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$n \times m-$valued Łukasiewicz-Moisil algebras with two modal operators

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Abstract

In 2000, Figallo and Sanza introduced the $n \times m-$valued Łukasiewicz-Moisil algebras, which are a particular case of Matrix Łukasiewicz algebras, and a non-trivial generalization of $n-$valued Łukasiewicz-Moisil algebras. Here we start a research on the class of $n \times m-$valued Łukasiewicz-Moisil algebras endowed with two modal operators (or $2\mathrm{mLM}_{n\times m}-$algebras). These algebras constitute a common generalization of both weak-tense Boolean algebras and weak-tense n-valued Łukasiewicz-Mosil algebras. Our most important result is a representation theorem for $2\mathrm{mLM}_{n\times m}-$algebras. In addition, as a corollary to the previous theorem, we obtain the representation theorem given by Chiriţă in 2012 for weak-tense $n-$valued Łukasiewicz-Moisil algebras.

Keywords: n-valued Łukasiewicz-Moisil algebras, weak-tense n-valued Łukasiewicz-Moisil, $n \times m$-valued Łukasiewicz-Moisil algebras, $n \times m$-valued Łukasiewicz-Moisil algebras with modal operators.

Introduction

In 1975, Suchoń ([24]) introduced the Matrix Łukasiewicz algebras as an algebraic version of a certain modal logic more general than the $n-$valued Moisil logic [24].

In 2000, Figallo and Sanza introduced the $n\times m-$valued Łukasiewicz-Moisil algebras ([20, 23, 22, 16, 17],which are a particular case of Matrix Łukasiewicz algebras and a non-trivial generalization of $n-$valued Łukasiewicz-Moisil algebras ([1]).

In 2007, Diaconescu and Georgescu, in their important work [8], started the algebraic study of the tense $n-$valued Moisil logic and introduced the tense MV-algebras as well. These two classes of algebras have aroused several authors interest lately (See [4, 5, 6, 7, 19, 3, 2, 12]). Chiriţă, in particular, in [4, 5], introduced tense θ–valued Łukasiewicz–Moisil algebras and proved an important representation theorem which allowed the completeness of the tense θ–valued Moisil logic to be shown (See [4]).

In [8], Diaconescu and Georgescu formulated an open problem about the representation of tense MV–algebras. This problem was solved in [19, 3] for semisimple tense MV–algebras. Also, in [2],tense basic algebras were studied, which are an interesting generalization of tense MV–algebras.

Following the ideas of Diaconescu y Georgescu, we have considered tense operators in several contexts since 2008. For more details the reader is referred to [9, 10, 11, 12, 13, 14, 15].

In 1979, Georgescu [18] introduced and investigated operators more general than tense operators, which he called weak-tense operators [18].

Recently, in [7], Chiriţă considered weak-tense operators on $\theta-$valued Łukasiewicz-Moisil algebras.

We can observe that in [6] a change was introduced by which weak-tense operators are named tense operators. In our opinion, this change is inadequate considering that weak-tense operators are not good models of tense logics. More precisely, the pair of weak-tense operators can be considered as a pair of modal operators without links with each other. However, it is possible to define tense operators taking into account weak tense operators, which is necessary for the algebraic version of the logic being studied. Besides, weak-tense $n-$valued Łukasiewicz-Moisil algebras were also studied in [7].

In this paper we consider and investigate the $n \times m-$ valued Łukasiewicz-Moisil algebras endowed with two modal operators. These algebras constitute a common generalization of weak tense Boolean algebras and weak tense $n-$valued Łukasiewicz-Moisil algebras. In other results we prove a representation theorem for the $n \times m-$ valued Łukasiewicz-Moisil with modal operators. We also obtain the representation theorem given by Chiriţă in [7] for weak-tense $n-$valued Łukasiewicz-Moisil algebras as a corollary to the previous theorem.

1 Preliminaries

1.1 Weak-tense Boolean algebras

In this subsection we will recall some basic definitions and results on the representation of weak-tense Boolean algebras [18, 7].

Definition 1.1 *A weak-tense Boolean algebra is a triple* $(\mathcal{B}, G, H)$ *such that* $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B \rangle$ *is a Boolean algebra and* $G, H : B \longrightarrow B$ *are two unary operations on* B *such that, for all* $x, y \in B$:

1. $G(1_B) = 1_B$ *and* $H(1_B) = 1_B$;
2. $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$.

Definition 1.2 *Let $(\mathcal{B}, G, H)$ be a weak-tense Boolean algebra. We define the operations $F, P : B \longrightarrow B$, by $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, for any $x \in B$.*

Remark 1.3 *Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B\rangle$ be a Boolean algebra. Then $(\mathcal{B}, 1_B, 1_B)$ is a weak-tense Boolean algebra, denoted by 1_B, the function $1_B : B \longrightarrow B$, $1_B(x) = 1_B$, for all $x \in B$.*

Definition 1.4 *A weak-frame is a triple (X, R, Q), where X is a nonempty set and R, Q are two binary relations on X.*

Let (X, R, Q) be a weak-frame and $\mathbf{2}$ be the standard Boolean algebra with two elements. We define the operations $G^*, H^* : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, for all $p \in \mathbf{2}$ and $x \in X$:

$$(G^*(p))(x) = \bigwedge\{p(y) \mid y \in X, xRy\} \tag{1}$$

$$(H^*(p))(x) = \bigwedge\{p(y) \mid y \in X, xQy\} \tag{2}$$

Proposition 1.5 *For any weak-frame (X, R, Q), $(\mathbf{2}^X, G^*, H^*)$ is a weak-tense Boolean algebra.*

In the weak-tense Boolean algebra $(\mathbf{2}^X, G^*, H^*)$, the weak-tense operators P^* and F^* are given by: for every $p \in \mathbf{2}^X$ and $x \in X$,

$$(P^*(p))(x) = \bigvee\{p(y) \mid y \in X, yRx\} \tag{3}$$

$$(F^*(p))(x) = \bigvee\{p(y) \mid y \in X, yQx\} \tag{4}$$

Definition 1.6 *Let $(\mathcal{B}, G, H)$ and $(\mathcal{B}', G', H')$ be two weak-tense Boolean algebras. A function $f : B \longrightarrow B'$ is a morphism of weak-tense Boolean algebras if f is a Boolean morphism and it satisfies the conditions: $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$, for any $x \in B$.*

By this definition, it follows that a morphism of weak-tense Boolean algebras commutes with the weak-tense operators F and P.

We will denote by $\mathfrak{WTB}$ the category of weak-tense Boolean algebras.

Theorem 1.7 *For any weak-tense Boolean algebra $(\mathcal{B}, G, H)$, there exist a weak-frame (X, R, Q) and an injective morphism of weak-tense Boolean algebras $\alpha : (\mathcal{B}, G, H) \longrightarrow (\mathbf{2}^X, G^*, H^*)$, where operators G^* and H^* are defined by relations 1 and 2.*

1.2 Weak-tense n-valued Łukasiewicz-Moisil algebras

In this section we will recall some basic definitions and results on the representation of weak-tense n-valued Łukasiewicz-Moisil algebras (See [7]).

Definition 1.8 *A weak-tense n-valued Łukasiewicz-Moisil algebra (or weak-tense LM_n–algebra) is a triple $(\mathcal{L}, G, H)$ such that*

$$\mathcal{L} = \langle L, \wedge, \vee, \sim, \sigma_1, \ldots, \sigma_n, 0_L, 1_L \rangle$$

is an n-valued Łukasiewicz-Moisil algebra (See [1]) *and $G, H : L \longrightarrow L$ are two unary operations on L such that, for all $x, y \in L$:*

1. *$G(1_L) = 1_L$ and $H(1_L) = 1_L$;*
2. *$G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$;*
3. *$G(\sigma_i(x)) = \sigma_i(G(x))$ and $H(\sigma_i(x)) = \sigma_i(H(x))$, for all $i = 1, \ldots, n-1$.*

Definition 1.9 *Let $(\mathcal{L}, G, H)$ be a weak-tense LM_n–algebra. We define the operations $F, P : L \longrightarrow L$, by $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$, for any $x \in L$.*

Let (X, R, Q) be a weak-frame and $\mathbf{Ł}_n$ the chain of n rational fractions $Ł_n = \{\frac{j}{n-1} \mid 1 \leq j \leq n-1\}$ endowed with the natural lattice structure and the unary operations $\sim$ and σ_i defined as follows: $\sim (\frac{j}{n-1}) = 1 - \frac{j}{n-1}$ and $\sigma_i(\frac{j}{n-1}) = 0$ if $i + j < n$ or $\sigma_i(\frac{j}{n-1}) = 1$ otherwise. . We define the operations $G^*, H^* : \mathbf{Ł_n}^X \longrightarrow \mathbf{Ł_n}^X$, for all $p \in \mathbf{Ł_n}$ and $x \in X$:

$$(G^*(p))(x) = \bigwedge \{p(y) \mid y \in X, xRy\} \tag{5}$$

$$(H^*(p))(x) = \bigwedge \{p(y) \mid y \in X, xQy\} \tag{6}$$

Proposition 1.10 *For any weak-frame (X, R, Q), $(\mathbf{Ł_n}^X, G, H)$ is a weak-tense LM_n–algebra.*

In the weak-tense LM$_n$–algebra $(\mathbf{Ł_n}^X, G^*, H^*)$, the weak-tense operators P^* and F^* are given by: for every $p \in \mathbf{Ł_n}^X$ and $x \in X$,

$$(P^*(p))(x) = \bigvee \{p(y) \mid y \in X, yRx\} \tag{7}$$

$$(F^*(p))(x) = \bigvee \{p(y) \mid y \in X, yQx\} \tag{8}$$

Definition 1.11 *Let $(\mathcal{L}, G, H)$ and $(\mathcal{L}', G', H')$ be two weak-tense LM_n–algebras. A function $f : L \longrightarrow L'$ is a morphism of weak-tense LM_n–algebras if f is an LM_n–algebra morphism and it satisfies the conditions: $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$, for any $x \in L$.*

By this definition, it follows that a morphism of weak-tense LM$_n$–algebras commutes with the weak-tense operators F and P.

Now we will recall a representation theorem for weak-tense LM$_n$–algebras that generalizes Theorem 1.7.

Theorem 1.12 ([7]) *For any weak-tense LM_n–algebra $(\mathcal{L}, G, H)$, there exist a weak-frame (X, R, Q) and an injective morphism of weak-tense LM_n–algebras*

$$\Phi : (\mathcal{L}, G, H) \longrightarrow (Ł_n^X, G^*, H^*)$$

where operators G^* and H^* are defined as in 5 and 6.

1.3 $n \times m$–valued Łukasiewicz–Moisil algebras

In this section we will recall some basic definitions and results on $n \times m$–valued Łukasiewicz–Moisil algebras (See [20, 22, 23, 21, 16, 17]).

Definition 1.13 *An $n \times m$–valued Łukasiewicz–Moisil algebra (or $LM_{n\times m}$–algebra), in which n and m are integers, $n \geq 2$, $m \geq 2$, is an algebra*

$$\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle,$$

where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0_L, 1_L\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfills the conditions:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,

(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,

(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,

(C5) $\sigma_{ij}x = \sigma_{ij}y$ *for all* $(i, j) \in (n \times m)$ *imply* $x = y$,

(C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1_L$,

(C7) $\sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x$.

Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$–algebra. In what follows we will denote by id_L, O_L and I_L the functions $id_L, O_L, I_L : L \longrightarrow L$, defined by $id_L(x) = x$, $O_L(x) = 0_L$ and $I_L(x) = 1_L$, respectively, for all $x \in L$.

The results announced here for $\mathrm{LM}_{n\times m}$–algebras will be used throughout the paper.

(LM1) $\sigma_{ij}(L) = C(L)$ for all $(i, j) \in (n \times m)$, where $C(L)$ is the set of all complemented elements of L ([20, Proposition 2.5]).

(LM2) Every $\mathrm{LM}_{n\times 2}$–algebra is isomorphic to an n–valued Łukasiewicz–Moisil algebra. It is worth noting that $\mathrm{LM}_{n\times 2}$–algebras constitute a non–trivial generalization of a n–valued Łukasiewicz–Moisil algebra. (see [22, Remark 2.1]).

(LM3) The class of $\mathrm{LM}_{n\times m}$–algebras is a variety; two equational bases for it can be found in [20, Theorem 2.7] and [22, Theorem 4.6].

(LM4) Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$–algebra. Let X be a non–empty set and let L^X be the set of all functions from X into L. Then L^X is an $\mathrm{LM}_{n\times m}$–algebra where the operations are defined componentwise (see [21]).

(LM5) Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$–algebra. We say that L is complete if the lattice $\langle L, \wedge, \vee, 0_L, 1_L\rangle$ is complete. Also, we say that L is completely chrysippian if, for every $\{x_s\}_{s\in S} \subseteq L$ such that $\bigwedge_{s\in S} x_s$ and $\bigvee_{s\in S} x_s$ exist, the following properties hold: $\sigma_{ij}(\bigwedge_{s\in S} x_s) = \bigwedge_{s\in S} \sigma_{ij}(x_s)$ for all $(i, j) \in (n \times m)$ and $\sigma_{ij}(\bigvee_{s\in S} x_s) = \bigvee_{s\in S} \sigma_{ij}(x_s)$ for all $(i, j) \in (n \times m)$ (see [21]).

(LM6) Let $C(L) \uparrow^{(n\times m)} = \{f : (n \times m) \longrightarrow C(L)$ such that for arbitrary i, j if $r \leq s$, then $f(r, j) \leq f(s, j)$ and $f(i, r) \leq f(i, s)\}$. Then

$$\langle C(L) \uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$$

is an $\mathrm{LM}_{n\times m}$–algebra where for all $f \in C(L) \uparrow^{(n\times m)}$ and $(i, j) \in (n \times m)$ the operations $\sim$ and σ_{ij} are defined as follows: $(\sim f)(i, j) = \neg f(n-i, m-j)$, where $\neg x$ denotes the Boolean complement of x, $(\sigma_{ij} f)(r, s) = f(i, j)$ for all $(r, s) \in (n \times m)$. The remaining operations are defined componentwise ([22, Proposition 3.2]). It is worth noting that this result can be generalized by substituting any Boolean algebra B for $C(L)$. Furthermore, if B is a complete Boolean algebra, it is simple to check that $B \uparrow^{(n\times m)}$ is also a complete $LM_{n\times m}$–algebra.

(LM7) Let $\mathcal{L}$ and $\mathcal{L}'$ be two $\mathrm{LM}_{n\times m}$–algebras. A morphism of $\mathrm{LM}_{n\times m}$–algebras is a function $f : L \longrightarrow L'$ such that the following properties hold for all $x, y \in L$:

(i) $f(0_L) = 0_{L'}$ and $f(1_L) = 1_{L'}$;

(ii) $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$;

(iii) $f(\sigma_{ij}(x)) = \sigma'_{ij}(f(x))$, for every $(i,j) \in (n \times m)$;

(iv) $f(\sim x) = \sim' f(x)$.

Let us observe that condition (iv) is a direct consequence of (C5), (C7) and the conditions (i) to (iii).

(LM8) Every $\mathrm{LM}_{n\times m}-$algebra L can be embedded into $C(L) \uparrow^{(n\times m)}$, ([22, Theorem 3.1]). Besides, L is isomorphic to $C(L) \uparrow^{(n\times m)}$ if and only if L is centered ([22, Corollary 3.1]), L being centered if for each $(i,j) \in (n\times m)$ there exists $c_{ij} \in L$ such that

$$\sigma_{rs} c_{ij} = \begin{cases} 0 & \text{if } i > r \quad \text{or} \quad j > s \\ 1 & \text{if } i \leq r \quad \text{and} \quad j \leq s \end{cases}$$

(LM9) Identifying the set $(n\times 2)$ with $\mathbf{n} = \{1, \ldots, n-1\}$, we have that $\tau_{Ł_n} : Ł_n \longrightarrow \mathbf{2} \uparrow^{\mathbf{n}}$ is an isomorphism which, in this case, is defined by $\tau_{Ł_n}(\frac{j}{n-1}) = f_j$, where $f_j(i) = 0$ if $i + j < n$ and $f_j(i) = 1$ otherwise (see [21]).

1.4 2mLM$_{n\times m}$-algebras

In this section we introduce 2mLM$_{n\times m}$–algebras. The notion of 2mLM$_{n\times m}-$algebra is obtained by endowing an LM$_{n\times m}$–algebra with two unary operations G and H, similar to the weak-tense operators on an n–valued Łukasiewicz-Moisil algebra. Below are the basic definitions and properties.

Definition 1.14 *An $n\times m$–valued Łukasiewicz–Moisil algebra with two modal operators (or 2mLM$_{n\times m}$–algebra) is a triple $(\mathcal{L}, G, H)$ such that*

$$\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L \rangle$$

is an LM$_{n\times m}$-algebra and $G, H : L \longrightarrow L$ are two unary operations on L such that, for all $x, y \in L$, the following conditions hold:

(T1) $G(1_L) = 1_L$ *and* $H(1_L) = 1_L$,

(T2) $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$,

(T3) $G(\sigma_{ij}(x)) = \sigma_{ij}(G(x))$ *and* $H(\sigma_{ij}(x)) = \sigma_{ij}(H(x))$, *for all* $(i,j) \in (n \times m)$.

In what follows, we will indicate the class of 2mLM$_{n\times m}$-algebras with $\mathbf{2mLM}_{n\times m}$ and denote its elements simply by L or (L, G, H) in case we need to specify the modal operators.

Remark 1.15

(i) *From Definition 1.14 and* (LM3) *we infer that* $\mathbf{2mLM}_{n\times m}$ *is a variety and two equational bases for it can be obtained.*

(ii) *If* (L, G, H) *is a* $2mLM_{n\times m}$*algebra, then from* (LM1) *and* (T3) *we have that* $(C(L), C(G), C(H))$ *is a weak-tense Boolean algebra, where the unary operations* $C(G) : C(L) \longrightarrow C(L)$ *and* $C(H) : C(L) \longrightarrow C(L)$*, are defined by* $C(G) = G\mid_{C(L)}$ *and* $C(H) = H\mid_{C(L)}$.

(iii) *Taking into account* (LM2)*, we infer that every* $2mLM_{n\times 2}-$*algebra is isomorphic to a weak-tense n-valued Łukasiewicz-Moisil algebra.*

Definition 1.16 *Let* $(\mathcal{L}, G, H)$ *be a* $2mLM_{n\times m}$*-algebra. Let us consider, for any* $x \in L$*, the unary operations* P*,* F*, defined by* $P(x) =\sim H(\sim x)$ *and* $F(x) =\sim G(\sim x)$.

Remark 1.17 *Let* $(\mathcal{L}, G, H)$ *be a* $2mLM_{n\times m}$*-algebra. For all* $x \in L$ *and* $(i, j) \in (n\times m)$*, we have:*

(i) $P(\sigma_{ij}x) = \sigma_{ij}P(x)$ *and* $F(\sigma_{ij}x) = \sigma_{ij}F(x)$,

(ii) *If* $x \in C(L)$*, then* $P(x) =\sim H(\sim x) = \neg H(\neg x)$*, where* $\neg x$ *is the complement of* $x \in C(L)$*. Similarly, we have that* $F(x) = \neg G(\neg x)$*. Thus, we can consider the operations* $C(P) = P\mid_{C(L)}$*,* $C(F) = F\mid_{C(L)}$*. It follows that* $C(P)$ *and* $C(F)$ *are the corresponding operations on weak-tense Boolean algebra* $(C(L), C(G), C(H))$.

Now we will indicate a list of properties of the modal operators G, H, P and F in a 2mLM$_{n\times m}$-algebra.

Proposition 1.18 *Let* $(\mathcal{L}, G, H)$ *be a* $2mLM_{n\times m}$*-algebra. The following properties hold, for all* $x, y \in L$:

1. $P(0_L) = 0_L$ *and* $F(0_L) = 0_L$,
2. $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,
3. $x \leq y$ *implies* $P(x) \leq P(y)$ *and* $F(x) \leq F(y)$,
4. $P(x \vee y) = P(x) \vee P(y)$ *and* $F(x \vee y) = F(x) \vee F(y)$,
5. $G(x) \vee G(y) \leq G(x \vee y)$ *and* $H(x) \vee H(y) \leq H(x \vee y)$,
6. $P(x \wedge y) \leq P(x) \wedge P(y)$ *and* $F(x \wedge y) \leq F(x) \wedge F(y)$
7. $G(x \vee y) \leq F(x) \vee G(y)$ *and* $H(x \vee y) \leq P(x) \vee H(y)$,

8. $G(x) \wedge F(y) \leq F(x \wedge y)$ *and* $H(x) \wedge P(y) \leq P(x \wedge y)$*.*

Proof. Conditions (1)-(8) hold in the weak-tense Boolean algebra $(C(L), C(G), C(H))$, so, by (C5), they are available in $(\mathcal{L}, G, H)$.

■

Based on the notion of weak-frame, we will give an example of a 2mLM$_{n\times m}$-algebra.

Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be a complete and completely chrysippian LM$_{n\times m}$-algebra.

Let us consider a weak-frame (X, R, Q). We will define on L^X the operations $\wedge, \vee, \sim, 0, 1$ and σ_{ij} for all $(i, j) \in (n \times m)$ as in (LM4) and the additional unary operations G^* and H^* as follows:

$$(G^*(p))(x) = \bigwedge\{p(y) \mid y \in X, xRy\} \tag{9}$$

$$(H^*(p))(x) = \bigwedge\{p(y) \mid y \in X, xQy\} \tag{10}$$

for all $p \in L^X$, $x \in X$.

Proposition 1.19 *For any weak-frame* (X, R, Q)*,* (L^X, G^*, H^*) *is a 2mLM$_{n\times m}$-algebra.*

Proof. Since L is an LM$_{n\times m}$–algebra, then by (LM4) we have that L^X is an LM$_{n\times m}$–algebra. Now we will prove that G and H satisfy properties (T1)–(T3) in Definition 1.14. Note that properties (T1) and (T2) have already been proved in the Boolean case. We will prove only (T3). Let $p \in L^X$, $x \in X$ and $(i, j) \in (n \times m)$. Considering that L is completely chrysippian, we have that: $G(\sigma_{ij}(p))(x) = \bigwedge\{\sigma_{ij}(p)(y) \mid y \in X, xRy\} = \sigma_{ij}(\bigwedge\{p(y) \mid y \in X, xRy\}) = \sigma_{ij}(G(p)(x)) = \sigma_{ij}(G(p))(x)$. In a similar way we can prove that $H(\sigma_{ij}(p))(x) = \sigma_{ij}(H(p))(x)$.

■

Remark 1.20 *In 2mLM$_{n\times m}$-algebra* (L^X, G^*, H^*) *the modal operators* P^* *and* F^* *are defined in the following way: for any* $p \in L^X$*,* $x \in X$*,*

$$(P^*(p))(x) = \bigvee\{p(y) \mid y \in X, yRx\} \tag{11}$$

$$(F^*(p))(x) = \bigvee\{p(y) \mid y \in X, yQx\} \tag{12}$$

Definition 1.21 *Let* $(\mathcal{L}, G, H)$ *and* $(\mathcal{L}', G', H')$ *be two 2mLM$_{n\times m}$–algebras. A function* $f : L \longrightarrow L'$ *is a morphism of 2mLM$_{n\times m}$–algebras if* f *is an LM$_{n\times m}$–algebra morphism and satisfies the conditions:* $f(G(x)) = G'(f(x))$ *and* $f(H(x)) = H'(f(x))$*, for any* $x \in L$*.*

Remark 1.22 *Let $f : L \longrightarrow L'$ be a morphism of $2mLM_{n\times m}$-algebras. Then, by* (C5), *we can prove that*

$$x \in C(L) \Longrightarrow f(x) \in C(L').$$

According to the previous remark we can consider the function $C(f) = f \mid_{C(L)}: C(L) \longrightarrow C(L')$. It follows that $C(f)$ is a morphism of weak-tense Boolean algebras.

We will denote by $\mathbf{2m\mathfrak{TLM}}_{n\times m}$ the category of $2\mathrm{mLM}_{n\times m}$-algebras. Then, the assignment $L \longmapsto C(L)$, $f \longmapsto C(f)$ defines a covariant functor $C : \mathbf{2m\mathfrak{LM}}_{n\times m} \longrightarrow \mathfrak{WTB}$.

2 Representation theorem for $2\mathrm{mLM}_{n\times m}$- algebras

In this section we give a representation theorem for $2\mathrm{mLM}_{n\times m}$–algebras. In order to prove it we use the representation theorem for weak-tense Boolean algebras.

Let $(\mathcal{B}, G, H)$ be a weak-tense Boolean algebra. We consider the set of all increasing functions in each component from $(n \times m)$ to B, that is,

$$D(B) = B\uparrow^{(n\times m)} = \{f : (n \times m) \longrightarrow B \text{ such that for arbitrary } i, j,$$
$$\text{if } r \leq s, \text{ then } f(r, j) \leq f(s, j) \text{ and } f(i, r) \leq f(i, s)\}.$$

We define the unary operations $D(G)$ and $D(H)$ on $D(B)$ by:

$$D(G)(f) = G \circ f \text{ and } D(H)(f) = H \circ f \text{ for all } f \in D(L).$$

The following result is necessary for the proof of Theorem 2.9.

Lemma 2.1 *If $(\mathcal{B}, G, H)$ is a weak-tense Boolean algebra then*

$$(D(B), D(G), D(H))$$

is a $2mLM_{n\times m}$–algebra.

Proof. By (LM6), $D(B)$ is an $\mathrm{LM}_{n\times m}$–algebra. We will prove that $D(G)$ and $D(H)$ verify (T1)–(T3) in Definition 1.14.

(T1): Let $f \in D(B)$ and $(i, j) \in (n \times m)$. Then $D(G)(1_{D(B)})(i, j) = (G \circ 1_{D(B)})(i, j) = G(1_{D(B)})(i, j) = G(1_B) = 1_B$; hence, $D(G)(1_{D(B)}) = 1_{D(B)}$.

(T2): Let $f, g \in D(B)$ and $(i, j) \in (n \times m)$. We have that: $D(G)(f \wedge g)(i, j) = (G \circ (f \wedge g))(i, j) = G((f \wedge g)(i, j)) = G(f(i, j) \wedge g(i, j)) = Gf(i, j) \wedge Gg(i, j) = (G \circ f)(i, j) \wedge (G \circ g)(i, j) = D(G)(f)(i, j) \wedge D(G)(g)(i, j) = (D(G)(f) \wedge D(G)(g))(i, j)$, so $D(G)(f \wedge g) = D(G)(f) \wedge D(G)(g)$.

(T3): Let $f \in D(B)$ and $(i,j),(r,s) \in (n \times m)$. Then $D(G)(\sigma_{rs}(f)(i,j)) = (G \circ (\sigma_{rs} f))(i,j) = G((\sigma_{rs} f)(i,j)) = Gf(r,s) = (G \circ f)(r,s) = D(G)(f)(r,s) = \sigma_{rs}(D(G)(f))(i,j)$, so $D(G)(\sigma_{rs}) = \sigma_{rs}(D(G))$.

■

Definition 2.2 *Let $(\mathcal{B}, G, H)$, $(\mathcal{B}', G, H)$ be two weak-tense Boolean algebras, $f : B \longrightarrow B'$ a weak-tense Boolean morphism and $D(B)$ and $D(B')$ the corresponding $2mLM_{n\times m}$–algebras. We will extend the function f to a function $D(f) : D(B) \longrightarrow D(B')$ in the following way: $D(f)(u) = f \circ u$, for every $u \in D(B)$.*

Lemma 2.3 *The function $D(f) : D(B) \longrightarrow D(B')$ is a morphism of $2mLM_{n\times m}$–algebras.*

Proof. Since f is a Boolean morphism, it is easy to prove that $D(f)$ is a bounded lattice homomorphism. Let $u \in D(B)$ and $(i,j),(r,s) \in (n \times m)$. Then, we have that

$$D(f)(\sigma_{rs} u)(i,j) = f((\sigma_{rs} u)(i,j)) = f(u(r,s))$$

and $\sigma_{rs}(D(f)(u))(i,j) = D(f)(u)(r,s) = f(u(r,s))$. It follows that $D(f) \circ \sigma_{rs} = \sigma_{rs} \circ D(f)$. Besides,

$D(f)(D(G)u)(r,s) = (f \circ (D(G)u))(r,s) = f((D(G)u)(r,s)$. ■

Lemma 2.4 *If $f : B \longrightarrow B'$ is an injective morphism of weak-tense Boolean algebras then $D(f) : D(B) \longrightarrow D(B')$ is an injective morphism of $2mLM_{n\times m}$–algebras.*

Proof. By Lemma 2.3, it remains to prove that $D(f)$ is injective. Let $u, v \in D(B)$ such that $D(f)(u) = D(f)(v)$, then $f(u(i,j)) = f(v(i,j))$ for all $(i,j) \in (n \times m)$. Since f is injective, we have that $u(i,j) = v(i,j)$, for all $(i,j) \in (n \times m)$; therefore, $u = v$. ■

Definition 2.5 *Let (L, G, H) be a $2mLM_{n\times m}$–algebra. We consider the function $\tau_L : L \longrightarrow D(C(L))$, defined by*

$$\tau_L(x)(i,j) = \sigma_{ij}(x)$$

for all $x \in L$, $(i,j) \in (n \times m)$.

Lemma 2.6 *τ_L is an injective morphism in* $\mathbf{2mLM}_{n\times m}$.

Proof. Taking into account [22, Theorem 3.1], the mapping $\tau_L : L \longrightarrow D(C(L))$ is a one-to-one $\mathrm{LM}_{n\times m}$–morphism. Besides, from (T3) it is simple to check that τ_L commutes with the modal operators G and H. ■

Definition 2.7 *Let (X, R, Q) be a weak-frame and $(\mathbf{2}^X, G, H)$ the weak-tense Boolean algebra of Proposition 1.5. We consider the function*

$$\beta : (D(\mathbf{2}^X), D(G), D(H)) \longrightarrow (D(\mathbf{2})^X, G', H')$$

, defined by: $\beta(f)(x)(i,j) = f(i,j)(x)$ for all $f \in D(\mathbf{2}^X)$, $x \in X$, $(i,j) \in (n \times m)$, where G' and H' are defined by: $G'(p)(x) = \bigwedge\{p(y) \mid y \in X, xRy\}$ and $H'(p)(x) = \bigwedge\{p(y) \mid y \in X, xQy\}$.

Lemma 2.8 *β is an isomorphism of 2mLM$_{n\times m}$–algebras.*

Proof. It is easy to see that β is an injective morphism of LM$_{n\times m}$–algebras. It remains to prove that β commutes with the modal operators.

Let $f \in D(\mathbf{2}^X)$, $x \in X$ and $(i,j) \in (n \times m)$. We have:

(a) $\beta(D(G)(f))(x)(i,j) = D(G)(f)(i,j)(x) = G(f(i,j))(x) = \bigwedge\{f(i,j)(y) \mid y \in X, xRy\}$.

(b) $G'(\beta(f))(x)(i,j) = \bigwedge\{\beta(f)(y)(i,j) \mid y \in X, xRy\} = \bigwedge\{f(i,j)(y) \mid y \in X, xRy\}$.

By (a) and (b), we obtain that $\beta(D(G)(f))(x)(i,j) = G'(\beta(f)(x))(i,j)$, so $\beta \circ D(G) = G' \circ \beta$. We define the function $\gamma : D(\mathbf{2})^X \to D(\mathbf{2}^X)$ by: $\gamma(g)(i,j)(x) = g(x)(i,j)$, for all $g \in D(\mathbf{2})^X$, $x \in X$, $(i,j) \in (n \times m)$. Let $r \leq s$. For all $x \in X$, we have that $g(x) \in D(\mathbf{2})$, so $g(x)(r,j) \leq g(x)(s,j)$ and $g(x)(i,r) \leq g(x)(i,s)$. It follows that $\gamma(g)(r,j)(x) \leq \gamma(g)(s,j)(x)$ and $\gamma(g)(i,r)(x) \leq \gamma(g)(s,j)(x)$ for all $x \in X$, so $\gamma(g)(r,j) \leq \gamma(g)(s,j)$ and $\gamma(g)(i,r) \leq \gamma(g)(s,j)$. Hence, γ is well defined. We will prove that β and γ are inverse to each other. Let $g \in D(\mathbf{2})^X$, $x \in X$ and $(i,j) \in (n\times m)$. We have that: $(\beta \circ \gamma)(g)(x)(i,j) = \beta(\gamma(g))(x)(i,j) = \gamma(g)(i,j)(x) = g(x)(i,j)$, hence $(\beta \circ \gamma)(g) = g$. Let $f \in D(\mathbf{2}^X)$, $(i,j) \in (n \times m)$ and $x \in X$. Then $(\gamma \circ \beta)(f)(i,j)(x) = \gamma(\beta(f))(i,j)(x) = \beta(f)(x)(i,j) = f(i,j)(x)$, so $(\gamma \circ \beta)(f) = f$.

■

Theorem 2.9 *For every 2mLM$_{n\times m}$–algebra (L, G, H) there exist a weak-frame (X, R, Q) and an injective morphism of 2mLM$_{n\times m}$–algebras $\alpha : (L, G, H) \longrightarrow (D(\mathbf{2})^X, G', H')$.*

Proof. Let (L, G, H) be a 2mLM$_{n\times m}$–algebra. By Remark 1.15 we have that

$$(C(L), C(G), C(H))$$

is a weak-tense Boolean algebra. By applying the representation theorem for weak-tense Boolean algebras, it follows that there exist a weak-frame (X, R, Q) and an injective morphism of weak-tense Boolean algebras $d : (C(L), C(G), C(H)) \longrightarrow (\mathbf{2}^X, G, H)$. Let $D(d) : D(C(L)) \longrightarrow D(\mathbf{2}^X)$ be the corresponding morphism of d by the morphism D. Then, by Lemma 2.4, $D(d)$ is an injective morphism. On the other hand, by using Lemma 2.6, we have an injective morphism of $2\mathrm{mLM}_{n\times m}$–algebras $\tau_L : L \longrightarrow D(C(L))$. Besides, by Lemma 2.8, $\beta : D(\mathbf{2}^X) \longrightarrow D(\mathbf{2})^X$ is an isomorphism of $2\mathrm{mLM}_{n\times m}$–algebras. Now, in the following diagram,

$$L \xrightarrow{\tau_L} D(C(L)) \xrightarrow{D(d)} D(\mathbf{2}^X) \xrightarrow{\beta} D(\mathbf{2})^X$$

if we consider the composition $\beta \circ D(d) \circ \tau_L$ we obtain the required injective morphism. ■

Corollary 2.10 *For every weak-tense LM_n–algebra (L, G, H) there exist a weak-frame (X, R, Q) and an injective morphism of weak-tense LM_n–algebras $\Phi : (L, G, H) \to (Ł_n^X, G', H')$.*

Proof. It is an immediate consequence of Remark 1.15 ii, Theorem 2.9 and (LM9). ■

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Łukasiewicz - Moisil Algebras*, volume 49 of *Annals of Discrete Mathematics*, North-Holland, 1991.

[2] M. Botur, I. Chajda, R. Halaš and M. Kolařik, Tense operators on Basic Algebras. *Internat. J. Theoret. Phys.*, 50(12):3737–3749, 2011.

[3] M. Botur, J.Paseka, Tense MV–algebras. *arXiv: 1305.3406 [math.AC].*

[4] C. Chiriţă, Tense θ–valued Moisil propositional logic, *Int. J. of Computers, Communications and Control*, 5:642–653, 2010.

[5] C. Chiriţă, Tense θ–valued Łukasiewicz–Moisil algebras. *J. Mult. Valued Logic Soft Comput.*, 17(1):1–24, 2011.

[6] C. Chiriţă, Polyadic tense θ–valued Łukasiewicz–Moisil algebras. *Soft Computing* 16(6):979–987, 2012.

[7] C. Chiriţă, Tense Multiple-Valued Logical Systems. PhD Thesis, University of Bucharest, Bucharest, 2012.

[8] D. Diaconescu and G. Georgescu, Tense operators on MV-algebras and Łukasiewicz-Moisil algebras. *Fund. Inform.* 81(4):379–408, 2007.

[9] A. V. Figallo and G. Pelaitay. Note on tense $SHn-$algebras. *An. Univ. Craiova Ser. Mat. Inform.*, 38(4):24–32, 2011.

[10] A. V. Figallo and G. Pelaitay. Remarks on Heyting algebras with tense operators. *Bulletin of the Section of Logic*, 41(1-2):71–74, 2012.

[11] A. V. Figallo and G. Pelaitay. Tense operators on De Morgan algebras. *Logic Journal of the IGPL*, 22(2): 255–267, 2014.

[12] A. V. Figallo and G. Pelaitay. Discrete duality for Tense Lukasiewicz-Moisil algebras. Accepted for publication in *Fundamenta Informaticae.*

[13] A. V. Figallo and G. Pelaitay. An algebraic axiomatization of Ewald's Intuitionistic Tense Logic. *Soft Computing* (2014), DOI 10.1007/s00500-014-1317-6.

[14] A. V. Figallo, G. Pelaitay and C. Sanza. Discrete duality for $TSH-$algebras. *Commun. Korean Math. Soc.* 27(1): 47–56, 2012.

[15] A. V. Figallo, C. Gallardo and G. Pelaitay. Tense operators on m-symmetric algebras. *International Mathematical Forum*, 6(41-44):2007–2014, 2011.

[16] A. V. Figallo and C. Sanza. The $\mathcal{NS}_{n\times m}-$propositional calculus. *Bulletin of the Section of Logic*, 37(2): 67–79, 2008.

[17] A. V. Figallo and C. Sanza, *Monadic $n \times m$-Lukasiewicz-Mosil Algebras. Mathematica Bohemica*, 137(4):425–447, 2012.

[18] G. Georgescu. A representation theorem for tense polyadic algebras. *Mathematica*, Tome 21 (44), No. 2, 131–138, 1979.

[19] J. Paseka, Operators on MV–algebras and their representations. *Fuzzy Sets and Systems*, 232:62–73, 2013. DOI 10.101/j.fss.2013.02.010.

[20] C. Sanza. Notes on $n\times m$-valued Lukasiewicz algebras with negation. *Logic Journal of the IGPL*, 12(6): 499–507, 2004.

[21] C. Sanza, Algebras de Łukasiewicz $n \times m-$valuadas con negación. PhD Thesis, Universidad Nacional del Sur, Bahía Blanca, Argentina, 2005.

[22] C. Sanza. $n \times m-$valued Łukasiewicz algebras with negation. *Reports on Mathematical Logic*, 40:83–106, 2006.

[23] C. Sanza. On $n \times m-$valued Lukasiewicz-Moisil algebras. *Cent. Eur. J. Math.*, 6(3):372–383, 2008.

[24] W. Suchoń, *Matrix Łukasiewicz Algebras. Reports on Mathematical Logic*, 4:91–104, 1975.

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