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Addendum to "*Model Complete Expansions of the Real Field by Modular Functions and Forms*"

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Abstract

We add a result to make explicit an implicit argumentation on the model completeness results of the cited paper, specifically [1, Theorem 3.1, pp. 326-327], and also comments on extensions of the main results to more general settings, which use the same kind of techniques.

1 Introduction

Jonathan Pila from Oxford University called the attention of the author to the omission of the result we prove in the following section. Although it is simple, it requires some tricky calculations which may not be obvious to the nonspecialist. The author wishes to thank him for this and also for pointing out the comments we reproduce in the end of this note.

The missing detais belong to the proof of [1, Theorem 3.1, pp. 326-327], on the strong definability of a complex analytic continuation of the real and imaginary parts of the given complex analytic functions.

2 The Results

The model completeness criterium [1, Theorem 2.3, p. 324] used in that paper requires that

> *(...) for each $\lambda \in \Lambda$, F_λ is the restriction to a compact polyinterval $D_\lambda \subseteq \mathbb{R}^{n_\lambda}$ of a real analytic function whose domain contains D_λ, and defined as zero outside D_λ, such that there exists a complex analytic function g_λ defined in a neighbourhood of a polydisk $\Delta_\lambda \supseteq D_\lambda$ and such that*
>
> 1. *g_λ is strongly definable in $\hat{R}$ and the restriction of g_λ to D_λ coincides with F_λ restricted to the same set; (...)*

The verification of this part is what was omitted and which we detail now.

Lemma 2.1 *The real and imaginary parts, $F_R(x, y)$ and $F_I(x, y)$, of the complex analytic function $F(z)$, $z = x+iy$, defined in a polydisk $\Delta_\rho = \{z \in \mathbb{C}^n : |z_i| < \rho,\ 1 \le i \le n\}$, are real analytic functions and admit complex analytic continuations*

$$\tilde{F}_R(z, w) = \frac{F(z + iw) + \overline{F(\bar{z} + i\bar{w})}}{2}, \quad \tilde{F}_I(z, w) = -\frac{iF(z + iw) - i\overline{F(\bar{z} + i\bar{w})}}{2},$$

with $z, w \in \Delta_{\rho/2}$, where the bar over F and the variables indicates complex conjugation.

Proof. We first show that $\tilde{F}_R$ and $\tilde{F}_I$ are complex analytic functions, and because the imaginary part of a complex analytic function f is the real part of $(-if)$, it is sufficient to prove that $\tilde{F}_R$ is complex analytic. Also, because

$$\tilde{F}_R(z, w) = \frac{F(z + iw) + \overline{F(\bar{z} + i\bar{w})}}{2},$$

it is enough to prove that if $f(z)$ is a complex analytic function, then $g(z) = \overline{f(\bar{z})}$ is complex analytic. We devlop $f(z)$ in a power series

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha,$$

where $z = (z_1, \ldots, z_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $c_\alpha \in \mathbb{C}$ and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. Then

$$\overline{f(\bar{z})} = \sum_{\alpha \in \mathbb{N}^n} \bar{c}_\alpha \bar{\bar{z}}^\alpha = \sum_{\alpha \in \mathbb{N}^n} \bar{c}_\alpha z^\alpha,$$

which is complex analytic.

Now, for $z, w \in \Delta_{\rho/2}$, we have that $z + iw \in \Delta_\rho$.

It is clear that the restrictions of $\tilde{F}_R$ and $\tilde{F}_I$ to real variables coincide with F_R and F_I, as required. ■

Theorem 2.2 *The functions $\tilde{F}_R$ and $\tilde{F}_I$ restricted to the real polyinterval $[-\rho/2, \rho/2]^{4n} \subset \mathbb{C}^{2n}$ (the real and imaginary parts of the variables) is strongly definable in the structure $\langle \mathbb{R}, 0, 1, +, \cdot, -, F_R, F_I \rangle$, where F_R and F_I are the real and imaginary parts of the complex analytic function F, restricted to the polyintervals $[-\rho, \rho]^{2n} \subset \mathbb{C}^n = \mathbb{R}^{2n}$.*

Proof. Write $F(z) = F(x + iy) = F_R(x, y) + iF_I(x, y)$, where $z \in \Delta_\rho$, and x, y are its real and imaginary part. Write $z = x + iy$ and $w = u + iv$, with real (vectors) x, y, u and v. Then $F(z + iw) = F_R(x + u, y + v) + iF_I(x + u, y + v)$, and

$\overline{F(\bar{z}+i\bar{w})} = F_R(x+u, -y-v) - iF_I(x+u, -y-v)$. This gives the strong definability of $\tilde{F}_R$ and $\tilde{F}_I$, as required. ■

We now return to the specific case of the paper [1, Theorem 3.1, pp. 326-327].

Completion of the proof of [1, Theorem 3.1, pp. 326-327]:
The main idea was to change variable $q = \exp(-2\pi i\tau)$, which maps the vertical strip $\{\tau \in \mathbb{C} : |\mathfrak{Re}(\tau)| \leq 1/2,\ \mathfrak{Im}(\tau) \geq \sqrt{3}/2\}$ onto the disk $\{q \in \mathbb{C} : |q| \leq \exp(-\pi\sqrt{3})\}$. The disk with twice the radius, that is $\{q \in \mathbb{C} : |q| \leq 2\exp(-\pi\sqrt{3}) = \exp(\ln 2 - \pi\sqrt{3})\}$ is the image of the vertical strip $\{\tau \in \mathbb{C} : |\mathfrak{Re}(\tau| \leq 1,\ \mathfrak{Im}(\tau) \geq (-\ln 2/2\pi + \sqrt{3}/2) > 0.75 = 3/4\}$, which in turn is contained in the set described in the shaded area of [1, Figure 1, pp. 328-329]. Therefore, the modularity formulas in the paragraph just above the statement of [1, Theorem 3.1, pp. 326-327] provide us with the strong definability of the appropriate extensions of the functions in that statement. Thus it is possible to define the complex analytic continuations of their real and imaginary parts. ■

3 Final Comments

The techniques used in that paper can be applied to more general settings, such as covering maps of $\mathcal{A}_g$ and also to general and mixed Shimura varieties; the appropriate functions are definable in $\mathbb{R}_{an}$, by results of Peterzil and Starchenko [4], Klinger, Ullmo and Yafaev [3], and Ziyang Gao [2], respectively. The main idea is that such automorphic forms are invariant under some translations and so they admit a Fourier transformation into a function defined in a compact domain in $\mathbb{C}^n$.

References

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