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Logic of deduction: Models of pre-order and maximal theories

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Abstract

In the decade of 1930 Alfred Tarski published several papers about a general concept of deductive consequence. As usual nowadays we choose only three conditions to characterize a consequence operator of Tarski: auto-deductibility, monotonicity and idempotency. From that we obtain the concept of Tarski space. With motivation started from Tarski spaces, we have introduced a formalization for that notion in a modal propositional logic, the logic *TK*, and showed its adequacy relative to an algebraic structure, the TK-algebras. In this paper we present two other types of model for *TK*, one given by the concept of pre-order and another by using maximal theories of *TK*.

Introduction

We begin with an abstract concept of consequence proposed by Alfred Tarski in the decade of 1930, and consider the associated Tarski spaces. Based on the Tarski spaces we present the logic *TK* and the TK-algebras. The logic *TK* can be presented also with the operator dual of the deductive closure. So we must consider the almost topological spaces where we can talk in a simple way about closed and open sets, pass from one to the other and compare this structure with the Tarski spaces. After this, we prove the adequacy of *TK* relative to almost topological spaces by using a particular almost topological space given by the maximal consistent theories of *TK*. As a last result we present a model for *TK* in pre-order style.

1 The space of a consequence operator

We present the consequence operator of Tarski and give emphasis to the structure generated by the consequence operator.

Definition 1.1 *A consequence operator on E is a function $^{-} : \mathcal{P}(E) \to \mathcal{P}(E)$ such that, for every $A, B \subseteq E$:*
(i) $A \subseteq \overline{A}$;
(ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;
(iii) $\overline{\overline{A}} \subseteq \overline{A}$.

The itens (i) and (iii) are enough to justify that, for every $A \subseteq E$, $\overline{\overline{A}} = \overline{A}$.

Definition 1.2 *A consequence operator $^{-} : \mathcal{P}(E) \to \mathcal{P}(E)$ is finitary when, for every $A \subseteq E$:*
(iv) $\overline{A} = \cup\{\overline{A}_f : A_f \; is \; a \; finite \; subset \; of \; A\}$.

Definition 1.3 *A Tarski space (or Tarski's deductive system or closure space) is a pair $(E, ^{-})$ such that E is a set and $^{-}$ is a consequence operator on E.*

Definition 1.4 *If $(E, ^{-})$ is a Tarski space and $A \subseteq E$, then $\overline{A}$ is the set of consequences of A or the closure of A.*

From (i), always $A \subseteq \overline{A}$, but the converse does not hold in general.

Definition 1.5 *If $(E, ^{-})$ is a Tarski space, then a set A is closed in $(E, ^{-})$ when $\overline{A} = A$; and A is open when its complement relative to E, denoted by A^C, is closed in $(E, ^{-})$.*

Proposition 1.6 *Any intersection of closed sets of $(E, ^{-})$ is also a closed set of $(E, ^{-})$.*

Clearly, $\overline{\emptyset}$ and E are the least and the greatest closed sets, respectively, associated to $(E, ^{-})$.

Proposition 1.7 *In any Tarski space $(E, ^{-})$, the set E is closed and the set $\emptyset$ is open.*

Definition 1.8 *A Tarski space $(E, ^{-})$ is 0-closed if it holds:*
(v) $\overline{\emptyset} = \emptyset$.

In general, $\emptyset$ is open, but not closed. In a space 0-closed we require that the empty set be clopen, that is, closed and open at the same time.

Proposition 1.9 *If $(E, ^{-})$ is Tarski space 0-closed and holds:*
(vi) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
then $(E, ^{-})$ is a topological space.

Definition 1.10 *A Tarski space $(E, ^{-})$ is* vacuous *when $\overline{\emptyset} = \emptyset$.*

The previous definition of topological space coincide with the Kuratowski characterization of topological spaces via the closure operator. Naturally every topological space is a Tarski space, but the converse does not hold. Each topological space is an instance of a vacuous Tarski space.

From this conception of Tarski space, we introduced in [13] an algebraic structure to formalize the consequence operator, and afterwards we introduced in [8] a propositional logic for the consequence operator. In the next section we recall these results.

2 TK-algebras and the Logic TK

The definition of TK-algebras can be considered as the algebraic counterpart of the notion of consequence operator.

Definition 2.1 *A* TK-algebra *is a sextuple* $\mathcal{A} = (A, 0, 1, \vee, \sim, \bullet)$ *where* $(A, 0, 1, \vee, \sim)$ *is a Boolean algebra and* $\bullet$ *is an unary operator, called* operator of Tarski*, such that:*
(i) $a \vee \bullet a = \bullet a$*;*
(ii) $\bullet a \vee \bullet(a \vee b) = \bullet(a \vee b)$*;*
(iii) $\bullet(\bullet a) = \bullet a$*.*

Examples 2.2
(a) The space of sets $\mathcal{P}(A)$ *with* $A \neq \emptyset$ *and* $\bullet a = a$*, for all* $a \in A$*, is a TK-algebra.*
(b) The space of sets $\mathcal{P}(\mathbb{R})$ *with* $\bullet X = X \cup \{0\}$ *is a TK-algebra.*
(c) The space of sets $\mathcal{P}(\mathbb{R})$ *with* $\bullet X = \cap\{I \ : I$ *is an interval and* $X \subseteq I\}$ *is a TK-algebra.*

Since we are working with Boolean algebras, the item (i) of the previous definition asserts that, for every $a \in A$, $a \leq \bullet a$.

We define in a TK-algebra:

$$a \to b =_{df} \ \sim a \vee b;$$

$$a - b =_{df} \ a \wedge \sim b.$$

Proposition 2.3 *In any TK-algebra the following assertions are valid:*
(i) $\sim\bullet a \leq \sim a \leq \bullet\sim a$*;*
(ii) $a \leq b$ *implies that* $\bullet a \leq \bullet b$*;*
(iii) $\bullet(a \wedge b) \leq \bullet a \wedge \bullet b$*;*
(iv) $\bullet a \vee \bullet b \leq \bullet(a \vee b)$*.*

The propositional logic TK is the logical system constructed over the propositional language of TK, that is, $L = (\neg, \vee, \to, \blacklozenge, p_1, p_2, p_3, \ldots)$, with the following axioms and rules:

(CPC) φ, if φ is a classical tautology;

(TK_1) $\varphi \to \blacklozenge\varphi$;

(TK_2) $\blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi$.

(MP) $\dfrac{\varphi \to \psi, \varphi}{\psi}$;

$(RM^{\blacklozenge})$ $\dfrac{\vdash \varphi \to \psi}{\vdash \blacklozenge\varphi \to \blacklozenge\psi}$.

As usual, we write $\vdash_S \varphi$ to indicate that the formula φ is a theorem of the axiomatic system S, and we drop the subscript when there is no risk of misunderstanding.

Definition 2.4 *Let $\Gamma \cup \{\psi\}$ be a set of formulas of TK. The set Γ deduces ψ, what is denoted by $\Gamma \vdash \psi$, if there is a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ of TK such that for every $i \in \{1, 2, \dots, n\}$:*
φ_i is an axiom of TK or
$\varphi_i \in \Gamma$ or
φ_i is obtained from previous formulas in the sequence by means of a deduction rule, and
$\varphi_n = \psi$.

Proposition 2.5 $\vdash \blacklozenge\varphi \to \blacklozenge(\varphi \vee \psi)$.

Proposition 2.6 $\vdash \varphi$ *implies that* $\vdash \blacklozenge\varphi$.

Proposition 2.7 $\Gamma \vdash \blacklozenge\varphi \vee \blacklozenge\psi \to \blacklozenge(\varphi \vee \psi)$

In [8] we have proved the adequacy of TK relative to TK-lagebras.

The operator $\blacklozenge$ is of course associated to the notion of closure, but we can define a dual operator, denoted by $\boxminus$, associated to the notion of interior.

3 A dual operator of closure

The operator $\boxminus$, a dual of $\blacklozenge$, is defined in the following way.

Definition 3.1 $\boxminus\varphi =_{df} \neg\blacklozenge\neg\varphi$.

We could, alternatively, consider the operator $\boxminus$ as primitive and substitute the axioms TK_1 and TK_2 by the following ones:

(TK_1^*) $\boxminus\varphi \to \varphi$;

(TK_2^*) $\boxminus\varphi \to \boxminus\boxminus\varphi$;

and the rule $RM^{\blacklozenge}$ is replaced by the rule $RM^{\boxminus}$:

$$(\text{RM}^{\boxminus})\ \frac{\vdash \varphi \to \psi}{\vdash \boxminus\varphi \to \boxminus\psi}.$$

Proposition 3.2 $\vdash \varphi \to \psi \Rightarrow \vdash \boxminus\varphi \to \boxminus\psi$.

Corollary 3.3 $\vdash \varphi \leftrightarrow \psi$ *implies that* $\vdash \boxminus\varphi \leftrightarrow \boxminus\psi$.

Proposition 3.4 $\vdash \boxminus\varphi \to \varphi$.

Proposition 3.5 $\vdash \boxminus\varphi \to \boxminus\boxminus\varphi$.

Proposition 3.6 $\vdash \boxminus(\varphi \wedge \psi) \to \boxminus\varphi$.

Corollary 3.7 $\vdash \boxminus(\varphi \wedge \psi) \to (\boxminus\varphi \wedge \boxminus\psi)$.

In the following we will show the adequacy of TK relative to some different models. We will use this version with the operator $\boxminus$ and the concept of interior.

4 Almost topological spaces and Tarski spaces

The almost topological spaces involve the Tarski space in the context of topological spaces.

Definition 4.1 *An almost topological space is a pair (E, Ω) such that E is a non empty set and $\Omega \subseteq \mathcal{P}(E)$ for which:*
(i) If $A \subseteq \Omega$, then $\cup A \in \Omega$.

Definition 4.2 *The collection Ω is called an almost topology, and each member of Ω is an open set of (E, Ω). A set $A \subseteq E$ is closed when its complement relative to E, denoted by A^C, is open in (E, Ω).*

Proposition 4.3 *In any almost topological space (E, Ω), the set $\emptyset$ is open and E is closed.*

Proposition 4.4 *In any almost topological space (E, Ω), the intersection of closed sets is a closed set.*

Definition 4.5 *Let (E, Ω) be an almost topological space. The closure of A is the set:*

$$\overline{A} =_{df} \cap\{X^C : A \subseteq X^C \text{ and } X \in \Omega\}.$$

The interior of A is the set:

$$\mathring{A} =_{df} \cup\{X \subseteq E : X \subseteq A \text{ and } X \in \Omega\}.$$

Proposition 4.6 *If (E, Ω) is an almost topological space and $A \subseteq E$, then $\overline{A}$ is closed and $\mathring{A}$ is open.*

Proposition 4.7 *If (E, Ω) is an almost topological space, then for every $A, B \subseteq E$:*
(i) $\mathring{A} \subseteq A \subseteq \overline{A}$;
(ii) $\mathring{\mathring{A}} = \mathring{A}$;
(iii) $\overline{A} = \overline{\overline{A}}$;
(iv) $\mathring{\emptyset} = \emptyset$;
(v) $\overline{E} = E$;
(vi) $A \subseteq B$ *implies that* $\mathring{A} \subseteq \mathring{B}$;
(vii) $A \subseteq B$ *implies that* $\overline{A} \subseteq \overline{B}$.

To describe the open sets in an almost topological space (E, Ω), we do not need to consider necessarily all the members of Ω. Thus we have the concept of basis.

Definition 4.8 *A basis for an almost topological space (E, Ω) is a subset $\mathcal{B} \subseteq \Omega$ such that for any $A \in \Omega$ there is a $B \subseteq \mathcal{B}$ such that:*

$$A = \cup B.$$

The elements of $\mathcal{B}$ are called basic open sets.

Proposition 4.9 *If $E \neq \emptyset$ and $\mathcal{B} \subseteq \mathcal{P}(E)$, then $\mathcal{B}$ is basis of an almost topological space (E, Ω), in which $\Omega = \{\cup B : B \subseteq \mathcal{B}\} \cup \{\emptyset\}$.*

Proof. Of course, $\emptyset \in \Omega$ and arbitrary unions of members of Ω belong to Ω. So Ω is an almost topology on E. ■

Now we will see how to evolve the concepts of almost topological space and Tarki space.

Proposition 4.10 *If (E, Ω) is an almost topological space and, for $A \subseteq E$, the set $\overline{A}$ is the closure of A, then $(E, ^-)$ is a Tarski space.*

On the other hand.

Proposition 4.11 *If $(E, ^-)$ is a Tarski space and $\Omega = \{X \subseteq E : X \text{ is open}\}$, then (E, Ω) is an almost topological space.*

5 Models given by maximal consistent theories

Now we set up the almost topological spaces as models for *TK*.

We denote the set of propositional variables of *TK* by Var(*TK*), and the set of its formulas by For(*TK*).

Definition 5.1 *A TK-theory is a set* $\Delta \subseteq$ *For(TK) such that* $\overline{\Delta} = \Delta$*, where* $\overline{\Delta} = \{\psi : \Delta \vdash \psi\}$.

Thus, a theory is a set of formulas closed relative to the deduction relation of *TK*. When $\Delta = \emptyset$, the theory correspond to the theorems of *TK*.

Definition 5.2 *Let* (E, Ω) *be an almost topological space. A restrict valuation is a function* $\langle . \rangle$*: Var(TK)* $\to \mathcal{P}(E)$ *that maps each variable of TK to an element of* $\mathcal{P}(E)$.

Definition 5.3 *Given an almost topological space* (E, Ω)*, a valuation is a function* $[.]$*: For(TK)* $\to \mathcal{P}(E)$ *that extends naturally and uniquely a restrict valuation* $\langle . \rangle$ *as follows:*
(i) $[p] = \langle p \rangle$
(ii) $[\neg\varphi] = E - [\varphi]$
(iii) $[\boxminus\varphi] = [\overset{\circ}{\varphi}]$
(iv) $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
(v) $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.

Thus, $[\top] = E$ and $[\bot] = \emptyset$, where $\top$ is any tautology and $\bot$ is any contradiction.

Definition 5.4 *Let* (E, Ω) *be an almost topological space. A valuation* $[.]$*: For(TK)* $\to \mathcal{P}(E)$ *is a model for a set* $\Gamma \subseteq$ *For(TK) when, for each formula* $\gamma \in \Gamma$, $[\gamma] = E$.

We denote this condition by $\langle (E, \Omega), [.] \rangle \vDash \Gamma$. In particular, a valuation $[.]$: For(*TK*) $\to \mathcal{P}(E)$ is a model for $\varphi \in$ For(*TK*) when $[\varphi] = E$. In this case we write $\langle (E, \Omega), [.] \rangle \vDash \varphi$. The notation shows the importance of both the space (E, Ω) and the valuation $[.]$.

Definition 5.5 *A subset* $\Gamma \subseteq$ *For(TK)* logically implies ψ*, what is denoted by* $\Gamma \vDash \psi$*, if every model of* Γ *is a model of* ψ.

Definition 5.6 *A formula* φ *is* valid *in the almost topological space* (E, Ω) *if for every valuation* $[.]$*: For(TK)* $\to (E, \Omega)$ *it follows that* $\langle (E, \Omega), [.] \rangle \vDash \varphi$.

Definition 5.7 *A formula* $\varphi \in$ *For(TK) is* valid *if it is valid in any almost topological space* (E, Ω).

Now we can show the easy direction of adequacy.

Lemma 5.8 $[\varphi \to \psi] = E \;\Leftrightarrow\; [\varphi] \subseteq [\psi]$.

Proof. $[\varphi \to \psi] = E \;\Leftrightarrow\; [\neg\varphi \vee \psi] = E \;\Leftrightarrow\; [\neg\varphi] \cup [\psi] = E \;\Leftrightarrow\; (E - [\varphi]) \cup [\psi] = E$ $\Leftrightarrow\; [\varphi] \subseteq [\psi]$. ■

Theorem 5.9 *(Soundness) If* $\Gamma \vdash \varphi$*, then* $\Gamma \vDash \varphi$*.*

Proof. The proof is by induction on the length of a given deduction.

For $n = 1$, we have that φ is an axiom of TK or $\varphi \in \Gamma$.

If $\varphi \in \Gamma$, of course, $\Gamma \vDash \varphi$. So let's consider that φ is an axiom of TK.

If φ is a classical tautology, by condition (vi), we have $[\varphi] = E$.

If φ is of the form $\boxminus\psi \to \psi$, then $[\boxminus\psi \to \psi] = [\neg(\boxminus\psi \wedge \neg\psi)] = E - [\boxminus\psi \wedge \neg\psi]) = E - ([\boxminus\psi] \cap [\neg\psi]) = E - ([\mathring{\psi}] \cap (E - [\psi])) = E - \emptyset = E$;

If φ is of the form $\boxminus\psi \to \boxminus\boxminus\psi$, then $[\boxminus\psi \to \boxminus\boxminus\psi] = [\neg\boxminus\psi \vee \boxminus\boxminus\psi] = (E - [\mathring{\psi}]) \cup [\mathring{\mathring{\psi}}] = (E - [\mathring{\psi}]) \cup [\mathring{\psi}] = E$.

In any case $\Gamma \vDash \varphi$.

By induction hypothesis, the enunciate holds for $k < n$.

If φ was obtained from Γ by MP, then we have $\Gamma \vdash \psi$ and $\Gamma \vdash \psi \to \varphi$. By induction hypothesis, $[\psi] = E$ and $[\psi \to \varphi] = E$ and by the previous lemma $[\psi] \subseteq [\varphi]$. As $[\psi] = E$, hence $[\varphi] = E$.

If φ was obtained from Γ by $RM^{\boxminus}$, we have $\Gamma \vdash \varphi \to \psi$ and by inductive hypothesis $[\varphi \to \psi] = E$. By the previous lemma $[\varphi] \subseteq [\psi]$ and, by Proposition 4.7 (vi), $[\mathring{\varphi}] \subseteq [\mathring{\psi}]$ and from that $\boxminus\varphi \subseteq \boxminus\psi$, therefore $[\boxminus\varphi \to \boxminus\psi] = E$.

Hence $\Gamma \vDash \varphi$. ■

Now the difficult direction, the Completeness. We characterize a particular almost topological space from collections of theories in TK. As usual, we name this model by canonical almost topological space.

Definition 5.10 *A set of formulas* Δ *is* maximal consistent *if it is consistent, and no proper extension of* Δ *is consistent.*

Proposition 5.11 $\Gamma \vdash \psi \;\Leftrightarrow\; \Gamma \cup \{\neg\psi\}$ *is inconsistent.*

Proposition 5.12 *If* Γ *is maximal consistent, then for every formula* ψ *of* $For(TK)$*, either* $\psi \in \Gamma$ *or* $\neg\psi \in \Gamma$*.*

Proposition 5.13 *If* Γ *is maximal consistent and* $\Gamma \vdash \psi$*, then* $\psi \in \Gamma$*.*

Theorem 5.14 *(Lindenbaum) Every consistent set of formulas* Γ *can be extended to a maximal consistent set* Δ.

Proof. The proof is standard; see, for instance, Fitting and Mendelsohn ([10], 1998, p. 76). ■

The domain of the canonical almost topological space is the set T of all maximal consistent theories of TK.

Now we need an almost topology over T. As in the Soundness Theorem we want to apply the interior to the operator $\boxminus$.

Definition 5.15 *For each* $\varphi \in For(TK)$, $\widehat{\varphi} = \{\Delta \in T : \varphi \in \Delta\}$ *and* $\mathcal{B} = \{\widehat{\boxminus\varphi} : \varphi \in For(TK)\}$.

Proposition 5.16 *The set* $\mathcal{B}$ *is a basis for an almost topology over* T.

Proof. It follows from Proposition 4.9. ■

In this way, we have an almost topological space (T, Ω) such that T is the set of maximal consistent theories of TK and each member of Ω is the set $\emptyset$ or any union of elements of $\mathcal{B}$. Each basic open set is of the form $V_\varphi = \{\Delta \in T : \boxminus\varphi \in \Delta\}$.

Definition 5.17 *A canonical valuation is any valuation* $[.]$: *For(TK)* $\rightarrow \mathcal{P}(T)$ *such that:*

(i) $[p] = \{\Delta \in T : p \in \Delta\} = \widehat{p}$.

Lemma 5.18 $\varphi \rightarrow \psi \ \Leftrightarrow\ \widehat{\varphi} \subseteq \widehat{\psi}$.

Proof. ($\Rightarrow$) If $\Delta \in \widehat{\varphi}$, then $\Delta \vdash \varphi$. Since $\Delta \vdash \varphi \rightarrow \psi$, then $\Delta \vdash \psi$. Hence, $\Delta \in \widehat{\psi}$.
($\Leftarrow$) If $\nvdash \varphi \rightarrow \psi$, then there exists $\Delta \in T$ such that $\neg(\varphi \rightarrow \psi) \in \Delta$. Hence, $\Delta \vdash \neg(\varphi \rightarrow \psi) \ \Leftrightarrow\ \Delta \vdash \varphi \wedge \neg\psi \ \Leftrightarrow\ (\Delta \vdash \varphi$ and $\Delta \vdash \neg\psi) \ \Leftrightarrow\ (\Delta \in \widehat{\varphi}$ and $\Delta \in \widehat{\neg\psi}) \ \Leftrightarrow\ (\Delta \in \widehat{\varphi}$ and $\Delta \notin \widehat{\psi}) \ \Leftrightarrow\ \widehat{\varphi} \nsubseteq \widehat{\psi}$. ■

Proposition 5.19 *For every* $\varphi \in For(TK)$, *it follows that* $[\varphi] = \widehat{\varphi}$.

Proof. By induction on the complexity of φ.

If φ is a propositional variable, then $[p] = \widehat{p}$, by the above definition.

If φ is of the form $\neg\psi$, then by induction hypothesis, $[\psi] = \widehat{\psi}$. So $[\varphi] = [\neg\psi] = T - [\psi] = T - \widehat{\psi} = \widehat{\neg\psi} = \widehat{\varphi}$.

If φ is of the form $\psi \vee \sigma$, then by induction hypothesis, $[\psi] = \widehat{\psi}$ and $[\sigma] = \widehat{\sigma}$. So $[\varphi] = [\psi \vee \sigma] = [\psi] \cup [\sigma] = \widehat{\psi} \cup \widehat{\sigma} = \widehat{\varphi}$.

If φ is of the form $\boxminus\psi$, then by induction hypothesis, $[\psi] = \widehat{\psi}$.

($\subseteq$) If $\Delta \in [\varphi] = [\boxminus\psi] = [\mathring{\psi}] \subseteq [\psi] = \widehat{\psi}$, then there is an open basic set $\widehat{\boxminus\sigma} \subseteq [\boxminus\psi]$ such that $\Delta \in \widehat{\boxminus\sigma}$ and $\widehat{\boxminus\sigma} \subseteq \widehat{\psi}$. From lemma above, we have that $\boxminus\sigma \to \psi$ and by the rule $RM^{\boxminus}$, $\boxminus\boxminus\sigma \to \boxminus\psi$. From axiom (TK_2^*) it follows that $\boxminus\sigma \to \boxminus\psi$ and by lemma above, $\widehat{\boxminus\sigma} \subseteq \widehat{\boxminus\psi} = \widehat{\varphi}$. Finally $\Delta \in \widehat{\varphi}$.

($\supseteq$) If $\Delta \in \widehat{\varphi} = \widehat{\boxminus\psi}$, then Δ belongs to the basic open set $\widehat{\boxminus\psi} \in \mathcal{B}$. From the axiom (TK_1^*) and the above lemma, it follows that $\widehat{\boxminus\psi} \subseteq \widehat{\psi} = [\psi]$. From Proposition 4.7 (vi), $\widehat{\boxminus\psi} = \mathring{\widehat{\boxminus\psi}} \subseteq [\mathring{\psi}] = [\boxminus\psi] = [\varphi]$. So, $\Delta \in [\varphi]$. ■

Theorem 5.20 *(Strong completeness) If $\Gamma \vDash \psi$, then $\Gamma \vdash \psi$.*

Proof. If $\Gamma \nvdash \psi$, then $\Gamma \cup \{\neg\psi\}$ is consistent. By Lindenbaum Theorem, $\Gamma \cup \{\neg\psi\}$ can be extended to a maximal consistent theory Δ. So, $\Delta \vDash \neg\psi$ and hence $\Delta \nvDash \psi$. Then $\Gamma \nvDash \psi$. ■

In this way the compactness is very simple.

Corollary 5.21 *(Compactness) If every finite $\Gamma_0 \subseteq \Gamma$ has model, then Γ has a model.*

Proof. If Γ does not have a model, then for every $\Delta \in T$, $\Gamma \nsubseteq \Delta$. So, by Lindenbaum Theorem, Γ is inconsistent. Hence, there is a formula ψ such that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$, that is, there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \psi$ and $\Gamma_0 \vdash \neg\psi$. So, Γ_0 does not have a model. ■

6 Pre-order models for TK

In this section it is proved the adequacy of *TK* relative to a kind of relational models given by pre-order relations.

Definition 6.1 *A pre-order is a structure $(E, \leq)$ such that $E \neq \emptyset$ and $\leq$ is a binary relation on E that is reflexive and transitive, that is, for all $x, y, z \in E$ we have: (i) $x \leq x$, and (ii) $x \leq y$ and $y \leq z$ implies that $x \leq z$.*

Example 6.2 *An example of pre-order non antisymmetric:*
Let $E = \mathcal{P}(\{0, 1, 2\})$ and define $A \leq B \Leftrightarrow A \subseteq B \cup \{0\}$. Hence, $(E, \leq)$ is a pre-order, $\{1\} \leq \{0, 1\}$, $\{0, 1\} \leq \{1\}$, but $\{1\} \neq \{0, 1\}$.

Definition 6.3 *Let $(E, \leq)$ be a pre-order. For each $x \in E$ the set of upper bounds of x is the set $\overrightarrow{x} = \{y \in E : x \leq y\}$.*

Let $(E, \leq)$ be a pre-order and $x, y \in E$. The reflexivity of $\leq$ ensures that $x \in \overrightarrow{x}$, and the transitivity of $\leq$ ensures that if $y \in \overrightarrow{x}$, then $\overrightarrow{y} \subseteq \overrightarrow{x}$.

Definition 6.4 *Let $(E, \leq)$ be a pre-order. A restrict valuation is a function $\langle . \rangle$: Var(TK) $\rightarrow \mathcal{P}(E)$ that interprets each variable of TK as an element of $\mathcal{P}(E)$.*

Definition 6.5 *A valuation in a pre-order $(E, \leq)$ is a function $[.]$: For(TK) $\rightarrow \mathcal{P}(E)$ that extends naturally and uniquely a restrict valuation $\langle . \rangle$ as follows:*
(i) $[p] = \langle p \rangle$
(ii) $[\neg\varphi] = E - [\varphi]$
(iii) $[\boxminus\varphi] = \{x \in E : \overrightarrow{x} \subseteq [\varphi]\}$
(iv) $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
(v) $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.

So $[\top] = E$ and $[\bot] = \emptyset$.

Proposition 6.6 *Given a valuation in a pre-order $(E, \leq)$, the following hold:*
(i) $[\boxminus\varphi] \subseteq [\varphi]$;
(ii) $[\boxminus\varphi] = [\boxminus \boxminus \varphi]$;
(iii) $[\varphi] \subseteq [\psi]$ *implies that* $[\boxminus\varphi] \subseteq [\boxminus\psi]$.

Proof. (i) If $y \in [\boxminus\varphi]$, as $\leq$ is reflexive, then $y \in \overrightarrow{y} \subseteq [\varphi]$ and hence $y \in [\varphi]$.
(ii) By (i) $[\boxminus \boxminus \varphi] \subseteq [\boxminus\varphi]$. Now, if $y \in [\boxminus\varphi]$ and $x \in \overrightarrow{y}$, then $\overrightarrow{x} \subseteq \overrightarrow{y} \subseteq [\varphi]$. Hence $x \in [\boxminus\varphi]$, therefore $\overrightarrow{y} \subseteq [\boxminus\varphi]$, that is, $y \in [\boxminus \boxminus \varphi]$.
(iii) If $y \in [\boxminus\varphi]$, then $\overrightarrow{y} \subseteq [\varphi] \subseteq [\psi]$. Thus $y \in [\boxminus\psi]$. ■

Proposition 6.7 *If $(E, \leq)$ a pre-order, then:*
(i) $[\varphi] = \emptyset \Rightarrow [\boxminus\varphi] = \emptyset$;
(ii) $[\varphi] = E \Rightarrow [\boxminus\varphi] = E$.

Proof. (i) $[\boxminus\varphi] = \{x \in E : \overrightarrow{x} \subseteq \emptyset\} = \emptyset$;
(ii) By the reflexivity of $\leq$, $[\boxminus\varphi] = \{x \in E : \overrightarrow{x} \subseteq E\} = E$. ■

Definition 6.8 *If $(E, \leq)$ is a pre-order, then a model for $\Gamma \subseteq$ For(TK) is a valuation $[.]$: For(TK) $\rightarrow \mathcal{P}(E)$ such that, for each formula $\gamma \in \Gamma$, $[\gamma] = E$.*

$\langle (E, \leq), [.] \rangle \vDash \Gamma$ denotes that $\langle (E, \leq), [.] \rangle$ is a model of Γ.

In particular, if $\Gamma = \{\varphi\}$ and the pre-order is clear from the context, then we can just write $[\varphi] = E$.

Definition 6.9 *A formula φ is* valid in a pre-order $(E, \leq)$ *if for every valuation over $(E, \leq)$ it holds that $[\varphi] = E$.*

We denote that by $\langle (E, \leq), [.] \rangle \vDash \varphi$.

Definition 6.10 *If for every pre-order $(E, \leq)$ and every valuation $[.]$: For(TK) $\rightarrow \mathcal{P}(E)$ it holds $\langle (E, \Omega), [.] \rangle \vDash \varphi$, then the formula φ is valid, which is denoted by $\vDash \varphi$.*

Definition 6.11 *A subset $\Gamma \subseteq$ For(TK) logically implies ψ, which is denoted by $\Gamma \vDash \psi$, if every model of Γ is a model of ψ too.*

Now, the easy direction of adequacy.

Lemma 6.12 $[\varphi \rightarrow \psi] = E \;\Leftrightarrow\; [\varphi] \subseteq [\psi]$.

Proof. Analogous to that for Lemma 5.8. ■

Theorem 6.13 *(Soundness) If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. The proof is by induction on the the length n of the deduction. Let φ with a deduction from Γ of length n.

For $n = 1$, we have that φ is an axiom of *TK* or $\varphi \in \Gamma$.

If $\varphi \in \Gamma$, of course, $\Gamma \vDash \varphi$. So let's consider that φ is an axiom of *TK*.

If φ is a classical tautology, by condition (vi) above, $[\varphi] = E$.

If φ is of the form $\boxminus\psi \rightarrow \psi$, then $[\boxminus\psi \rightarrow \psi] = E$, by Proposition 6.6 (i) and Lemma 6.12.

If φ is of the form $\boxminus\psi \rightarrow \boxminus\boxminus\psi$, then $[\boxminus\psi \rightarrow \boxminus\boxminus\psi] = E$ by Proposition 6.6 (ii) and Lemma 6.12.

In any case $\Gamma \vDash \varphi$.

By induction hypothesis, the proposition holds for $k < n$.

If φ was obtained from Γ by MP, then we have $\Gamma \vdash \psi$ and $\Gamma \vdash \psi \rightarrow \varphi$. By induction hypothesis, $[\psi] = E$ and $[\psi \rightarrow \varphi] = E$ and by Lemma 6.12 $[\psi] \subseteq [\varphi]$. As $[\psi] = E$, hence $[\varphi] = E$.

If φ was obtained from Γ by $RM^{\boxminus}$, we have $\Gamma \vdash \varphi \rightarrow \psi$ and by induction hypothesis $[\varphi \rightarrow \psi] = E$. By Lemma 6.12 $[\varphi] \subseteq [\psi]$ and, by Proposition 6.6 (iii), $[\boxminus\varphi] \subseteq [\boxminus\psi]$ and, again, from Lemma 6.12, $[\boxminus\varphi \rightarrow \boxminus\psi] = E$.

Hence $\Gamma \vDash \varphi$. ■

Theorem 6.14 *If* Γ *is consistent, then* Γ *has a model.*

Proof. By Lindenbaum Theorem, every consistent set Γ can be extended to a maximal consistent set Δ. Thus, we will verify that Δ has a model and, since $\Gamma \subseteq \Delta$, also Γ has a model.

We define the following valuation $[.]$ for $For(TK)$ in the pre-order $(E, \leq)$ such that $E = \{0\}$ and $\leq$ is given by $0 \leq 0$. Then, $\mathcal{P}(E) = \{\emptyset, E\}$ and we obtain a natural Boolean model.

For each propositional variable p, $[p] = E \;\Leftrightarrow\; p \in \Delta$.

Now, we show that for every formula ψ, $[\psi] = E \;\Leftrightarrow\; \psi \in \Delta$, that is, $[.]$ is a model for Δ.

The proof is by induction on the number of operators, and the Boolean part is very easy. So let us develop the relevant part.

($\Leftarrow$) If $\psi \in \Delta$ and ψ is of the form $\boxminus\sigma$, then $\boxminus\sigma \in \Delta$ and, by Axiom TK_1^*, it follows that $\sigma \in \Delta$. By induction hypothesis, $[\sigma] = E$ and, by Proposition 6.7, $[\boxminus\sigma] = E$. Hence, $[\psi] = E$.

($\Rightarrow$) If $\psi \notin \Delta$, then $\neg\psi \in \Delta$. By ($\Leftarrow$), $[\neg\psi] = E$. Hence, $[\psi] = \emptyset \neq E$. ■

Corollary 6.15 *(Strong completeness) If* $\Gamma \vDash \psi$*, then* $\Gamma \vdash \psi$*.*

Proof. If $\Gamma \vDash \psi$, then every model of Γ is a model of ψ, that is, there is no model of $\Gamma \cup \{\neg\psi\}$. By the previous theorem, $\Gamma \cup \{\neg\psi\}$ is inconsistent and, by Lemma 5.11, $\Gamma \vdash \psi$. ■

7 Final considerations

The Logic *TK* was originally motivated by the definition of Tarski operators. So it can be thought as a modal logic of deductibility. Due to similarity between Tarski spaces and topological spaces we have tried to develop models for *TK* in the topological style. This paper show us that the almost topological spaces are adequate models to *TK*. Particularly, we construct an almost topological space of maximal consistent theories of *TK*. After this, we present a new model for *TK* given by pre-order structures. It is interesting to observe that the Boolean part of *TK* is interpreted in the Boolean set algebra of this structures. The pre-order aspects are relevant only for the modal part of *TK*.

Of course there are common aspects in all these mathematical structures that are models of *TK*. In next steps we plan to investigate the relations among these structures.

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