is called *the real spectrum of* A and has a natural spectral topology: it has as subbase the set $\{S_a\subseteq Sper(A) : a \in A\}$ where $S_a := \{P \in Sper(A) : a \in P\setminus -P\}$, $a \in A$. For $T \in po(A)$ we consider also the subspace $Sper_T(A) = \{P \in Sper(A) : T\subseteq P\}$, then $Sper_{\sum A^2}(A) = Sper(A)$. There is a spectral map $\Pi_A\colon Sper(A) \to Spec(A)$, $P \mapsto P \cap -P$.

Now consider a ring homomorphism $f : A\longrightarrow A'$.

(iii) The *inverse image* function induced by f is a *contravariant* increasing function between the complete lattices $f^\star : po(A')\longrightarrow po(A) \ : \ T' \mapsto f^\star(T') := f^{-1}[T']$. The inverse image of a proper preorder is a proper preorder. Moreover, the image inverse function gives a well defined (spectral) function: $f^\star : Sper_{S'}(A')\longrightarrow Sper_T(A)$, for each $T \in po(A), S' \in po(A'), T\subseteq f^\star(S')$. The *direct image* function induced by f is a *covariant* increasing function between complete lattices $f_\star : po(A)\longrightarrow po(A')$: $T \mapsto f_\star(T) := \sum A'^2 f[T]$ $(= \bigcap\{S' \in po(A') : f[T]\subseteq S'\})$.

(iv) The pair of functions given by direct and inverse image $(f_\star, f^\star) : po(A) \rightleftarrows po(A')$ forms an *adjunction*: for each $T \in po(A), S' \in po(A')$, ($f[T]\subseteq S'$ iff) $f_\star(T)\subseteq S' \Leftrightarrow T\subseteq f^\star(S')$, in particular $T\subseteq f^\star \circ f_\star(T)$, $f_\star \circ f^\star(S')\subseteq S'$ and $f_\star(T) = f_\star \circ f^\star \circ f_\star(T)$, $f^\star \circ f_\star \circ f^\star(S') = f^\star(S')$.

(v) The direct and inverse image constitute, respectively, a covariant and a contravariant functor $Rings\longrightarrow CompleteLattices$. I.e. $(id_A)_\star = id_{po(A)} = (id_A)^\star$, and if $f' : A'\longrightarrow A''$ is a ring homomorphism, then $(f' \circ f)_\star = (f')_\star \circ (f)_\star$ and $(f' \circ f)^\star = (f)^\star \circ (f')^\star$.

□

Fact 35 *Consider the category poRings whose objects are the poRings (A,T) and arrows $h : (A,T)\longrightarrow(A',T')$ are the ring homomorphisms $h : A\longrightarrow A'$ such that $h[T]\subseteq T'$. Then:*

(a) If $h : A\longrightarrow A'$ is a ring homomorphism and $T \in po(A), T' \in po(A')$, then we have the following equivalences: $h[T]\subseteq T'$ *iff* $h_\star(T)\subseteq T'$ *iff* $T\subseteq h^\star(T')$.

(b) Let $L_{poRings} = (+,\cdot,-,0,1,T(\ ))$ be the first-order language that extends the language of rings by the addition of a unary predicate symbol $T(\ )$. Then poRings is a full reflective subcategory of the category $L_{poRings} - Str$ of all $L_{poRings}$-structures and its $L_{poRings}$-homomorphisms.

(c) poRings $\hookrightarrow L_{poRings} - Str$ is an elementary subclass that is closed under upward directed colimits, substructures, products, reduced products.

□

Proposition 36 *Let A be a ring. Then the vN-hull, $\eta_A \colon A \to R(A)$ induces a (spectral) bijection between spectral spaces $Sper(R(A)) \to Sper(A)$. This bijection fits into a commutative diagram with the maps Π_A, $\Pi_{R(A)}$ of Remark 34(ii), and the bijection $(\eta_A)^\star : Spec(R(A)) \to Spec(A)$.*

Proof. There is a natural bijection between $Sper(B)$ and equivalence classes of ring homomorphisms from B to a real closed field: where $f : B \longrightarrow F$, $f' : B \longrightarrow F'$ are equivalent if there is a real closed field K and homomorphisms $j : F \longrightarrow K$, $j' : F \longrightarrow K'$ such that $j \circ f = j' \circ f'$ [8]. Using this identification, the first claim follows from the universal property of η_A.

The second claim follows simply from the definitions of Π_A, $\Pi_{R(A)}$ and the fact that taking preimage of a set under a ring homomorphism commutes with finite intersection and negation. ■

Remark 37 Consider the vN-Hull $\eta_A : A \longrightarrow R(A)$.

- If $T \in po(A)$ is proper, then there is an order $P \in Sper_T(A)$. Since $(\eta_A)^\star : Sper(R(A)) \longrightarrow Sper(A)$ is bijective (by Prop. 36), there is an order $Q \in Sper(R(A))$ with $T \subseteq (\eta_A)^\star(Q)$, i.e. $(\eta_A)_\star(T) \subseteq Q$. Thus $(\eta_A)_\star(T) \in po(R(A))$ is a proper preorder, too. On the other hand, as $T \subseteq (\eta_A)^\star((\eta_A)_\star(T))$, if $(\eta_A)_\star(T)$ is a proper preorder on $R(A)$, then T is a proper preorder on A.

- If $P \in Sper(A)$ is a *maximal* order, then $P \in po(A)$ is proper so $(\eta_A)_\star(P) \in po(R(A))$ is proper and $(\eta_A)^\star((\eta_A)_\star(P)) \in po(A)$ is proper, too. In this case, as $P \subseteq (\eta_A)^\star((\eta_A)_\star(P))$, we get $P = (\eta_A)^\star((\eta_A)_\star(P))$; since $(\eta_A)^\star : Sper(R(A)) \longrightarrow Sper(A)$ is a increasing bijection, $P = (\eta_A)^\star(Q)$ for some (unique) $Q \in maxSpec(R(A))$ and, as $P = (\eta_A)^\star((\eta_A)_\star(P))$, then $(\eta_A)_\star(P) = Q \in po(R(A))$ is a maximal order. Summing up, we have the pair of inverse bijections $\{Q \in Sper(R(A)) : Q \text{ is maximal}\} \underset{(\eta_A)_\star}{\overset{(\eta_A)^\star}{\leftrightarrow}} \{P \in Sper(A) : P \text{ is maximal }\}$.

□

The remarks above *suggest* that the association $(A, T) \mapsto (R(A), (\eta_A)_\star(T))$ has a privileged role. For simplicity, we will write $T_\star := (\eta_A)_\star(T)$.

Proposition 38 *If poRegRings denotes the full subcategory of poRings formed by pairs (V, S) where V is a vN-regular ring, then the inclusion functor poRegRings $\hookrightarrow$ poRings has a left adjoint. I.e., for each poring (A, T), the poring-morphism η_A :*

[8] This relation is transitive because the elementary class of real closed fields is model complete and has the *amalgamation property* for (mono)morphisms.

$(A,T)\longrightarrow(R(A),T_\star)$ has the following universal property: Given a poring (V,S) with V a vN-regular ring and a poring-morphism $f:(A,T)\longrightarrow(V,S)$, there exists a unique poring-morphism $F:(R(A),T_\star)\longrightarrow(V,S)$ that extends f through η_A i.e. $F\circ\eta_A=f$.

Proof. As the underlying functors *poRings*$\longrightarrow$*Rings*, *poRegRings*$\longrightarrow$*RegRings* are faithful and $\eta_A : A\longrightarrow R(A)$ has the universal property relatively to the inclusion *RegRings* $\hookrightarrow$ *Rings*, it only remains verify that $F[T_\star]\subseteq S$. This can be seen as follows:

$$F[T_\star]\subseteq S \text{ iff } F_\star((\eta_A)_\star(T))=F_\star(T_\star)\subseteq S \text{ iff } (F\circ\eta_A)_\star(T)\subseteq S \text{ iff } f_\star(T)\subseteq S \text{ iff } f[T]\subseteq S$$

■

Corollary 39 *(***Relative version of** *Prop. 36) For each ring A there is a natural bijection between the orders in A and the equivalence class of ring homomorphisms from A to a real closed field. Using these identification, it follows from the universal property of η_A that it induces a continuous bijection $(\eta_A)^\star : Sper_{T_\star}(R(A))\longrightarrow Sper_T(A)$.*

□

6 Applications to the theory of Quadratic Forms

In this section, we consider some applications of the previous constructions to some abstract codifications of the theory of quadratic forms over rings, mainly to Special Groups Theory ([**DM1**]).

For any ring A here, we will assume $2\in A^\bullet$. By preorder here we will always mean proper preorder.

40 On real semigroups of porings:
In [**DP1**], [**DP2**] the authors introduce the concept of *real semigroup*. This is a first order axiomatizable concept intended to deal with quadratic forms over general preordered rings. The category of real semigroups is dual to the category of abstract real spectra.

In [**DP1**, 9.1(A)] the authors describe a covariant functor S from the category of porings into the category of *real semigroups*. In particular, the (canonical) poring-morphism $\eta_A : (A,T)\longrightarrow(R(A),T_\star)$ induces a canonical morphism of real semigroups $S(\eta_A) : S(A,T)\longrightarrow S(R(A),T_\star)$.

The *Post hull* of a real semigroup M is defined as the algebra of continuous functions $\mathcal{C}(X_M,3)$ where $3=\{1,0,-1\}$ and X_M is the *boolean space* $Hom(M,3)$ with the constructible topology, see [**DP2**, III.4]. □

Theorem 41 *The morphism of real semigroups $S(\eta_A) : S(A,T) \longrightarrow S(R(A), T_\star)$ induces an isomorphism between Post hulls. In particular it is a complete embedding of real semigroups and thus reflects the Witt-equivalence of forms.*

Proof. When $M = S(A, \sum A^2)$, then X_M is the booleanization of the spectral space $Sper(A) = Sper_{\sum A^2}(A)$; this result can be easily extended to the relative version $T \subseteq A$ a preorder. It follows from Corollary 39 above that $\eta_A^\star : X_{S(R(A),T^\star)} \longrightarrow X_{S(A,T)}$ is a homeomorphism of boolean spaces. Therefore $Post(\overline{\eta}_A) : Post(S(A,T)) \longrightarrow Post(S(R(A), T^\star)$ is an isomorphism of Post algebras.

By [**DP2**, Thm. III.4.5] a morphism of semigroups is a complete embedding (i.e. it reflects reflects Witt-equivalence [**DP2**, I.2.7(c)], [**DP2**, III.4.3]) iff it induces an injective map on Post-hulls, which proves the second claim. ■

We will focus now on applications of the vN-hull construction to the first-order theory of Special Groups in the language $L_{SG} = (\cdot, 1, -1, \equiv)$ ([**DM1**]). We begin by registering the following facts:

Lemma 42 *([**MS**]) For each $n \in \mathbb{N}$, the functor k-theory functor $k_n : protoSG \longrightarrow ptGr$ ([**DM6**]), preserves pure embeddings, where the language of pointed groups is $L_{ptGr} = (\cdot, 1, -1)$.* □

Corollary 43 *If $f : G \rightarrowtail G'$ is a pure embedding of a protoSG, G, into a RSG, G', that satisfies the property [SMC] (Special Marshall's conjecture, [**DM6**]), then G is a RSG that satisfies [SMC] and [MC] (Marshall's signature conjecture), [MWRC] (Milnor's conjecture for the graded Witt ring) ([**DM7**]).*

Proof. As each of the axioms for RSGs is either a negation of an atomic formula or the universal closure of $\phi \to \psi$, where ϕ and ψ are positive primitive formulas, we can conclude from the hypothesis that G is a RSG. Since G' satisfies [SMC], $\omega_n(G') : k_n(G') \longrightarrow k_{n+1}(G')$ is injective ($\omega_n(G') = l(-1) \otimes -$), $\forall n \in \mathbb{N}$. By Lemma 42, the homomorphism $k_n(f) : k_n(G) \longrightarrow k_n(G')$ is injective for each $n \in \mathbb{N}$ and, as $\omega_n(G') \circ k_n(f) = k_{n+1}(f) \circ \omega_n(G)$, we conclude that $\omega_n(G) : k_n(G) \longrightarrow k_{n+1}(G)$ is injective $\forall n \in \mathbb{N}$, i.e. G satisfies [SMC]. The equivalence between [SMC] and the conjunction of [MC] and [MWRC] is established in Lemma 1.2 in [**DM7**]. ■

44 On (proto-)special groups of porings:
Denote by $\mathcal{G} : poRings \longrightarrow protoRSG$ the functor

$$((A,T) \xrightarrow{h} (A',T')) \in poRings \qquad \mapsto \qquad (A^\bullet/T^\bullet \xrightarrow{\bar{h}} A'^\bullet/T'^\bullet) \in protoRSG$$

(i) It is straightforward that $\mathcal{G}$ preserves directed inductive limits, products and reduced products. Moreover, as a consequence, $\mathcal{G}$ preserves pure embeddings.

(ii) Theorem 7.2 in [**DM7**] establishes that if V is a vN-regular ring (with $2 \in V^\bullet$) and $S \subseteq V$ is a (proper) preorder in V, then $\mathcal{G}(V,S)$ is a RSG that satisfies [MC], [SMC] and [MWRC]. □

Now we will apply the previous results on *poRegRings* to describe classes of porings and RSGs that are interesting under different criteria:
(I) *Categorial*: they are full subcategories of, respectively, *poRings* and *RSG*, closed under many constructions.
(II) *Logical:* they are (relatively simple) first-order elementary classes in the appropriate languages.
(III) *Quadratic form theory:* the associated RSGs satisfy the interesting properties [MC], [SMC] and [MWRC].

Theorem 45 *Consider the class*

$$vNpur := \{(A,T) \in poRings \mid \mathcal{G}(\eta_A) : \mathcal{G}(A,T) \to \mathcal{G}(R(A), T_\star) \text{ is an } L_{SG}\text{-pure embedding}\}$$

(i) (Alternative descriptions of *vNpur.) Let $(A,T) \in poRings$. Then the following are equivalent:*

(i1) $(\mathcal{G}(\alpha_p^A))_{p\in Spec(A)} : \mathcal{G}(A,T) \longrightarrow \prod_{p\in Spec(A)} \mathcal{G}(k_p(A), (\alpha_p^A)_\star(T))$ *is an L_{SG}-pure embedding*

(i2) $\mathcal{G}((\alpha_p^A)_{p\in Spec(A)}) : \mathcal{G}(A,T) \longrightarrow \mathcal{G}(\prod_{p\in Spec(A)}(k_p(A), (\alpha_p^A)_\star(T)))$ *is an L_{SG}-pure embedding*

(i3) $\mathcal{G}(\delta_A) : \mathcal{G}(A,T) \longrightarrow \mathcal{G}((\prod_{p\in Spec(A)} k_p(A)), (\delta_A)_\star(T))$ *is an L_{SG}-pure embedding*

(i4) $\mathcal{G}(\eta_A) : \mathcal{G}(A,T) \longrightarrow \mathcal{G}(R(A), (\eta_A)_\star(T))$ *is an L_{SG}-pure embedding*

(i5) $\exists (V,S) \in poRegRings, \exists h : (A,T) \longrightarrow (V,S)$ *poRings − morphism, such that* $\mathcal{G}(h) : \mathcal{G}(A,T) \longrightarrow \mathcal{G}(V,S)$ *is an L_{SG}-pure embedding.*

(ii) The full subcategory $vNpur \subseteq poRings$ is closed under: (a) isomorphisms; (b) pure substructures; (c) directed inductive limits; (d) (non-empty) products; (e) proper reduced products; (f) elementary equivalence.

*(iii) vNpur is an $L_{poRings}$-elementary class that is axiomatizable by ∀∃-sentences and also by Horn sentences ([**CK**]).*

(iv) If $(A,T) \in vNpur$, then $\mathcal{G}(A,T)$ is a RSG that satisfies [SMC] and [MC], [MWRC].

Proof. We already remarked in 32 that $2 \in A^\bullet$ iff $2 \in R(A)^\bullet$ and, in 37.(i), we saw that T is a proper preorder on A iff $(\eta_A)_\star(T)$ is a proper preorder on $R(A)$.

(i) The equivalence (i1) $\Leftrightarrow$ (i2) follows from the fact the $\mathcal{G}$ is a functor that preserves products. (i2) $\Leftrightarrow$ (i3): follows from the characterization of products in *poRings* and of direct image of preorders. (i4) $\Rightarrow$ (i5): is clear. (i5) or (i3) $\Rightarrow$ (i4): since whenever a composition $g \circ f$ is pure, then f is pure, the result follows from the universal property of the *poRing*-morphism $\eta_A) : (A, T) \longrightarrow (R(A), T_\star)$. (i3) $\Rightarrow$ (i4): it follows from Proposition 3.2.(d) in [**DM7**] that the arrow $\tilde{\delta}_A : (R(A), T_\star) \longrightarrow (\prod_{p \in Spec(A)} k_p(A)), (\delta_A)_\star(T)$ is a $L_{poRings}$-pure embedding. Since $\mathcal{G}$ is a functor that preserves pure embeddings, the result then follows from the fact that if g, f are pure then $g \circ f$ is pure.

(ii) We use the characterization in (i5). The closure under (a) and (b) are clear because the composition of pure embeddings is a pure embedding.

(c) Let $(I, \leqslant)$ be an upward directed poset and let $\{((A_i, T_i) \xrightarrow{h_{ij}} (A_j, T_j) : i \leqslant j \in I\} \subseteq vNpur$ be a directed system; consider the pure embeddings $\mathcal{G}(\eta_{A_i}) : \mathcal{G}(A_i, T_i) \longrightarrow \mathcal{G}(R(A_i), (T_i)_\star)$. Since the directed colimit of pure embeddings is a pure embedding ([**MM2**]), the colimit arrow $\varinjlim_{i \in I} \mathcal{G}(\eta_{A_i}) : \varinjlim_{i \in I} \mathcal{G}(A_i, T_i) \longrightarrow \varinjlim_{i \in I} \mathcal{G}(R(A_i), (T_i)_\star)$ is a pure embedding. Since $poRegRings \subseteq poRings$ is closed under directed colimits and both functors R and $\mathcal{G}$ preserve directed colimits,

$$\varinjlim_{i \in I} \mathcal{G}(R(A_i), (T_i)_\star) \cong \mathcal{G}(R(\varinjlim_{i \in I} A_i), \varinjlim_{i \in I} (T_i)_\star),$$

there is a pure embedding $F : \mathcal{G}(\varinjlim_{i \in I} (A_i, T_i)) \longrightarrow \mathcal{G}(R(\varinjlim_{i \in I} A_i), \varinjlim_{i \in I} (T_i)_\star)$, with $\mathcal{G}(R(\varinjlim_{i \in I} A_i), \varinjlim_{i \in I} (T_i)_\star) \in poRegRing$, i.e. $\varinjlim_{i \in I} (A_i, T_i) \in vNpur$.

(d) Let I be a non-empty set and $\{(A_i, T_i) : i \in I\} \subseteq vNpur$. For each $i \in I$ choose a pure embedding $\mathcal{G}(f_i) : \mathcal{G}(A_i, T_i) \longrightarrow \mathcal{G}(V_i, S_i)$, with V_i a vN-regular ring. Since the product of pure embeddings is a pure embedding ([**MM2**]), the product arrow $\prod_{i \in I} \mathcal{G}(f_i) : \prod_{i \in I} \mathcal{G}(A_i, T_i) \longrightarrow \prod_{i \in I} \mathcal{G}(V_i, S_i)$ is a pure embedding. Since $poRegRings \subseteq poRings$ is closed under products and $\mathcal{G}$ preserves products, there is a pure embedding $F := \mathcal{G}(\prod_{i \in I} f_i) : \mathcal{G}(\prod_{i \in I} A_i, \prod_{i \in I} T_i) \longrightarrow \mathcal{G}(\prod_{i \in I} V_i, \prod_{i \in I} S_i)$, with $(\prod_{i \in I} V_i, \prod_{i \in I} S_i) \in poRegRing$, i.e. $\prod_{i \in I}(A_i, T_i) \in vNpur$.

(e) Let $\mathcal{F}$ be a proper filter over a set $I \neq \emptyset$ and $\{(A_i, T_i) : i \in I\} \subseteq vNpur$; it is well-known that the reduced product $(\prod_{i \in I}(A_i, T_i))/\mathcal{F}$ is isomorphic to the inductive limit of the direct system $\langle\ (\prod_{i \in J}(A_i, T_i)) \overset{proj_{JK}}{\twoheadrightarrow} (\prod_{i \in K}(A_i, T_i)) : (J \supseteq K) \in \mathcal{F}\ \rangle$ (see, for instance [**MM1**]), and the conclusion follows from items (a), (c) and (d).

(f) By Fraine's Lemma (Lemma 8.1.1 in [**BS**]), $(A, T) \equiv (A', T')$ iff (A, T) is elementary embeddable in some ultrapower of (A', T') and the conclusion follows from (a), (b) and (e).

(iii) All the statements follow from well-known model-theoretic results applied to item (ii). By Theorem 4.1.12 in [**CK**], a subclass of first-order structures is elementary if and only if it is closed under ultraproducts and elementary equivalence, conditions guaranteed by items (e) and (f) in (ii). By Theorem 6.2.5 in [**CK**], an elementary class of structures is axiomatizable by Horn-sentences if and only if it is closed under reduced products and this condition is assured by (ii).(e). By Theorem 5.2.6 in [**CK**], an elementary class of structures is axiomatizable by $\forall\exists$-sentences if and only if it is closed under direct inductive limits of embeddings: the desired conclusion comes from (ii).(d).

(iv) By 44.(ii) and Corollary 43. ■

Corollary 46 *The full subcategory vNpur is weakly reflective in poRings, i.e. for every preordered ring* R *there exist a preordered ring* $\tilde{R} \in vNpur$ *and a morphism* $f\colon R \to \tilde{R}$ *such that every morphism* $R \to P$ *with* $P \in vNpur$ *factors through* f *(not necessarily uniquely). Moreover it is weakly cocomplete, i.e. for every diagram in vNpur there is a weakly initial cocone (i.e. for every cocone over the diagram there is a, not necessarily unique, morphism of cones from this one).*

Proof. The category *vNpur* is accessible, since it is first order axiomatizable by Thm. 45(iii). Moreover it is accessibly embedded into *poRings* by Thm. 45(ii)(c) and closed under products by Thm. 45(ii)(d). By the equivalence of [**AR**, Thm. 4.8 (ii)] and [**AR**, Thm. 4.8 (iii)], these properties guarantee that *vNpur* is weakly reflective in *poRings*. [**AR**, Thm. 4.8 (ii)]. Finally, the weak cocompleteness follows from [**AR**, Thm. 4.11]. ■

Remark 47 The notion of "Faithfully Quadratic Ring" introduced and developed in [**DM9**] gives an axiomatic approach to "well-behaved" quadratic form theory of porings. Instead, we provide in the Theorem 45 above an approach that selects a class of porings that are relatively "well-behaved": i.e., its objects have a nice relation with a very well-behaved class of porings (*poRegRings*), that is stable under many constructions. In the sequel of the present research, we intend to:

- Provide explicit axiomatizations of *vNpur* by Horn sentences and $\forall\exists$-sentences.
- Understand the relation between representation and transversal representation of forms defined on porings in *vNpur*.
- Establish precise connections of *vNpur* with the class of faithfully quadratic rings.

□

Theorem 48 *Consider the class $RSGvN := \{G \in RSG : \exists (V,S) \in poRegRings, \exists j : G \longrightarrow \mathcal{G}(V,S)$ a L_{SG}-pure embedding $\}$. Then:*

(i) The class $RSGvN$ contains the class $\{\mathcal{G}(A,T) : (A,T) \in vNpur\}$.

(ii) The full subcategory $RSGvN \subseteq RSG$ is closed under: (a) isomorphisms; (b) pure substructures; (c) (non-empty) products; (d) proper reduced products; (e) elementary equivalence.

*(iii) $RSGvN$ is an L_{SG}-elementary class that is axiomatizable by Horn sentences ([**CK**]).*

(iv) If $G \in RSGvN$, then G is a RSG that satisfies [SMC] and [MC], [MWRC].

Proof. Item **(i)** follows from Theorem 45.(i). Item **(iv)** follows from Corollary 43.

(ii) The closure under (a) and (b) are clear because the composition of pure embeddings is a pure embedding.

(c) Let I be a non-empty set and $\{G_i : i \in I\} \subseteq RSGvN$; select a pure embedding $f_i : G_i \longrightarrow \mathcal{G}(V_i, S_i)$, with V_i a vN-regular ring. Since the product of pure embeddings is a pure embedding ([**MM2**]), the product arrow $\prod_{i\in I} f_i : \prod_{i\in I} G_i \longrightarrow \prod_{i\in I} \mathcal{G}(V_i, S_i)$ is a pure embedding. Since $poRegRings \subseteq poRings$ is closed under products and $\mathcal{G}$ preserves products $\prod_{i\in I} \mathcal{G}(V_i, S_i) \cong \mathcal{G}(\prod_{i\in I} V_i, \prod_{i\in I} S_i)$, thus there is a pure embedding $F : \prod_{i\in I} G_i \longrightarrow \mathcal{G}(\prod_{i\in I} V_i, \prod_{i\in I} S_i)$, with $(\prod_{i\in I} V_i, \prod_{i\in I} S_i) \in poRegRing$, i.e. $\prod_{i\in I} G_i \in RSGvN$.

(d) Let $\mathcal{F}$ be a proper filter over a set $I \neq \emptyset$ and $\{G_i : i \in I\} \subseteq RSGvN$. Select a pure embedding $f_i : G_i \longrightarrow \mathcal{G}(V_i, S_i)$, with V_i a vN-regular ring. The reduced product $(\prod_{i\in I} G_i)/\mathcal{F}$ is isomorphic to the inductive limit of the directed system $\langle\, (\prod_{i\in J} G_i \overset{proj_{JK}}{\twoheadrightarrow} \prod_{i\in K} G_i) : (J \supseteq K) \in \mathcal{F} \,\rangle$ and the reduced product $(\prod_{i\in I} \mathcal{G}(V_i, S_i)/\mathcal{F}$ is isomorphic to the inductive limit of the direct system $\langle\, (\prod_{i\in J} \mathcal{G}(V_i, S_i) \overset{proj_{JK}}{\twoheadrightarrow} \prod_{i\in K} \mathcal{G}(V_i, S_i)) : (J \supseteq K) \in \mathcal{F} \,\rangle$. As the product and directed inductive limit of pure embeddings is a pure embedding, then we have a pure embedding

$$\underset{J\in\mathcal{F}}{\underrightarrow{lim}} \, (\prod_{i\in J} f_i) : \underset{J\in\mathcal{F}}{\underrightarrow{lim}} \, (\prod_{i\in J} G_i) \longrightarrow \underset{J\in\mathcal{F}}{\underrightarrow{lim}} \, (\prod_{i\in J} \mathcal{G}(V_i, S_i)).$$

As the functor $\mathcal{G}$ preserves products and directed inductive limits and $poRegRings \subseteq poRings$ is closed under products and directed inductive limits, we have an isomorphism $\underset{J\in\mathcal{F}}{\underrightarrow{lim}} \, (\prod_{i\in J} \mathcal{G}(V_i, S_i)) \cong \mathcal{G}(\underset{J\in\mathcal{F}}{\underrightarrow{lim}} \, (\prod_{i\in J} (V_i, S_i))$. Thus there is a pure embedding

$$F : (\prod_{i\in I} G_i)/\mathcal{F} \longrightarrow \mathcal{G}(\prod_{i\in I} (V_i, T_i)/\mathcal{F})$$

and $\prod_{i\in I} (V_i, T_i)/\mathcal{F} \in poRegRings$, i.e. $(\prod_{i\in I} G_i)/\mathcal{F} \in RSGvN$.

(e) By Fraine's Lemma (Lemma 8.1.1 in [**BS**]), $G \equiv H$ iff G is elementary embeddable in some ultrapower of H and the conclusion follows from (a), (b) and (d).

(iii) All the statements follow from well-known model-theoretic results applied to item (ii) . By Theorem 4.1.12 in [**CK**], a subclass of first-order structures is elementary if and only if it is closed under ultraproducts and elementary equivalence, conditions guaranteed by items (e) and (f) in (ii). By Theorem 6.2.5 in [**CK**], an elementary class of structures is axiomatizable by Horn-sentences if and only if it is closed under reduced products and this condition is assured by (ii).(e). ■

Remark 49 One of the main open tasks in the theory of Special Groups is to determine the extent to which this concept generalizes the theory of quadratic forms over fields and rings. For instance:
(i) The Representation Conjecture means: Every reduced special group (rsg) is isomorphic to the rsg given by a pythagorean field, via the functor $\mathcal{G}$.
(ii) The Weak Representation Conjecture means: Every rsg can be embedded, by an L_{SG}-elementary embedding (or, instead, by a L_{SG}-pure morphism), into a representable reduced special group (i.e., the rsg associated, via $\mathcal{G}$, to a pythagorean field).
(iii) In [**DM8**] a relaxed version of (i) is established: Every rsg is isomorphic to a rsg associated to (a part of) a ring real of continuous functions.
We provide in the Theorem 48 above a new approach to the representation problem under a intermediary balance that selects a new and very well-behaved class of porings, the class *poRegRings*, that contains the Pythagorean fields is stable under many constructions. Besides the very difficult representation question (is $RSG = RSGvN$?) we intend, in the sequel of our research:

- to analyze if the class $RSGvN$ is closed under other constructions, like directed inductive limits and extensions.

- provide explicit axiomatizations of $RSGvN$ by Horn sentences.

□

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References

[**AR**] J. Adámek, J. Rosický, **Locally Presentable and Accessible Categories**, Lecture Notes Series of the LMS **189**, Cambridge University Press, Cambridge, 1994.

[**BS**] J. L. Bell, A. B. Slomson, **Models and Ultraproducts: an Introduction**, North-Holland Publishing Company, Amsterdam, Netherlands, 1971.

[**Car**] A.B. Carson, *The model completion of the theory of commutative regular rings*, Journal of Algebra **27** (1973), 136-146.

[**CK**] C. C. Chang, H. J. Keisler, **Model Theory**, third edition, North-Holland Publishing Company, Amsterdam, Netherlands, 1990.

[**Cos**] M. Coste, *Localization, spectra and sheaf representation*, in: Applications of sheaves, Springer Lecture Notes in Math. **753** (1979), 212-238.

[**DM1**] M. Dickmann, F. Miraglia, **Special Groups: Boolean-Theoretic Methods in the Theory of Quadratic Forms**, Memoirs of the AMS **689**, American Mathematical Society, Providence, USA, 2000.

[**DM2**] M. Dickmann, F. Miraglia, *On Quadratic Forms whose total signature is zero mod* 2^n*. Solution to a problem of M. Marshall*, Inventiones Mathematicae **133** (1998), 243-278.

[**DM3**] M. Dickmann, F. Miraglia, *Lam's Conjecture*, Algebra Colloquium **10** (2003), 149-176.

[**DM4**] M. Dickmann, F. Miraglia, *Elementary Properties of the Boolean Hull Functor and Reduced Quotient Functors*, The Journal of Symbolic Logic **68** (2003), 946-971.

[**DM5**] M. Dickmann, F. Miraglia, *Bounds for the representation of quadratic forms*, Journal of Algebra **268** (2003), 209-251.

[**DM6**] M. Dickmann, F. Miraglia, *Algebraic K-theory of Special Groups*, Journal of Pure and Applied Algebra **204** (2006), 195-234

[**DM7**] M. Dickmann, F. Miraglia, *Quadratic form theory over preordered von Neumann-regular rings*, Journal of Algebra **319(4)** (2008), 1696-1732.

[**DM8**] M. Dickmann, F. Miraglia, *Representation of reduced special groups in algebras of continuous functions*, Contemporary Mathematics **493** (2009), 83-97.

[**DM9**] M. Dickmann, F. Miraglia, *Faithfully Quadratic Rings*, Memoirs of AMS **1128**, American Mathematical Society, Providence, USA, 2015.

[**DMM**] M. Dickmann, M. Marshall, F. Miraglia, *Lattice-Ordered Reduced Special Groups*, Annals of Pure and Applied Logic **132** (2005), 27-49.

[**DP1**] M. Dickmann, A. Petrovich, *Real Semigroups and Abstract Real Spectra*, Contemporary Mathematics AMS **344** (2004), 99-119.

[**DP2**] M. Dickmann, A. Petrovich, *Real Semigroups, Real Spectra and Quadratic Forms over Rings*, preprint (2012), 271 pp., available at www.maths.manchester.ac.uk/raag/preprints/0339.pdf .

[**DST**] M. Dickmann, N. Schwartz, M. Tressl, **Spectral Spaces**, in preparation.

[**EGA**] A. Grothendieck, J. Dieudonné, **Éléments de géométrie algébrique: I. Le langage des schémas**, Grundlehren der Mathematischen Wissenschaften **166** (2nd ed.) Springer-Verlag, Berlin - New York, 1971.

[**Gil**] R. Gilmer, *Zero-dimensional extension rings and subrings*, in: Zero-dimensional commutative rings, edited by D. Anderson and D. Dobbs, Lecture notes in pure and appl. mathematics, vol. 171, M. Dekker, New York, 1995

[**Hoc**] M. Hochster, *Prime ideal structure in commutative rings*, Transactions of AMS **142** (1969), 43-60.

[**Joh**] P. T. Johnstone, *Rings, fields, and spectra*, J. Algebra, **49(1)** (1977), 238-260.

[**Ken**] J. F. Kennison, *Integral Domain Type Representations in Sheaves and Other Topoi*, Math. Z. **151** (1976), 35-56.

[**Lam1**] T. Y. Lam, **The algebraic theory of quadratic forms**, W. A. Benjamin, Massachusetts, USA, 1973.

[**Lam2**] T. Y. Lam, *An Introduction to Real Algebra*, Rocky Mountain Journal of Mathematics **14** (1984), 767-814.

[**LS**] L. Lipshitz, D. Sarracino, *The model companion of the theory of commutative rings without nilpotent elements*, Proceedings of AMS **37** (1973), 381-387.

[**Mac**] S. Mac Lane, **Categories for the working mathematician**, Graduate texts in Mathematics **27**, Springer Verlag, Berlin, Germany, 1998.

[**Mar1**] M. Marshall, **Abstract Witt Rings**, Queen's Papers in Pure and Applied Mathematics **57**, Queen's University, Ontario, Canada, 1980.

[**Mar2**] M. Marshall, **Spaces of Orderings and Abstract Real Spectra**, Lecture Notes in Mathematics **1636**, Springer Verlag, Berlin, Germany, 1996.

[**Mar3**] M. Marshall, *Real reduced multirings and multifields*, Journal of Pure and Applied Algebra **205** (2006), 452-468.

[**MM1**] H. L. Mariano, F. Miraglia, *Profinite Structures are Retracts of Ultraproducts of Finite Structures*, Reports on Mathematical Logic, **42** (2007), 169-181.

[**MM2**] H. L. Mariano, F. Miraglia, *On Profinite Structures*, The Many Sides of Logics, Studies in Logic **21** (2009), College Publications, 201-224.

[**MM3**] H. L. Mariano, F. Miraglia, *The Profinite Hull of Special Groups and Local-Global Principles*, Studia Logica **97(1)** (2011), 127-160.

[**MM4**] H. L. Mariano, F. Miraglia, *The Boolean and Profinite hulls of reduced special groups*, Logic Journal of IGPL **21(2)** (2013), 160-182.

[**MM5**] H. L. Mariano, F. Miraglia, *Model-theoretic applications of the Profinite hull functor of special groups*, South American Journal of Logic **1(1)** (2015), 111-133.

[**MS**] H. L. Mariano, D. F. Santos, *Special Groups and Graded Rings*, in preparation.

[**Oli**] J. P. Olivier, *Anneaux absolument plats universels et épimorphismes à buts réduits*, Séminaire Samuel. Algèbre Commutative, tome 2 , exp. no. 6 (1968), 1-12.

[**Pie**] R.S. Pierce, **Modules over Commutative Regular Rings**, Memoirs of the AMS **70**, American Mathematical Society, Providence, USA, 1967.

[**PV**] N. Popescu, C. Vraciu, *Some remarks about the regular ring associated to a commutative ring*, Rev. Roumaine Math. Pures et Appl. **23** (1978), 269-277.

[**Smi**] K. Smith *Commutative regular rings and Boolean-valued fields*, The Journal of Symbolic Logic **49** (1984), 281-297.

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