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On Semi-fusions and Semi-negations

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Abstract

Inspired by the adjective-noun combination in natural language, the paper introduces new kind of logical products, referred to as '*semi-fusions*', differing in their deductive capacity from classical conjunction and its known sub-structural variants such as fusion. The presentation is in terms of suitable natural-deduction systems.

1 Introduction

Within the area of substructural logics (e.g., see [5]), there is a well-known distinction between *additive* rules (known also as shared-context rules) and *multiplicative* rules (independent-context rules). This distinction gives rise to two well-known products, differing in their rules: *additive* (to be referred to as conjunction) and *multiplicative* (to be referred to as fusion). Both originate from some interpretation of *sentential conjunction* in natural language.

However, sub-structurality is more flexible, and gives rise to two additional natural products (to be referred to as semi-fusions), originating from interpreting the *adjective-noun combination (ANC)* in natural language (here - English). For the facts about ANC, in particular the entailments they induce, the reader is referred to [4]. For a comprehensive proof-theoretic treatment of ANC according to the proof-theoretic semantics the reader is referred to [2]. Here, my purpose is to present the additional products purely logically[1]. A distinctive characteristic of semi-fusions is is that they invalidate *full* conjunction-simplification, according to which *both conjuncts* (either separately in the additive case or jointly in the multiplicative case) are inferable from a conjunction. A semi-fusion makes available both conjuncts jointly, and *one of the conjuncts* (only) on its own. By this feature, the semi-fusions resemble the conjunction in certain systems of *connexive logic* [1, 3, 8], projecting *none* of its conjuncts (see [7]).

[1]While in English the various products combining adjective and nouns are *covert*, in the logic presented here they are *overt* operators.

A notable fact about the semi-fusions is that they need a finer view of structural rules. While typically the view in sub-structural logics is that a structural rule is either included or excluded, uniformly for all connectives, I assume that a structural rule (weakening in the case) can be confined to specific operators. Thus, additive conjunction is assumed to adhere to weakening, while fusion and the semi-fusions - not.

2 Natural deduction

In this section, I present natural deduction (ND) rules for the semi-fusions, starting by recapitulating the familiar ND-rules for conjunction ('$\wedge$') and fusion ('$\circ$'). The rules are formulated in Gentzen's *logistic* style over sequents of the form $\Gamma \vdash \varphi$, where Γ is a multi-set of formulas.

$\wedge$: additive product (conjunction)

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ (\wedge I) \qquad \frac{\Gamma \vdash \varphi \wedge \psi \quad \Gamma, \varphi \vdash \chi}{\Gamma \vdash \chi}\ (\wedge GE_1) \qquad \frac{\Gamma \vdash \varphi \wedge \psi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}\ (\wedge GE_2) \tag{1}$$

Since weakening is assumed to be applicable to conjunction, the $(\wedge GE)$-rules have as special cases (when choosing the arbitrary conclusion χ as one of the conjuncts) the usual E-rules:

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ (\wedge \hat{E}_1) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ (\wedge \hat{E}_2) \tag{2}$$

In addition to being inspired by sentential conjunction, this conjunction reflects also the combination of *intersective* adjectives and nouns.

To guide the intuition regarding left semi-fusions, one may consider the following sentence, corresponding to additive conjunction:

$$\textsf{b is a red ball} \tag{3}$$

As is well known (see [4]), (3) should entail *both*

$$\textsf{b is red} \tag{4}$$

and

$$\textsf{b is a ball} \tag{5}$$

Sentence (3) should be contrasted with the sentences (8) and (12) below, that correspond to semi-fusions according to their entailments.

$\circ$: multiplicative product (fusion)

$$\frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1\Gamma_2 \vdash \varphi \circ \psi}\ (\circ I) \qquad \frac{\Gamma_1 \vdash \varphi \circ \psi \quad \Gamma_2, \varphi, \psi \vdash \chi}{\Gamma_1\Gamma_2 \vdash \chi}\ (\circ GE) \tag{6}$$

Consequently, $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$, but both $\varphi \circ \psi \nvdash \varphi$ and $\varphi \circ \psi \nvdash \psi$.

The main point here is, that for the multiplicative product, *both* conjuncts need to be assumed (and later discharged) in order to draw a conclusion from it, whereas each conjunct suffices on its own for drawing a conclusion from an additive conjunction.

As mentioned in the Introduction, the flexibility of the control over contexts allows for the definition of additional[2] products. The motivation for these additional products originates from adjective-noun composition ([2]). Their distinctive feature is that their E-rules "reveal" unconditionally one component of the product only, in addition to revealing both components jointly. Consequently, both operators have two E-rules each, one multiplicative and one additive.

$\oslash$**: left semi-fusion** This operator adheres to the I-rule $(\oslash I)$ and the GE-rule $(\oslash GE_1)$ equal to the fusion $(\circ GE)$-rule, but also to the following $(\oslash \hat{E}_2)$ rule.

$$\frac{\Gamma_1, \varphi \vdash \psi \quad \Gamma_2 \vdash \varphi}{\Gamma_1\Gamma_2 \vdash \varphi \oslash \psi}\ (\oslash I) \quad \frac{\Gamma_1 \vdash \varphi \oslash \psi \quad \Gamma_2, \varphi, \psi \vdash \chi}{\Gamma_1\Gamma_2 \vdash \chi}\ (\oslash GE_1) \quad \frac{\Gamma_1\Gamma_2 \vdash \varphi \oslash \psi \quad \Gamma_1, \varphi \vdash \psi}{\Gamma_2 \vdash \varphi}\ (\oslash \hat{E}_2) \tag{7}$$

To guide the intuition regarding left semi-fusion, one may consider the following sentence

$$\textsf{g is a fake gun} \tag{8}$$

As is well known (see [4]), (8) should entail

$$\textsf{g is fake} \tag{9}$$

but not

$$\textsf{g is a gun} \tag{10}$$

as in general any fake artefact is not real. Adjective like fake are known as *privative*. The role of the minor premise of $(\oslash \hat{E}_2)$ is to take care of the assumptions on which the conclusion depends. A detailed proof-theoretic linguistic study can be found in [2].

$\oslash$**: right semi-fusion** This operator adheres to the I-rule $(\oslash I)$ and the GE-rule $(\oslash GE_1)$ equal to the fusion $(\circ GE)$-rule, but also to the following $(\oslash \hat{E}_2)$ rule.

$$\frac{\Gamma_1, \psi \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1\Gamma_2 \vdash \varphi \oslash \psi}\ (\oslash I) \quad \frac{\Gamma_1 \vdash \varphi \oslash \psi \quad \Gamma_2, \varphi, \psi \vdash \chi}{\Gamma_1\Gamma_2 \vdash \chi}\ (\oslash GE_1) \quad \frac{\Gamma_1\Gamma_2 \vdash \varphi \oslash \psi \quad \Gamma_1, \psi \vdash \varphi}{\Gamma_2 \vdash \psi}\ (\oslash \hat{E}_2) \tag{11}$$

[2]As far as I know, those additional products were not considered in the literature before.

To guide the intuition regarding right semi-fusion, one may consider the following sentence

$$\textsf{Jumbo is a small elephant} \tag{12}$$

As is well known (see [4]), (12) should entail

$$\textsf{Jumbo is a(n) elephant} \tag{13}$$

but not

$$\textsf{Jumbo is small} \tag{14}$$

as it might be a rather large animal. Adjective such as small are known as *subsective* (but not *intersective*). Once again, the minor premise of the E-rule takes care of having the conclusion depend on the correct assumptions.

To complete the picture, the following additional $\hat{E}_3$-rules hold to, but they are not interesting in terms of ANC.

$$\frac{\Gamma_1\Gamma_2 \vdash \varphi \otimes \psi \quad \Gamma_2 \vdash \varphi}{\Gamma_1, \varphi \vdash \psi}\ (\otimes\hat{E}_3) \qquad \frac{\Gamma_1\Gamma_2 \vdash \varphi \otimes \psi \quad \Gamma_2 \vdash \psi}{\Gamma_1, \psi \vdash \varphi}\ (\oslash\hat{E}_3) \tag{15}$$

Thus, while projecting one component *unconditionally*, like a conjunction, semi-fusions project their other component *conditionally*. If the object language contains a conditional satisfying a property akin to the deduction theorem, the conclusions of those additional E-rules could be expressed by means of this conditional. The material implication does not seem fit, though. I will not pursue this conditional projection any further here.

Proposition 2.1 (reduction) *Both semi-fusions admit reduction of maximal formulas.*

Proof.

left semi-fusion (‘$\otimes$’):

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \varphi \vdash \psi\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash \varphi\end{array}}{\Gamma_1\Gamma_2 \vdash \varphi \otimes \psi}\ (\otimes I) \quad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \varphi \vdash \psi\end{array}}{\Gamma_2 \vdash \varphi}\ (\otimes\hat{E}_2) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash \varphi\end{array} \tag{16}$$

right semi-fusion (‘$\oslash$’):

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \psi \vdash \varphi\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash \psi\end{array}}{\Gamma_1\Gamma_2 \vdash \varphi \oslash \psi}\ (\oslash I) \quad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \psi \vdash \varphi\end{array}}{\Gamma_2 \vdash \psi}\ (\oslash\hat{E}_2) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash \psi\end{array} \tag{17}$$

∎

The following proposition summarises some simple properties of semi-fusions that follow directly by the I/E-rules (but see Section 3). Those properties manifest clearly some notable differences between additive conjunction, fusion and semi-fusions. All associations are to the right.

This proposition states, in addition to commutativity etc., properties of *iterated* ANCs. For example (again, see [4]),

$$\mathbf{j}\ \textsf{is a round brown cake} \tag{18}$$

entails all of

$$\mathbf{j}\ \textsf{is a cake},\ \mathbf{j}\ \textsf{is round}\ \text{and}\ \mathbf{j}\ \textsf{is brown} \tag{19}$$

while

$$\mathbf{j}\ \textsf{is a experienced skilful teacher} \tag{20}$$

entails only

$$\mathbf{j}\ \textsf{is a teacher}\ \text{and}\ \mathbf{j}\ \textsf{is a skilful teacher} \tag{21}$$

but *not*, for example,

$$\mathbf{j}\ \textsf{is experienced} \tag{22}$$

Proposition 2.2 (properties of semi-fusions)

1.
$$\begin{gathered} \varphi \oslash \varphi \vdash \varphi \\ \varphi \vdash \varphi \oslash \varphi \\ \varphi \obslash \varphi \vdash \varphi \\ \varphi \vdash \varphi \obslash \varphi \end{gathered} \tag{23}$$

2.
$$\begin{gathered} \varphi \oslash \psi \nvdash \psi \oslash \varphi \\ \varphi \obslash \psi \nvdash \psi \obslash \varphi \\ \varphi \oslash \psi \nvdash \varphi \obslash \psi \\ \varphi \obslash \psi \nvdash \varphi \oslash \psi \end{gathered} \tag{24}$$

3.
$$\begin{gathered} \varphi \oslash \psi \oslash \chi \vdash \chi \\ \varphi \oslash \psi \oslash \chi \nvdash \psi \\ \varphi \oslash \psi \oslash \chi \nvdash \varphi \\ \varphi \oslash \varphi \oslash \psi \vdash \varphi \oslash \psi \\ \varphi \oslash \psi \nvdash \varphi \oslash \varphi \oslash \psi \end{gathered} \tag{25}$$

4.

$$\begin{gathered}\varphi \obslash \psi \obslash \chi \nvdash \chi \\ \varphi \obslash \psi \obslash \chi \vdash \varphi \\ \varphi \obslash \psi \obslash \chi \nvdash \psi \\ \varphi \obslash \varphi \obslash \psi \nvdash \varphi \obslash \psi \\ \varphi \obslash \psi \nvdash \varphi \obslash \varphi \obslash \psi\end{gathered} \tag{26}$$

5.

$$\begin{gathered}\varphi \wedge \psi \oslash \chi \vdash \varphi \\ \varphi \wedge \psi \oslash \chi \vdash \varphi \wedge \chi \\ \varphi \wedge \psi \oslash \chi \vdash \chi \\ \varphi \wedge \psi \oslash \chi \nvdash \psi \ \ \oslash \chi \\ \varphi \wedge \psi \oslash \chi \nvdash \varphi \ \ \oslash \psi\end{gathered} \tag{27}$$

6.

$$\begin{gathered}\varphi \oslash \psi \wedge \chi \vdash \psi \\ \varphi \oslash \psi \wedge \chi \nvdash \psi \wedge \chi \\ \varphi \oslash \psi \wedge \chi \nvdash \chi \\ \varphi \oslash \psi \wedge \chi \nvdash \varphi \oslash \chi \\ \varphi \oslash \psi \wedge \chi \nvdash \varphi\end{gathered} \tag{28}$$

7.

$$\begin{gathered}\varphi \wedge \psi \obslash \chi \vdash \varphi \\ \varphi \wedge \psi \oslash \chi \vdash \varphi \wedge \chi \\ \varphi \wedge \psi \oslash \chi \vdash \chi \\ \varphi \wedge \psi \oslash \chi \nvdash \psi \ \ \oslash \chi \\ \varphi \wedge \psi \oslash \chi \vdash \psi\end{gathered} \tag{29}$$

8.

$$\begin{gathered}\varphi \oslash \psi \wedge \chi \vdash \psi \\ \varphi \oslash \psi \wedge \chi \vdash \psi \wedge \chi \\ \varphi \oslash \psi \wedge \chi \vdash \chi \\ \varphi \oslash \psi \wedge \chi \nvdash \varphi \oslash \chi \\ \varphi \oslash \psi \wedge \chi \nvdash \varphi\end{gathered} \tag{30}$$

9.

$$\begin{gathered}\varphi \oslash \psi \obslash \chi \nvdash \varphi \\ \varphi \oslash \psi \obslash \chi \nvdash \varphi \oslash \chi \\ \varphi \oslash \psi \obslash \chi \nvdash \chi \\ \varphi \oslash \psi \obslash \chi \nvdash \varphi \obslash \chi \\ \varphi \oslash \psi \obslash \chi \vdash \psi\end{gathered} \tag{31}$$

10.

$$\begin{array}{c} \varphi \oslash\!\!\!\!\!\bigcirc \psi \oslash \chi \nvdash \psi \\ \varphi \oslash\!\!\!\!\!\bigcirc \psi \oslash \chi \nvdash \psi \oslash \chi \\ \varphi \oslash\!\!\!\!\!\bigcirc \psi \oslash \chi \nvdash \chi \\ \varphi \oslash\!\!\!\!\!\bigcirc \psi \oslash \chi \nvdash \varphi \oslash\!\!\!\!\!\bigcirc \chi \\ \varphi \oslash\!\!\!\!\!\bigcirc \psi \oslash \chi \vdash \varphi \end{array} \tag{32}$$

3 A finer analysis

The analysis of semi-fusions above is made under the common convention in logic that the formulas of the object language are freely generated from propositional variables (atomic propositions) by the connectives. A more interesting analysis can be obtained by assuming a *diversified*, two-sorted object language, that restricts the well-formedness of formulas obtained with a semi-fusion as a main operator. This diversification reflects the fact that in natural language, adjective and nouns belong to two different categories (that for the current concerns can be assumed disjoint). In an ANC, the one formula (in English, the left one) has to be an adjective, while the second one has to be a noun.

Suppose the set of propositional variables is partitioned into two classes, ranged over by p_α (adjectival) and p_ν (nominal). This partition gives rise to the following two-sorted object language, the formulas of which are ranged over by φ_α and φ_ν, defined by mutual recursion.

Definition 3.1 (two-sorted object language)

$$\begin{array}{c} \varphi_\alpha ::= p_\alpha \mid \varphi_\alpha \wedge \varphi_\alpha \\ \varphi_\nu ::= p_\nu \mid \varphi_\alpha \oslash \varphi_\nu \mid \varphi_\alpha \oslash\!\!\!\!\!\bigcirc \varphi_\nu \end{array} \tag{33}$$

(one might consider also $\varphi_\nu ::= \varphi_\nu \wedge \varphi_\nu$). Thus, combining an adjectival formula by means of a semi-fusion with a nominal formula gives rise to another nominal formula, that can serve again as a right argument of a semi-fusion.

Appealing to this diversified object language deprives some of the usual properties of products from being expressible. Thus, commutativity, idempotence and left associativity are not well-formed, and the question whether they hold or for semi-fusions or not cannot rise at all. Such an analysis reflects more accurately the underlying ANC in natural language than the analysis based on a one-sorted, freely-generated object language.

4 Preliminary remarks on semi-negations

ANC leads naturally to two unary connectives[3] of non-classical *negations*, which I refer to as *semi-negations*, in view of their interaction with semi-fusions. Below I briefly list the distinctive properties of semi-negation. Similarly to the relationship of a connexive implication to "negation as cancelation" [6], one may relate semi-negations to a paradigm of "semi-negation as partial cancelation". First, recall that classical negation satisfies

$$\neg(\varphi \wedge \psi) \nvdash \varphi, \qquad \neg(\varphi \wedge \psi) \nvdash \psi \tag{34}$$

right semi-negation ($\overset{\rightarrow}{\neg}$):

$$\overset{\rightarrow}{\neg}(\varphi \oslash \psi) \nvdash \varphi, \qquad \overset{\rightarrow}{\neg}(\varphi \oslash \psi) \vdash \psi \tag{35}$$

Right semi-negation can be seen as reflecting what is known as *pragmatic implicature*, according to which from $\overset{\rightarrow}{\neg}(\varphi \oslash \psi)$ one may infer ψ, while φ is not inferable. For example, from Jumbo is not a small elephant one might infer Jumbo is an elephant (presumably, a large one).

left semi-negation ($\overset{\leftarrow}{\neg}$):

$$\overset{\leftarrow}{\neg}(\varphi \obslash \psi) \vdash \varphi \qquad \overset{\leftarrow}{\neg}(\varphi \obslash \psi) \nvdash \psi \tag{36}$$

Right semi-negation can also be seen as reflecting *pragmatic implicature*, but has to be articulated with a focal emphasis of the second argument. For example, from x is not a fake gun (with gun emphasised for focus) one might infer x is fake (maybe a fake submachine-gun).

A similar phenomenon occurs with *adverbial modification*. A pragmatic implicature of John does not drive fast would be that John drives, but presumably slowly.

Another interesting negation, also mimicking some pragmatic implicature, is one that distributes over semi-fusions by producing *exclusive or*, where from Jumbo is not a small elephant one could derive that either Jumbo is not small (for an elephant), or Jumbo is not an elephant, but not both (e.g., because Jumbo is a flying saucer ...).

More work is needed for establishing a neat proof-theory for those connectives.

[3]Currently, I do not have a precise definition of such connectives.

5 Conclusions

Based on some inspiration from the adjective-noun combination in natural language, the paper distinguishes between four products, two of which not having been considered before, differing on their component projecting behaviour. Also, semi-negations are identified, acting on semi-fusions differently than the De Morgan rule of (classical) negation acting on conjunction.

conjunction (‘$\wedge$’): projects each component unconditionally, *on its own*.

fusion ($\circ$’): projects unconditionally *both components jointly*.

semi-fusion (‘$\oslash, \varnothing$’): project both components jointly unconditionally, projects one component unconditionally on its own and projects the other component conditionally. A semi-negation of a semi-fusion (equi-directed) also projects unconditionally one component only.

The definition of the new semi-fusions is based on taking advantage of more possibilities allowed by sub-structurality in the definition of the rules, where additivity and multiplicativity are combined for the E-rules, as well as the structural rule of weakening being prohibited for those connectives.

As for the diversified object language, its use deserves further study. One possibility might be the introduction of unary operators of *nominalization* and its inverse, taking α-formulas to ν-formulas, and vice versa. Thus, one could move from small to smallness and vice versa.

Finally, a task needing pursuing is the devising of proof-terms for semi-fusions, as it is clear that pairing is not the correct Curry-Howard correspondent of semi-fusions.

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