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On Inverse Limits of Compact Structures

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For Francisco Miraglia on occasion of his 70-th birthday

Abstract

We prove that if $\mathfrak{A}$ is the inverse limit of a system of compact structures $\mathfrak{A}_i$ and continuous homomorphisms f_{ij} over an downward directed poset I, then the natural map into the ultraproduct $h : \mathfrak{A} \to \Pi_i \mathfrak{A}_i/_{\mathcal{U}}$ is pure, provided that $\mathcal{U}$ is a directed ultrafilter over I. This implies by general properties of atomic compact structures that $\mathfrak{A}$ is a retract of the ultraproduct, result which generalizes the main theorem of Mariano and Miraglia in [MM07] for profinite structures. It yields also a strengthening of known results on the preservation of first order properties under these limits. We discuss the definability and continuity of the obtained retractions.

Keywords: Inverse limits, compact structures, atomic compactness, retractions, ultraproducts.

Introduction

Inverse directed limits of first-order compact structures play a central role in several branches of mathematics as group theory [RZ10], the theory of continua [IM12], and the theory of dynamical systems [Kee08], and they are a main tool to build compactifications [HK99]. Profinite structures are the best known and most studied case of these limits.

Mariano and Miraglia proved in [MM07] that given an inverse directed system of finite structures $\mathfrak{A}_i$ and homomorphisms f_{ij}, $i \leq j$ in I, and a directed ultra filter $\mathcal{U}$ over $(I, \leq)$, then its profinite limit $\mathfrak{A}$ is a retract of the ultraproduct $\Pi_i \mathfrak{A}_i/_{\mathcal{U}}$, fact which they have applied to the study of special groups [MM13]. In this paper we generalize their result to any system of compact Hausdorff structures and continuous homomorphism. For this purpose we prove that the natural map $\mathfrak{A} \hookrightarrow \Pi_i \mathfrak{A}_i \to \Pi_i \mathfrak{A}_i/_{\mathcal{U}}$ is pure and utilize the known fact that an atomic compact structure is a retract of any of its pure extensions [Weg66].

An interesting model theoretic question is the preservation of first order properties by limits. For example, it is shown in [PW68] that any positive formula is preserved by inverse directed limits of compact structures when the bonding homomorphism f_{ij} are onto. It follows from our result that the surjectivity condition is unnecessary and the preservation is achieved for the larger family of formulas preserved by retracts. This family has been characterized by Keisler in [Kei65] and includes, among others, all formulas of the form $\forall\vec{x}(\phi(\vec{x}) \to \psi(\vec{x}))$ where $\phi(\vec{x})$ is existential and $\psi(\vec{x})$ is positive.

In the next section we present some basic facts about atomic compact structures and retracts, in the following one we explain inverse limits and the main result (Theorem 2.3) as well as some of its consequences, and in the last one we discuss the continuity and definability of the retractions. General reference for the concepts utilized in this note are [Grä08] and [Will68].

1 Preliminaries

Throughout this paper L denotes a fixed first order language. A first order L-formula ϕ is *existential* (*universal*) if it has the form $\exists\vec{x}\psi(\vec{x},\vec{y})$ (respectively, $\forall\vec{x}\psi(\vec{x},\vec{y})$) where $\psi(\vec{x},\vec{y})$ is quantifier free; ϕ is *positive* if it is constructed from atomic formulas using $\wedge$, $\vee$ and the quantifiers $\forall$ and $\exists$, and *positive existential* (*positive universal*) if it is positive but does not contain $\forall$ (respectively, $\exists$). Here, $\vec{x}, \vec{y}$ denote finite lists of variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ (perhaps empty) and $\exists\vec{x}\psi(\vec{x},\vec{y})$ abbreviates $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n, \vec{y})$; similarly for $\forall\vec{x}\psi(\vec{x},\vec{y})$.

Let $\mathfrak{A}, \mathfrak{B}$ be first order L-structures. A function $f : \mathfrak{A} \to \mathfrak{B}$ is a *L-homomorphism* if $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for each symbol of constant $c \in L$, $f\left(h^{\mathfrak{A}}(\vec{a})\right) = h^{\mathfrak{B}}\left(f(\vec{a})\right)$ for each n-ary functional symbol $h \in L$, and $\mathfrak{A} \models R(\vec{a})$ implies $\mathfrak{B} \models R(f(\vec{a}))$ for each n-ary relational symbol $R \in L$, where $\vec{a} = (a_1, \ldots, a_n)$ is any n-tuple of elements of A and $f(\vec{a})$ denotes the tuple $(f(a_1), \ldots, f(a_n))$.

Note that an homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ preserves any positive existential formula $\phi(\vec{x})$. This means that $\mathfrak{A} \models \phi(\vec{a})$ for a tuple $\vec{a}$ in A implies $\mathfrak{B} \models \phi(f(\vec{a}))$. Moreover, if f is onto it preserves all positive formulae.

Definition 1.1 An homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ is *pure* if it *reflects* positive existential formulas with parameters in A. That is, for any positive existential formula $\phi(\vec{x})$ and tuple $\vec{a}$ in $\mathfrak{A}$, if $\mathfrak{B} \models \phi(f(\vec{a}))$ then $\mathfrak{A} \models \phi(\vec{a})$.

Since it reflects the identity and the basic relations in L a pure homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ must be an *embedding*; that is, it establishes an isomorphism between $\mathfrak{A}$ and the substructure induced by $f(A)$ in $\mathfrak{B}$.

Definition 1.2 A *retraction* of an homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ is an homomorphism $g : \mathfrak{B} \to \mathfrak{A}$ such that $g \circ f$ is the identity function on $\mathfrak{A}$. In this case we say that $\mathfrak{A}$ (and f too) is a *retract* of $\mathfrak{B}$.

It's easy to check that if f admits a retraction then it is a pure embedding. However the converse is not always true, in fact the structures which are retracts of all their pure extensions are exactly the atomic compact ones (see Lemma 1.7).

Keisler has characterized in [Kei65] the sentences preserved by retractions as those which are equivalent to a prenex form $Q_1 y_1 \dots Q_n y_n \theta(y_1, \dots, y_n)$ where θ is quantifier free in conjunctive (or disjunctive) normal form, and whenever Q_i is the existential quantifier the atomic subformulas containing the variable y_i do not appear negated in θ. Call these formulas, with possible additional free variables $\vec{x}$, *retract formulas.*

Lemma 1.3 *Any retract formula $\varphi(\vec{x})$ is reflected by retracts.*

Proof. In a retract situation $\mathfrak{A} \underset{f}{\overset{g}{\leftrightarrows}} \mathfrak{B}$, $\varphi(\vec{x})$ is reflected by the retract f if and only the retraction $g : (\mathfrak{B}, f(\vec{a})) \to (\mathfrak{A}, \vec{a})$ preserves the sentence $\varphi(c_1, \dots, c_n)$ for any $\vec{a}$ in $\mathfrak{A}$. ■

We turn now to topological conditions on structures.

Definition 1.4 $(\mathfrak{A}, \tau)$ is a *topological structure* if τ is a topology on A such that $R^{\mathfrak{A}}$ is a closed subset of A^n (with the product topology) for any n-ary relation symbol $R \in L \cup \{=\}$, and $f^{\mathfrak{A}} : A^k \to A$ is a continuous for any k-ary function symbol $f \in L$.

We will write $\mathfrak{A}$ for $(\mathfrak{A}, \tau)$, the topology being understood from the context. As the diagonal must be closed, topological structures are always Hausdorff. A simple induction shows that if $\varphi(\vec{x}, \vec{y})$ is a universal positive formula then for any $\vec{b}$ in $\mathfrak{A}$ the set of realizations $\{\vec{a} : \mathfrak{A} \models \varphi(\vec{a}, \vec{b})\}$ is a closed subset of A^m.

Definition 1.5 $\mathfrak{A}$ *is a* compact *structure if it is a topological structure with a compact topology.*

In this case the projections $A^k \to A$ are closed and then the set $\{\vec{a} : \mathfrak{A} \models \varphi(\vec{a}, \vec{b})\}$ is closed in A^m for any positive formula $\varphi(\vec{y}, \vec{x})$.

A weaker notion of compactness for structures is atomic compactness, actually a form of model theoretic saturation introduced originally by Kaplansky [Kap54] for abelian groups and generalized by J. Mycielski [Myc64] to algebraic structures, where it is usually called *equational compactness*, and to general relational structures by [Weg66]. Note that in the next definition we do not limit the number of variables or parameters appearing on the set of atomic formulas.

Definition 1.6 A structure $\mathfrak{A}$ is *atomic compact* if any set of atomic formulas with parameters in A which is finitely satisfiable in $\mathfrak{A}$ is satisfiable in $\mathfrak{A}$.

A compact structure is atomic compact by an straightforward application of Tychonoff's theorem. But the reciprocal is not always true and, in fact, it is an open question whether it holds for many classes of structures. The next fact yields an useful characterization of atomic compact structures.

Lemma 1.7 ([Weg66]) *For any structure $\mathfrak{A}$ the following are equivalent:*
1. $\mathfrak{A}$ *is atomic compact.*
2. $\mathfrak{A}$ *is retract of any pure extension.*

Condition (2) in the lemma may be strengthened to: $\mathfrak{A}$ is *pure injective*; that is, for any pure homomorphism $f : \mathfrak{B} \to \mathfrak{C}$, the map $\hom(\mathfrak{C},\mathfrak{A}) \to \hom(\mathfrak{B},\mathfrak{A})$ induced by composition with f is surjective. This characterization is well known in the theory of modules and has been noticed for arbitrary structures by several people (see Appendix by G.H. Wenzel in [Grä08]).

2 Inverse directed limits of compact structures

Let $I = (I, \leq)$ be a partial ordered set (poset), we say that I is *downward directed* if for any pair i, j of elements of I, there exists $k \in I$ such that $k \leq i$ and $k \leq j$.

If $\mathcal{C}$ is any category and $(I, \leq)$ is a downward directed poset, an (*inverse directed*) *system* in $\mathcal{C}$ over I is a set of objects of $\mathcal{C}$ labeled by elements of I, say $\langle \mathfrak{A}_i : i \in I \rangle$, together with a set of morphisms

$$\langle f_{ij} : \mathfrak{A}_i \to \mathfrak{A}_j : \text{ with } i \leq j \text{ in } I \rangle$$

called *bonding maps* such that f_{ii} is the identity on $\mathfrak{A}_i$ for each $i \in I$ and $f_{kj} = f_{ij} \circ f_{ki}$ whenever $k \leq i \leq j$ in I. We denote such a system $\mathcal{A} = \langle \mathfrak{A}_i, f_{ij} \rangle_{i.j \in I}$.

Definition 2.1 *A* cone *over $\mathcal{A}$ is a pair $(\mathfrak{A}, \{f_i\}_{i \in I})$ where $\mathfrak{A}$ is an object of $\mathcal{C}$ and for any $i \in I$, $f_i : \mathfrak{A} \to \mathfrak{A}_i$ is a morphism such that $f_j = f_{ij} \circ f_i$ whenever $i \leq j$ are elements of I. An* inverse limit *of $\mathcal{A}$ is a cone $(\mathfrak{A}, \{f_i\}_{i \in I})$ over $\mathcal{A}$ such that for any other cone $(\mathfrak{B}, \{h_i\}_{i \in I})$ over $\mathcal{A}$ there exist a unique $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $h_i = f_i \circ \phi$ for any $i \in I$.*

These limits may not exist in a given category but it is straightforward to show that they are unique up to isomorphism when they do, in which case we talk about *the inverse limit* of $\mathcal{A}$, and denote it $\varprojlim \mathcal{A}$.

In the category of L-structures with L-homomorphisms, the inverse limit of any system $\mathcal{A} = \langle \mathfrak{A}_i, f_{ij} \rangle_{i,j \in I}$ always exists, if we allow empty structures and morphisms, and it is given by

$$\mathfrak{A} = \{X \in \Pi_{i \in I} \mathfrak{A}_i : \ f_{ij}\left(X(i)\right) = X\left(j\right) \text{ for any } i \leq j \text{ in } I\}$$

together with the restrictions of the projections $\pi_i : \Pi_{i \in I} \mathfrak{A}_i \to \mathfrak{A}_i$, that is, $f_i(X) = X(i)$. It is easy to see that $\varprojlim \mathcal{A} = (\mathfrak{A}, \{f_i\}_{i,j \in I})$ satisfies the defining universal property. If L does not have constants the inverse directed limit of a family of non-empty structures may be empty even for surjective bonding maps. Since it determines completely $\varprojlim \mathcal{A}$, we may identify the limit with the substructure $\mathfrak{A}$ of $\Pi_{i \in I} \mathfrak{A}_i$.

In the category of topological structures with continuous homomorphism the same construction works. In this case the limit object $\mathfrak{A}$ is a closed subset of $\Pi_i \mathfrak{A}_i$ endowed with the product topology, due to the continuity of the maps f_{ij} and the projections of the product. Therefore, if each $\mathfrak{A}_i$ is compact, $\Pi_i \mathfrak{A}_i$ and hence $\mathfrak{A}$ are compact too. Moreover, it is well known that in the compact case $\mathfrak{A}$ is non-empty (cf. [Will68]).

Definition 2.2 If I is a downward directed poset, an ultrafilter $\mathcal{U}$ over I is *directed* if for any $j \in I$ the set $(j] = \{i \in I : i \leq j\}$ is an element of $\mathcal{U}$.

Such ultrafilters exists because the directness condition ensures that for any $j_1, \ldots, j_n$ elements of I the set $(j_1] \cap, \ldots, \cap (j_n]$ is not empty. Now we are ready to state and prove our main theorem.

Theorem 2.3 *Let $\mathcal{A} = \langle \mathfrak{A}_i, f_{ij} \rangle_{i,j \in I}$ be system of compact structures and continuous homomorphism over a downward directed poset I, and suppose that $\mathcal{U}$ is a directed ultrafilter over I. If $\mathfrak{A}$ is the limit of $\mathcal{A}$ then the composite map*

$$h : \mathfrak{A} \overset{\iota}{\hookrightarrow} \Pi_i \mathfrak{A}_i \overset{\pi}{\to} \Pi_i \mathfrak{A}_i /_{\mathcal{U}}$$

is pure. Here, ι is the inclusion and π is the natural projection from $\Pi_i \mathfrak{A}_i$ to the ultraproduct $\Pi_i \mathfrak{A}_i /_{\mathcal{U}}$.

Proof. For any $X \in \Pi_i \mathfrak{A}_i$ denote $\pi(X) = X/_{\mathcal{U}}$ the class of X in the ultraproduct. For any finite $J \subseteq I$, define

$$D_J := \{X \in \Pi_i \mathfrak{A}_i : f_{jk}\left(X(j)\right) = X(k) \text{ for all } j, k \in J \text{ such that } j \leq k\}$$

which is a closed subset of the product. Given $Y_1/_{\mathcal{U}}, \dots, Y_n/_{\mathcal{U}} \in \mathfrak{A}$, suppose $\phi(y_1, \dots, y_n, x_1, \dots, x_m)$ is a quantifier free positive L-formula such that

$$\Pi_i \mathfrak{A}_i/_{\mathcal{U}} \models \exists x_1 \dots \exists x_m \phi\left(Y_1/_{\mathcal{U}}, \dots, Y_n/_{\mathcal{U}}, x_1, \dots, x_m\right)$$

Then there are elements $X_1/_{\mathcal{U}}, \dots, X_m/_{\mathcal{U}}$ in the ultraproduct such that, by Łoś theorem,

$$\{i \in I : \mathfrak{A}_i \models \phi\left(Y_1(i), \dots, Y_n(i), X_1(i), \dots, X_m(i)\right)\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is directed there is $i \leq \min J$ such that

$$\mathfrak{A}_i \models \phi\left(Y_1(i), \dots, Y_n(i), X_1(i), \dots, X_m(i)\right)\},$$

and thus, because ϕ is positive quantifier free,

$$\mathfrak{A}_j \models \phi\left(f_{ij}Y_1(i), \dots, f_{ij}Y_n(i), f_{ij}X_1(i), \dots, f_{ij}X_m(i)\right)$$

for all $j \in J$. That is,

$$\mathfrak{A}_j \models \phi\left(Y_1(j), \dots, Y_n(j), X_1'(j), \dots, X_m'(j)\right) \text{ for all } j \in J$$

where $X_t'(j) = f_{ij}(X_t(i))$ for $j \in J$ and $X_t'(j) \in \mathfrak{A}_j$ is chosen arbitrarily otherwise. Note that $X_1', \dots, X_m' \in D_J$ because for $j \leq k$ in $J : f_{jk}X_t'(j) = f_{jk}f_{ij}(X_t(i)) = f_{ik}(X_t(i)) = X_t'(j)$. This means that the set

$$E_J := \{(X_1, \dots, X_m) \in D_J^m : \mathfrak{A}_j \models \phi\left(Y_1(j), \dots, Y_n(j), X_1(j), \dots, X_m(j)\right) \text{ for all } j \in J\}$$

is non-empty. In addition, this set is closed in $(\Pi_i \mathfrak{A}_i)^m$ because the realizations of $\phi(Y_1(j), \dots, Y_n(j), x_1, \dots, x_m)$ in $\mathfrak{A}_j^m$ for $j \in J$ are closed. Besides the family $\{E_J : J \subseteq_{fin} I\}$ has the finite intersection property because $E_{J_1} \cap \dots \cap E_{J_n} = E_{\cup_i J_i}$. Then using compactness of $(\Pi_i \mathfrak{A}_i)^m$ we get

$$\{(X_1, \dots, X_m) \in \mathfrak{A}^m : \mathfrak{A} \models \phi\left(Y_1, \dots, Y_n, X_1, \dots, X_m\right)\} = \cap_{J \subseteq_{fin} I} E_J \neq \emptyset.$$

Hence, $\mathfrak{A} \models \exists x_1 \dots \exists x_m \phi\left(Y_1, \dots, Y_n, x_1, \dots, x_m\right)$. ■

Clearly, the purity of h implies non-emptyness of $\mathfrak{A}$, therefore the compactness hypothesis can not be dropped. Combining Theorem 2.3 with Lemma 1.7 we obtain:

Corollary 2.4 *The natural map h of Theorem 2.3 is an embedding and has a retraction $g : \Pi_i \mathfrak{A}_i/_{\mathcal{U}} \to \mathfrak{A}$.*

Our corollary does not give a specific construction of the retraction g while a categorical construction is provided in [MM07] for the profinite case. We give later a topological construction for the general case (Proposition 3.4).

Corollary 2.5 *Under the hypothesis of Theorem 2.3,* $\lim\limits_{\leftarrow}\mathcal{A}$ *preserves retract formulas; that is, if* $\phi(\vec{x})$ *is a retract L-formula,* $X_j \in \mathfrak{A}$*, and* $\mathfrak{A}_i \models \phi[X_1(i), \ldots, X_m(i)]$ *for all* $i \in I$ *then* $\mathfrak{A} \models \phi[X_1, \ldots, X_m]$.

Proof. Łoś theorem implies $\Pi_i\mathfrak{A}_i/_{\mathcal{U}} \models \phi\left(X_1/_{\mathcal{U}}, \ldots, X_m/_{\mathcal{U}}\right)$, and since $h(X_i) = X_i/_{\mathcal{U}}$ is a retract then Lemma 1.3 applies. ■

Retract formulas include all universal formulas, all positive formulas, and all formulas of the form $\forall\vec{x}(\phi(\vec{x}) \to \psi(\vec{x}))$ where $\phi(\vec{x})$ is quantifier free or existential and $\psi(\vec{x})$ is positive.

Examples 2.6 The limit of an inverse directed system of compact graphs where each vertex has out-degree bounded by N has the same property because this is described by the retract sentence: $\forall x\exists y_1 \ldots \exists y_N \forall y(Rxy \to y = y_1 \vee \ldots \vee y = y_N)$. Similarly, an inverse directed limit of compact planar graphs is locally planar (each finite subgraph is planar) because planarity is characterized by a set of universal sentences expressing the absences of Kuratowski's subgraphs (subdivisions of K_5 or $K_{3,3}$). Notice that these properties are not preserved by products.

3 Continuity and definability of the retraction

Notice first that it is an immediate consequence of Corollary 2.4 that $g \circ \pi$ is a retraction of the embedding $\mathfrak{A} \hookrightarrow \Pi_i\mathfrak{A}_i$. This fact may be extended to certain reduced filters $\Pi_i\mathfrak{A}_i/_{\mathcal{F}}$. Call a filter $\mathcal{F}$ over I *compatible* with a downward direct poset I if $S \cap (i] \neq \emptyset$ for all $S \in \mathcal{F}$ and $i \in I$. For example, the smallest filter $\mathcal{F} = \{I\}$, the Fréchet filter $\mathcal{F} = \{S \subseteq I : I \setminus S \text{ is finite}\}$ if I does not have a minimum, or any directed ultrafilter are compatible with I. Then we have the following generalization of 2.3

Theorem 3.1 *Let* $\mathcal{A} = \langle \mathfrak{A}_i, f_{ij}\rangle_{i,j\in I}$ *be a system of compact structures and continuous homomorphism over a downward direct poset* I*. Then the limit object* $\mathfrak{A}$ *is a retract of the product* $\Pi_i\mathfrak{A}_i$ *and, more generally, from any reduced product* $\Pi_i\mathfrak{A}_i/_{\mathcal{F}}$ *where* $\mathcal{F}$ *is compatible with* I*.*

Proof. $\mathcal{F}$ may be extended to a directed ultrafilter $\mathcal{U}$ over I and thus the projection π factors in the form: $\Pi_i\mathfrak{A}_i \xrightarrow{\pi'} \Pi_i\mathfrak{A}_i/_{\mathcal{F}} \xrightarrow{\rho} \Pi_i\mathfrak{A}_i/_{\mathcal{U}}$. By Corollary 2.4, $g \circ \rho$ is a retraction of $\mathfrak{A} \xhookrightarrow{\iota} \Pi_i\mathfrak{A}_i \xrightarrow{\pi'} \Pi_i\mathfrak{A}_i/_{\mathcal{F}}$. ■

A natural question is wether the retraction $g : \Pi_i\mathfrak{A}_i/_{\mathcal{U}} \to \mathfrak{A}$ may be chosen continuous when $\Pi_i\mathfrak{A}_i/_{\mathcal{U}}$ is endowed with the quotient topology induced by π. The answer to this question is in general negative. Notice first that by definition of quotient topology

g is continuous if and only if the induced retraction $g \circ \pi : \Pi_i \mathfrak{A}_i \to \mathfrak{A}$ is continuous, but the next examples show that in very simple situations there is no continuous map from $\Pi_i \mathfrak{A}_i$ onto the limit $\mathfrak{A}$.

Examples 3.2 The "topologist's sine curve" T is an inverse limit of the system

$$\cdots \xrightarrow{f} [0,1] \xrightarrow{f} [0,1] \xrightarrow{f} [0,1]$$

where $f(x) = 2x$ in $[0, \frac{1}{2}]$ and $f(x) = \frac{3}{2} - x$ in $[\frac{1}{2}, 1]$ (Example 16, [IM12]). Moreover, $[0,1]^\omega$ is path connected while T is a favorite example of a non path connected space, and continuous images preserve path connectedness; hence, there is no continuous surjection $[0,1]^\omega \to T$. A similar situation is obtained for compact groups since the limit of the inverse system

$$\cdots \xrightarrow{z^2} \mathbb{S}^1 \xrightarrow{z^2} \mathbb{S}^1 \xrightarrow{z^2} \mathbb{S}^1,$$

where $\mathbb{S}^1$ is the circle group, is a "solenoid" which is not locally connected (see [Kee08]), but a continuous image of the locally connected space $(\mathbb{S}^1)^\omega$ should be locally connected (Th. 27.12, [Will68]).

These examples show that path connectedness or local connectedness are not preserved by limits of compact Hausdorff systems. On the other hand it is well known that connectedness is preserved (Th. 116, [IM12]).

To finish, we give a topological description of the retraction $g : \Pi_i \mathfrak{A}_i/_{\mathcal{U}} \to \mathfrak{A}$. For this purpose we recall the definition of *$\mathcal{U}$-limit* for an I-family $\{a_i : i \in I\}$ in a topological space X and an ultrafilter over I (see [S78]):

Definition 3.3 $\{a_i\} \to_{i \in I\ \mathcal{U}} x$ *if and only if* $\{i \in I : a_i \in V\} \in \mathcal{U}$ *for any open neighborhood* V *of* x.

The following properties are well known or part of the folklore on the subject:
- In Hausdorff spaces $\mathcal{U}$-limits are unique, then we may write $x = \lim_{i \in I\ \mathcal{U}} \{a_i\}$.
- In a compact space any I-family has a $\mathcal{U}$-limit.
- In product spaces $\mathcal{U}$-limits are computed componentwise.
- $\mathcal{U}$-limits are preserve by continuous functions.
- Closed sets are closed under $\mathcal{U}$-limits.
- If $S \in \mathcal{U}$ then $\lim_{i \in I\ \mathcal{U}} \{a_i\} = \lim_{i \in S\ \mathcal{U} \restriction S} \{a_i\}$, where $\mathcal{U} \restriction S = \mathcal{U} \cap P(S)$.

Proposition 3.4 *Under the hypothesis of Theorem 2.3, define* $g_{\mathcal{U}}(X/_{\mathcal{U}}) = Y \in \Pi_i \mathfrak{A}_i$ *coordinatewise as*

$$Y(j) = \lim_{i \leq j\ \mathcal{U} \restriction (j]} \{f_{ij} X(i)\} \text{ in } \mathfrak{A}_j$$

Then the map $g_{\mathcal{U}}$ *is a retraction of the embedding* $h : \mathfrak{A} \to \Pi_i \mathfrak{A}_i/_{\mathcal{U}}$.

Proof. To see that $g_{\mathcal{U}}$ is well defined assume $X/_{\mathcal{U}} = X'/_{\mathcal{U}}$, let $S = \{i \in I : X(i) = X'(i)\} \in \mathcal{U}$ and fix $j \in I$. As $(j] \cap S \in \mathcal{U}$ we have:

$$\begin{aligned} \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{f_{ij}X(i)\} &= \lim_{i\in(j]\cap S\ \mathcal{U}\restriction(j]\cap S} \{f_{ij}X(i)\} \\ &= \lim_{i\in(j]\cap S\ \mathcal{U}\restriction(j]\cap S} \{f_{ij}X'(i)\} = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{f_{ij}X'(i)\}. \end{aligned}$$

$g_{\mathcal{U}}$ is a retraction because by continuity of $f_{jj'}$:

$$f_{jj'}Y(j) = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{f_{jj'}f_{ij}X(i)\} = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{f_{ij'}X(i)\} = \lim_{i\leq j'\ \mathcal{U}\restriction(j']} \{f_{ij'}X(i)\} = Y(j').$$

and thus $Y \in \mathfrak{A}$. Moreover, if $X \in \mathfrak{A}$ then

$$Y(j) = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{f_{ij}X(i)\} = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{X(j)\} = X(j)$$

and thus $g_{\mathcal{U}}(h(X)) = g_{\mathcal{U}}(X/_{\mathcal{U}}) = X$. To see that $g_{\mathcal{U}}$ is an homomorphism assume $(X_1/_{\mathcal{U}}, \ldots, X_m/_{\mathcal{U}}) \in R^{\Pi_i\mathfrak{A}_i/_{\mathcal{U}}}$, without loss of generality $(X_1(i), \ldots, X_m(i)) \in R^{\mathfrak{A}_i}$ for all i, and fixing $j : (f_{ij}X_1(i), \ldots, f_{ij}X_m(i)) \in R^{\mathfrak{A}_j}$ for any $i \leq j$.Thus

$$(Y_1(j), \ldots, Y_m(j)) = \lim_{i\leq j\ \mathcal{U}\restriction(j]} \{(f_{ij}X_1(i), \ldots, f_{ij}X_m(i))\} \in R^{\mathfrak{A}j},$$

because $R^{\mathfrak{A}j}$ is a closed in $\mathfrak{A}_j^m$. Therefore, $(g(X_1/_{\mathcal{U}}), \ldots, g(X_m/_{\mathcal{U}})) = (Y_1, \ldots, Y_m) \in R^{\Pi_i\mathfrak{A}_i}$. ■

With a little patience it may be verified that for profinite limits our construction yields the same retraction that the categorical construction given in [MM07]. Other constructions we have attempted yield the same map thus we may wonder if the retraction is unique in some sense with respect to $\mathcal{U}$. Clearly, the induced retraction $g_{\mathcal{U}} \circ \pi : \Pi_i\mathfrak{A}_i \to \mathfrak{A}$ is not unique as there is one for each directed ultrafilter.

Although it may be impossible to have $g_{\mathcal{U}}$ continuous for the Tychonoff topology, the following last observation may be useful. Assume $\mathcal{A} = \langle \mathfrak{A}_i, f_{ij}\rangle_{i,j\in I}$ is an inverse directed system of equi-bounded compact metric structures $\mathfrak{A}_i = (\mathfrak{A}_i, d_i)$ with 1-Lipschitz bonding maps; that is, $d_j(f_{ij}x, f_{ij}y) \leq d_i(x, y)$, then $\Pi_i\mathfrak{A}_i$ and the limit $\mathfrak{A}$ may be endowed with the *sup* metric: $d(X, X') = \sup_{i\in I} d_i(X(i), X'(i))$, which induces the *uniform* topology. The ultraproduct $\Pi_i\mathfrak{A}_i/_{\mathcal{U}}$ inherits then a not necessarily metrizable quotient topology. Compactness of the limit may be lost, but in this circumnstances we have:

Proposition 3.5 *The retraction $g_{\mathcal{U}} : \Pi_i\mathfrak{A}_i/_{\mathcal{U}} \to \mathfrak{A}$ defined in Proposition 3.4 is continuous for the uniform topology, and so is the retraction $g_{\mathcal{U}} \circ \pi : \Pi_i\mathfrak{A}_i \to \mathfrak{A}$.*

Proof. It is enough to show that $g_{\mathcal{U}} \circ \pi$ is continuous. Assume $g_{\mathcal{U}}\pi(X) = Y$ and $g_{\mathcal{U}}\pi(X') = Y'$ as in Proposition 3.4 and $d(X, X') < \varepsilon$ in $\Pi_i \mathfrak{A}_i$. Then $d_i(X(i), X'(i)) < \varepsilon$ for all i and thus, for fixed $j : d_j(f_{ij}X(i), f_{ij}X'(i)) < \varepsilon$ for any $i \leq j$. Since d_j is continuous in $\mathfrak{A}_j \times \mathfrak{A}_j$ then $d_j(Y(j), Y'(j)) = d_j(\lim_{i\leq j\ \mathcal{U}\upharpoonright(j)}\{f_{ij}X(i)\}, \lim_{i\leq j\ \mathcal{U}\upharpoonright(j)}\{f_{ij}X'(i)\}) = \lim_{i\leq j\ \mathcal{U}\upharpoonright(j)}\{d_j(f_{ij}X(i), f_{ij}X'(i))\} \leq \varepsilon$, which implies $d(Y, Y') \leq \varepsilon$. ∎

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