South American Journal of Logic
Vol. 2, n. 1, pp. 41–55, 2016
ISSN: 2446-6719

The Interplay between Logic, Mathematics and Philosophy

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Abstract

Applications of logic to mathematics are well known, as are also those of elementary logic in the refutation of the various empiricist criteria of cognitive significance and of Riemann's conception of n-extended magnitudes in the refutation of Kant's conception of space. However, the interplay between the three disciplines is much richer, and not only philosophy can serve to the conceptual clarification in the other two disciplines, but model theory can be used to refute views in the philosophy of logic and the philosophy of mathematics

In my opinion, mathematics, physics and philosophy form an interconnected scientific system, and I have always seen it a part of my life's work especially to cultivate the relationship between mathematics and philosophy.

David Hilbert, Letter to the Undersecretary of the Ministry of Culture of 30 July 1918, quoted in Ortiz Hill and da Silva, The Road Not Taken, *p. 391*

1 Introduction *et alia*

Probably the most important contribution of so-called analytic philosophy to philosophy is the application of logical tools to the study of philosophical problems. The development of logic in the last one hundred fifty years has been simply extraordinary, though usually only some of its most elementary parts have found application in philosophy. An example thereof was the refutation of the different proposals of cognitive significance by the logical empiricists and their associates in the 1930s. On the other hand, mathematics has also been applied during that same period to elucidate central philosophical problems, like that of the nature of physical space and time. As is well known, the development of non-Euclidean geometries helped refute the well-entrenched Kantian theory of space and time as a priori forms of our intuition. Though there was a staunch resistance by some philosophers, Riemann's view of the empirical nature of physical space and time finally triumphed after the development of general relativity.

In this paper we will be concerned not only with some applications of logic and mathematics to philosophy, and applications of logic to mathematics and of mathematics to logic, but also with the application of philosophy to logic and mathematics, something that at first sight looks as unreal.

As a sort of footnote to this Introduction, let us consider a very elementary application of a mathematical theory to philosophy, namely, to a philosophical assertion by Georg Wilhelm Friedrich Hegel that certainly has irritated more than one philosopher with some knowledge of post-Cantorian mathematics. In his book *Wissenschaft der Logik*,[1] in English: *The Science of Logic* –though the book is neither about science nor about logic, but a metaphysical treatise–, Hegel asserts that there are two sorts of infinite, namely, an infinite that opposes the finite and that he calls "false infinite", and an infinite that synthesizes the finite and the so-called false infinite, which is supposed to be the "true infinite". However, as is now well known from the rudiments of set theory, adding finite to any sort of infinite leaves us with the same infinite. To put it more precisely: for any infinite cardinal κ and any finite n, $\kappa + n = \kappa = n + \kappa$. In fact, the following is true: $\kappa + \kappa = \kappa$, for any infinite κ. Of course, Hegel preceded the birth of set theory and, thus, one should not be too hard on him for the blooper. Nonetheless, it should serve as a reminder for any serious philosopher not to ignore the results of the most exact sciences: logic and mathematics (and also physics!). Let us consider in what follows less trivial and more interesting cases of the interplay between logic, mathematics and philosophy.

[1] *Wissenschaft der Logik* 1812–1813, second revised edition 1834, reprint, Felix Meiner, Hamburg 1967–1969. See volume I, Chapter II Part C(c), pp. 132ff., especially p. 133.

2 On the Application of Group Theory to Semantic Theories

The best-known semantic theory of sense and referent is that of Gottlob Frege,[2] and many authors either consider that it is the only one, or prefer not to consider any alternative. Two expressions have the same sense if they are essentially synonymous. Hence, expressions in different languages that are perfect translations of each other are said to have the same sense. The same happens in the rare occasions in which two expressions of the same language are perfectly synonymous. Thus, 'The morning star is a planet' and 'Der Morgenstern ist ein Planet' are statements expressing the same sense, in one case in English, in the other one in German. On the other hand, different expressions can have different senses but refer to the same entity. Thus, for example, 'the least prime number', 'the least even number' and 'the only number that is both even and prime' are expressions referring uniquely to the number 2, though expressing different senses. On the other hand, the expression 'die kleinste Primzahl' has the same sense as the expression 'the least prime number'. But let us confine ourselves to only one language. The statements (i) 'the least even number is smaller than 3' and (ii) 'the least prime number is smaller than 3' express different senses. According to Frege, the referents of statements are truth-values, of which there are only two, namely, truth and falsity. Statements (i) and (ii), though having different senses, refer both to the truth. However, (iii) 'Paris is the capital of France on the 1st of January 2014; and (iv) 'The morning star is a planet' are also true statements and, thus, also refer to the truth.

A mapping from an English language statement to another can be called a transformation. One can consider a set of transformations in this mathematical sense of English statements to English statements in such a way that true statements are always assigned to true statements and false statements are always assigned to false ones, thus, dividing the set of statements into two equivalence classes. It can be easily seen that such a set of transformations is a group in the precise mathematical sense, since (a) any transformation has an inverse transformation, (b) there is the identical transformation assigning each statement to itself, and (c) the composition of transformations is associative. This group can very well be called "the Fregean Group".

However, if we examine statements (i)-(iv) above, we immediately observe that statements (i) and (ii) seem to be much nearly related to each other than to statements (iii) and (iv). An alternative semantic theory of sense and referent was propounded by Husserl in 1900 in the First Logical Investigation.[3] Husserl had already obtained

[2]See his classic paper 'Über Sinn und Bedeutung' 1892, reprint in Gottlob Frege, *Kleine Schriften*, edited by I. Angelelli, 1967, second edition, 1990, pp. 143–162, as well as, among others, *Grundgesetze der Arithmetik I*, 1893, reprint 1962.

[3]*Logische Untersuchungen II* 1901, Hua XIX(1) 1984.

the distinction between sense and referent in 1890[4] and made use of it in his review of Schröder's *Vorlesungen über die Algebra der Logik I*,[5] as Frege himself acknowledged in a letter to Husserl dated 24 May 1891.[6] However, only in *Logische Untersuchungen* did Husserl's semantic theory with respect to statements reach its complete maturity. As Frege, Husserl distinguished between the sense and the referent of expressions, but besides a general agreement, radically differed both on the sense and referent of what Frege called 'conceptual words' and, more importantly for our present purpose, on the issue of the referents of statements. For Husserl, statements (i) and (ii) above have much more in common than with (iii) and (iv), namely, they refer to the same state of affairs, whereas (iii) and (iv) refer to completely different states of affairs. Thus, for Husserl, not truth-values, but states of affairs are the referents of statements. Both sentences (i) and (ii) refer to the mathematical fact that the number 2 is smaller than the number 3. There should be no doubt that Husserl's choice of a referent for statements is by far more informative than Frege's.

Nonetheless, in the First Logical Investigation Husserl had still not distinguished between his preferred candidate for the referent of statements and its supporting referential basis, namely, what from the Sixth Logical Investigation[7] on he called "the situation of affairs". Let us consider the following inequalities: (v) '$5 + 2 < 7 + 1$' and (vi) '$7 + 1 > 5 + 2$'. Contrary to the difference existing between (i) and (ii) above or between '$5 + 2 < 7 + 1$' and (vii) '$4 + 3 < 6 + 2$', which is a mere semantic difference, referring to the same states of affairs by means of expressions with different senses, the distinction between (v) and (vi) has a more ontological nature. Nonetheless, there is a common ontological basis for both inequalities, namely, that the number 8 occurs later in the natural number sequence –is, hence, a larger number– than the number 7. This Husserlian distinction between state of affairs and situation of affairs seems to be mostly limited to mathematical contexts, though a colloquial language case could be the following. Consider the following pairs of statements: (a) 'My neighbour, the car merchant Joe, has sold an expensive car' and (b) 'My childhood friend Peter has bought an expensive car'. Though I am not aware of it, it could very well be the case that statements (a) and (b), that clearly refer to different states of affairs, have the same situation of affairs as referential basis, in case Peter bought the car from Joe.[8]

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[4] In the posthumously published paper 'Zur Logik der Zeichen (Semiotik)', published as appendix to the Husserliana edition of *Philosophie der Arithmetik* 1891, Hua XI 1970, pp. 340–373.

[5] 'Besprechung von Ernst Schröders *Vorlesungen über die Algebra der Logik I*', reprint in *Aufsätze und Rezensionen 1890–1910*, Hua XXII, pp. 3–43.

[6] See Frege's *Wissenschaftlicher Briefwechsel* 1974, pp. 94–98.

[7] *Logische Untersuchungen II*, Hua XIX(2), 1984, §48. See also his *Vorlesungen über Bedeutungslehre*, Hua XXVI, 1987 §7.

[8] I obtained this example when I was nineteen and had my first acquaintance with the sense-referent distinction in Husserl's states of affairs version, only to learn some three years later that Husserl had obtained the distinction at least some sixty-five years before me.

Both the sameness of state of affairs and the sameness of situation of affairs give rise to groups of transformations between statements, as also does the sameness of sense. Hence, we now have four different groups of transformations of the statements of a natural language, as we sketched in an old paper a long time ago.[9] This is a clear case of the application of the mathematical theory of groups of transformations to philosophical semantics. Moreover, since states of affairs can be seen as equivalence classes of senses, situations of affairs as equivalence classes of states of affairs, and truth-values as equivalence classes of situations of affairs, the groups of transformations determined by the sameness of truth-value, of situation of affairs, of state of affairs and that determined by the sameness of sense are clearly formally related. Nonetheless, only the groups of transformations determined by sameness of state of affairs and by sameness of situation of affairs are non-trivial, since the group of transformations determined by truth-value allows only for two equivalence classes, whereas the group of transformations determined by sameness of sense allows for as many equivalence classes as there are statements in the language, in fact, countably many. The group determined by sameness of state of affairs and the group determined by sameness of situation of affairs can be called the Husserlian Group I and the Husserlian Group II, respectively.

3 The Application of Husserlian Philosophical Semantics to Logic

Though analytic philosophers and philosophical logicians are usually so fond of Fregean semantics, the fact of the matter is that only in propositional logic does Fregean semantics seem to be useful. As is well known, in that very elementary part of logic in some sense all differences between statements seem to be reduced to the sameness of truth-value.

The situation is certainly much different when we move up to first order logic, not to mention more complicated systems. Already in the intuitive and informal interpretation of first-order formulas and statements, what is referred to is not a truth-value, but a state of affairs. This is still clearer when we consider first-order semantics, namely classical model theory. Statements are interpreted by states of affairs or classes of states of affairs in the different models, not simply by truth-values.

Nonetheless, situations of affairs –although less conspicuously than states of affairs-also admit an important application to first order semantics. As already mentioned, situations of affairs can be seen as equivalence classes of states of affairs. Now, in classical model theory there are metatheorems establishing the equivalence between

[9]'Remarks on Sense and Reference in Frege and Husserl' 1982, reprint in Claire Ortiz Hill and Guillermo E. Rosado Haddock, *Husserl or Frege?: Meaning, Objectivity and Mathematics* 2000, paperback edition, 2003, pp. 23–40. For the groups of transformations, see especially pp. 36–37.

statements that seem at first sight not to be related, since they talk about very different states of affairs. Nonetheless the statements are equivalent. A foremost example of this issue is Abraham Robinson's Model-Completeness Test, of which we will say much more below. The necessary and sufficient conditions for model-completeness made explicit in the formulation of that important result in classical model theory, though equivalent, clearly refer to different states of affairs. On the other hand, such statements are more strongly related with each other than with statements with which they only share the same truth-value. It seems pertinent to say that the necessary and sufficient conditions for model–completeness mentioned in the Model-Completeness Test have in common the same situation of affairs.

4 The Application of Husserlian Philosophical Semantics to Mathematics

In his *Vorlesungen über Bedeutungslehre*[10] the mathematician turned philosopher Edmund Husserl considered –though did not elaborate– the possibility of applying his distinction between states of affairs and situations of affairs to physics. So far as we know, he did not envisage its application to mathematics –as the present author has done in some older papers.[11] In any case, as we mentioned above, equivalences between statements in classical model theory can be fruitfully rendered as sameness of situation of affairs. On the other hand, not only in logic, but also in mathematics, statements seem to refer to states of affairs. And as happens with meta-statements in model theory about equivalences between statements, meta-statements in mathematics can very well be interpreted as saying that two or more equivalent statements have the same situation of affairs as referential base. In earlier papers we have argued that one can fruitfully use the notion of situation of affairs to render both what dual statements in mathematics –like the Prime Ideal Theorem and the Ultrafilter Theorem– have in common and, more generally, what statements like the Axiom of Choice and its many equivalents in different areas of mathematics have in common. Thus, for example, the Axiom of Choice, Zorn's Lemma and Tychonoff's Theorem talk about very different things, that is, they refer to different states of affairs. But they have much more in common than any of them with the also true statement that 'Paris is the capital of France the 1$^{\text{st}}$ of January of 2014' or with '$2 + 2 = 4$'. What the Axiom of Choice, Zorn's Lemma and Tychonoff's Theorem have in common is the situation of affairs.[12]

[10] *Vorlesungen über Bedeutungslehre*, Hua XXVI, 1987, §30 (b), pp. 101–102.

[11] See on this issue our 'On Husserl's Distinction between State of Affairs (Sachverhalt) and Situation of Affairs (Sachlage)', 'Interderivability of Seemingly Unrelated Mathematical Statements' and 'Platonism, Phenomenology and Interderivability', all included in the references.

[12] As pointed out in 'On Frege's Two Notions of Sense', we do not exclude the possibility of finer semantic distinctions in mathematics or elsewhere. In some sense, the semantic scheme of statement

5 Logic Applied to Mathematics and Mathematics Applied to Logic

The application of one of the sister disciplines to the other is a much better known fact than the already mentioned. Among the many instances of application of logic to mathematics one can mention the applications of recursive function theory to applied mathematics and, more specifically, to computer science. One can also mention the multiple applications of classical model theory to algebra, a field that is especially alive and fruitful. Since one can consult almost any book on model theory to find multiple examples of those applications, it is unnecessary to dwell on this issue.[13]

A less studied case is that of the applications of mathematics to logic. Probably, the logicist prejudice that mathematics is derivable from logic –which, by the way, is not the case– has prevented logicians to explicitly discuss this issue. The fact of the matter is that mathematics makes possible the establishment of many metatheorems of logic. Both mathematical induction and its variant complete induction are of a mathematical nature, but they are used extensively in the proof of logical metatheorems. Hence, it was Husserl –and also Hilbert–, not Frege, who was right with respect to the relation between logic and mathematics. If mathematics were derivable from logic, one could not make use of mathematical theorems to prove logical theorems or metatheorems.

6 Excursus on a non-logical Refutation of Kantian Constructivism

Before considering the application of logic to philosophical issues, specifically, issues in the philosophy of mathematics and the philosophy of logic, let us consider arguments not originating in logic that can be used to refute one of the most popular views in the philosophy of mathematics, namely, Kantian constructivism.

As is well known, there seem to be as many constructivisms as there are constructivists. Thus, the first problem with constructivism is that people like Kant, Brouwer, Markov, Bishop, Griss and others do not understand the same thing under mathematical constructivism. Moreover, in any of its versions constructivism has difficulties dealing with uncountable cardinalities. But even for the application of mathematics to physics you have to deal with real numbers and with functions on real numbers, thus, you have to admit at least two uncountable cardinalities, namely, the cardinality of the set of real numbers and the cardinality of the set of functions on real numbers.

$\mapsto$ proposition $\mapsto$ state of affairs $\mapsto$ situation of affairs $\mapsto$ truth-value should be seen as minimal.

[13] We have just recently learnt that there are applications of model theory by Jean-Louis Krivine and others to functional analysis. See José Iovino's *Applications of Model Theory to Functional Analysis*.

Now, the origin of constructivism as a serious philosophy of mathematics is due mostly to the groundwork of Immanuel Kant, and his extraordinary influence on philosophy in the last two centuries. Nonetheless, as many others, Kant was a child of his century's theoretical limitations. Not only had alternative geometries not been seriously considered in Kant's times, but Kant was also limited by the philosophical tradition. As we have shown in our paper 'Intuitions, Concepts and Wholes',[14] Kant's arguments in the Transcendental Aesthetic on the intuitive nature of space and time, and on which the whole constructivist conception of Kant is built –according to which we construct mathematical entities in pure intuition–, is based on an exceptionally narrow view of concepts, a view entrenched in the philosophical tradition almost until the twentieth century. According to Kant, pure space (and also pure time) is an intuition because it is given to us as a unity such that all possible partial spaces are seen as delimitations of the one space, whereas concepts are given to us as contained in each of its presumed parts, for example, as the concept of a horse is contained in any representation of a concrete horse, or the concept of number is in some sense contained in the representation of any of the natural numbers. Nonetheless, it should be clear that Kant is presupposing that concepts are concerned only with discrete objects, each of which is a sort of instantiation of the concept. That is exactly what happens with discrete collections of objects, be they empirical and also finite, as that of horses, or non-empirical and also infinite, as that of natural numbers. Since partial spaces and partial times are not discrete instantiations of a general concept, but delimitations of the unique space or time, respectively, obtained by means of division –a division that can be iterated indefinitely and we always obtain smaller and smaller spaces (or times)–, space and time cannot be concepts, but intuitions.

However, since at least Bernhard Riemann's epoch-making monograph *Über die Hypothesen, welche der Geometrie zugrunde liegen*,[15] in which the great mathematician, among many other things, subsumed Euclidean and non-Euclidean geometries with zero, respectively, negative or positive curvature under a general concept of an n-fold extended magnitude and then subsumed the concept of an n-fold extended continuous magnitude under the more general concept of a continuous manifold, contrasting this notion with that of a discrete manifold, while operating only with concepts and without referring to any sort of intuition, the ground for Kant's mathematical constructivism disappeared. There is no need to refer to intuition when considering the different geometric manifolds. Space and time are simply continuous manifolds, whereas the set of natural numbers is a manifold of discrete entities as is that of the set of horses. In fact, the development of mathematics in the nineteenth century, and not only that of geometry, helped expand considerably our notion of concept, and now the

[14] 'Intuitions, Concepts and Wholes', in *Notae Philosophicae Scientia Formalis* 2 (1), 2013, pp. 45–53.

[15] *Über die Hypothesen welche der Geometrie zugrunde liegen* 1867, reprint, 1923, 1960.

mathematician talks about the concept of group or the concept of topological space,[16] though such concepts have little to do with concepts as understood by the philosophical tradition, including the great Kant.

7 Classical Model Theory versus Constructivism

According to a well-known Quinean dogma,[17] a logic's ontological commitment is determined by the application of the quantifiers: to be is to be a possible value of a quantifier. Hence, since second- and higher-order logics quantify over properties and relations, they are committed to abstract entities. On the contrary, since first-order logic only quantifies over individuals, it is not committed to the existence of abstract entities, and is safe for nominalists and other sorts of anti-Platonists. Carnap basically, though not in the details, agrees with Quine –of course, Carnap wrote it first– when, after postulating his Principle of Tolerance in *Logische Syntax der Sprache*,[18] he discusses both an example of a restricted logic presumably without commitment to abstract entities, and a more liberal logic with such commitments. And as his Principle of Tolerance puts it, it is a matter of choice or taste to choose between logical languages with different commitments. In the present and the next section we will show that both Quine's and Carnap's assertions are false. In both cases we will be dealing with a first-order language with its classical semantics, that is, with what is usually called 'classical model theory'.

In a philosophical paper of dubious quality[19] published in *The Journal of Symbolic Logic* when he was presiding over the *Association for Symbolic Logic*, Hillary Putnam defended a sort of mild constructivism for mathematics. He seemed not to be aware that the moment one accepts classical model theory one has to abandon any sort of constructivism. As stated by Wilfrid Hodges[20] and also by the present author,[21] there are multiple theorems in model theory that directly belie any sort of constructivism. The best-known and probably most dramatic example is Tarski's Upward Löwenheim-Skolem Theorem, according to which, if a first-order theory has an infinite model of cardinality α, it has models of any infinite cardinality $\beta \geq \alpha$. In combination with the usual Löwenheim-Skolem Theorem, that means that when a first order theory has an

[16] A topological space is a family $\mathcal{T}$ of subsets of a set X such that both X and the empty set $\emptyset$ are members of the family, as are also arbitrary unions and finite intersections of members of the family.

[17] 'On what there is', 1948, reprinted in his *From a Logical Point of View*, pp. 1–19.

[18] *Logische Syntax der Sprache* 1934, enlarged English edition, *Logical Syntax of Language* 1937, §17, pp. 51–52.

[19] 'Models and Reality' 1980, reprinted in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics*, revised edition, pp. 421–444.

[20] See his 'Elementary Logic', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic* I, p. 34.

[21] 'Why and How Platonism?', *JIGPAL* 15 (5/6), 2007, pp. 185–218.

infinite model, it has models of any infinite cardinality. Such results –and many others– in classical model theory clearly belie any idea of constructing models. They are simply forced on us the moment we accept classical, that is, first-order model theory.

As mentioned above, there are many other theorems in model theory that counter any attempt to interpret model theory constructively. An example is the famous theorem of Morley, according to which any (deductively) complete first-order theory with infinite models in a countable language that is α-categorical for a non-countable α is also β-categorical for any non-countable β. This surprising powerful result is also a slap in the face of any constructivism in mathematics.

8 Classical Model Theory versus Nominalism and Conventionalism

If nominalism in mathematics were true, any existential statement about mathematical entities would be false. Moreover, if nominalism were true, any universal statement about mathematical entities would be vacuously true. On the other hand, if conventionalism in mathematics were true, the truth or falsity of a mathematical statement, be it existential, universal or whatever, would not depend on how things are in the mathematical universe, but on our arbitrary conventions. We will show that classical model theory is incompatible both with nominalism and with conventionalism, refuting once and for all mathematical nominalism and mathematical conventionalism, and showing, by the way, that Quine's contention that first-order logic is only committed to the existence of individuals is false, and that, contrary to Carnap's Principle of Tolerance, one cannot arbitrarily choose a logic free of ontological commitments –of course, except when you renounce completely to classical model theory, depriving logic of any talk about mathematics.

Let us first fix some terminology. Let $\mathcal{L}$ be a first-order language and $\mathcal{T}$ a theory –set of closed formulas– in that language. Let $\mathcal{M}$ and $\mathcal{M}^*$ be two models of the theory $\mathcal{T}$. The structure $\mathcal{M}$ is an elementary substructure of the structure $\mathcal{M}^*$ if and only if the universe M of $\mathcal{M}$ is a subset of the universe M^* of $\mathcal{M}^*$, in symbols $M \subseteq M^*$, and for any well-formed formula $\varphi \in \mathcal{L}$ in the variables $x_1, \dots, x_n$ and any $d_1, \dots, d_n \in M$, $\models_{\mathcal{M}} \varphi[d_1, \dots, d_n]$ if and only if $\models_{\mathcal{M}^*} \varphi[d_1, \dots, d_n]$. $\mathcal{M}$ is only a substructure of $\mathcal{M}^*$ if the above is valid only for atomic φ (and its Boolean combinations, that is, its quantifier-free combinations). Moreover, a theory $\mathcal{T}$ in a first-order language $\mathcal{L}$ is model complete if for any two models $\mathcal{M}$ and $\mathcal{M}^*$ of $\mathcal{T}$, if $\mathcal{M}$ is a substructure of $\mathcal{M}^*$, then $\mathcal{M}$ is an elementary substructure of $\mathcal{M}^*$. Abraham Robinson's Model-Completeness Test then says, among other things, that a consistent first-order theory $\mathcal{T}$ in the language $\mathcal{L}$ is model-complete if and only if for any existential sentence φ in $\mathcal{L}$, there exists a universal sentence ψ in $\mathcal{L}$ such that $\mathcal{T} \models \varphi \leftrightarrow \psi$, that is, on the basis of $\mathcal{T}$, φ and ψ are logically equivalent.

Now let us suppose, for the sake of the argument, that we have a nominalist first-order language, that is, one in which any existential statements are false and any universal statements are vacously true. However, an immediate consequence of Robinson's Model-Completeness Test is precisely that not all existential statements in a first-order language can be false or all universal statements true. For any (presumably false) existential statement in the language, there is a universal one logically equivalent to it, and thus, false. Hence, not all universal statements can be true. But their negations are also logically equivalent though true, and are also logically equivalent to a universal, respectively, to an existential statement. Thus, not all existential statements can be false. Hence, nominalism has been refuted. Moreover, since we cannot make all existential statements false, or all of them true, or all universal statements true, or all of them false, conventionalism is also false. It is also false that first-order logic does not have an ontological commitment to abstract entities. Thus, Quine's contention about ontological commitment of logical languages is false. Moreover, it is also false that we can freely choose between an ontologically loaded and an ontologically virgin quantifier language. Hence, Carnap's Principle of Tolerance is also false.

9 Abstract Model Theory and the First-Order versus Second-Order Issue

Mostly motivated by their nominalist prejudices some philosophers of logic have tried more or less to disqualify second-order logic, arguing that since it is semantically incomplete, that is, there is not a complete adequacy between truth and theoremhood, it lacks a very important and desirable metalogical property of logical systems. However, their selection of semantic completeness –a property had by first-order and propositional logic–, as playing such a decisive role as to make logical systems lacking that property as affected by a plague, seems unwarranted. There are other desirable metalogical properties of logical systems under whose test first-order logic does not fare well. Take for example the property of decidability, namely, that there is a mechanical decision procedure to determine whether any given statement of the logic's language is a theorem or not. That metalogical property is had by propositional logic and by the first-order monadic fragment of first-order logic, but not by full first-order logic. Full first-order logic is as undecidable as second-order logic. On the other hand, there is the very important metalogical and metamathematical property of categoricity, that is, a theory $\mathcal{T}$ is categorical when all its models are isomorphic. With regard to this property, it is first-order logic, when compared to second-order logic that fares badly. There are mathematical theories known to be categorical, which when expressed in the language of first-order logic are not categorical, but when expressed in the language of second-order logic are categorical. The best-known example of such theories is Dedekind-Peano arithmetic: first-order Dedekind-Peano arithmetic is non-categorical –in virtue both of the Löwenheim-Skolem theorems and of Skolem's non-standard countable model (and its uncountable elementary extensions)–, whereas second-order Dedekind-Peano arithmetic is categorical. Of course, the expressive poverty of first-order languages has resulted in an extraordinary richness of its semantics. Non-categoricity, the Löwenheim-Skolem theorems and the Compactness theorem have made possible the unsuspected richness of classical model theory, whereas the categoricity of second-order theories, together with the failure both of the Löwenheim-Skolem theorems and the Compactness theorem do not allow for an interesting model theory of second-order logic.

In any case, the fact of the matter is that philosophers of logic have argued against second-order logic and on behalf of first-order logic adducing that second-order logic, besides not being semantically complete, does not have a unique semantics. They have argued that, besides full second-order semantics, in whose models there are as many relations and functions as there can possibly be, there are two other semantics for second-order logic, namely, Henkin's semantics and multi-sorted semantics. In multi-sorted semantics there are multiple non-hierarchisized domains, whereas in some models of Henkin's semantics there are not as many relations and functions as they can possibly be, for example, there are models with countable domain and also countable sets of functions and relations.

However, the fact of the matter is that one can prove that multi-sorted semantics and Henkin's semantics are equivalent, what reduces the number of possible semantics to two. Let us then leave multi-sorted semantics aside and examine Henkin's semantics. Henkin built his deviant second-order semantics in order to make possible the obtainment of a sort of Weak-Semantic Completeness Theorem for second-order logic. However, from this Weak Semantic Completeness Theorem follow as corollaries both a Compactness theorem for second-order logic as well as a sort of Löwenheim-Skolem theorem, thus, precisely the two fundamental metatheorems which second-order logic with its usual full semantics lacks. It was the great Swedish logician Per Lindström who offered a correct explanation for this presumed incongruence in the first and most important of his characterization theorems of first-order logic in contrast to any of its extensions.[22] The First Characterization theorem says that any extension of first-order logic –second-order logic is its most natural one– for which both the Compactness theorem and the Löwenheim-Skolem theorem are valid is equivalent to first-order logic. Hence, what Henkin really did when building a deviant semantics for second-order logic and then proving a Weak Semantic Completeness Theorem for second-order logic based on that semantics was really a reduction of second-order logic to first-order logic. Second-order logic with Henkin's semantics is no second-order logic at all, but a version

[22]See, for example, his 'On Characterizing Elementary Logic', in Sören Stenlud (ed.), *Logical Theory and Semantic Analysis*, Reidel, Dordrecht 1974, pp. 129–146.

of first-order logic. Therefore, the argument of the plurality of semantics used against second-order logic has been deflated. Second-order logic has only one semantics, namely, its classical full semantics for which neither semantic completeness, compactness nor the Löwenheim-Skolem theorem are valid. Thus, once more, logic has helped to solve a philosophical problem, this time in the philosophy of logic, not in the philosophy of mathematics, as in the previous section.

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