South American Journal of Logic Vol. 2, n. 1, pp. 57–81, 2016 ISSN: 2446-6719

# Counterfactuals in Temporal Alethic-Deontic Logic

Daniel Rönnedal

## Abstract

The purpose of this paper is to describe a set of counterfactual temporal alethic-deontic systems, i.e. systems that include counterfactual, temporal, alethic and deontic operators. All systems are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called  $T \times W$  semantics, to characterise our systems semantically and semantic tableaux to characterise them proof theoretically. Our models contain several different accessibility relations and a similarity relation between possible worlds, which are used in the definitions of the truth conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of them.

**Keywords:** conditional logic;  $T \times W$  logics, temporal logic; modal logic; deontic logic; semantic tableaux; counterfactuals; subjunctive conditionals.

# 1 Introduction

The purpose of this paper is to describe a set of counterfactual temporal alethic-deontic systems, i.e. systems that include counterfactual, temporal, alethic and deontic operators. All systems are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called  $T \times W$  semantics (see Section 3), to characterise our systems semantically and semantic tableaux (see Section 4) to characterise them proof theoretically. Our models contain several different accessibility relations and a similarity relation between possible worlds, which are used in the definitions of the truth conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of them.

The systems developed in this essay are combinations of the counterfactual systems and temporal alethic-deontic systems introduced by Rönnedal [37] and Rönnedal [38], respectively. (See also Rönnedal [39] and Rönnedal [40].) However, the results in the present paper are not only a straightforward extension of previous results. This

paper also contains some major novelties. The most important difference is that [37] only described some ordinary frames and models (see Section 3.1), without a similarity relation between possible worlds. In this paper, I will introduce a set of so-called supplemented frames and models (see Section 3.1) that includes a ternary similarity relation between possible worlds. In supplemented models, the truth conditions of the counterfactual operators can be defined in terms of this similarity relation (see Section 3.1.3). This kind of semantics seems to be intuitively more plausible than the kind of semantics used in [37]. Furthermore, I will establish several interesting relationships between ordinary and supplemented models (see Theorem 1 in Section 3.2). Therefore, the results in this paper are logically significant and non-trivial.

Pioneering contributions to counterfactual logic can be found in Robert Stalnaker's [42] and David Lewis's works [31]. The semantic conditions C - cc5 and C - cc7 play an important part in systems inspired by Stalnaker, and the semantic condition C - cc6 plays an important part in systems inspired by Lewis (see Section 3.2). Many logics described by Stalnaker and Lewis are included in our counterfactual temporal alethic-deontic systems.

Several philosophers and logicians have developed logical systems that deal with various combinations of the conditions governing temporal, alethic and deontic elements (e.g. Chellas [14], Bailhache [3, 4, 5, 6], van Eck [20], Thomason [43, 44], Åqvist and Hoepelman [48], Åqvist [51], Bartha [7], Horty [27], Belnap, Perloff and Xu [8], Brown [9, 10, 11]). However, as far as I know, no one has developed any systems that include temporal, alethic and deontic operators, as well as counterfactuals. Consequently, all of the systems in this essay are entirely new.

Most temporal alethic-deontic logicians use some kind of tree-like structure to describe their systems semantically, for instance the so-called Ockhamist frames perhaps first hinted at in Prior [35], and they try to find different axioms that correspond to different conditions that may be imposed on these structures. In this paper, we use semantic tableaux instead. The tableau approach makes our systems unique.

Other interesting works that deal with combinations of tense and modality include Ciuni and Zanardo [16], DiMaio and Zanardo [19], von Kutschera [30], Zanardo [47], Åqvist [50] and Wölfl [46]. For some informal philosophical reflections, see e.g. [2] and [21]. For more information on various mono-modal systems and on how to combine different systems, see e.g. [4, 5, 6], [12], [15], [17], [25], [29], [36], [49].

There are basically three kinds of semantics that have been used by temporal alethicdeontic logicians,  $T \times W$  semantics (e.g. [14], [3], [4], [5], [6], [45], [48], [51]), moment based (branching time) semantics (e.g. [7], [27], [8]) and branch based semantics (e.g. [9], [10]). We use a kind of  $T \times W$  semantics in this essay. According to this approach, truth is relativised to world-moment pairs. A sentence may be true in one possible world at a time and false in another possible world at the same time, or true in one possible world at a time and false in the same world at another time. This leads to quite rich and flexible semantics.

There are many good philosophical reasons to be interested in the systems in this paper. One reason is that we seem to be able to use them to symbolise many different forms of conditionals and principles that are difficult or impossible to formalise adequately in other systems. Another reason is that there are many arguments that are intuitively valid that cannot be shown to be valid in any existing systems. I will now consider one example of such an argument. Let us call this deduction "The Test Argument".

#### The Test Argument

1. If it ought to be the case that it is always going to be the case that she is completely honest and courageous, then if you were in her situation it would be the case that it is always going to be the case that you are completely honest and courageous.

2. It ought to be the case that it is always going to be the case that she is completely honest and courageous.

3. It is absolutely necessary that if you are completely honest, then you do not lie. It follows that:

4. If you were in her situation, then it would not be permitted that some time in the future it will be the case that you lie.<sup>1</sup>

This argument is intuitively valid. It seems impossible for the premises to be true while the conclusion is false. In other words, it appears to be necessary that if the premises are true, then the conclusion is true also. I am only speaking about the validity of the argument, not about its soundness. I am not claiming anything about the truth-value of the premises. As is widely recognised, an argument can be valid even if the premises are not true. However, it seems impossible to prove that this argument is valid in any known logical systems. To be able to formalise this argument adequately it seems necessary to have a logical system that includes deontic operators, temporal operators, alethic operators and counterfactual operators, as well as ordinary truth functional connectives. I.e. we seem to need systems of the kind used in this paper. This is a good reason to be interested in counterfactual temporal alethic-deontic logic.

Premise 1 in The Test Argument is an instance of one kind of universalisability principle: If A is obligatory for x, then A is obligatory for anyone that is in x's situation.<sup>2</sup> It has proved very difficult to find exact and plausible formalisations of this principle. To me it seems that we need some kind of counterfactual operator to be able to formalise different versions of this principle.<sup>3</sup> I am not in your situation and you are not in my

<sup>&</sup>lt;sup>1</sup>The sentences in this argument are not particularly "natural"; some might even want to question their grammaticality. However, they are not much different from other similar sentences that are often used by logicians, and it seems possible, and not too difficult, to understand the thoughts that they express.

 $<sup>^{2}</sup>$ See e.g. Adler and Gowans [1] and Potter and Timmons [33] for more information about this well-known principle in ethics and for some relevant references.

<sup>&</sup>lt;sup>3</sup>The interpretation of "the" universalisability principle that entails premise 1, is not the only

situation. In fact, it appears to be historically impossible for me to be in your position and for you to be in my position. Yet, it seems reasonable for me to think about what would be the case if I were in your situation etc. It is an important ethical skill to be able to see things from another person's perspective, to "walk in someone else's shoes", to project oneself into another individual's situation. In Section 5.1, I will show how we can prove that the conclusion in The Test Argument is entailed by the premises.

The essay is divided into 6 sections. In Section 2, I describe the syntax of our systems and in Section 3 their semantics. Section 4 deals with the proof theoretic characterisation of our logics and Section 5 includes some examples of theorems. Finally, Section 6 contains soundness proofs for every system and completeness proofs for a subclass of them. My conjecture is that at least all of the systems based on ordinary models are complete, but I have not been able to prove this.

# 2 Syntax

## 2.1 Alphabet

(i) A denumerably infinite set Prop of proposition letters  $p, q, r, s, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \ldots$ , (ii) a denumerably infinite set NT of names of times  $t_0, t_1, t_2, t_3 \ldots$ , (iii) the primitive truth-functional connectives  $\neg$  (negation),  $\land$  (conjunction),  $\lor$  (disjunction),  $\rightarrow$  (material implication) and  $\leftrightarrow$  (material equivalence), (iv) the alethic operators  $U, M, \Box, \Diamond, \Box$ , and  $\Diamond$ , (v) the temporal operators  $R, \underline{A}, \underline{S}, \underline{G}, \underline{H}, \underline{F}$  and  $\underline{P}$ , (vi) the deontic operators O and P, (vii) the counterfactual operators  $\Box \rightarrow$  and  $\Diamond \rightarrow$ , and (viii)  $\top$  (verum),  $\bot$  (falsum) and the brackets (, ).

## 2.2 Language

The language L is the set of well-formed formulas (wffs) generated by the usual clauses for proposition letters and propositionally compound sentences, and the following clauses: (i) if A is a wff, then UA ("it is universally (or absolutely) necessary that A"), MA("it is universally (or absolutely) possible that A"),  $\Box A$  ("it is historically necessary (or settled) that A"),  $\diamondsuit A$  ("it is historically possible that A"),  $\Box A$  ("it is temporally necessary that A"),  $\diamondsuit A$  ("it is temporally possible that A"),  $\underline{A}A$  ("it is always the case that A"),  $\underline{S}A$  ("it is some time the case that A"),  $\underline{G}A$  ("it is always going to be the case that A"),  $\underline{H}A$  ("it has always been the case that A"),  $\underline{F}A$  ("it will some time in the future be the case that A"),  $\underline{P}A$  ("it was some time in the past the case that A"), OA ("it ought to be the case that A") and PA ("it is permitted that A") are wffs, (ii)

one that is possible, and it is not necessarily the best. Again, I am not primarily interested in the truth-value of the premises. The point of the argument is to illustrate the usefulness of the systems introduced in this paper.

if A is a wff and t is in NT, then RtA ("it is realised at time t that A") is a wff, (iii) if A and B are wffs, so are  $(A \Box \rightarrow B)$  ("If it were the case that A, then it would be the case that B") and  $(A \Leftrightarrow B)$  ("If it were the case that A, then it might be the case that B"), and (iv) nothing else is a wff.

Capital letters  $A, B, C \dots$  are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter  $\Gamma$  represents an arbitrary set of formulas. Brackets around sentences are usually dropped if the result is not ambiguous.

## 2.3 Definitions

 $\Leftrightarrow A$  ("it is historically impossible that A") =  $\neg \Leftrightarrow A$ , FA ("it is forbidden that A") =  $\neg PA$ ,  $\bigtriangledown A$  ("it is historically contingent that A") =  $\diamondsuit A \land \diamondsuit \neg A$ ,  $\triangle A$  ("it is historically non-contingent that A") =  $\neg \bigtriangledown A$  (or  $\Box A \lor \Box \neg A$ ),  $[G]A = A \land \underline{G}A$ ,  $\langle F \rangle A = \neg [G] \neg A$ (or  $A \lor \underline{F}A$ ),  $[H]A = A \land \underline{H}A$ ,  $\langle P \rangle A = \neg [H] \neg A$  (or  $A \lor \underline{P}A$ ),  $A \Rightarrow B = \Box (A \Rightarrow B)$ ,  $A \Leftrightarrow B = \Box (A \Leftrightarrow B)$ ,  $A \Longrightarrow B = (A \diamondsuit \top) \land (A \Box \Rightarrow B)$  ("If A were the case, then B would be the case"),  $A \Leftrightarrow B = \neg (A \Box \Rightarrow \neg B)$  (or  $(A \Box \rightarrow \bot) \lor (A \Leftrightarrow B)$ ) ("If A were the case, then B might be the case").

 $A \square \Rightarrow B$  is an alternative to  $(A \square \Rightarrow B)$ , and  $A \Leftrightarrow B$  is an alternative to  $(A \Leftrightarrow B)$ .

# **3** Semantics

## **3.1** Basic concepts

## 3.1.1 Counterfactual temporal alethic-deontic frame

We will consider two kinds of frames in this essay: ordinary and supplemented (counterfactual temporal alethic-deontic) frames. An ordinary (counterfactual temporal alethicdeontic) frame F is a relational structure  $\langle W, T, \langle R, S, \{R_A : A \in L\} \rangle$ , where W is a non-empty set of possible worlds, T is a non-empty set of times,  $\langle$  is a binary relation on T ( $\langle \subseteq T \times T$ ), R and S are two ternary (alethic or deontic) accessibility relations ( $R \subseteq W \times W \times T$  and  $S \subseteq W \times W \times T$ ), and { $R_A : A \in L$ } is a set of ternary counterfactual accessibility relations, one for each sentence, A, in L ( $R_A \subseteq W \times W \times T$ ).

A supplemented (counterfactual temporal alethic-deontic) frame  $F_s$  is a relational structure  $\langle W, T, \langle R, S, \{R_A : A \in L\} \rangle$ , where  $W, T, \langle R, S \rangle$  and  $\{R_A : A \in L\}$  are exactly as in an ordinary frame, and  $\geq$  is a ternary similarity relation defined over the elements in  $W (\geq \subseteq W \times W \times W)$ . If it is clear that we are talking about a supplemented frame, we will sometimes drop the subscript.

R "corresponds" to the alethic operators  $\Box$  and  $\diamondsuit$ , < to the temporal operators  $\underline{G}, \underline{F}, \underline{H}$  and  $\underline{P}, R_A$  and  $\geq$  to the counterfactual operators  $\Box \rightarrow$  and  $\diamondsuit$ , and S to the deontic

operators O and P. Informally,  $\tau < \tau'$  says that the time  $\tau$  is before the time  $\tau'$  (or that  $\tau'$  is later than  $\tau$ ),  $R\omega\omega'\tau$  says that the possible world  $\omega'$  is alethically accessible from the possible world  $\omega$  at time  $\tau$ ,  $S\omega\omega'\tau$  says that  $\omega'$  is deontically accessible from  $\omega$  at  $\tau$ ,  $R_A\omega\omega'\tau$  says that the possible world  $\omega'$  is A-accessible from the possible world  $\omega$  at time  $\tau$ , and  $\omega \geq_{\omega'} \omega''$  says that the possible world  $\omega$  is at least as similar to ("near" to) world  $\omega'$  as is world  $\omega''$ . To "decide" whether  $\omega \geq_{\omega'} \omega''$ , we place ourselves in possible world  $\omega'$ , and ask which world is more similar to ours,  $\omega$  or  $\omega''$ . We can think of  $\omega \geq_{\omega'} \omega''$  as a dyadic similarity relation relativised to  $\omega'$ . In temporal aletic-deontic logic it is also possible to relativise our similarity relation to points in time (or just points in time or both). However, we will not do this in the present paper. We are here interested in "global" similarity between possible worlds, not in similarity between possible worlds at moments in time or between world-moment pairs.

#### 3.1.2 Counterfactual temporal alethic-deontic model

We will use two kinds of models in this essay: ordinary and supplemented counterfactual temporal alethic-deontic models. An ordinary model M is a triple  $\langle F, V, v \rangle$  where: (i) F is a counterfactual temporal alethic-deontic frame; (ii) V is a valuation or interpretation function, which to every proposition letter p in Prop assigns a subset of  $W \times T$ , i.e. a set of ordered pairs  $\langle \omega, \tau \rangle$ , where  $\omega \in W$  and  $\tau \in T$ ; and (iii) v is a function which to each temporal name in NT assigns a time in T.

A supplemented model  $M_s$  is a triple  $\langle F_s, V, v \rangle$  where:  $F_s$  is a supplemented frame, and V and v are exactly as in an ordinary model.

We will sometimes drop the subscript if it is clear from the context that we are talking about supplemented models (or if we are talking about any model whatsoever, ordinary or supplemented).

Let  $M, \omega, \tau \Vdash A$  abbreviate "A is true in the possible world  $\omega$  at the time  $\tau$  in the model M" (or "A is true at the pair  $\langle \omega, \tau \rangle$  in M"), and let x, y, z, w etc. range over possible worlds in W, and t over times in T. In a supplemented model the counterfactual accessibility relations can be defined in terms of the similarity relation over our possible worlds in the following way:

 $\gamma 0$ . For every A,  $R_A xyt$  if and only if (iff)  $M, y, t \Vdash A \land \forall z (M, z, t \Vdash A \rightarrow y \geq_x z)$ .

Intuitively, this condition says that the possible world y is A-accessible from the possible world x at the time t iff y is one of the closest worlds to x in which A is true at t.

Note that M stands for a class of models and F for a class of frames.

#### 3.1.3 Truth in a model, validity, satisfiability, logical consequence etc.

Let M be any (ordinary or supplemented) counterfactual temporal alethic-deontic model based on a frame  $\langle W, T, \langle R, S, \{R_A : A \in L\} \rangle$  ( $\langle W, T, \langle R, S, \{R_A : A \in L\} \geq \rangle$ ). Let

 $\omega \in W$ ,  $\tau \in T$  and let A be a well-formed sentence in L. The truth conditions for proposition letters and complex sentences are given in the following list. Those for truth-functional connectives are the usual ones (illustrated by conjunction):

(i)	$M, \omega, \tau \Vdash p$	iff	$\langle \omega, \tau \rangle \in V(p)$ for any p in Prop,
(ii)	$M, \omega, \tau \Vdash A \wedge B$	$\operatorname{iff}$	$M, \omega, \tau \Vdash A \text{ and } M, \omega, \tau \Vdash B,$
(iii)	$M, \omega, \tau \Vdash \Box A$	iff	$\forall \omega' \in W \text{ s.t. } R\omega \omega' \tau \colon M, \omega', \tau \Vdash A,$
(iv)	$M, \omega, \tau \Vdash \diamondsuit A$	$\operatorname{iff}$	$\exists \omega' \in W \text{ s.t. } R\omega \omega' \tau \colon M, \omega', \tau \Vdash A,$
(v)	$M, \omega, \tau \Vdash \boxdot A$	$\operatorname{iff}$	$\forall \omega' \in W: \ M, \omega', \tau \Vdash A,$
(vi)	$M, \omega, \tau \Vdash {\mathbin{\diamondsuit}} A$	$\operatorname{iff}$	$\exists \omega' \in W: \ M, \omega', \tau \Vdash A,$
(vii)	$M, \omega, \tau \Vdash \underline{A}A$	$\operatorname{iff}$	$\forall \tau' \in T: \ M, \omega, \tau' \Vdash A,$
(viii)	$M, \omega, \tau \Vdash \underline{S}A$	$\operatorname{iff}$	$\exists \tau' \in T: \ M, \omega, \tau' \Vdash A,$
(ix)	$M, \omega, \tau \Vdash \underline{G}A$	$\operatorname{iff}$	$\forall \tau' \in T \text{ s.t. } \tau < \tau' \colon M, \omega, \tau' \Vdash A,$
(x)	$M, \omega, \tau \Vdash \underline{F}A$	$\operatorname{iff}$	$\exists \tau' \in T \text{ s.t. } \tau < \tau' \colon M, \omega, \tau' \Vdash A,$
(xi)	$M, \omega, \tau \Vdash \underline{H}A$	$\operatorname{iff}$	$\forall \tau' \in T \text{ s.t. } \tau' < \tau \colon M, \omega, \tau' \Vdash A,$
(xii)	$M, \omega, \tau \Vdash \underline{P}A$	$\operatorname{iff}$	$\exists \tau' \in T \text{ s.t. } \tau' < \tau \colon M, \omega, \tau' \Vdash A,$
(xiii)	$M, \omega, \tau \Vdash Rt'A$	$\operatorname{iff}$	$M, \omega, v(t') \Vdash A$ , for all $t' \in NT$ ,
(xiv)	$M, \omega, \tau \Vdash OA$	iff	$\forall \omega' \in W \text{ s.t. } S \omega \omega' \tau \colon M, \omega', \tau \Vdash A,$
(xv)	$M, \omega, \tau \Vdash PA$	$\operatorname{iff}$	$\exists \omega' \in W \text{ s.t. } S \omega \omega' \tau \colon M, \omega', \tau \Vdash A,$
(xvi)	$M, \omega, \tau \Vdash UA$	$\operatorname{iff}$	$\forall \omega' \in W \text{ and } \forall \tau' \in T: \ M, \omega', \tau' \Vdash A,$
(xvii)	$M, \omega, \tau \Vdash MA$	$\operatorname{iff}$	$\exists \omega' \in W \text{ and } \exists \tau' \in T: \ M, \omega', \tau' \Vdash A,$
(xviii)	$M, \omega, \tau \Vdash A \dashrightarrow B$	$\operatorname{iff}$	$\forall \omega' \in W \text{ s.t. } R_A \omega \omega' \tau \colon M, \omega', \tau \Vdash B,$
(xix)	$M, \omega, \tau \Vdash A \diamondsuit B$	$\operatorname{iff}$	$\exists \omega' \in W \text{ s.t. } R_A \omega \omega' \tau \colon M, \omega', \tau \Vdash B.$

 $A \square \rightarrow B$  is true in a world  $\omega$  at time  $\tau$  iff B is true in all possible worlds that are A-accessible from  $\omega$  at  $\tau$ . And  $A \Leftrightarrow B$  is true in a world  $\omega$  at time  $\tau$  iff B is true in at least one possible world that is A-accessible from  $\omega$  at  $\tau$ . According to a popular view, a would-counterfactual is true (roughly) just in case the consequent is true in the nearest possible world(s) where the antecedent is true. If we define the counterfactual accessibility relations in terms of the similarity relation as in  $\gamma 0$ , we can extend this basic idea to our temporal alethic-deontic systems. The following truth conditions can then be derived.

 $M, \omega, \tau \Vdash A \square \to B \text{ iff } \forall \omega' \in W \text{ s.t. } M, \omega', \tau \Vdash A \land \forall \omega''(M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega'') : M, \omega', \tau \Vdash B.$ 

 $M, \omega, \tau \Vdash A \Leftrightarrow B \text{ iff } \exists \omega' \in W \text{ s.t. } M, \omega', \tau \Vdash A \land \forall \omega''(M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega''): M, \omega', \tau \Vdash B.$ 

Intuitively, this means that  $A \Box \rightarrow B$  is true in the world  $\omega$  at the time  $\tau$  iff in every world  $\omega'$  that is as close to  $\omega$  as possible in which A is true at  $\tau$ , B is true at  $\tau$ . And  $A \Leftrightarrow B$  is true in the world  $\omega$  at the time  $\tau$  iff there is a world  $\omega'$  that is as close to  $\omega$  as possible in which A is true at  $\tau$ .

Other basic semantic concepts like validity, satisfiability, logical consequence etc. are defined as in [38].

## **3.2** Conditions on frames and models

In [38] several different frame- and model-conditions were introduced. All of these conditions can be used to characterise and classify our counterfactual temporal alethic-deontic frames and models. We can also use the conditions on the temporal accessibility relation < mentioned by [40]. In this section, I will explore some conditions that say something about the formal properties of the counterfactual accessibility relations. Some of these conditions are modifications of conditions introduced by [37]. The conditions on the similarity relation are "new".

In Table 1 and Table 2 the symbols  $\land$ ,  $\rightarrow$ ,  $\leftrightarrow$ ,  $\forall$  and  $\exists$  are used as metalogical symbols in the standard way. Let F be a counterfactual temporal alethic-deontic frame, M a counterfactual temporal alethic-deontic model based on F,  $\{R_A : A \in L\}$  the set of counterfactual accessibility relations and  $\geq$  the similarity relation in F. Intuitively,  $\|A\|^{M,t}$  is the set of all A-worlds at t (in M), i.e. the set of all worlds where A is true at t (in M). If for all  $R_A$  in  $\{R_A : A \in L\}$ ,  $\forall t \forall x \forall y (R_A xyt \rightarrow M, y, t \Vdash A)$ , we say that  $R_A$  satisfies or fulfills condition C - c1 and also that M satisfies or fulfills condition C - c1 and so that M satisfies of all models that satisfy condition C - c1 and the sentence T1 is valid in the class of all models that satisfy condition C - c1 and similarly in all other cases. Let C be any of the conditions we explore. Then a C-model is a model that satisfies condition, the initial C will often be dropped.

	For all $A$ and for all $B$		
C - c0	$\forall t \forall x \forall y (( \ A\ ^{M,t} = \ B\ ^{M,t}) \to (R_A xyt \leftrightarrow R_B xyt))$		
C-c0'	$\forall t \forall x ((\forall y (R_A xyt \to M, y, t \Vdash B) \land \forall y (R_B xyt \to M, y, t \Vdash A))$		
	$\rightarrow (R_A xyt \leftrightarrow R_B xyt))$		
C-c1	$\forall t \forall x \forall y (R_A xyt \to M, y, t \Vdash A)$		
C – $c2$	$\forall t \forall x \forall y ((R_A xyt \land M, y, t \Vdash B) \to R_{A \land B} xyt)$		
C – $c3$	$\forall t \forall x ((\ A\ ^{M,t} \neq \emptyset) \rightarrow \exists y R_A x y t)$		
C-c4	$\forall t \forall x \forall y \forall z ((R_A xyt \land M, y, t \Vdash B) \rightarrow (R_{A \land B} xzt \rightarrow (R_A xzt \land M, z, t \Vdash B)))$		
C – $c5$	$\forall t \forall x (M, x, t \Vdash A \to R_A x x t)$		
C-c6	$\forall t \forall x \forall y ((R_A xyt \land M, x, t \Vdash A) \to x = y)$		
C-c7	$\forall t \forall x \forall y \forall z ((R_A xyt \land R_A xzt) \rightarrow y = z)$		
Table 1			

	For all A and for all B		
$\gamma 0$	$\forall t \forall x \forall y (R_A x y t \leftrightarrow M, y, t \Vdash A \land \forall z (M, z, t \Vdash A \to y \ge_x z))$		
C – $cc2$	$\forall w \forall x \forall y ((x \ge_w y \lor y \ge_w x) \lor (x \ge_w y \land y \ge_w x))$		
C – $cc3$	If $  A  ^{M,t} \neq \emptyset$ then $\forall x \exists y (M, y, t \Vdash A \land \forall z (M, z, t \Vdash A \to y \geq_x z))$		
C – $cc4$	$\forall w \forall x \forall y \forall z ((x \ge_w y \land y \ge_w z) \to x \ge_w z)$		
C – $cc5$	$\forall x \forall yx \ge_x y$		
C – $cc6$	$\forall x \forall y (y \ge_x x \to y = x)$		
C-cc7	$\forall x \forall y \forall z ((y \ge_x z \land z \ge_x y) \to y = z)$		
Table 2			

Let us say a few words about the conditions in Table 2. We have already mentioned the condition  $\gamma 0$  (see Section 3.1.2). In any  $\gamma 0$ -model the following sentences are valid:  $\Box(p \leftrightarrow q) \rightarrow ((p \Box \rightarrow r) \leftrightarrow (q \Box \rightarrow r)), (p \Box \rightarrow p), \Box(p \rightarrow q) \rightarrow (p \Box \rightarrow q) \text{ and } ((p \land q) \Box \rightarrow r) \rightarrow (p \Box \rightarrow (q \rightarrow r)).$ 

Intuitively, C - cc2 means that  $\geq_w$  is complete (strongly connected, total), i.e. world x is as close to world w as is world y or y is as close to w as is x (or x and y are equally close to w).  $\forall w \forall x \forall y ((x \geq_w y \lor y \geq_w x) \lor (x \geq_w y \land y \geq_w x))$  is equivalent to  $\forall w \forall x \forall y (x \geq_w y \lor y \geq_w x)$ .

C - cc3, or the limit assumption, roughly says that there is always a closest A-world (for any A), or in other words, any sentence A that is true in some world w at t, is true in some closest-to-x world at t (for any x). More precisely the condition says that if A is true in some possible world w at t, then no matter what world x you're in, there will always be some world y in which A is true at t that is at least as close to your world x as is any other world where A is true at t. The assumption prohibits that there are no closest A-worlds, only an infinite sequence of closer and closer A-worlds, so to speak. If there are only finitely many worlds in a model, the limit assumption holds automatically.  $\Diamond p \rightarrow ((p \Box \rightarrow q) \rightarrow (p \diamond \rightarrow q))$  is valid in any  $\gamma 0cc3$ -model.

According to C - cc4,  $\geq_w$  is transitive, i.e. if world x is at least as close to world w as is y and y is at least as close to w as is world z, then x is at least as close to w as is z.

According to C - cc5 ("Stalnaker's base-condition"), every world x is at least as similar to itself as is any world z. In other words, there is no world z distinct from world x that is closer to x than is x itself, i.e. each world is at least as close to itself as is every other.  $(p \Box \rightarrow q) \rightarrow (p \rightarrow q)$  is valid in any  $\gamma 0cc5$ -model.

C - cc6 ("Lewis's base-condition") says that if world y is at least as similar to world x as is x, then y is identical to x. In other words, there is no world y distinct from world x that is as similar to x as is x, i.e. each world is closer to itself than any other is.  $(p \land q) \rightarrow (p \Box \rightarrow q)$  is valid in any  $\gamma 0cc6$ -model.

Intuitively, C - cc7 says that  $\geq_x$  is anti-symmetric. If world y is at least as close to world x as is world z and z is at least as close to x as is y, then y is identical to z, i.e. there are no two distinct worlds that are equally similar to a possible world x. This condition

is connected to systems of the kind developed by Stalnaker.  $(p \Box \rightarrow q) \lor (p \Box \rightarrow \neg q)$  and  $(p \Box \rightarrow (q \lor r)) \rightarrow ((p \Box \rightarrow q) \lor (p \Box \rightarrow r))$  are valid in any  $\gamma 0cc7$ -model.

Note that  $\{C - c5, C - c7\}$  entails C - c6. If every world is at least as close to itself as is every other and there are no two distinct worlds that are equally similar to any possible world, then obviously each world is closer to itself than any other is.

The following theorem tells us something about the relations between these conditions.

**Theorem 3.1** (i) Let M be a supplemented  $\gamma 0$ -model. Then M satisfies C - c0, C - c1and C - c2. (ii) Let M be a supplemented  $\gamma 0cc3$ -model. Then M satisfies C - c3. (iii) Let M be a supplemented  $\gamma 0cc4$ -model. Then M satisfies C - c4. (iv) Let M be a supplemented  $\gamma 0cc5$ -model. Then M satisfies C - c5. (v) Let M be a supplemented  $\gamma 0cc6$ -model. Then M satisfies C - c6. (vi) Let M be a supplemented  $\gamma 0cc7$ -model. Then M satisfies C - c7.

**Proof.** In the following proof, "CL" means that the step is valid in "classical logic". (i) (C - c0). 1.  $||A||^{M,t} = ||B||^{M,t}$  [Assumption]. 2.  $R_A xyt$  [Assumption]. 3.  $M, y, t \models A \land \forall z (M, z, t \models A \to y \ge_x z)$  [2,  $(\gamma 0)$ ]. 4.  $M, y, t \models B \land \forall z (M, z, t \models B \to y \ge_x z)$  [1, 3, CL, Definition of  $||A||^{M,t}$ ]. 5.  $R_B xyt$  [4,  $(\gamma 0)$ ]. 6.  $R_A xyt \to R_B xyt$  [2-5, CL]. 7.  $R_B xyt \to R_A xyt$  [similarly]. 8.  $R_A xyt \leftrightarrow R_B xyt$  [6, 7, CL]. 9. If  $||A||^{M,t} = ||B||^{M,t}$  then  $R_A xyt \leftrightarrow R_B xyt$  [1-8, CL].

(C-c1). 1.  $R_A xyt$  [Assumption]. 2.  $M, y, t \Vdash A \land \forall z (M, z, t \Vdash A \rightarrow y \ge_x z)$  [1,  $(\gamma 0)$ ]. 3.  $R_A xyt \rightarrow M, y, t \Vdash A$  [1, 2, CL]. 4.  $\forall t \forall x \forall y (R_A xyt \rightarrow M, y, t \Vdash A)$  [3, CL].

(C-c2). 1.  $R_Ayxt$  [Assumption]. 2.  $M, x, t \Vdash B$  [Assumption]. 3. not  $R_{A \land B}yxt$ [Assumption]. 4.  $M, x, t \Vdash A$  [1,  $(\gamma 0)$ ]. 5.  $M, x, t \Vdash A \land B$  [2, 4, CL]. 6. not  $(M, x, t \Vdash A \land B \land \forall z(M, z, t \Vdash A \land B \rightarrow x \geq_y z))$  [3,  $(\gamma 0)$ ]. 7.  $\exists z(M, z, t \Vdash A \land B \land \neg (x \geq_y z))$ [5, 6, CL]. 8.  $M, z, t \Vdash A \land B \land \neg (x \geq_y z)$  [7, CL]. 9.  $M, z, t \Vdash A$  [first conjunct in 8, CL]. 10.  $M, z, t \Vdash A \rightarrow x \geq_y z$  [1,  $(\gamma 0)$ ]. 11.  $x \geq_y z$  [9, 10, CL]. 12. Contradiction [second conjunct in 8 and 11]. 13.  $(R_Ayxt \land M, x, t \Vdash B) \rightarrow R_{A \land B}yxt$  [1-12, CL]. 14.  $\forall t \forall x \forall y((R_Axyt \land M, y, t \Vdash B) \rightarrow R_{A \land B}xyt)$  [13, CL].

(ii) (C - c3). 1.  $||A||^{M,t} \neq \emptyset$  [Assumption]. 2.  $\exists y(M, y, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow y \geq_x z))$  [1, C - cc3, CL]. 3.  $\exists yR_Axyt$  [2,  $(\gamma 0)$ ]. 4.  $||A||^{M,t} \neq \emptyset \rightarrow \exists yR_Axyt$  [1-3, CL]. 5.  $\forall t \forall x((||A||^{M,t} \neq \emptyset) \rightarrow \exists yR_Axyt)$  [4, CL].

(iii) (C - c4). Note that  $\forall t \forall x \forall y \forall z ((R_A xyt \land M, y, t \Vdash B) \rightarrow (R_{A \land B} xzt \rightarrow (R_A xzt \land M, z, t \Vdash B)))$  is equivalent to  $\forall t \forall x \forall y (\exists z (R_A yzt \land M, z, t \Vdash B) \rightarrow (R_{A \land B} yxt \rightarrow (R_A yzt \land M, x, t \Vdash B)))$ . So to prove the former it suffices to prove the latter. 1.  $\exists z (R_A wzt \land M, z, t \Vdash B)$  [Assumption]. 2.  $R_{A \land B} wxt$  [Assumption]. 3. not  $(R_A wxt \land M, x, t \Vdash B)$  [Assumption]. 4.  $M, x, t \vDash A \land B \land \forall z (M, z, t \vDash A \land B \rightarrow x \geq_w z)$  [2, ( $\gamma$ 0)]. 5.  $M, x, t \vDash B$  [first conjunct in 4, CL]. 6. not  $R_A wxt$  or not  $M, x, t \vDash B$  [3, CL]. 7. not  $R_A wxt$  [5, 6, CL]. 8. not  $(M, x, t \vDash A \land \forall z (M, z, t \vDash A \rightarrow x \geq_w z))$  [7, ( $\gamma$ 0)]. 9.  $M, x, t \vDash A$  [first conjunct in 4, CL]. 10. not  $\forall z (M, z, t \vDash A \rightarrow x \geq_w z)$  [8, 9, CL]. 11.  $\exists z (M, z, t \vDash$ 

 $\begin{array}{l} A \wedge \neg(x \geq_w z) \quad [10, \text{ CL}]. \ 12. \ M, y, t \Vdash A \wedge \neg(x \geq_w y) \quad [11, \text{ CL}]. \ 13. \ R_A wvt \wedge M, v, t \Vdash B \\ [1, \text{ CL}]. \ 14. \ M, v, t \Vdash A \wedge B \quad [13, (\gamma 0), \text{ CL}]. \ 15. \ x \geq_w v \quad [14, \text{ second conjunct in 4, CL}]. \\ 16. \ \forall u(M, u, t \Vdash A \rightarrow v \geq_w u) \quad [\text{first conjunct in 13, } (\gamma 0)]. \ 17. \ v \geq_w y \quad [16, \text{ first conjunct in 12, CL}]. \\ 18. \ x \geq_w y \quad [15, 17, (C - cc4)]. \ 19. \ \text{Contradiction [second conjunct in 12 and 18]}. \\ 20. \ \forall t \forall x \forall y (\exists z (R_A yzt \wedge M, z, t \Vdash B) \rightarrow (R_{A \wedge B} yxt \rightarrow (R_A yxt \wedge M, x, t \Vdash B))) \quad [1-19, \text{CL}]. \end{array}$ 

(iv) C - c5. 1.  $M, x, t \Vdash A$  [Assumption]. 2.  $R_A xxt$  iff  $M, x, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow x \geq_x z)$  [ $\gamma 0$ ]. 3.  $x \geq_x z$  [C - cc5]. 4.  $\forall z(M, z, t \Vdash A \rightarrow x \geq_x z)$  [3, CL]. 5.  $M, x, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow x \geq_x z)$  [1, 4, CL]. 6.  $R_A xxt$  [2, 5, CL]. 7.  $M, x, t \Vdash A \rightarrow R_A xxt$  [1-6, CL]. 8.  $\forall t \forall x(M, x, t \Vdash A \rightarrow R_A xxt)$  [7, CL].

(v) C-c6. 1.  $R_A xyt \land M, x, t \Vdash A$  [Assumption]. 2.  $R_A xyt$  [From 1]. 3.  $M, x, t \Vdash A$  [From 1]. 4.  $R_A xyt$  iff  $M, y, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow y \geq_x z)$  [ $\gamma 0$ ]. 5.  $M, y, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow y \geq_x z)$  [ $\gamma 0$ ]. 5.  $M, y, t \Vdash A \land \forall z(M, z, t \Vdash A \rightarrow y \geq_x z)$  [2, 4, CL]. 6.  $M, y, t \Vdash A$  [5, CL]. 7.  $\forall z(M, z, t \Vdash A \rightarrow y \geq_x z)$  [5, CL]. 8.  $M, x, t \Vdash A \rightarrow y \geq_x x$  [7, CL]. 9.  $y \geq_x x$  [3, 8, CL]. 10.  $y \geq_x x \rightarrow x = y$  [C-cc6]. 11. x = y [9, 10, CL]. 12.  $\forall t \forall x \forall y ((R_A xyt \land M, x, t \Vdash A) \rightarrow x = y)$  [1-11, CL].

(vi) C - c7. 1.  $R_A xyt \wedge R_A xzt$  [Assumption]. 2.  $R_A xyt$  [1, CL]. 3.  $R_A xzt$  [1, CL]. 4.  $R_A xyt$  iff  $M, y, t \Vdash A \wedge \forall z (M, z, t \Vdash A \rightarrow y \geq_x z)$  [ $\gamma 0$ ]. 5.  $R_A xzt$  iff  $M, z, t \Vdash A \wedge \forall w (M, w, t \Vdash A \rightarrow z \geq_x w)$  [ $\gamma 0$ ]. 6.  $M, y, t \Vdash A \wedge \forall z (M, z, t \Vdash A \rightarrow y \geq_x z)$  [2, 4, CL]. 7.  $M, z, t \Vdash A \wedge \forall w (M, w, t \Vdash A \rightarrow z \geq_x w)$  [3, 5, CL]. 8.  $M, y, t \Vdash A$  [6, CL]. 9.  $\forall z (M, z, t \Vdash A \rightarrow y \geq_x z)$  [6, CL]. 10.  $M, z, t \Vdash A$  [7, CL]. 11.  $\forall w (M, w, t \Vdash A \rightarrow z \geq_x w)$ [7, CL]. 12.  $M, z, t \Vdash A \rightarrow y \geq_x z$  [9, CL]. 13.  $M, y, t \Vdash A \rightarrow z \geq_x y$  [11, CL]. 14.  $y \geq_x z$ [10, 12, CL]. 15.  $z \geq_x y$  [8, 13, CL]. 16.  $y \geq_x z \wedge z \geq_x y$  [14, 15, CL]. 17.  $(y \geq_x z \wedge z \geq_x y)$ [1-18, CL].

# 3.3 Classification of model classes and the logic of a class of models

The conditions on our models listed in Tables 1 and 2 can be used to obtain a categorisation of the set of all models into various kinds. I shall say that  $\mathbf{M}(C_1, \ldots, C_n)$  is the class of (all) models that satisfy the conditions  $C_1, \ldots, C_n$ . E.g.  $\mathbf{M}(C - dD, C - aT, C - MO, C - c1)$  is the class of all models that satisfy C - dD, C - aT, C - MO (see [38]) and C - c1.

The set of all sentences (in L) that are valid in a class of models  $\mathbf{M}$  is called the logical system of (the system of or the logic of)  $\mathbf{M}$ , in symbols  $S(\mathbf{M}) = \{A \in L : \mathbf{M} \Vdash A\}$ . E.g.  $S(\mathbf{M}(C - dD, C - aT, C - MO, C - c1))$  is the set of all sentences that are valid in the class of all models that satisfy C - dD, C - aT, C - MO and C - c1.

# 4 Proof theory

## 4.1 Semantic tableaux

The kind of semantic tableau systems I use is inspired by e.g. Melvin Fitting and Graham Priest (Fitting [22], Fitting and Mendelsohn [23], Priest [34]). The propositional part is similar to systems introduced by Raymond Smullyan [41] and Richard Jeffrey [28]. The concepts of semantic tableau, branch, open and closed branch etc. are essentially defined as in [38], [39] and [40] (see also Priest [34]). For more on semantic tableaux, see D'Agostino, Gabbay, Hähnle and Posegga [18].

## 4.2 Tableau rules

I use the same (temporal, alethic, deontic etc.) tableau rules as in [38] and [40]. In addition, I will consider a set of tableau rules for the counterfactual operators and some rules for the "temporal" necessity operators.

## 4.2.1 New basic alethic rules

$\Box - pos \ (\Box)$	$\diamond - pos \ (\diamond)$	$\Box - neg (\neg \Box)$	$\diamond - neg (\neg \diamond)$
$\odot A, w_i t_k$	$\diamond A, w_i t_k$	$\neg \odot A, w_i t_k$	$\neg \diamondsuit A, w_i t_k$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A, w_j t_k$	$A, w_j t_k$	$\Diamond \neg A, w_i t_k$	$\Box \neg A, w_i t_k$
	where $w_j$ is new		

## 4.2.2 Basic counterfactual rules

$\Box \rightarrow -pos \ (\Box \rightarrow)$	$\Leftrightarrow -pos \ (\diamondsuit)$	$\Box \rightarrow -neg \ (\neg \Box \rightarrow)$	$\Leftrightarrow -neg \ (\neg \Leftrightarrow)$	
$A \square B, w_i t_k$	$A \Leftrightarrow B, w_i t_k$	$\neg (A \square B), w_i t_k$	$\neg (A \Leftrightarrow B), w_i t_k$	
$r_A w_i w_j t_k$	$\downarrow$	$\downarrow$	$\downarrow$	
$\downarrow$	$r_A w_i w_j t_k$	$A \Leftrightarrow \neg B, w_i t_k$	$A \Box \rightarrow \neg B, w_i t_k$	
$B, w_j t_k$	$B, w_j t_k$			
	where $w_j$ is new			

Table 4

## 4.2.3 CUT, CId(I), CId(II)

I will include CUT, CId(I) and CId(II) in every tableau system; however, in many systems these rules are redundant (see [37]). I call the identity rules in this section C-identity rules, CId(I) and CId(II), to distinguish them from the temporal identity rules in [38].

CUT	CId (CIdI)	CId (CIdII)	
*	$\alpha(w_i)$	$\alpha(w_i)$	
$\checkmark$ >	$w_i = w_j$	$w_j$ = $w_i$	
$\neg A, w_i t_k  A, w_i t_k$	$\downarrow$	$\downarrow$	
for every A	$\alpha(w_j)$	$\alpha(w_j)$	
Table 5			

## 4.2.4 Counterfactual (accessibility) rules (c-rules)

Τ-	c0			T-c0'		
If $D$ is of	If $D$ is of the form			If $D$ is of the form		
$\boxdot(A \leftrightarrow B) \rightarrow$			$((A \Box \rightarrow B) \land (B \Box \rightarrow A)) \rightarrow$			
$((A \square C) \leftrightarrow$	$(B \Box \rightarrow C)),$		$((A \square C))$	$() \leftrightarrow (B \Box \rightarrow C)$	′)),	
$D, w_i t_l$ can be added to any open		$D, w_i t_l$ can be added to any open				
branch on whic	ch $w_i t_l$ occurs.		branch on which $w_i t_l$ occurs.			
		Tab	le 6			
				1		
T-c1	T-c2		T-c3	T-c4		
$r_A w_i w_j t_l$	$r_A w_i w_j t_l$		$A, w_i t_l$	$r_A w_i w_j t_l$		
$\downarrow$	$B, w_j t_l$		$\downarrow$	$B, w_j t_l$		
$A, w_j t_l$	$\downarrow$		$r_A w_j w_k t_l$	$r_{A \wedge B} w_i w_k t_l$		
	$r_{A \wedge B} w_i w_j t_l$	wh	ere $w_k$ is new	$\downarrow$		
				$r_A w_i w_k t_l$		
				$B, w_k t_l$		
T-c5	T - c6		T-c7			
$A, w_i t_l$	$A, w_i t_l$		$r_A w_i w_j t_l$			
$\downarrow$	$r_A w_i w_j t_l$		$r_A w_i w_k t_l$			
$r_A w_i w_i t_l$	$\downarrow$		$\downarrow$			
	$w_i = w_j$		$w_j$ = $w_k$			
Table 7						

# 4.3 Tableau systems

A tableau system is a set of tableau rules. A counterfactual temporal alethic-deontic tableau system includes all propositional rules, all basic alethic rules, all basic deontic rules, all basic temporal rules (including the rules for <u>A</u> and <u>S</u> (see [40])), all basic counterfactual rules (and CUT and the C-identity rules) (Sections 4.2.1 - 4.2.3, Tables 3 - 5). The minimal counterfactual temporal alethic-deontic tableau system is called "T". The temporal identity rules, Id(I) and Id(II), are added to every system that

includes T - FC, T - PC ([38]) or T - C ([40]) (they are redundant in every other system). By adding any subset of the rules introduced in Section 4.2.4 (Tables 6 - 7) or various accessibility rules introduced by [38] or [40], we obtain an extension of T(note that some of these are deductively equivalent). I use the following conventions for naming systems. " $aA_1 \dots A_i dB_1 \dots B_j cC_1 \dots C_k adD_1 \dots D_l tE_1 \dots E_m adtF_1 \dots F_n$ " is a system, where  $A_1 \dots A_i$  is a list (possibly empty) of a-rules,  $B_1 \dots B_j$  is a list (possibly empty) of d-rules,  $C_1 \dots C_k$  is a list (possibly empty) of c-rules (counterfactual rules),  $D_1 \dots D_l$  is a list (possibly empty) of ad-rules,  $E_1 \dots E_m$  is a list (possibly empty) of t-rules, and  $F_1 \dots F_n$  is a list (possibly empty) of adt-rules. I sometimes abbreviate by omitting "redundant" letters in a name, if it doesn't lead to any ambiguity. E.g. aTdDc1235adOCt4adtSP is the counterfactual temporal alethic-deontic system that includes the rules T - aT, T - dD, T - c1, T - c2, T - c3, T - c5, T - OC, T - t4 and T - SP ([38]).

## 4.4 Some proof-theoretical concepts

The concepts of proof, theorem, derivation, consistency, inconsistency in a system etc. are defined as in [38] and [40]. Let S be a tableau system. Then the logic (or the (logical) system) of S, L(S), is the set of all sentences (in L) that are provable in S, in symbols  $L(S) = \{A \in L : \vdash_S A\}$ . E.g. L(aTdDc125t4) is the set of all sentences that are provable in the system aTdDc125t4, i.e. in the system that includes the basic rules and the (non-basic) rules T - aT, T - dD (see [38]), T - c1, T - c2, T - c5 and T - t4 ([38]).

## 5 Examples of theorems

In this section, I will consider some examples of theorems in some systems. The proofs are usually straightforward and are left to the reader.

**Theorem 5.1** The sentences in Table 8 are theorems in the indicated systems.

T0	$\boxdot(p \leftrightarrow q) \rightarrow ((p \Box \rightarrow r) \leftrightarrow (q \Box \rightarrow r))$	C - c0	
T0'	$((p \Box \rightarrow q) \land (q \Box \rightarrow p)) \rightarrow ((p \Box \rightarrow r) \leftrightarrow (q \Box \rightarrow r))$	C - c0'	
T1	$p \Box \rightarrow p$	C - c1	
T2	$((p \land q) \Box \to r) \to (p \Box \to (q \to r))$	C-c2	
T3	$\Diamond p \to ((p \Box \to q) \to (p \diamond \to q))$	C-c3	
T4	$(p \diamondsuit q) \to ((p \Box \to (q \to r)) \to ((p \land q) \Box \to r))$	C-c4	
T5	$(p \Box \rightarrow q) \rightarrow (p \rightarrow q)$	C-c5	
T6	$(p \land q) \to (p \Box \to q)$	C - c6	
T7	$(p \Box \rightarrow q) \lor (p \Box \rightarrow \neg q)$	C-c7	
Table 8			

**Theorem 5.2** Let S be a counterfactual (tableau) system in [37] that includes e.g. the accessibility rules Tc1, Tc3, and Tc4. Then the corresponding counterfactual temporal alethic-deontic (tableau) system S' is the counterfactual temporal alethic-deontic (tableau) system that includes the rules T - c1, T - c3, and T - c4, and similarly in every other case. Furthermore, let A be a sentence in any of the tables 9 - 15 in [37] and let t(A) be the result of replacing every occurrence of  $\Box$  by  $\Box$ , every occurrence of  $\diamond$  by  $\neg \diamond$ . Then if A is a theorem in the counterfactual system S, t(A) is a theorem in the corresponding counterfactual temporal alethic-deontic system S'.

**Theorem 5.3** The following sentences are theorems in every system described in this essay:  $(p \Box \rightarrow q) \leftrightarrow \neg (p \Leftrightarrow \neg q), \neg (p \Box \rightarrow q) \leftrightarrow (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow \neg (p \Leftrightarrow q), \neg (p \Box \rightarrow \neg q) \leftrightarrow (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow \neg (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow \neg (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow \neg (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow \neg (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow (p \Leftrightarrow \neg q), (p \Box \rightarrow \neg q) \leftrightarrow (p \Leftrightarrow \neg q) \leftrightarrow ((p \Leftrightarrow \bot) \lor (p \Box \rightarrow q)) and (p \Leftrightarrow q) \leftrightarrow ((p \Leftrightarrow \top) \land (p \Leftrightarrow q)).$ 

A counterpossible is a counterfactual with an impossible antecedent. In many counterfactual systems the following sentences are valid:  $\Leftrightarrow p \to (p \Box \to q)$  and  $\Leftrightarrow p \to \neg (p \Leftrightarrow \to q)$ q). They are not theorems in any of our systems. However, we do have  $\neg \otimes p \rightarrow (p \Box \rightarrow q)$ ,  $\neg \Diamond p \rightarrow \neg (p \Diamond \neg q), \neg Mp \rightarrow (p \Box \neg q) \text{ and } \neg Mp \rightarrow \neg (p \Diamond \neg q) \text{ in every system that}$ includes T - c1. Hence, it is not the case that every would counterfactual with an (historically) impossible antecedent is vacuously true. And it is not the case that every might counterfactual with an (historically) impossible antecedent is vacuously false. In other words, in the systems in this paper not all would counterpossibles are vacuously true and not all might counterpossibles are vacuously false. This seems to be intuitively plausible. Consider the following sentences: 1. If I were a very famous philosopher, many people would read my papers; and 2. If I were a very famous philosopher, very few people would read my papers. Suppose 1 and 2 are uttered by a person who in fact is not a very famous philosopher. Since the person is not in fact a very famous philosopher, the antecedent in 1 and 2 is historically impossible (not just false) at the time of the utterance, even though she might have been a very famous philosopher and she might become a very famous philosopher in the future. Then both 1 and 2 are counterpossibles. However, 1 seems true and 2 false. Countless similar examples are conceivable. This suggests that counterpossibles of this kind should not be vacuously true.

All of the following sentences are valid in T and every stronger system:  $Up \to \Box p$ ,  $\Box p \to \Box p, \Diamond p \to \Diamond p, \Diamond p \to Mp, \Box p \to \Box \Box p, \Diamond p \to \Box \Diamond p.$ 

If the system includes T - c1, then  $U(p \to q) \to (p \Box \to q)$  and  $\Box(p \to q) \to (p \Box \to q)$  are theorems.

If the system includes  $T - c^2$ , then  $((p \Box \rightarrow q) \land ((p \land q) \Box \rightarrow r)) \rightarrow (p \Box \rightarrow r)$  and  $((p \Box \rightarrow (p \land q)) \land ((p \land q) \Box \rightarrow r)) \rightarrow (p \Box \rightarrow r)$  are theorems.

If the system includes T - c7, then  $(p \square (q \lor r)) \to ((p \square q) \lor (p \square r))$  is a theorem.

If the system includes  $T - c^2$  and  $T - c^4$ , then  $(p \Leftrightarrow q) \to ((p \Box \to (q \to r)) \leftrightarrow ((p \land q) \Box \to r))$  is a theorem.

Let S be a counterfactual temporal alethic-deontic system. Then if S includes T-c5and T-c6, then  $A \to ((A \Box \to B) \leftrightarrow (A \to B))$  and  $A \to ((A \Leftrightarrow B) \leftrightarrow (A \to B))$  are theorems in S. If S includes T-c3, T-c5 and T-c6, then  $A \to ((A \Box \Rightarrow B) \leftrightarrow (A \to B))$ and  $A \to ((A \Leftrightarrow B) \leftrightarrow (A \to B))$  are theorems in S. In other words, counterfactuals with true antecedent reduce to material conditionals. This is true for both would and might counterfactuals of both kinds. In fact, a counterfactual with true antecedent is true iff the consequent is true in the indicated systems, i.e. the following sentences are theorems in the relevant systems:  $A \to ((A \Box \to B) \leftrightarrow B), A \to ((A \Leftrightarrow B) \leftrightarrow B), A \to ((A \Leftrightarrow B) \leftrightarrow B), A \to ((A \Box \Rightarrow B) \leftrightarrow B), A \to ((A \to A \to A) \to ((A \to A \to A$ 

In every system we have  $U(A \to B) \to \Box(A \to B)$ , in every system that includes T-c1 we have  $\Box(A \to B) \to (A \Box \to B)$  and in every system that includes T-c5 we have  $(A \square B) \rightarrow (A \rightarrow B)$ . The implications in the other direction do not hold. So, in these systems  $U(A \to B)$  is stronger than  $\Box (A \to B)$  that is stronger than  $A \Box \to B$ that is stronger than  $A \to B$ . In every system we have  $\Box(A \to B) \to \Box(A \to B)$  and in every system that includes T - aT we have  $\Box(A \to B) \to (A \to B)$ , but not the converse implications. So,  $\Box(A \to B)$  is stronger than  $\Box(A \to B)$  that is stronger than  $A \to B$ . It is neither the case that  $\Box (A \to B)$  entails  $A \Box \to B$  nor that  $A \Box \to B$  entails  $\Box(A \to B)$ . In every system,  $(A \Box \Rightarrow B) \to (A \Box \to B)$  and  $(A \Leftrightarrow B) \to (A \Leftrightarrow B)$ are theorems, but not vice versa, i.e. it is not the case that  $A \square \to B$  entails  $A \square \to B$ and it is not the case that  $A \Leftrightarrow B$  entails  $A \Leftrightarrow B$ . So,  $A \Longrightarrow B$  is stronger than  $A \square \rightarrow B$  and  $A \Leftrightarrow B$  is weaker than  $A \Leftrightarrow B$ . In every system that includes T - c1and  $T - c^3$ ,  $A \Leftrightarrow \top$  is equivalent to  $\diamond A$ . Hence, in these systems  $A \implies B$  is equivalent to  $\Diamond A \land (A \Box \rightarrow B)$  and  $A \Diamond \Rightarrow B$  is equivalent to  $\Diamond A \rightarrow (A \diamond \Rightarrow B)$ . This entails that the following sentences are theorems in every system that includes T - c1 and T - c3:  $(A \hookrightarrow B) \leftrightarrow (A \hookrightarrow B)$  and  $(A \hookrightarrow B) \leftrightarrow (A \hookrightarrow B)$ .

Suppose the system contains T - c0, T - c5 and T - dD. Then all of the following sentences are theorems.

$$O(A \square B) \rightarrow O(A \rightarrow B)$$
$$O(A \square B) \rightarrow (OA \rightarrow OB)$$
$$(A \land (A \square OB)) \rightarrow OB$$
$$(A \square OC) \rightarrow ((A \land B) \rightarrow OC)$$
$$(A \square OC) \rightarrow ((A \land B) \rightarrow OC)$$
$$(A \square OB) \rightarrow (A \square O(B \lor C))$$
$$(OA \land O(A \square B)) \rightarrow OB$$
$$O(A \square C) \rightarrow O((A \land B) \rightarrow C)$$
$$O(A \square B) \rightarrow O(A \square (B \lor C))$$
$$O(A \square B) \rightarrow (PA \rightarrow PB)$$

$$\begin{array}{c} (PA \land O(A \Box \rightarrow B)) \rightarrow PB \\ (FB \land O(A \Box \rightarrow B)) \rightarrow FA \\ O(A \Box \rightarrow B) \rightarrow (FB \rightarrow FA) \\ ((OA \land OB) \land O((A \land B) \Box \rightarrow C)) \rightarrow OC \\ (O(A \Box \rightarrow (B \lor C)) \land (FB \land FC)) \rightarrow FA \\ ((OA \lor OB) \land O((A \lor B) \Box \rightarrow C)) \rightarrow OC \\ (OA \land O(A \Box \rightarrow (B \land C)) \rightarrow (OB \land OC) \\ (FC \land O((A \lor B) \Box \rightarrow C)) \rightarrow (FA \land FB) \\ ((PA \lor PB) \land O((A \Box \rightarrow C)) \rightarrow (FA \land FB) \\ ((PA \lor PB) \land O((A \Box \rightarrow CC)) \rightarrow (B \Box \rightarrow OC)) \\ \Box(A \leftrightarrow B) \rightarrow ((A \Box \rightarrow OC) \leftrightarrow (B \Box \rightarrow OC)) \\ \Box(A \leftrightarrow B) \rightarrow ((C \Box \rightarrow OA) \leftrightarrow (C \Box \rightarrow OB)) \\ \Box(A \leftrightarrow B) \rightarrow (O(A \Box \rightarrow C) \leftrightarrow O(B \Box \rightarrow C)) \\ \Box(A \leftrightarrow B) \rightarrow (O(A \Box \rightarrow C) \land O(B \Box \rightarrow C))) \rightarrow OC \\ (O(A \lor B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow C))) \rightarrow OC \\ (O(A \lor B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow C))) \rightarrow O(C \lor D) \\ (OA \land (O(A \Box \rightarrow B) \land O(A \Box \rightarrow C))) \rightarrow (OB \land OC) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow D))) \rightarrow O(C \lor D) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow O(B \lor C) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow D))) \rightarrow (OC \land OD) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow D))) \rightarrow (OC \land OD) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow D))) \rightarrow (PC \lor OD) \\ (OA \land (O(A \Box \rightarrow B) \land O(A \Box \rightarrow C))) \rightarrow (PB \land PC) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \lor O(B \Box \rightarrow D))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PB \land PC) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \lor O(B \Box \rightarrow D))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PB \land PC) \\ (O(A \land B) \land (O(A \Box \rightarrow C) \lor O(B \Box \rightarrow D))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \lor PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \rightarrow C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow C) \land O(B \Box \rightarrow D))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \land C))) \rightarrow (PC \land PD) \\ (OA \land (O(A \Box \rightarrow B) \lor O(A \Box \land$$

## 5.1 The Test Argument

In this section, I will prove that The Test Argument that was described in the introduction is valid. I will use the following sentences: H = She is completely honest; C = She is courageous, I = You are completely honest; D = You are courageous, L =You lie, and S = You are in her situation. To prove this I will create a closed semantic tableau that begins with the premises (1)-(3) and the negation of the conclusion (4). This tableau shows that it is impossible that the premises are true while the conclusion is false, and hence that the argument is valid. We do not use any special accessibility rules in our proof. This means that the conclusion follows from the premises in every system in this paper. And this entails that the argument is valid in the class of all models. Here is our proof (at step 11 and at the last step (step 20) we use the derived rule MP (Modus Ponens)).

## 6 Soundness and completeness theorems

The soundness and completeness proofs in this section are modifications and extensions of proofs found in [37] and [38] (see also [39] and [40]).

Let " $aA_1 \ldots A_i dB_1 \ldots B_j cC_1 \ldots C_k adD_1 \ldots D_l tE_1 \ldots E_m adt F_1 \ldots F_n$ " be a counterfactual temporal alethic-deontic tableau system as defined above. Then I shall say that the class of models, **M**, corresponds to S just in case **M** = **M**(C - A\_1, \ldots, C - A\_i, C - B\_1, \ldots, C - B\_j, C - C\_1, \ldots, C - C\_k, C - D\_1, \ldots, C - D\_l, C - E\_1, \ldots, C - E\_m, C - F\_1, \ldots, C - F\_n).

S is strongly sound with respect to  $\mathbf{M}$  iff  $\Gamma \vdash_S A$  entails  $\mathbf{M}, \Gamma \Vdash A$ . S is strongly complete with respect to  $\mathbf{M}$  just in case  $\mathbf{M}, \Gamma \Vdash A$  entails  $\Gamma \vdash_S A$ .

## 6.1 Soundness theorems

Let M be any model and b any branch of a tableau. Then b is satisfiable in M iff there is a function f from  $w_0, w_1, w_2, \ldots$  to W and a function g from  $t_0, t_1, t_2, \ldots$  to T such that (i) A is true in  $f(w_i)$  at  $g(t_j)$  in M, for every node  $A, w_i t_j$  on b, (ii) if  $rw_i w_j t_k$  is on b, then  $Rf(w_i)f(w_j)g(t_k)$  in M, (iii) if  $sw_i w_j t_k$  is on b, then  $Sf(w_i)f(w_j)g(t_k)$  in M, (iv) if  $r_A w_i w_j t_k$  is on b, then  $R_A f(w_i)f(w_j)g(t_k)$  in M, (v) if  $t_i < t_j$  is on b, then  $g(t_i) < g(t_j)$  in M, (vi) if  $w_i = w_j$  is on b, then  $f(w_i) = f(w_j)$  in M, (vii) if  $t_i = t_j$  is on b, then  $g(t_i) = g(t_j)$  in M. If these conditions are fulfilled, we say that f and g show that b is satisfiable in M.

**Lemma 6.1** (Soundness lemma). Let b be any branch of a tableau and M be any counterfactual temporal alethic-deontic model. If b is satisfiable in M and a tableau rule is applied to it, then it produces at least one extension, b', of b such that b' is satisfiable in M.

**Proof.** The proof proceeds by going through all the tableau rules. I will only sketch some parts to illustrate the method.

(T-c2). Suppose that  $r_A w_i w_j t_l$  and  $B, w_j t_l$  are on b, and that we apply T-c2 to give an extended branch of b containing  $r_{A \wedge B} w_i w_j t_l$ . Since b is satisfiable in M,  $R_A f(w_i) f(w_j) g(t_l)$  and B is true in  $f(w_j)$  at  $g(t_l)$ . Accordingly,  $R_{A \wedge B} f(w_i) f(w_j) g(t_l)$ , since M satisfies condition C-c2.

(T - c4). Suppose that  $r_A w_i w_j t_l$ ,  $B, w_j t_l$  and  $r_{A \wedge B} w_i w_k t_l$  are on b, and that we apply T - c4 to give an extended branch of b containing  $r_A w_i w_k t_l$  and  $B, w_k t_l$ . Since b is satisfiable in M,  $R_A f(w_i) f(w_j) g(t_l)$ ,  $R_{A \wedge B} f(w_i) f(w_k) g(t_l)$  and B is true in  $f(w_j)$  at  $g(t_l)$ . Accordingly,  $R_A f(w_i) f(w_k) g(t_l)$  and B is true in  $f(w_k)$  at  $g(t_l)$ , since M satisfies condition C - c4.

(T-c5). Suppose that  $A, w_i t_l$  is on b and that we apply T-c5 to obtain an extended branch, b', of b containing  $r_A w_i w_i t_l$ . Since b is satisfiable in M, A is true in  $f(w_i)$  at  $g(t_l)$ . Hence,  $R_A f(w_i) f(w_i) g(t_l)$ . For M satisfies condition C-c5. Consequently, T-c5 produces at least one extension, b', of b such that b' is satisfiable in M.

**Theorem 6.2** (Soundness theorem I). Let S be any of the tableau systems discussed in this essay and let  $\mathbf{M}$  be the class of models that corresponds to S. Then S is strongly sound with respect to  $\mathbf{M}$ .

**Proof.** Once the Soundness lemma is established the proof is an easy modification of similar proofs found elsewhere (see e.g. [34], [37], [38], [39] and [40]). ■

**Theorem 6.3** (Soundness theorem II). (i) Tc012 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$ . (ii) Tc0123 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$  and C - cc3. (iii) Tc01234 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$ , C - cc3 and C - cc4. (iv) Tc012345 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$ , C - cc3, C - cc4 and C - cc5. (v) Tc0123456 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$ , C - cc3, C - cc5 and C - cc6. (vi)

Tc01234567 is sound with respect to the class of all supplemented models that satisfy  $\gamma 0$ , C - cc3, C - cc4, C - cc5, and C - cc7 (and hence also C - cc6). (Soundness results for other combinations of these conditions are easily obtained.)

**Proof.** This follows from Soundness theorem I and Theorem 1 in Section 3.2. ■

## 6.2 Completeness theorems

Let b be an open complete branch of a tableau and let I be the set of numbers on b immediately preceded by a "t". We shall say that  $i \rightleftharpoons j$  just in case i = j, or " $t_i = t_j$ " or " $t_j = t_i$ " occur on b.  $\rightleftharpoons$  is an equivalence relation and [i] is the equivalence class of i. Furthermore, let K be the set of numbers on b immediately preceded by a "w". We shall say that  $k \approx l$  just in case k = l, or " $w_k = w_l$ " or " $w_l = w_k$ " occur on b.  $\approx$  is an equivalence relation and [k] is the equivalence class of k.

**Definition 6.4** (Induced model) The counterfactual temporal alethic-deontic model,  $M = \langle W, T, \langle R, S, \{R_A : A \in L\}, V, v \rangle$ , induced by b is defined as follows.  $W = \{\omega_{[k]} : k \in K\}$ ,  $T = \{\tau_{[i]} : i \in I\}$ ,  $\tau_{[i]} < \tau_{[j]}$  iff  $t_i < t_j$  occurs on b,  $R\omega_{[i]}\omega_{[j]}\tau_{[k]}$  iff  $rw_iw_jt_k$  occurs on b,  $S\omega_{[i]}\omega_{[j]}\tau_{[k]}$  iff  $sw_iw_jt_k$  occurs on b,  $R_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$  iff  $r_Aw_iw_jt_k$  occurs on b. If  $p, w_it_j$ occurs on b, then p is true in  $\omega_{[i]}$  at  $\tau_{[j]}$  (i.e. then  $\langle \omega_{[i]}, \tau_{[j]} \rangle \in V(p)$ ); if  $\neg p, w_it_j$  occurs on b, then p is false in  $\omega_{[i]}$  at  $\tau_{[j]}$  (i.e. then it is not the case that  $\langle \omega_{[i]}, \tau_{[j]} \rangle \in V(p)$ ). If  $t_i$  occurs on b, then  $v(t_i) = \tau_{[i]}$ .

If our tableau system neither includes T - FC, T - PC ([38]) nor T - C ([40]),  $\Rightarrow$  is reduced to identity and [i] = i. Hence, in such systems, we may take T to be  $\{\tau_i : t_i \text{ occurs on } b\}$  and dispense with the equivalence classes. Likewise, if our tableau system neither includes T - c6 nor T - c7,  $\approx$  is reduced to identity and [k] = k. Accordingly, in such systems, we may take W to be  $\{\omega_i : w_i \text{ occurs on } b\}$  and dispense with the equivalence classes.

**Lemma 6.5** (Completeness lemma). Let b be an open branch in a complete tableau and let M be a temporal alethic-deontic model induced by b. Then: (i) A is true in  $\omega_{[i]}$ at  $\tau_{[j]}$ , if A,  $w_i t_j$  is on b, and (ii) A is false in  $\omega_{[i]}$  at  $\tau_{[j]}$ , if  $\neg A, w_i t_j$  is on b.

**Proof.** The proof is by induction on the complexity of A.

I will go through one example to illustrate the method.

 $A = \Box B$ . Suppose  $A, w_i t_k$  occurs on b, i.e.  $\Box B, w_i t_k$  is on b. Since b is complete  $(\Box)$  has been applied to  $\Box B, w_i t_k$ . Thus, for all  $w_j$  on  $b, B, w_j t_k$  is on b. By the induction hypothesis, for all  $\omega_{[j]}$  such that  $R\omega_{[i]}\omega_{[j]}\tau_{[k]}$ , B is true in  $\omega_{[j]}$  at  $\tau_{[k]}$ . Hence,  $\Box B$  is true in  $\omega_{[i]}$  at  $\tau_{[k]}$ . Suppose that  $\neg A, w_i t_k$  occurs on b, i.e.  $\neg \Box B, w_i t_k$  is on b. Then  $\Diamond \neg B, w_i t_k$ 

is on b (by  $(\neg \Box)$ ). For b is complete. Furthermore, since b is complete ( $\diamondsuit$ ) has been applied to  $\diamondsuit \neg B, w_i t_k$ . Thus, for some  $w_j, rw_i w_j t_k$  and  $\neg B, w_j t_k$  are on b. By the induction hypothesis,  $R\omega_{[i]}\omega_{[j]}\tau_{[k]}$  and B is false in  $\omega_{[j]}$  at  $\tau_{[k]}$ . Hence,  $\Box B$  is false in  $\omega_{[i]}$  at  $\tau_{[k]}$ .

**Theorem 6.6** (Completeness theorem). Let S be any of the tableau systems discussed in this essay, not including T - c0 or T - c0', and let **M** be the class of models that corresponds to S. Then S is strongly complete with respect to **M**.

**Proof.** The proof is a modification of similar proofs in [37] and [38] (see also e.g. [34], [39] and [40]).

First we show that the weakest system is complete. Then we have to check that the model induced by the open branch, b, is of the right kind in every case. I will only consider some cases to illustrate the method.

(C-c4). Suppose that  $R_A\omega_{[i]}\omega_{[j]}\tau_{[l]}$ ,  $R_{A\wedge B}\omega_{[i]}\omega_{[k]}\tau_{[l]}$  and that B is true in  $\omega_{[j]}$  at  $\tau_{[l]}$ . Then  $r_Aw_iw_jt_l$  and  $r_{A\wedge B}w_iw_kt_l$  occur on b [by the definition of an induced model]. Since the tableau is complete CUT has been applied and either  $B, w_jt_l$  or  $\neg B, w_jt_l$  is on b. Assume that  $\neg B, w_jt_l$  is on b. Then B is false in  $\omega_{[j]}$  at  $\tau_{[l]}$  [by the completeness lemma]. But this is absurd. So,  $B, w_jt_l$  is on b. Since the tableau is complete T - c4 has been applied and  $r_Aw_iw_kt_l$  and  $B, w_kt_l$  occur on b. It follows that  $R_A\omega_{[i]}\omega_{[k]}\tau_{[l]}$  and that B is true in  $\omega_{[k]}$  at  $\tau_{[l]}$ , as required [by the definition of an induced model and the completeness lemma].

(C-c5). Suppose that A is true in  $\omega_{[i]}$  at  $\tau_{[j]}$ . Since the tableau is complete CUT has been applied and either  $A, w_i t_j$  or  $\neg A, w_i t_j$  is on b. Suppose  $\neg A, w_i t_j$  is on b. Then A is false in  $\omega_{[i]}$  at  $\tau_{[j]}$  [by the Completeness Lemma]. But this is impossible. Hence,  $A, w_i t_j$  is on b. Since the tableau is complete T - c5 has been applied and  $r_A w_i w_i t_j$  is on b. Accordingly,  $R_A \omega_{[i]} \omega_{[i]} \tau_{[j]}$ , as required [by the definition of an induced model].

(C-c7). Assume that  $R_A \omega_{[i]} \omega_{[j]} \tau_{[l]}$  and  $R_A \omega_{[i]} \omega_{[k]} \tau_{[l]}$ . Then  $r_A w_i w_j t_l$  and  $r_A w_i w_k t_l$  occur on b [by the definition of an induced model]. Since the tableau is complete T-c7 has been applied and  $w_j = w_k$  is on b. Hence,  $j \approx k$ . So, [j] = [k]. It follows that  $\omega_{[j]} = \omega_{[k]}$ , as required.

**Problem 6.7** Whether or not any other system discussed in this paper is complete is left as an open question.

Acknowledgement: I want to express my thanks to the reviewers for some good suggestions.

# References

- J. E. Adler and C. W. Gowans (2013). Universalizability. The International Encyclopedia of Ethics. Edited by Hugh LaFollette, pp. 5251-5258.
- [2] C. E. Alchourrón and E. Bulygin, Normative Knowledge and Truth, In J. J. E. Garcia et al. (eds.). Philosophical Analysis in Latin America, D. Reidel Publishing Company, (1984), pp. 25-45.
- [3] P. Bailhache, Les normes dans le temps et sur l'action, Essai de logique déontique, Université de Nantes, 1986.
- [4] P. Bailhache, Essai de logique déontique, Paris, Librarie Philosophique, Vrin, Collection Mathesis, 1991.
- [5] P. Bailhache, The Deontic Branching Time: Two Related Conceptions, Logique et Analyse 36 (1993) 159-175.
- [6] P. Bailhache, Canonical Models for Temporal Deontic Logic, Logique et Analyse 149 (1995) 3-21.
- [7] P. Bartha, Moral Preference, Contrary-to-Duty obligation and Defeasible Oughts, In: McNamara and Prakken, (eds.), 1999, pp. 93-108.
- [8] N. Belnap, M. Perloff and M. Xu, Facing the Future: Agents and Choices in Our Indeterminist World, Oxford, Oxford University Press, 2001.
- [9] M. A. Brown, Agents with Changing and Conflicting Commitments: A Preliminary Study, In: McNamara and Prakken, (eds.), 1999, pp. 109-125.
- [10] M. A. Brown, Conditional Obligation and Positive Permission for Agents in Time, Nordic Journal of Philosophical Logic vol. 5 number 2 (2000) 83-112.
- [11] M. A. Brown, Rich deontic logic: a preliminary study, Journal of Applied Logic 2 (2004) 19-37.
- [12] J. P. Burgess, Basic Tense Logic, In: Gabbay and Guenthner, (eds.), Handbook of Philosophical Logic, 1984, Vol. 2, pp. 89-133, 2nd Edition, 2002, Vol. 7, pp. 1-42.
- [13] W. Carnielli and C. Pizzi. Modalities and Multimodalities. Springer, 2009.
- [14] B. F. Chellas, The Logical Form of Imperatives, Stanford, Perry Lane Press, 1969.
- [15] B. F. Chellas, Modal Logic: An Introduction, Cambridge, Cambridge University Press, 1980.

- [16] R. Ciuni and A. Zanardo, Completeness of a Branching-Time Logic with Possible Choices, Studia Logica 96 (2010) 393-420.
- [17] N. B. Cocchiarella and M. A. Freund. Modal Logic: An Introduction to Its Syntax and Semantics. New York: Oxford University Press, 2008.
- [18] M. D'Agostino, D. M. Gabbay, R. Hähnle and J. Posegga (eds.), Handbook of Tableau Methods, Dordrecht, Kluwer Academic Publishers, 1999.
- [19] M. C. DiMaio and A. Zanardo, A Gabbay-Rule Free Axiomatization of T × W Validity, Journal of Philosophical Logic 27 (1998) 435-487.
- [20] J. E. van Eck, A System of Temporally Relative Modal and Deontic Predicate Logic and its Philosophical Applications, Department of Philosophy, University of Groningen, The Netherlands, 1981.
- [21] F. Feldman. Doing The Best We Can: An Essay in Informal Deontic Logic. Dordrecht: D. Reidel Publishing Company, 1986.
- [22] M. Fitting, Proof Methods for Modal and Intuitionistic Logics, Dordrecht, D. Reidel Publishing Company, 1983.
- [23] M. Fitting and R. L. Mendelsohn, First-Order Modal Logic, Kluwer Academic Publishers, 1998.
- [24] D. M. Gabbay and F. Guenthner (eds.), Handbook of Philosophical Logic 2nd Edition, Vol. 3, Dordrecht, Kluwer Academic Publishers, 2001.
- [25] D. M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev, Many-Dimensional Modal Logics: Theory and Applications, Amsterdam, Elsevier, 2003.
- [26] R. Hilpinen (Ed.), New Studies in Deontic Logic: Norms, Actions, and the Foundation of Ethics, Dordrecht, D. Reidel Publishing Company, 1981.
- [27] J. F. Horty, Agency and Deontic Logic, Oxford, Oxford University Press, 2001.
- [28] R. C. Jeffrey, Formal Logic: Its Scope and Limits, McGraw-Hill, New York, 1967.
- [29] M. Kracht, Tools and Techniques in Modal Logic, Amsterdam, Elsevier, 1999.
- [30] F. von Kutschera, T  $\times$  W Completeness, Journal of Philosophical Logic 26 (1997) 241-250.
- [31] D. Lewis. *Counterfactuals*. Oxford: Basil Blackwell, 1973.

- [32] P. McNamara and H. Prakken (eds.), Norms, Logics and Information Systems: New Studies in Deontic Logic and Computer Science, Amsterdam, IOS Press, 1999.
- [33] N. T. Potter and M. Timmons (eds.). (1985). Morality and Universality: Essays on Ethical Universalizability. Dordrecht, Holland: D. Reidel.
- [34] G. Priest, An Introduction to Non-Classical Logic, Cambridge, Cambridge University Press, 2008.
- [35] A. Prior, Past, Present and Future, Oxford, Clarendon, 1967.
- [36] N. Rescher and A. Urquhart, Temporal logic, Wien, Springer-Verlag, 1971.
- [37] D. Rönnedal. Counterfactuals and Semantic Tableaux. Logic and Logical Philosophy, Vol. 18, No. 1, 2009, pp. 71-91.
- [38] D. Rönnedal. Temporal Alethic-Deontic Logic and Semantic Tableaux. Journal of Applied Logic, 10, 2012, pp. 219-237.
- [39] D. Rönnedal. Extensions of Deontic Logic: An Investigation into some Multi-Modal Systems, Department of Philosophy, Stockholm University, 2012.
- [40] D. Rönnedal. Quantified Temporal Alethic Deontic Logic. Logic and Logical Philosophy. Vol 24, No 1 (2015), pp. 19-59.
- [41] R. M. Smullyan, First-Order Logic, Heidelberg, Springer-Verlag, 1968.
- [42] R. C. Stalnaker. A Theory of Conditionals. In N. Rescher, ed., Studies in Logical Theory, Blackwell: Oxford, 1968.
- [43] R. Thomason, Deontic Logic as Founded on Tense Logic, In: Hilpinen, (Ed.), 1981, pp. 165-176.
- [44] R. Thomason, Deontic logic and the role of freedom in moral deliberation, In: Hilpinen, (Ed.), 1981, pp. 177-186.
- [45] R. Thomason, Combinations of Tense and Modality, In: Gabbay and Guenthner, (eds.), Handbook of Philosophical Logic, 1984, Vol. 2, pp. 135-165, 2nd Edition, 2002, Vol. 7, pp. 205-234.
- [46] S. Wölfl, Combinations of Tense and Modality for Predicate Logic, Journal of Philosophical Logic 28 (1999) 371-398.

- [47] A. Zanardo, Branching-Time Logic with Quantification over Branches: the Point of View of Modal Logic, The Journal of Symbolic Logic vol. 61 number 1 (1996) 1-39.
- [48] L. Åqvist and J. Hoepelman, Some theorems about a "tree" system of deontic tense logic, In: Hilpinen, (Ed.), 1981, pp. 187-221.
- [49] L. Åqvist, Deontic Logic, In: Gabbay and Guenthner, (eds.), Handbook of Philosophical Logic, Vol. 2, 1984, pp. 605-714, 2nd Edition, Vol. 8, 2002, pp. 147-264.
- [50] L. Åqvist, The Logic of Historical Necessity as Founded on Two-Dimensional Modal Tense Logic, Journal of Philosophical Logic 28 (1999) 329-369.
- [51] L. Åqvist, Combinations of tense and deontic modality: On the Rt approach to temporal logic with historical necessity and conditional obligation, Journal of Applied Logic 3 (2005) 421-460.

Daniel Rönnedal Department of Philosophy Stockholm University 106 91 Stockholm, Sweden *E-mail:* daniel.ronnedal@philosophy.su.se