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Anatomy of a Nonidentity Paradox

Dale Jacquette

Abstract

A logical paradox is described for a nonidentity predication, applied to the property of being identical to a particular named object, in conjunction with the universal reflexivity of identity. Intuitive criteria in four conditions are proposed for an attention-worthy logical paradox, which the inference appears to satisfy. The paradox highlights in a striking way the existence presuppositions of classical logic.

Keywords: Classical (extensional, existence presuppositional) logic; identity; nonidentity; paradox; Tarski, Alfred.

1 Three Assumptions in Contradiction

A nonidentity paradox is built out of: (1) the reflexivity of identity, $\forall x[x = x]$; (2) the stipulative definition of property F as the property of being identical with object a, $F = \lambda x[x = a]$; and (3) the proposition that conditionally only a self-non-identical object has property F, $\forall x[Fx \rightarrow x \neq x]$. The paradox states:

1.	$\forall x[x = x]$	Reflexivity of $=$
2.	$F = \lambda x[x = a]$	F is the property of being $= a$
3.	$\forall x[Fx \rightarrow x \neq x]$	Only self-non-identical objects have property F
4.	$a = a$	(1)
5.	$\lambda x[x = a]a$	[4]
6.	Fa	(2)[5]
7.	$Fa \rightarrow a \neq a$	(3)
8.	$a \neq a$	[6, 7]
9.	$a = a \leftrightarrow a \neq a$	[4, 8]

More compactly, at the price of relying on a formally unstated background principle of λ-equivalence and taking a couple of forgivable deductively valid inference shortcuts, as in the final step:

1.	$\forall x[x = x]$	Reflexivity of $=$
2.	$F = \lambda x[x = a]$	F is the property of being $= a$
3.	$\forall x[Fx \rightarrow x \neq x]$	Only self-non-identical objects have property F
4.	$a = a$	(1)
5.	$Fa \rightarrow a \neq a$	(3)
6.	$a = a \rightarrow a \neq a$	$[2, 5]$ (λ-equivalence presupposed)
7.	$a = a \leftrightarrow a \neq a$	$[4, 6]$

We find in the inference the following four requirements for an *attention-worthy* logical paradox. The construction should: (i) involve the deductively valid derivation of a logical inconsistency; (ii) represent a potentially unexpected consequence that does not trivially result from immediately transparently self-contradictory sentences; in which (iii) the denial of any assumption is made only at the cost of counterintuitive consequences; and finally (iv) support a philosophically interesting moral, in what we can learn from the way it is solved or the reasons for supposing it unsolvable.

The implication is that object a is self-non-identical, since property F is just the property of being identical to a. Why, however, should that be paradoxical? Why should a self-non-identical object nevertheless not have the property of being identical to a? The reason presumably depends on the assumption that every object we can name in the universal domain of objects is self-identical, as proposition (1) declares, against the background of classical extensional existence-presuppositional first-order predicate-quantificational logic. Are we never entitled in conventional logic, then, to classify a named object as nonexistent? Can we not intelligibly say that Zeus and Sherlock Holmes do not exist?

2 Essentials of Paradox Analysis

The paradox raises these and other questions. The first task is to understand its logical structure. Assumption (3) from a mile away is dubious, but not obviously objectionable. It is the counterpart of the liar sentence in the derivation of the liar paradox.

A paradox, after all, has to come from somewhere. In discerning exactly why assumption (3) causes trouble of which (1) and (2) by themselves would be incapable, we are pointed toward the ultimate semantic presuppositions of classical predicate-quantificational logic that are the real root source of the nonidentity paradox. The accomplishment of that task fulfills a major part of an attention-worthy paradox's obligatory moral. We must learn something valuable from an attention-worthy paradox, regardless of whether and how it is solved. The nonidentity paradox arises from below the surface of the inference, which is part of its logical and semantic interest, in the extensionalist existence presuppositions of classical logic by which a is required logically to exist in order to serve as a true predication subject.

There is an antinomy here because assumptions (1) or (1) and (2) taken by themselves logically imply $\exists x[x = a]$, whereas assumptions (2) and (3) logically imply the negation, $\neg\exists x[x = a]$. That is why, and, indeed, the only reason why, a logical contradiction later appears in paradox inference steps [8] and [9]. Diagnosing the paradox in this way reveals that classical logic is not entirely in sync with its own syntax, or with the comprehensive expressive possibilities that its formal language ostensibly affords. In one sense, the diagnosis of the paradox is surprising if we were not expecting contradiction arising in precisely this way from these assumptions. The element of surprise or unexpectedness is supposed to be a mark of genuine paradox, so perhaps the nonidentity inference terminating in logical contradiction is paradoxical.

In another sense, however, the nonidentity paradox is straightforwardly predictable from the three assumptions. We should already know from an understanding of its extensionalist semantics that in classical logic we can only name objects for true predications in a referential domain of existent entities. We cannot flatly assert that a named object is non-self-identical. We come closest by understanding the paradox assumptions as implying that nothing has property F, that if anything has F then it is self-nonidentical, where to have property F is to be identical to a particular named object. Which seems rather a different thing. Even with such a relatively innocent-seeming conditional assumption as (3), taken by itself, we are still immediately drawn into explicit contradiction in a classical extensionalist existence presuppositional semantic environment by the preposterous idea that a named object might not be self-identical.

3 Classical Existence Presuppositions

Few will quarrel with the universal reflexivity of identity in (1), although the paradox logically depends on the assumption. Without (1), (3) by itself or together with (2) is powerless to produce an inconsistency. Assumption (1), moreover, is existentially loaded, since it implies that every named object is self-identical, thereby precluding the possibility of a named object being existent or nonexistent by virtue of being self-nonidentical.

Assumption (2), formulated as $F = \lambda x[x = a]$, may appear to violate Alfred Tarski's syntactical constraint on definitions. We are not permitted to define terms by allowing any object variables to appear in the definition's *definiens* that do not already appear in its *definiendum*.[1] The rule can be understood to apply to both definitions of terms marked by *definiendum* $=_{df}$ *definiens*' or *definiendum* $\leftrightarrow$ *definiens*', serving as the definition of a concept. In assumption (2) there is a variable x' that appears in the *definiens* of concept or property F that does not appear in the *definiendum*, which consists only of the predicate term F'.

We can remedy the situation by explaining that $F = \lambda x[x = a]$ conveniently abbreviates the more complete predicate-quantificational expression, $\forall x[Fx \leftrightarrow \lambda y[y = a]x]$. Granted, the expanded formula appears in another way to violate Tarski's syntactical constraint, by virtue of variable y' appearing in the *definiens* but not in the *definiendum*. That cosmetic problem is also rectified by rewriting assumption (2) as $\forall x[Fx \leftrightarrow \lambda x[x = a]x]$. There is no risk of quantifier collision here, because λ is not a quantifier, and the respective scopes of universal quantifier and lambda operator are unambiguous. Tarski's constraint is thereby satisfied, since the only object variable appearing in the *definiens*, the variable x', appears also in the *definiendum*. The example further incidentally illustrates the fact that Tarski's restriction is too generally and inexactly formulated. Paradox is not always averted by observing the restriction, especially when as sometimes happens it can be trivially satisfied. We should not expect Tarski's syntactical constraint on definitions to forestall all paradoxical sentences as demonstrated by the above permutations of nonidentity paradox assumption (2). Assumption (3), as previously observed, perhaps the most individually suspicious of the lot, is not to be faulted on grounds of Tarski's syntactical constraint, which it can easily be made defiantly to satisfy.

Perhaps the most dramatic way of presenting the relationship between paradox assumption (3) and the existence presuppositions of classical predicate-quantificational logic is to consider what happens when assumption (3) is denied. The following elementary but still instructive inference then results:

1.	$\neg\forall x[Fx \rightarrow x \neq x]$	Hypothesis
2.	$\exists x\neg[Fx \rightarrow x \neq x]$	[1]
3.	$\exists x[Fx \wedge x = x]$	[2]
4.	$\exists xFx$	[3]
5.	$\exists x[x = a]$	[4] Paradox assumption (2)

Whereas, paradox assumption (1) by itself is sufficient to imply:

1.	$\forall x[x = x]$	Paradox assumption (1)
2.	$a = a$	(1)
3.	$\exists x[x = a]$	[2]

[1] Tarski 1944, 61: [N]o free variable may occur in the *definiens* which does not also occur in the *definiendum*.

If we want to get paradox assumption (2) also in on the act, then with (1) it jointly supports derivation to the same conclusion in a different way:

1.	$\forall x[x = x]$	Paradox assumption (1)
2.	$\forall x[Fx \leftrightarrow \lambda x[x = a]x]$	Paradox assumption (2)
3.	$a = a$	(1)
4.	$Fa \leftrightarrow a = a$	(2) λ-equivalence
5.	Fa	[3,4]
6.	$\exists x[Fx]$	[5]
7.	$\exists x[x = a]$	(2) [6]

In the following inference, assumption (3), together with the conditional making F the property of being identical to object a in assumption (2), validly implies that a does not exist, by paradox of strict implication, where anything follows from a contradiction. Here universal reflexivity of identity in the original paradox assumption (1) is unnecessary. Intuitively, if F is the property of being identical to a and something has the property of being identical to a only if it is not self-identical, then no such object as a exists, and property F, thanks entirely to assumption (3), is uninstantiated. That is to say that there is or exists no such object as a, $\neg\exists x[x = a]$.

1.	$\forall x[Fx \leftrightarrow \lambda x[x = a]x]$	Paradox assumption (2)
2.	$\forall x[Fx \rightarrow x \neq x]$	Paradox assumption (3)
3.	$\exists x[x = a]$	Hypothesis for *Reductio*
4.	$Fa \rightarrow a \neq a$	[2]
5.	$a = a \rightarrow a \neq a$	[1]
6.	$a \neq a$	[5]
7.	$b = a$	Hypothesis from [3]
8.	$b = a \wedge b \neq a$	Substitution of identicals
9.	$\neg\exists x[x = a]$	[3,8] *Reductio ad absurdum*

The paradox results because paradox assumption (3) wants to talk about nonexistent objects, which classical logical *syntax* appears to permit, while the extensional existence presuppositional *semantics* of classical predicate-quantificational logic strictly forbids the attempt as meaningless, permitting logic to make intelligible statements only about existent objects occurring in the logic's referential domain. To avoid the nonidentity paradox we must retract at least one of the three assumptions that jointly imply the logical inconsistency in the paradox's concluding propositions. The interest of the paradox lies largely in correctly understanding its philosophical meaning for standard issue semantics.

Effectively, $\forall x[Fx \leftrightarrow \lambda x[x = a]x]$ and $\forall x[Fx \rightarrow x \neq x]$ say that $\neg\exists x[x = a]$, while $\forall x[x = x]$ by itself or with $\forall x[Fx \leftrightarrow \lambda x[x = a]x]$ says that $\exists x[x = a]$. Whether the implication is paradoxical depends on how closely we are wedded to classical extensionalist existence presuppositional objectual referential semantics for predicate-quantificational logic. We can make such commitments explicit simply by espousing the universal reflexivity of identity in paradox assumption (1), using standard quantifier inference rules, or (1) with (2), where being identical to an object is among the object's properties. Nothing in classical logic truly has a property unless it exists, which is precisely the existence presupposition of extensional semantics that assumptions (2) and (3) have the effrontery to challenge, with a proposition that looks to all intents and purposes as though it had a legitimate role to play in explaining that the property in question could be exemplified only by non-self-identical objects, which is of course to say by no objects whatsoever. To avoid the paradox, it is too late to withhold existence from object a. The object already exists in the logic's referential domain by virtue of the fact that we can truthfully say on the strength of the reflexivity of identity that $a = a$. By default, object a truly has the property of being identical to something, and, in particular, to itself, object a. The paradox consequently brings home that in classical logic $\forall x\exists y[x = y]$ is deductively provable from the universal reflexivity of identity in $\forall x[x = x]$.

Who would be so bold or mad as to dispute the universal reflexivity of identity? Certainly it would be preposterous to do so merely by asserting its negation instead, that $\exists x[x \neq x]$. At least it would be folly by itself, without also replacing classical logic's purely extensionalist existence presuppositional referential and quantifier semantics. Which turns out finally to be the point. Paradox assumptions (2) and (3) are perfectly respectable, syntactically unprohibited and arguably even stipulatively true propositions of classical logical notation. Their only sin is to jointly imply that a named object under the perfectly intelligible conditions they impose cannot exist.

Classically, all named objects exist and there should be a property of being identical to any of them by name; so right away there is a problem. The logic's syntax permits indirect reference to named nonexistents, which it is the point of the nonidentity paradox to show. The paradox nevertheless only arises in further conjunction with assumption (1), the innocent-looking universal reflexivity of identity, and cannot be derived from (2) and (3) alone. The reflexivity of identity in (1), as far as it goes, is in fact logically and conceptually innocuous. It is the background semantics of the universal quantifier in the expression of the universal reflexivity of identity in assumption (1), that is the ultimate source of logical inconsistency in the nonidentity paradox. The paradox demonstrates moreover that it is the universal quantification specifically in assumption (1), and not the same classically interpreted universal quantification in assumption (3) (or universally formulated expansions of (2)) that is responsible for the resulting antinomy.

Who goes up against the universal reflexivity of identity, without simply affirming its negation? Someone, perhaps, who thinks that universal reflexivity is too restrictive, and hence too fragile in its classical universality. It is contradicted by the mere experimental syntactically well-formed mention of a nonexistent named object. The damage is done by ostensibly referring to a nonexistent object that, moreover, is only incidentally implied as belonging to a classically syntactically well-defined general category of self-non-identicals, by another, third, and logically independent syntactically well-formed stipulative paradox assumption.

The paradox, accordingly, lies deep in the semantic referential domain assumptions of any classical purely extensionalist semantics. If the nonidentity paradox is to be fixed or forestalled, the effort must ultimately progress by revisiting the existence presuppositions of classical predicate-quantificational logic.

References

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Dale Jacquette
Institute of Philosophy
University of Bern
Länggassstrasse 49a, Postfach, 3000 Bern 9, Switzerland