South American Journal of Logic
Vol. 2, n. 2, pp. 165–174, 2016
ISSN: 2446-6719

A Quarter of a Century of Joint Work with Chico Miraglia

Max Dickmann

Abstract

This paper is a slightly enlarged write-up of my talk in the workshop "Logic and Applications; in honor to F. Miraglia's 70-th birthday", held in São Paulo, September 16-17, 2016. It gives an overview of our scientific collaboration during the years 1992–2016.

Keywords: Algebraic theory of quadratic forms, special group, Marshall's conjecture, Lam's conjecture, K-theory, Pfister index, realisability problem, T-faithfully quadratic ring, real semigroup.

Meeting Chico.

I met Chico Miraglia in Brazilia in 1972. At that time Chico was Professor at the Institute of Mathematics of the University of São Paulo, and I was Professor of Mathematics at the Catholic University of Santiago, Chile, where Rolando Chuaqui had invited me to take up a position (the Pinochet coup d'état of September 1973 put a sure end to that appointment).

Already in that first meeting both Chico and I had the pleasure, through long and substantive talks, of realising that, beyond a common interest in mathematics, and far more importantly, we also shared a "Weltanschauung", a vision of the world.

For 20 years after that first meeting our lives went on separate tracks. We met again in 1992, when Chico was visiting Oxford for a couple of years and, for personal reasons, came to Paris with a certain regularity. It was then that our collaboration in the field of mathematics really began —needless to say, without forgetting all the rest (that remains, today, as alive as in the day we first met). It was through this process of collaboration that I became aware of Chico's outstanding mathematical talent, to which our joint work owes so much.

I think the best gift I can offer Chico on his 70-th birthday is a brief revival of our joint work through a quarter of a century. An attempt follows.

Our Joint Work; Some Highlights.

(1) **Reduced special groups.** In the early 1990s I was interested in and had began work on the algebraic theory of quadratic forms, i.e., homogeneous quadratic polynomials in many variables with coefficients in *arbitrary* fields [1]. My interest on this area of research went back to personal exchanges with Murray Marshall —who, unfortunately, passed away in 2015— during a stay in Berkeley in 1991. Marshall made me acquainted with his work on abstract spaces of orders (hereafter called AOS), an axiomatic theory of an algebraic-topological nature, inspired on, and generalizing, that of spaces of orders of fields. During these exchanges I got the strong impression that Marshall's AOSs were the topological leg of a duality —modeled on Stone's famous duality Boolean algebras/Stone spaces— whose algebraic side remained to be discovered. Shortly afterwards I found the axioms for the algebraic structures whose duals were Marshall's AOS, and proved the duality [2]. I baptised these algebraic structures **reduced special groups** (abbreviated RSG) and studied their basic algebraic properties, which later appeared, in a more complete form, in [DM3].

To be sure, RSGs were not the first algebraic axiomatisation of the (reduced) theory of quadratic forms. Previous attempts include the *quadratic form schemes* (see [Ku]), and the *abstract Witt rings* of Marshall himself [Ma1]. However, these axiomatisations are based on primitive notions that appear only after a rather long travel through the algebraic theory of quadratic forms (e.g., the notion of equivalence of binary Pfister forms). On the contrary, RSGs begin at simple first principles, and their formulation does not require any prior knowledge of the algebraic theory of quadratic forms; even more, this theory can be entirely developed in the framework of RSGs (this is actually done in the monograph [DM3]).

All this was in ferment when Chico appeared in Paris (1992); he got immediately interested in the subject. I also suceeded to arise the interest of Arileide Lira de Lima, a Brazilian student working on her Ph. D. under my direction in Paris. [3]

(2) **Marshall's conjecture.** The reception of these ideas in the community of practitioners of quadratic form theory was luckworm ("Oh!; one more axiomatic version!"). Chico and I realized that one —admittedly radical— way to credit our approach was to try to solve, using the technology of RSGs, a significant open problem in the field. An interesting problem of that type was posed by Marshall in 1974, and was still was open in the middle of the 1990s. Namely:

> To give an arithmetical characterization of the n-th powers of the fundamental ideal of the Witt ring $W(F)$ of quadratic forms of a formally real

[1] Most frequently, of characteristic $\neq 2$; the characteristic 2 case is a different science.
[2] In fact, this duality *literally* generalizes Stone's.
[3] Her Ph. D. dissertation, [Li], was defended in 1995.

> Pythagorean field F. This ideal, denoted by $I^n(F)$, is generated by Pfister forms of degree n, i.e., forms of the shape $\bigotimes_{i=1}^{n}\langle\, 1, a_i\,\rangle$, with $a_i \in F\setminus\{0\}$. Therefore, the forms $\varphi \in I^n(F)$ have signature $\mathrm{sgn}_\sigma(\varphi)\equiv 0 \pmod{2^n}$ at every order σ of F. The open question was *whether the converse holds*.

In the absence of a manageable criterion, proving that a given quadratic form lies in $I^n(F)$ is a practically impossible task, except for very obvious cases; this is due to the intractable nature of cancellation in sums of quadratic forms modulo Witt-equivalence.

In December 1996 I thought I had an affirmative proof using the circle of ideas and methods of RSGs. At once I sent my argument to Chico, who, very quickly, found a gap. Having pinpointed the difficulty, in January 1997, together with Chico in Paris, we looked at a rather cryptic manuscript of Voevodsky, [Vo], giving some hints towards the solution of a famous conjecture of Milnor, [Mi]; we really DID NOT understand much [4]. We decided to consult an accredited translator —Bruno Kahn, whose office in the Jussieu building in Paris was accross mine. We asked him if a certain statement in Voevodsky's manuscript had the meaning we believed it had. Kahn 's reply was "yes"; at that very moment we realized that this result filled up our gap and that we had solved Marshall's conjecture, later published in [DM1] [5].

(3) **Lam's conjecture.** Pursuing the same line of research we later turned our attention to a "torsion" variant of Marshall's conjecture, called Lam's conjecture, dating from 1976 (see [La], Open Problem B). This second conjecture widely generalizes the field of application of the arithmetical characterization quoted above, to *arbitrary* formally real fields, by the simple (and necessary) expedient of weakening the conclusion $\varphi \in I^n(F)$ to $\varphi \in I^n(F) + W_t(F)$, where $W_t(F)$ denotes the torsion subgroup of $W(F)$. [6]

We succeeded in proving Lam's conjecture, [DM5]; this required a refinement of Voevodsky's result due to Orlov-Vishik-Voevodsky [OVV]. Besides this result, our proof of Lam's conjecture involves the following main ingredients:

- A K-theory of (not necessarily reduced) special groups which we developed in [DM8].
- The technical, but crucially important notion of RSGs satisfying a K-theoretic condition that we call "strong Marshall conjecture"; we introduced this notion in [DM5] and further studied it in [DM8].

[4] After a lot of additional (and later) work, Voevodsky's manuscript turned out to lay the basis of his celebrated solution to Milnor's conjecture that led to his Fields Medal (2002). However, at the time we looked at it, we had to accept that Voevodsky spoke Russian, while we expressed ourselves in Portunhol (a hybrid of Portuguese and Spanish).

[5] Proof also included as Chapter 9 in [DM3] at the referee's request. Accessible expositions of the solution to Milnor's conjecture are [Ka] or [Mo].

[6] Note that $\varphi \in W_t(F)$ implies $\mathrm{sgn}_\sigma(\varphi) = 0$ for every order σ of F; thus, the addition of the torsion term is a necessary condition.

Our joint work in this period also dealt with the following issues:

(4) **Bounds.** As indicated above (see item (2) above), every quadratic form $\varphi \in I^n(F)$ ($n \geq 1, F$ a Pythagorean field) is a linear combination (with coefficients in F) of Pfister forms of degree n. In [DM6] we considered the problem of whether there is a uniform bound, depending on n and (possibly) other parameters, on the number of Pfister forms sufficient to represent forms $\varphi \in I^n(F)$ as linear combinations. Rather surprisingly, the answer turned out to be "yes" and the only additional parameter required is the dimension (= number of variables) of the form φ. Explicitly:

Main Results 1. *With notation as above, we have:*
(i) Fix integers $n, m \geq 1$. We prove that if $\varphi \in I^n(F)$ and $\dim(\varphi) = m$, there is a uniform bound on the number k of Pfister forms $\psi_1, \ldots, \psi_k$, of degree n such that, in $W(F)$ we have $\varphi \approx \bigotimes_{i=1}^{k} a_i\psi_i$, with $a_i \in F \setminus \{0\}$ [7]. "Uniform" means that the bound k does not depend on the form φ (of dimension m) nor on the field F. The bound k is a bounded recursive function *of m and n.*
(ii) Similar results hold for the following classes of RSGs:

- *The RSGs $G_T(F)$ associated to a preorder T of F.*
- *The RSGs associated to formally real fields whose Pythagoras number is bounded by a fixed integer.*

(iii) We discovered a class of RSGs for which the bound above has the form cm^{n-1} (c a constant). This class has interesting closure properties under certain constructions.
(iv) For the class of RSGs of finite stability index s [8], the bound k above is of the form $c\,[m/2^n]([m/2^n]+1)$, where c is a constant depending on s. □

However, in our view the most important feature of this work consists in having introduced and dealt with the notion of **Pfister index** of a quadratic form $\varphi \in I^n(G)$ (G a special group):

$$I(n, \varphi, G) = \text{the least integer } k \text{ such that } \varphi \text{ is a linear combination with coefficients in } G \text{ of at most } k \text{ Pfister forms of degree } n.$$

$$I(n, m, G) = \sup\{I(n, \varphi, G) \mid \varphi \in I^n(G) \text{ and } \dim(\varphi) = m\} \in \mathbb{N} \cup \{\infty\}.$$

The existence of these bounds follows from the solutions to the Marshall and Lam conjectures by an argument using ultraproducts.

[7] $\approx$ stands for Witt-equivalence of the given forms, i.e., equality in the ring $W(F)$.
[8] Cf. [Ma2], § 3.4, pp. 47 ff., for the meaning of this notion.

(5) **The realisability problem.** This problem, to my knowledge first posed by Marshall, asks whether

> Every RSG is isomorphic to the RSG $G_{\rm red}(K) = K/K^2$ associated to some Pythagorean, formally real field K.

One may also consider variants where, e.g., $G_{\rm red}(K)$ is replaced by $G_T(K) = K/T$, with K a formally real field and T a preorder of K.

Since the category of RSGs has rich functorial properties while that of fields DOES NOT, the realisability problem formulated, as above, in terms of "isomorphisms" may lead to a negative answer or even may turn out to be independent of the axioms of set theory. A more tractable reformulation may be obtained by replacing the word "isomorphic" by "elementary equivalent" (or even weaker logical notions).

In fact, our paper [DM10] substantiates a cautious feeling towards an affirmative solution to the realisability question in Marshall's original terms. We show that, by moderately enlarging the class of representation objets, the answer to the problem, thus reformulated, *is positive.* Precisely, we prove :

Theorem 2. *Every* RSG *is isomorphic to the* RGS $G(S) = S/(\mathcal{C}(X)^{\times})^2$ *associated to a subgroup* S *of the multiplicative group* $\mathcal{C}(X)^{\times}$ *of invertible continuous real valued functions on a Boolean space* X.[9] □

Note that the "representation universe" $\mathcal{C}(X)$ is:

- A Pythagorean ring.
- A ring with "many units" (cf. [DM10], Ch. 6, pp. 61 ff., or [Ma2], p. 153).
- A real closed ring in the sense of Schwartz, [Sch].

In the case of countable RSGs, the Boolean space X can uniformly be chosen as Cantor's space.

(6) **A change of paradigm. Faithfully quadratic rings.**

In 2004-2005, almost imperceptibly, a change set in our line of research, a change that resulted, 10 years later, in the monograph [DM12] published in November 2015. Which were the guidelines of this new direction of work ?

The main idea was to apply the tools of RSGs beyond the realm of fields, to study quadratic form theory over certain classes of rings. We extend the algebraic theory of quadratic forms over fields of characteristic $\neq 2$ to diagonal quadratic forms [10] with

[9] S contains $1, -1$, is closed under product and $(\mathcal{C}(X)^{\times})^2 \subseteq S$.

[10] Recall that, in general, quadratic forms with entries in a ring *are not* diagonizable.

invertible coefficients over various classes of commutative, unitary rings, where -1 is not a sum of squares and 2 is invertible. The main ingredients required for this extension are:

- An extension of the classical notion of isometry of quadratic forms via matrices to a suitable notion of T-**isometry**, where $T = A^2$ or T is a preorder of A [11].
- The introduction of three axioms that express simple properties of the representation of elements of the ring by quadratic forms; these properties are well-known to hold over fields of characteristic $\neq 2$. [12]

Our main results are:

Main Results 3. *In rings* $\langle A, T \rangle$ *satisfying these three axioms, the following hold:*
(i) *The associated structure* $G_T(A) = A^\times/T$ *is a special group.*
(ii) *Quadratic form theory in* A *based on the* (*intrinsic*) *notion of* T*-isometry is identical to that based on* (*formal*) *isometry in the RSG* $G_T(A)$.
(iii) *In case* $T = A^2$, *the mod* 2 *Milnor K-theory of* A (*cf.* [*Gu*]) *is naturally isomorphic to that of the special group* $G(A)$. □

Item (ii) in these results justifies the name ***T*-faithfully quadratic** that we have chosen for rings satisfying the three axioms alluded to above. Conversely, for rings satisfying a mild additional assumption, the faithful representation of T-isometry given by item (ii) implies the validity of the mentioned axioms.

Item (ii) of the preceding results is a very powerful tool. It guarantees that the following structural results —the backbone of the algebraic theory of quadratic forms in the case of fields of characteristic $\neq 2$ (or $= 0$)— are still valid for T-faithfully quadratic rings under T-isometry:

Main Results 4. *The following results are valid for* T*-faithfully quadratic rings,* A*, under* T*-isometry, where* $T = A^2$ *or* T *is a preorder of* A*:*
(i) *The Arason-Pfister Haupsatz.*
(ii) *Milnor's Witt ring conjecture mod 2.*
(iii) *Marshall's signature conjecture.*
(iv) *Uniform upper bounds for the Pfister index of quadratic forms* (*see Main Results 1.*(i)).
(v) *A generalization of Sylvester's local-global principle.* □

[11] A^2-isometry is just matrix isometry.

[12] **Warning.** These axioms involve several notions of "representation of elements by quadratic forms" which are different in the context of rings, but coincide in that of fields.

Once these structural results have been established, the largest part of the monograph [DM12] is devoted to prove that several important classes of rings are T-faithfully quadratic for various (possibly all) T. Thus, we have:

Main Results 5. *The following results hold:*
(i) *T-quadratic faithfulness is preserved under arbitrary products* ([*DM12*], *Thm.* 4.6)
(ii) *With a few exceptions, rings with many units are* completely *faithfully quadratic, i.e., T-faithfully quadratic for $T = A^2$ and all preorders T* ([*DM12*], *Thm.* 6.5).
(iii) *Reduced f-rings, A, are T-faithfully quadratic for all preorders T containing the natural partial order $T_\sharp^A$ of A. Further, the RSG $G_T(A)$ associated to the preordered ring $\langle A, T \rangle$ is a Boolean algebra: it is a quotient of the Boolean algebra of idempotents of A* ([*DM12*], *Thm.* 8.21).
(iv) *In the important special case of* (iii) *where $A = \mathcal{C}(X)$ = the ring of continuous real-valued functions on a topological space X, it follows that $\mathcal{C}(X)$ is* completely *faithfully quadratic* ([*DM12*], *Prop.* 8.25).
(v) Strictly representable *preordered rings $\langle A, T \rangle$* [13] *are T-faithfully quadratic and their associated RSGs are Boolean algebras* ([*DM12*], *Thm.* 9.9). □

The logical form of the axioms for (T-)quadratic faithfulness.

Main Results 6. (i) *The axioms defining the notion of a T-faithfully quadratic ring are first-order in the language $\{+, \cdot, 0, 1, -1\}$ for unitary rings, augmented with a unary predicate T. They are geometric sentences.*
(ii) *Hence, the class of T-faithfully quadratic rings is closed under right-directed inductive limits; since it is also closed under arbitrary products* ([*DM12*], *Thm.*4.6)*, it is closed under reduced products. It follows that,*
(iii) *The class of T-faithfully quadratic rings admits an axiomatization by Horn-geometric sentences.* However,
(iv) *We only have an explicit Horn-geometric axiomatization for faithfully quadratic rings* (*i.e., when $T = A^2$*). □

(7) **New horizons. Real semigroups.**

In the last years Chico joined —bringing along his proverbial energy and talent— a line of research on which A. Petrovich (Buenos Aires) and I started to work around 2000: the theory of **real semigroups** (see [DP]). This is an approach to the theory of quadratic forms with *arbitrary* (i.e., not necessarily invertible) coefficients, over **semi-**

[13] I.e., preordered rings having a representation dense in an algebra $\mathcal{C}(X)$ for some compact Hausdorff space X, where the elements of T are represented by non-negative functions and those of $T^\times$ by strictly positive functions. Equivalently, Archimedean preordered rings with bounded inversion.

real rings, i.e., rings where -1 is not a sum of squares, based on their real spectra [14]. Though much has been done (and even more remains to be done), real semigroups have already had an important role in our joint work with Chico: for example, in [DM11], in Ch. 6 of [DM12], in the forthcoming paper [DMP], and in ongoing work.

References

[**1**] **Publications by Max Dickmann and Francisco Miraglia in refereed journals**

[DM1] On quadratic forms whose total signature is zero modulo 2^n. Solution to a problem of M. Marshall. *Inventiones Mathematicae* 133(2):243–278, 1998.

[DM2] Algebraic K-theory of Fields and Special Groups. In: *Advances in Contemporary Logic and Computer Science* (W. Carnielli and I. D'Ottaviano, eds.), *Contemporary Mathematics*, 235:83–98, Amer. Math. Soc., 1999.

[DM3] *Special Groups; Boolean-Theoretic Methods in the Theory of Quadratic Forms. Memoirs of the American Mathematical Society*, vol. 145, n. 689, 247 pp., 2000.

[DM4] Orders and Relative Pythagorean Closures. *Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo*, 5:63–84, 2001.

[DM5] Lam's Conjecture. *Algebra Colloquium*, 10(2):149–176, 2003.

[DM6] Bounds for the Representation of Quadratic Forms. *Journal of Algebra*, 268(1):209–251, 2003.

[DM7] Elementary Properties of the Boolean Hull and Reduced Quotient Functors. *The Journal of Symbolic Logic*, 68(3):946–971, 2003.

[DMM] (With M. Marshall) Lattice-Ordered Reduced Special Groups. *Annals of Pure and Applied Logic*, 132(1):27–49, 2005.

[DM8] Algebraic K-theory of Special Groups. *Journal of Pure and Applied Algebra*, 204(1):195–234, 2006.

[DM9] Quadratic Form Theory over Preordered von Neumann-Regular Rings. *Journal of Algebra*, 319(4):1696–1732, 2008.

[14] Again, this approach was initiated by Marshall in Chs. 6–8 of [Ma2] from the dual perspective of **abstract real spectra**.

[DM10] Representation of Reduced Special Groups in Algebras of Continuous Functions. In: *Quadratic Forms−Algebra, Arithmetic and Geometry* (R. Baeza, W. K. Chan, D. W. Hoffmann, R. Schulze-Pillot, eds.), *Contemporary Mathematics*, 493:83–97, 2009.

[DM11] Rings and Real Semigroups. In: *Logic Without Frontiers* (J.-Y. Béziau and M. E. Coniglio, eds.), *Tribute Series*, vol. 27., pp. 23–51. College Publications, London, 2011.

[DM12] *Faithfully Quadratic Rings. Memoirs of the American Mathematical Society*, vol. 238, n. 1128, 142 pp., 2015.

[DM13] Faithfully Quadratic Rings. A Summary of Results. In: *Algebra, Logic and Number Theory*, Banach Center Publ. 108, pp. , 37–48. Institute of Mathematics, Polish Acad. of Sciences, Warszawa 2016.

[DMP] (With A. Petrovich) Constructions in the Category of Real Semigroups. In: *Ordered Algebraic Structures and Related Topics* (F. Broglia et al., eds.), *Contemporary Mathematics*, to appear, 2017.

[**2**] Further references

[DP] M. Dickmann and A. Petrovich. Real Semigroups and Abstract Real Spectra, I. *Contemporary Mathematics*, 344:99–119, 2004.

[Gu] D. Guin. Homologie du groupe linéaire et K-théorie de Milnor des anneaux. *Journal of Algebra*, 123(1):27–59, 1989.

[Ka] B. Kahn. La conjecture de Milnor (d'après Voevodsky). *Séminaire Bourbaki* 1996/97, Exp. No. 834, *Astérisque*, 245:379–418, 1997.

[Ku] M. Kula. Fields with prescribed quadratic form schemes. *Mathematische Zeitschrift*, 167:201–212, 1979.

[La] T. Y. Lam. Ten lectures on quadratic forms over fields. In: *Conference on Quadratic Forms – 1976* (G. Orzech, ed.), *Queen's Papers in Pure and Applied Mathematics*, 46, pp. 1–102. Queen's Univ., Ontario, Canada, 1977.

[Li] A. Lira de Lima. *Les groupes spéciaux*. Ph. D. thesis, Univ. of Paris VII, 1995.

[Ma1] M. Marshall. Abstract Witt Rings. *Queen's Papers in Pure and Applied Mathematics*, 57, Queen's Univ., Ontario, Canada, 1980.

[Ma2] M. Marshall. *Spaces of Orderings and Abstract Real Spectra. Lecture Notes in Mathematics*, vol. 1636. Springer, Berlin, 190 pp., 1996.

[Mi] J. Milnor. Algebraic K-theory and quadratic forms. *Inventiones Mathematicae*, 9:318–344, 1970.

[Mo] F. Morel. Voevodsky's proof of Milnor's Conjecture. *Bulletin of the American Mathematical Society* (N.S.), 35:123–143, 1998.

[OVV] D. Orlov, A. Vishik and V. Voevodsky. Motivic cohomology of Pfister quadrics and Milnor's conjecture on quadratic forms. Preprint.

[Sch] N. Schwartz. Real Closed Rings. In: *Algebra and Order* (Luminy-Marseille, 1984), *Res. Exp. Math.*, 14, pp. 175–194. Heldermann, Berlin, 1986.

[Vo] V. Voevodsky. The Milnor Conjecture. Preprint, December 1996.

Max Dickmann
Institut de Mathématiques de Jussieu – Paris Rive Gauche
CNRS, Universités Paris 6 et 7
Paris, France
E-mail: dickmann@math.univ-paris-diderot.fr