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Choice and Change in an Epistemic Context

Sven Ove Hansson

Abstract

The use of choice functions to model belief change is one of the seminal contributions of Carlos Alchourrón and his collaborators. Several variants of this methodology have been developed. They differ in the objects on which the choice is performed. We can for instance choose which sentences to remove, choose directly among potential outcomes, choose which maximal outcomes the actual outcome should be included in, or which possible worlds it should be compatible with, etc. The relations between these modes of selection are investigated. Some but not all of them are equivalent, and some of them can be seen as more general than some of the others. Furthermore, the philosophical basis for the choice approach to belief change is discussed. It could potentially be supported by doxastic voluntarism, i.e. the view that our beliefs result from deliberate choices. However, the version of doxastic voluntarism needed to substantiate this interpretation of belief change is strong and not very plausible. Instead it is argued that the choice approach to belief change can be supported by the assumption that rational belief change should be reconstructible as choice, i.e. performed "as if" we choose our beliefs.

Keywords: Carlos Alchourrón, belief change, AGM theory, doxastic voluntarism, inferential-preferential method, choice function, selection function.

Introduction

One of the lasting contributions of Carlos Alchourrón and his collaborators is the use of choice mechanisms to model belief change. Since choice is usually conceived in this context as based on preferences, this method has been called the "inferential-preferential" approach to logical modelling. (Hansson and Gärdenfors 2014) It was used by David Lewis (1973) in his pioneering work on conditional logic, but transferred to belief change and considerably refined and elaborated in two seminal papers by Carlos Alchourrón and co-workers (Alchourrón and Makinson 1985; Alchourrón, Gärdenfors, and Makinson 1985) In this form it has had a major influence on other areas of logic, not least the study of non-monotonic reasoning.

In belief change theory, a person's beliefs are represented by a set of sentences, the belief set, usually denoted K. The belief set is assumed to be closed under logical consequence, i.e. $K = \mathrm{Cn}(K)$ where Cn is a function that takes us from a set of sentences to the set consisting of all those sentences that can be logically derived from it. The central problem in belief change is to find, given a belief set and an input, a new belief set that satisfies the new condition(s). In *contraction*, the input is a sentence p to be removed. Therefore, the new belief set should not include (or imply) the given sentence p, and furthermore, no new beliefs should be added. Belief contraction can therefore be conceived as the search for some belief set $K \div p$ such that[1]:

$$p \notin K \div p = \mathrm{Cn}(K \div p) \subseteq K \tag{1}$$

In *revision*, the other operation of classical belief change theory, the input is a sentence p to be consistently added to the original belief set. Therefore, the outcome should be a belief set $K * p$ such that[2]:

$$p \in K * p = \mathrm{Cn}(K * p) \neq \mathrm{Cn}(\{\perp\}), \tag{2}$$

where $\perp$ is a logically inconsistent sentence. In both contraction and revision we have to choose which sentences in K to retain and which to give up. An obvious question arises in this context: Does it make a difference how we perform that selection? In particular, does it matter on precisely what objects we perform the choice? We can for instance choose which sentences to remove, choose directly among potential outcomes, choose which maximal outcomes the actual outcome should be included in, or which possible worlds it should be compatible with, etc.

The main purpose of the present contribution is to investigate to what extent it matters on which type of objects the choice is made. As a preparation, the philosophical adequacy of representing belief change as choice-driven will be discussed in Section 1. In Section 2, the standard tools for a precise description of choices are introduced. In Section 3, seven potential objects of choice in belief change are investigated, and the conclusion of this investigation is presented in Section 4.

1 How relevant is choice for belief change?

How should we interpret choice in the context of belief change? Should we interpret it in the same way as in social choice theory, where choices are actions performed by persons? Are our beliefs the products of deliberate choices that we make as epistemic

[1] (unless p is a tautology in which case this is impossible)
[2] (unless p is inconsistent in which case this is impossible)

agents? Such an interpretation would connect belief change in an interesting way to a much debated standpoint in epistemology, namely doxastic (or epistemic) voluntarism according to which beliefs originate in voluntary choices.

There are several variants of doxastic voluntarism. (Nottelmann 2006; Hansson 2013a) For our present purposes, the following three distinctions are particularly important:

(1) According to *behavioural* doxastic voluntarism, the act that we can voluntarily choose is that of believing, i.e. holding a belief. According to *genetic* doxastic voluntarism, that status is instead assigned to the formation or acquisition of belief. (Audi 2001)

(2) Doxastic voluntarism is *complete* if it claims that all beliefs are held or formed voluntarily and *partial* if the claim is limited to only some of our beliefs.

(3) Doxastic voluntarism is *direct* if it claims that we directly choose to hold or form beliefs and *indirect* if the claim is only that we perform will-controlled actions that in their turn cause, in ways that may not be will-controlled, the holding or formation of beliefs.

These are three independent distinctions. (In the literature, the term "strong" doxastic voluntarism is used for both complete and direct variants and the term "weak" for both partial and indirect variants; the use of more specific terms is much to be preferred.)

What type of doxastic voluntarism, if any, is needed to support the use of choice functions in belief change? Since belief change is concerned with the formation rather than the holding of beliefs, it may be suggested that we need a *genetic* variant. Since choice functions are assumed to cover all belief changes, we need a *total* variant and cannot do with a partial one. Finally, since belief changes are assumed to follow directly in response to the input, we have use for a *direct* rather than an indirect version.[3]

However, doxastic voluntarism that is both total and direct is not epistemologically credible (in either the behavioural or the genetic form). The majority of doxastic voluntarists admit that total direct voluntarism is implausible not least since many belief changes result from sensory evidence that we accept without even reflecting on doing so. Hence, Ronney Mourad (2008) concedes that "most beliefs are involuntary" (p. 60) and that this applies in particular when we have conclusive evidence in support of either belief or disbelief (p. 62). Similarly, Philip Nickel acknowledges that "conclusive evidence, when grasped by a doxastic subject, must induce belief". (Nickel 2010, p. 313) These and most other defenders of direct doxastic voluntarism claim that some beliefs are formed at will, but stop short of the claim that all of them are.

[3] With the possible exception of non-prioritized belief change. See Makinson 1997b; Hansson et al 2001; Fermé and Hansson 2001.

To make a long story short, if we interpret the choice functions of belief change models as representing actual voluntary choice by the agent, then we have to accept highly questionable standpoints on the voluntary control of beliefs. Fortunately, there is a way out, similar to a move that is common in decision theory. Proponents of Bayesian decision theory do not claim that all rational decision-making takes the form of actually performing expected utility calculations, only that rational agents act *as if* they performed such calculations. This was clearly stated by Harsanyi:

> "[H]e simply cannot help acting *as if* he assigned numerical *utilities*, at least implicitly, to alternative possible outcomes of his behavior, and assigned numerical *probabilities*, at least implicitly, to alternative contingencies that may arise, and *as if* he then tried to maximize his expected utility in terms of these utilities and probabilities chosen by him... [W]e may very well decide to choose these utilities and probabilities in a fully *conscious* and *explicit* manner, so that we can make fullest possible use of our conscious intellectual resources, and of the best information we have about ourselves and about the world. But the point is that the basic claim of Bayesian theory does not lie in the suggestion that we *should* make a conscious effort to maximize our expected utility rather, it lies in the mathematical theorem telling us that *if* we act in accordance with a few very important rationality axioms then we *shall* inevitably maximize our expected utility." (Harsanyi 1977, 381-382) .

A similar approach is plausible in belief revision. Just as we can classify an intuitive or impulsive decision as rational if it could have been the outcome of a rational procedure or mechanism, we can classify a belief change as rational under similar conditions. This approach to rationality has the advantage of being more practically useful than an approach focusing on the actual procedure or mechanism, since the latter is often inaccessible to the analyst. I propose, therefore, that the tradition of modelling belief change as rational choice that was developed by Alchourrón and his collaborators should be interpreted according to the decision-theoretical notion of rational behaviour as behaviour developing "as if" it was ruled by deliberate, intentional choice, rather than according to a more literal interpretation of choice.

2 Choice functions and preference relations

A choice is a process in which we begin with a collection of objects, and end up with some (but usually not all) of those objects. We can represent this process mathematically with a *choice function*. A choice function is defined over a set $\mathcal{A}$ of alternatives. It can be used to make a selection among the elements of any subset of $\mathcal{A}$. The formal definition is as follows:

C is a choice function for a set $\mathcal{A}$ if and only if for each subset $\mathcal{B}$ of $\mathcal{A}$:

(1) $C(\mathcal{B}) \subseteq \mathcal{B}$, and

(2) $C(\mathcal{B}) \neq \varnothing$ if $\mathcal{B} \neq \varnothing$.

As can be seen from this definition, $C(\mathcal{B})$ can have more than one element. In everyday talk about choice, choices sometimes have this property, sometimes not:

Example 1:
"I am going to throw away these old LP records unless you want some of them. Choose those you want to have, and then I will throw away the rest."

Example 2:
"Since you have done so much for me I want to give you an LP record from my collection. You are free to choose whichever you like."

Choice functions, as defined above, represent the type of choice instantiated in the first of these examples. This applies not only to the use of choice functions in logic but also to their use in social choice theory where they have a much longer tradition. In social choice theory, when the choice yields a multi-element outcome, this is interpreted as meaning that reduction of the choice to that set is as far as rationality considerations will take us. In most applications of social choice theory, the decision-maker will have to further narrow down the choice to one single object. Which object she ends up with is presumed to be arbitrary from the viewpoint of rationality, as long as it is among those selected by the choice function. This second-stage narrowing-down is often described as a matter of picking rather than choosing. (Ullmann-Margalit and Morgenbesser 1977) Hence, if a, b, and c are three potential marriage partners, then $C(\{a, b, c\}) = \{a, b\}$ does not indicate a bigamous proclivity but equal (rational) propensities to choose a or b.

In belief revision, the use of choice functions with multiple outputs is interpreted differently. A few studies in belief revision have been devoted to indeterministic operations that specify, for some inputs, a set containing more than one possible outcome, without telling us which of these will eventuate. (Galliers 1992; Lindström and Rabinowicz 1991) Obviously, an interpretation of multi-element outputs similar to that in social choice theory is then possible; if there are several alternative outputs, then we may treat each of them as an admissible output. However, the vast majority of belief revision studies are devoted to deterministic operations. As we will see in Sections 3.1 and 3.5, a single (deterministic) outcome can be obtained from a choice function that yields multiple outputs, but of course this requires a further step in which the outcome is constructed from the chosen objects (usually through intersection).

For some purposes (such as direct choice among potential outcomes of a belief change[4]) we have use for choice functions that directly narrow down the set of alterna-

[4] See Section 3.7.

tives to only one. A choice function is *monoselective* (Hansson 2013b) if and only if its outcome never has more than one element, or equivalently:

$$\text{If } x \in C(\mathcal{B}) \text{ and } y \in C(\mathcal{B}) \text{ then } x = y. \tag{3}$$

(Obviously, we can alternatively define a monoselective choice function as one whose output is an element rather than a subset of the input set, i.e. as a function $\widehat{C}$ such that $\widehat{C}(\mathcal{B}) \in \mathcal{B}$ unless $\mathcal{B} = \varnothing$ in which case $\widehat{C}(\mathcal{B})$ is undefined.)

Among the rationality properties that have been proposed for choice functions, the following two are arguably the most important ones (Sen 1970):

Chernoff (property α) (Chernoff 1954)

If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then $\mathcal{B}_1 \cap C(\mathcal{B}_2) \subseteq C(\mathcal{B}_1)$.

Property β

If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $x, y \in C(\mathcal{B}_1)$, then: $x \in C(\mathcal{B}_2)$ if and only if $y \in C(\mathcal{B}_2)$

As already mentioned, choices are often assumed to be closely related to preferences. Preferences are represented by the binary relations $\geq$ ("at least as good as") and $>$ ("better than"). In what follows we will assume that the latter of these two relations is definable in terms of the former in the standard way:

$$x > y \text{ if and only if } x \geq y \text{ and } \textit{not } y \geq x \tag{4}$$

The following properties of the two relations will be referred to in what follows:

Either $x \geq y$ or $y \geq x$ (completeness)

If $x \geq y$ and $y \geq z$ then $x \geq z$ (transitivity of weak preference)

If $x > y$ and $y > z$ then $x > z$ (transitivity of strict preference)

If $x_1 > x_2$, $x_2 > x_3$,... and $x_{n-1} > x_n$ then *not* $x_n > x_1$ (acyclicity)

(See Hansson 2001 for an introduction to the logic of preferences.)

A choice function C is *based on* a preference relation $\geq$ if and only if it always chooses the most preferred elements of its input set, i.e. if and only if for all $x \in \mathcal{B}$:

$$x \in C(\mathcal{B}) \text{ if and only if } x \geq y \text{ for all } y \in \mathcal{B} \tag{5}$$

Furthermore, C is *relational* if and only if it is based on some preference relation $\geq$ in this way, and *transitively relational* if and only if it is based on some transitive preference relation $\geq$ in this way.

It is possible to base a choice function on a given preference relation $\geq$ if and only if $\geq$ satisfies completeness and acyclicity.[5] All relational choice functions satisfy Chernoff. A relational choice function satisfies property β if and only if it is also transitively relational. (Sen 1970)

3 Alternative objects of choice for belief contraction

As already indicated, choice functions have been used in different ways in belief change theory. In what follows, these usages will be classified according to their *objects of choice*, in order to clarify the relationships between choices made on different such objects. Since belief revision is usually defined in terms of belief contraction[6], the focus will be on belief contraction. (The picture is largely the same for belief revision.) We will begin with the two types of contraction that bear Alchourrón's mark.

3.1 Choosing remainders for intersection

The 1985 paper by Carlos Alchourrón, Peter Gärdenfors, and David Makinson in the *Journal of Symbolic Logic* is the most quoted and no doubt the most influential paper in the literature on belief change. When constructing the contraction of a belief set K by some sentence p, they started with the observation that among the many subsets of K not implying p, some are inclusion-maximal, i.e. they are as large as they can be without implying p. These sets are called p-remainders, and the set of p-remainders of K is denoted $K \perp p$.

Intuitively, when contracting K by p we want to keep as much of K as we can while still removing p. This could lead us to take one of the elements of $K \perp p$ as the contraction outcome. However, it may be impossible to single out one of these elements as epistemically preferable to all the others. If several p-remainders share the top position, then our post-contraction beliefs will be those that are held in all the top-ranked p-remainders. Formally, this is achieved by introducing a selection function (choice function) γ that takes us from $K \perp p$ to a subset $\gamma(K \perp p)$. (Since the 1985 paper the tradition in the belief change literature is to denote choice functions by γ instead of C as in social choice theory, and to call them "selection functions".) The outcome of contracting K by p is the intersection of all elements of $\gamma(K \perp p)$, i.e.

[5] The hierarchies used in safe contraction do not in general satisfy completeness. Relations of epistemic entrenchment satisfy transitivity and completeness, but the standard way to base contraction on epistemic entrenchment does not employ a choice function based on the entrenchment relation. (However, see Rott 2001, pp. 223-267 on choice-related properties of entrenchment relations.)

[6] Via the Levi identity, $K * p = \mathrm{Cn}((K \div \neg p) \cup \{p\})$.

$$K \div p = \bigcap \gamma(K \perp p) \tag{6}$$

This construction is called *partial meet contraction*. One way to construct γ is to base it on a transitive relation covering all remainders of K.[7] If γ selects the elements of $K \perp p$ that are highest ranked according to such a relation, then the resulting contraction is a *transitively relational partial meet contraction*.

3.2 Choosing what to remove

1985 was the *annus mirabilis* of belief change theory. In that year, in addition to their joint paper with Peter Gärdenfors in the *Journal of Symbolic Logic*, Carlos Alchourrón and David Makinson published an article in *Studia Logica* where they applied a choice mechanism in a quite different way. Instead of focusing on what to retain as in partial meet contraction, they focused on what to remove. When constructing $K \div p$ from K, in order to make sure that $K \div p$ does not imply p we have to remove something from every subset of K that implies p. Since the logic is compact[8], in order for p not to be implied by a subset $K \div p$ of K it is both necessary and sufficient that at least one element has been removed from each minimal p-implying subset of K. These minimal p-implying sets have later been called the p-kernels of K. In analogy with remainder sets, the kernel set $K \perp\!\!\!\perp p$ consists of all subsets of K that imply p but have no proper subset implying p. Following Alchourrón's and Makinson's idea, we can contract p from K by removing at least one element from each element of $K \perp\!\!\!\perp p$.

The basic, non-relational variant of this *choice of sentences to be removed* makes use of a function σ ("incision function") that chooses at least one sentence from each element of $K \perp\!\!\!\perp p$. All the sentences selected to be removed from some element of $K \perp\!\!\!\perp p$ are removed from K, and what remains is closed under logical consequence in order to make it a belief set, i.e.:

$$K \div p = \mathrm{Cn}(K \setminus \sigma(K \perp\!\!\!\perp p)) \tag{7}$$

This operation is called *kernel contraction*. It was developed later than the relational variant that was the topic of the 1985 paper. (Hansson 1994a) However, its application to belief sets is arguably not very exciting, since the operations definable in this way coincide exactly with the partial meet contractions.[9]

The relational variant that was developed in the 1985 paper is called *safe contraction*. The relation in question is denoted $\prec$ and has to satisfy the above-mentioned property

[7] By a remainder of K is meant a set that is an element of $K \perp p$ for some p.

[8] By this is meant that for all sentences p and all sets X of sentences, if $p \in \mathrm{Cn}(X)$, then there is a finite subset X' of X such that $p \in \mathrm{Cn}(X')$.

[9] The relationship between incision functions and selection functions was investigated in Falappa et al 2006.

of acyclicity. Let X be a p-kernel, i.e. $X \in K \perp\!\!\!\perp p$. Then an element q of X is selected for removal from X if and only if it has a minimal position in the hierarchy, i.e. if and only if there is no sentence r in X such that $r \prec q$.

The following are additional properties of interest for hierachies:

If $q \prec r$ then $p\&q \prec r$ (continuing-down)

If $p \prec r$ then either $p \prec q$ or $q \prec r$ (virtual connectivity)

It was shown by Hans Rott (1992) that an operator $\div$ on a belief set K is a safe contraction, based on a virtually connected hierarchy that satisfies continuing-down, if and only if it is a transitively relational partial meet contraction. In other words, the one-to-one correspondence between (plain) partial meet contraction and (plain) kernel contraction referred to above can be reconstructed between transitively relational partial meet contraction and a class of safe contractions based on orderly hierarchies.[10]

3.3 Choosing base-remainders for intersection

Partial meet contraction on belief sets has been criticized for attaching importance to "merely derived" beliefs. For instance, Agustina believes that her husband is faithful (h). She also believes that Buenos Aires is the capital of Argentina (b). Since she has a logically closed set of beliefs, she also believes that Buenos Aires is the capital of Argentina if and only if her husband is faithful ($h \leftrightarrow b$). Now suppose that she makes observations inducing her to give up her belief in h. She cannot then retain both her belief in b and her belief in $h \leftrightarrow b$. In the framework presented in Section 3.1 she has a choice between giving up only b, giving up only $h \leftrightarrow b$, or giving up both.[11] Arguably, this is a frivolous choice that does not correspond to any actual thought processes. It may seem more plausible to represent her original belief state with a belief base B that contains her seriously held beliefs such as h and b but not her merely derived beliefs such as $h \leftrightarrow b$. When contracting B by h, the option of removing b in order to retain $h \leftrightarrow b$ will not even arise. (Hansson 1994b)[12]

This approach, in which operations of change take place primarily on belief bases that are not logically closed, was foreseen by Alchourrón and Makinson already in 1982:

> "We suggest, finally, that the intuitive processes themselves, contrary to casual impressions, are never really applied to theories as a whole, but

[10] For a more extensive presentation of safe contraction, see Rott and Hansson (2014).

[11] She will give up b if and only if at least one of the chosen remainders does not contain it, and similarly for $h \leftrightarrow b$.

[12] This was expressed by Fuhrmann (1991) as a filtering condition, according to which if a sentence is removed, then other sentences that were believed "just because" of that sentence should also be removed. See also Rott (2000).

> rather to more or less clearly identified bases for them. For a theory is an infinite object, having as it does an infinite number of elements, and it is only by working on some finite generator or representative of the theory that the outcome of a process such as contraction can ever in practice be determined." (Alchourrón and Makinson 1982, 21-22)

The expulsion of merely derived beliefs from the objects of change may seem to be just a cosmetic operation. Does it really matter if the merely derived beliefs are included in the belief representation? Cannot we just let them be there and remove them in the operations of change whenever that is called for? The answer is that for some operations of change, including partial meet contraction, their presence has a considerable impact on the properties of the operation.

To see this, let X be a p-remainder of the belief set K, i.e. $X \in K \perp p$, and let q be any sentence in K. Now consider the sentence $p \rightarrow q$. Since it follows logically from K it is also an element of K. We are going to prove that $p \rightarrow q$ is also included in X, i.e. $p \rightarrow q \in X$. Since X is logically closed[13] this follows directly if $q \in X$. For the other case, when $q \notin X$, suppose to the contrary that $p \rightarrow q \notin X$. Then, since X is a p-remainder of K, the addition of $p \rightarrow q$ to X will have to be a set that implies p. Thus $X \cup \{p \rightarrow q\}$ implies p, thus X implies $(p \rightarrow q) \rightarrow p$. Since $(p \rightarrow q) \rightarrow p$ is logically equivalent with p, this means that X implies p, contrary to our assumption that X is a p-remainder. This contradiction is sufficient to prove that our supposition $p \rightarrow q \notin X$ was wrong, and with this we have proved that $p \rightarrow q \in X$.

Note that this holds for all p-remainders of K. Since the partial meet contraction $\bigcap \gamma(K \perp p)$ is the intersection of a collection of p-remainders of K, it follows that $p \rightarrow q \in \bigcap \gamma(K \perp p)$ for all $q \in K$. From this we can derive the following property of a partial meet contraction operator $\div$:

$$\text{If } p \in K, \text{ then } \mathrm{Cn}((K \div p) \cup \{p\}) = K \text{ (recovery)} \tag{8}$$

Recovery is a controversial property. (For discussions, see: Gärdenfors 1982; Hansson 1991; 1999b; Makinson 1987; 1997a; Glaister 2000.) For our present purposes, the important point is that although recovery is satisfied by all partial meet contractions on belief sets, it does not hold if the contraction is instead performed on a (logically non-closed) belief base.

More precisely, in order to perform *base-generated partial meet contraction* on a belief set K, we need to assign to it a belief base, i.e. a set B such that $K = \mathrm{Cn}(B)$. We also need a selection function γ that operates on the remainders of B, rather than those of K. Partial meet contraction on B is then defined as follows:

[13] This is easily proved. Suppose that X implies some sentence r that is not one of its elements. Then $X \cup \{r\}$ is a subset of K that does not imply p, contrary to our assumption that X is an inclusion-maximal subset of K that does not imply p.

$$B \div p = \bigcap \gamma(B \perp p), \tag{9}$$

and the corresponding base-generated partial meet contraction $\widehat{\div}$ on K is defined as follows:

$$K \,\widehat{\div} p = \mathrm{Cn}(\bigcap \gamma(B \perp p)) \tag{10}$$

This operation can easily be shown not to satisfy recovery. For a simple counter-example, let $B = \{p\&q\}$.

In the context of a specific base B for K we can refer to the sentences of B as *base-sentences* and the remainders of B as *base-remainders* in relation to K. It should be obvious that technically speaking, (direct) partial meet contraction is a special case of base-generated partial meet contraction (namely the case when $B = K$).

3.4 Choosing base-sentences to be removed

Kernel contraction, as introduced in Section 3.2, can straightforwardly be generalized to belief bases. For any belief base B, its p-kernels are the inclusion-minimal subsets of B that imply p, and $B \perp\!\!\!\perp p$ is the set of all p-kernels of B. Incision functions are defined in the same way as for belief sets, and kernel contraction on the base is defined as follows:

$$B \div p = B \cap \mathrm{Cn}(B \setminus \sigma(B \perp\!\!\!\perp p)) \tag{11}$$

Kernel contraction on belief sets, as introduced in Section 3.2, is a special case of this operation, namely the case when $B = K$.[14]

Base-generated kernel contraction is defined as follows:

$$K \,\widehat{\div} p = \mathrm{Cn}(B \cap \mathrm{Cn}(B \setminus \sigma(B \perp\!\!\!\perp p))) \tag{12}$$

Base-generated partial meet contraction is a proper special case of base-generated kernel contraction. In other words, if an operator on a belief set is a base-generated partial meet contraction then it is also a base-generated kernel contraction, but the converse relationship does not hold. (Hansson 1994a)

3.5 Choosing possible worlds for curtailment of the belief set

In an article published in 1988 Adam Grove showed that belief changes can be constructed as choices among possible worlds rather than among remainders, kernels, or

[14] Equation (7) in Section 3.2 is equivalent with (11) when B is logically closed.

sentences.[15] Both his work and most of the subsequent work on possible worlds-based belief change has had its focus on belief revision, but in order to achieve comparability the focus here will be on belief contraction.

By a possible world, in the logical sense, is meant a maximally consistent set of sentences. If $\mathcal{L}$ is the (set of all sentences in the) language, then W is a possible world if and only if $W \in \mathcal{L} \perp \bot$. For all possible worlds W and all sentences p, either p or its negation is included in W, i.e. $p \in W$ or $\neg p \in W$. In other words, each possible world is maximally specified in the sense that each sentence is either true or false in it.

There is a close connection between belief sets and possible worlds. A belief set K is *compatible with* a possible world W if and only if $K \subseteq W$. Furthermore, each belief set is completely determined by the possible worlds that it is compatible with; it is in fact their intersection. It holds for all logically closed sets K that:

$$K = \bigcap \{W \in \mathcal{L} \perp \bot \mid K \subseteq W\} \tag{13}$$

Therefore, from a technical point of view we can always replace talk about belief sets by talk about sets of possible worlds, and vice versa.[16] When modelling contraction of K by some sentence p, we are looking for a subset $K \div p$ of K that does not contain p. This means that $K \div p$ should be compatible with (1) all the worlds that K is compatible with, and (2) in addition some world(s) containing $\neg p$. To choose among the latter worlds we can use a choice function C:

$$\{W \in \mathcal{L} \perp \bot \mid K \div p \subseteq W\} = \{W \in \mathcal{L} \perp \bot \mid K \subseteq W\} \cup C(\{W \in \mathcal{L} \perp \bot \mid \neg p \in W\}) \tag{14}$$

We can describe this process as one of *curtailment*: We add more possible worlds that the original belief set has to be intersected with. As a result of this, the new belief set is further curtailed (made smaller) so that it becomes compatible with more possible worlds than the original one. This recipe gives rise to exactly the same operations as partial meet contraction. Furthermore, a similar relationship holds in the transitively relational case. Let the choice function C in the above equation be based on a complete and transitive relation on $\mathcal{L} \perp \bot$, such that C always chooses those $\neg p$-including possible worlds that have the highest rank according to this underlying transitive relation.[17] The

[15] Another way to construct choices among worlds are the faithful assignments introduced in Katsuno and Mendelzon 1991.

[16] A less cursory introduction to possible worlds and their relations to belief sets can be found in Hansson 1999a, pp. 51-57 and 287-304.

[17] Following Grove (1988), such complete and transitive relations on possible worlds are usually represented by concentric circles around the belief set. The innermost circle contains the possible worlds that are compatible with the belief set; these are also the worlds with the highest rank. The next circle includes the worlds with the second-highest rank, etc. The choice function selects all the $\neg p$-including worlds in the innermost sphere containing any such worlds.

operations of contraction that are obtainable in this way coincide exactly with those that are obtainable as transitively relational partial meet contractions.

These exact correspondences between choice (or transitively relational choice) on remainders and possible worlds are among the most beautiful results obtained in belief change theory. They may give the impression that it makes no difference what objects of choice we employ, but as we have already seen that is not true. Furthermore, after a study of the underlying logic, these one-to-one correspondences stand out as much less surprising than what first impressions might make us believe. The underlying mechanism is a neat one-to-one correspondence called "Grove's bijection" between the remainder set $K \perp p$ and the set of possible worlds not containing p.[18]

3.6 Choosing sentences for eradication

Partial meet contraction was introduced as a modification of a much simpler operation that was first introduced by Carlos Alchourrón and David Makinson (1982), namely full meet contraction:

$$K \sim p = \bigcap (K \perp p) \tag{15}$$

It seems to have been realized from the beginning that full meet contraction removes too much from the belief set. In fact, when we contract K by p in this way, we also remove all the non-tautological consequences of p. (Hansson 2007) The term *eradication* can be used for the removal of a sentence along with all its non-tautological consequences. Eradication is obviously a too harsh form of contraction; I should be able to give up my belief that Shakespeare wrote Hamlet in 1602 without giving up my belief that Shakespeare wrote Hamlet.

Partial meet contraction was constructed as an improvement of full meet contraction, allowing us to retain more of the belief set. As we have seen partial meet contraction inserts a selection mechanism operating on the set of remainders. Alternatively, we can insert a selection mechanism at another point in (15), namely on the sentence to be contracted. (See Figure 1.) We can introduce a function f (a *sentential selector*) that takes us from the sentence to be contracted to the one that is "really" going to be eliminated. This procedure is related to the common experience of giving up a composite belief by relinquishing only the least plausible part of it.

> I realized that $p\&q$ cannot be true, so I had to give up either my belief in p or my belief in q, or perhaps both. After some deliberation I chose to give up only p.

[18] A rough idea of the nature of this bijection can perhaps be conveyed by mentioning (1) that if K is a belief set, $p \in K$, and $X \in K \perp p$, then $\mathrm{Cn}(X \cup \{\neg p\})$ is a possible world, and (2) that if K is a belief set, $p \in K$, and $p \notin Y \in \mathcal{L} \perp \perp$, then $Y \cap K \in K \perp p$. For details on the bijection, see Grove 1988 or Hansson 1999a, pp. 53-55.

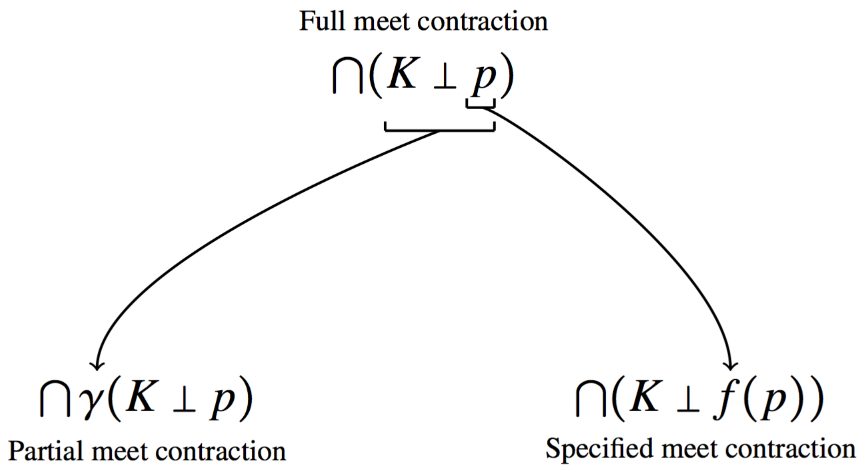

Figure 1: Two ways to modify full meet contraction by the introduction of a selection mechanism.

A contraction based in this way on the selection of a sentence for eradication is called *specified meet contraction*, and its defining formula is $\bigcap(K \perp f(p))$. To justify it we can assume that contraction by a sentence p consists in the eradication of some (typically small) part of the belief that p represents. (Hansson 2008) This supposition gains much in plausibility from a result showing that specified meet contraction is a very general operation, provided that we are dealing with finite-based belief sets. (K is finite-based if and only if there is some finite set B such that $K = \mathrm{Cn}(B)$.) A sentential operation $\div$ on a finite-based K can be constructed as a specified meet contraction if and only if it satisfies the following three conditions (Hansson 2007):

$K \div p = \mathrm{Cn}(K \div p)$ (closure),

$K \div p \subseteq K$ (inclusion), and

$K \div p$ is finite-based (finite-based outcome)

The first two of these conditions are uncontroversially taken for given in studies of contractions on belief sets. It follows that, as long as we operate in a framework with only finite-based belief sets, all contraction operators including the ones treated in Sections 3.1-3.5 can be represented as specified meet contraction.

It has also been shown that specified meet contraction satisfies the recovery postulate if and only if the sentential selector f satisfies the simple property that $f(p)$ is a logical consequence of p,[19] which is another way of saying that the eradicated belief $f(p)$ is a part of the belief p that is contracted through its eradication.

[19] More precisely: if and only if this holds for all $p \in K$. See Hansson 2007.

A sentential selector is of course not a choice function in the traditional sense, but it can be reconstructed as one. This is particularly clear when we reconstruct partial meet contraction as specified meet contraction. $f(p)$ is then an element of the set of non-tautological sentences that are implied by p, and we can let f be a monoselective choice function $\widehat{C}$, as defined in Section 2:[20]

$$f(p) = \widehat{C}(\mathrm{Cn}(\{p\}) \setminus \mathrm{Cn}(\varnothing)) \tag{16}$$

3.7 Choosing among potential outcomes

Some of the objects of choice that we have discussed thus far may seem rather indirect or perhaps even contrived. For instance, the mental processes that take place when we have to give up some belief do not refer to remainders or possible worlds. An interesting alternative approach is to instead select directly among the potential outcomes of the operation. This type of operation is called *repertoire contraction.* (Hansson 2013c)[21] It is based on the assumption that only some of the logically closed subsets of K are at all viable as contraction outcomes. The set of potential outcomes is called the *repertoire* and can be denoted $\mathcal{K}$.

When contracting K by p we make a selection among those elements of $\mathcal{K}$ that do not contain p.[22] This selection can be modelled with a monoselective choice function:

$$K \div p = \widehat{C}(\{X \in \mathcal{K} \mid p \notin X\}) \tag{17}$$

There are several ways in which repertoire contraction can be based on a relation. The most obvious would be to apply a strict linear ordering to $\mathcal{K}$ and then construct $\widehat{C}$ such that for all sentences p it selects the highest-ranked element of $\mathcal{K}$ that does not contain p. However, this approach has the implausible property that for all sentences p and q, either $K \div (p\&q) = K \div p$ or $K \div (p\&q) = K \div q$.[23] The following example shows why

[20] For contraction operators satisfying success ($p \notin K \div p$ whenever p is a non-tautology) but not necessarily recovery, we can instead define $f(p)$ as $\widehat{C}(\mathcal{L} \setminus \mathrm{Cn}(\{\neg p\})$. Cf. Hansson 2008.

[21] The choice principle in repertoire contraction, namely to choose directly among potential outcomes that satisfy the success condition for the operation, can also be applied to other operations of change, including revision. A more general class of operations that applies this recipe to a wide variety of belief change operations is introduced in Hansson 2014.

[22] The case when p is included in all elements of $\mathcal{K}$ will not be treated here. It is excluded for non-tautologous p if $\mathcal{K}$ is required to satisfy the condition $\bigcap \mathcal{K} = \mathrm{Cn}(\varnothing)$. (Hansson 2013c)

[23] To see why this holds, let X be the highest-ranked element of $\mathcal{K}$ that does not contain $p\&q$. Since X is logically closed, either $p \notin X$ or $q \notin X$. In the former case, suppose that there is some X' in $\mathcal{K}$ that is higher ranked than X and does not contain p. Since X' is logically closed, it does not either contain $p\&q$. This contradicts the assumption that X is the highest-ranked element of $\mathcal{K}$ not containing $p\&q$, and we can conclude that there is no such X'. Consequently X is the highest-ranked element of $\mathcal{K}$ that does not contain p, thus $K \div p = X$. In the other case, when $q \notin X$, we can show in the same way that $K \div q = X$.

that property should preferably be avoided:

> Pauline and Quentin are my neighbours. Based on what I have seen, I believe both that Pauline is a safe and careful driver (p) and that Quentin is a safe and careful driver (q).
> *Case i:* I see their car passing the zebra crossing outside my children's school at very high speed. I get a glimpse of the driver's face. It is a woman with long black hair, and it might very well be Pauline. I therefore give up my belief that Pauline is a safe driver (p), but I still believe that Quentin is a safe driver (q).
> *Case ii:* I see the car driven in the same way. I get a glimpse of the driver's face. It is a man with a beard, and it might very well be Quentin. I give up q but retain p.
> *Case iii:* I see the car driven in the same way, but I cannot see the driver. I give up both p and q, suspending my judgment on who is the reckless driver.

Due to the first two cases we have $q \in K \div p$ and $p \in K \div q$. It would follow from the principle we are discussing that either $p \in K \div (p\&q)$ or $q \in K \div (p\&q)$, which seems absurd.

However, there are other ways to make the choice function in repertoire contraction relational. For instance, we can allow ties (two elements of $\mathcal{K}$ can have the same ranking) and furthermore assume that if $X_1, X_2, \ldots, X_n$ are elements of $\mathcal{K}$ that are all equally ranked, then there is a unique inclusion-maximal element of $\mathcal{K}$ that is a subset of each of them. (This is satisfied for instance if the set $X_1 \cap X_2 \cap \cdots \cap X_n$ is an element of $\mathcal{K}$, presumably representing a state of hesitation among $X_1, X_2, \ldots, X_n$.) We can furthermore construct $\widehat{C}$ such that for any subset $\mathcal{N}$ of $\mathcal{K}$, $\widehat{C}(\mathcal{N})$ is the unique inclusion-maximal element of $\mathcal{N}$ that is a subset of all its top-ranked elements. This construction is called *perimaximal contraction*. (Hansson 2013b) It is quite similar to transitively relational partial meet contraction (which is also a limiting case), but philosophically it differs in that the objects of choice are potential outcomes of contraction.

4 Conclusion

The relationships among the different ways to apply choice functions in belief contraction are summarized in Figure 2. We can conclude that it actually makes a difference to which objects we apply choice functions. Their application to remainders of belief sets and to possible worlds yield the same outcomes, but there are several other plausible objects of choice, and for some of these the resulting operations will have quite different properties.

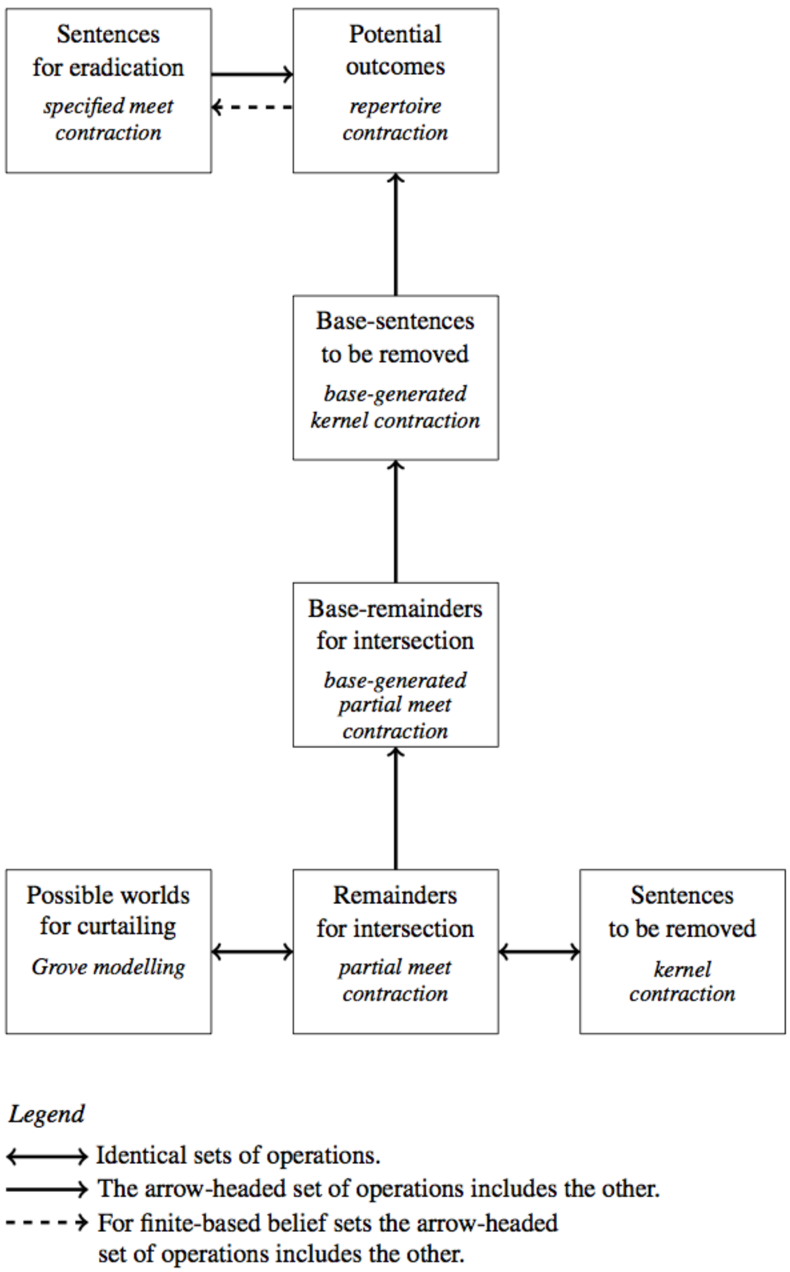

Figure 2: The relationships among the classes of contraction operations on belief sets that are obtainable by applying choice functions to different objects of choice.

The choice approach to belief change that Carlos Alchorrón and his collaborators developed is surprisingly rich. Many if not most of its potential variants and ramifications remain to be explored.

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Sven Ove Hansson
Division of Philosophy
Royal Institute of Technology
Brinellvägen 32, 100 44 Stockholm, Sweden
E-mail: soh@kth.se