South American Journal of Logic
Vol. 1, n. 1, pp. 249–265, 2015
ISSN: 2446-6719

On Existence in Set Theory, Part III: Applications to New Axioms

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Abstract

The present paper applies the conceptual apparatus developed in [1] and [2] to new axioms, and devises new tools for an analysis of the search for new axioms. Beginning with a discussion of the role of the axioms for a set theory, the distinction between existence axioms and axioms of the nature of sets is reconsidered, and definition 1.1 gives a precise meaning for "axiom of the nature of sets". After this preliminary discussion, the present paper treats the comparison among new axioms, and among collections of new axioms, from the point of view of existence. The conceptual apparatus is then applied to analyzing the theory obtained from ZFC by adding the axiom *"there is a measurable cardinal"*, and the axiom $\mathbf{V} = \mathbf{L}[U]$, where $\mathbf{L}[U]$ is the unique inner model of measurability associated with the first measurable cardinal, if there is one.

1 Introduction

A presentation of a formal system for ZFC set theory is usually intended to describe a hierarchy of sets. According to a well-known intuitive description, in an iterative, cumulative hierarchy, sets are "produced" in levels, and there is no maximum for the plurality of levels. At a given level, all sets produced in earlier levels, and only those, are available as elements for the production of new sets. One can easily identify two separate aspects of this intuitive description: (i) that the passage from one level to the next is given through the production of all sets that can be constituted with the elements already available; and (ii) that there is no bound for these levels.

According to the above description, the axiomatization of set theory is articulated in terms of the notion of "production" of sets in levels, within a hierarchy. In other words, the axioms for set theory can be understood as an attempt at specification (a) of sets as objects related by a well-founded and extensional membership relation, and (b) of the production of sets. The axioms of extensionality and regularity are those directly related to (a), and, because of this, they are understood as "axioms of the nature of sets". The other axioms stand in relation to (b), and are usually understood as "existence axioms". In this way, if the first group of axioms is changed, then, intuitively, the resulting axiomatization talks about objects of a different nature, whereas if the first group is fixed and the second one is changed, then the resulting axiomatization only gives a different answer to the question "what sets are there?". In [6], page 103, Löwe and Steel explain the axiomatization of set theory as follows:

> Zermelo-Fraenkel set theory with choice, or ZFC, is the commonly accepted system of axioms for set theory, and hence for all of mathematics. Most of the axioms of ZFC express closure properties of the universe of sets. (The exceptions are Extensionality and Foundation, which in effect limit the objects under consideration.) Although all mathematical assertions can be expressed in the language of ZFC, and "most" of them can be decided using only the axioms of ZFC, there are nevertheless interesting mathematical assertions which cannot be decided using ZFC alone. The most famous of these is the Continuum Hypothesis.

The analysis of existence assertions contained in Parts I ([1]) and II ([2]) showed that although the distinction "nature vs. existence" is meaningfull, and can be made precise, there is no dichotomy here. Extensionality is both an axiom of the nature of sets and a weak existence axiom. Consider the following:

Definition 1.1 *The sentence A in $L(ZF)$ is a sentence* of the nature of sets *if A is consistent with ZFC, and admits conditional degree* 1 *of existence requirement relative to the simple context A.*[1]

According to this definition, to say that A is a sentence of the nature of sets means that, assuming A, it follows that A is valid in every transitive domain, including the empty one. This is motivated by the fact that a sentence, assumed to be valid, expresses no more than an aspect of the nature of sets if and only if it holds in every domain of (set) existence that is coherent with this very nature, that is, in every "model" of this nature. In order to see how this motivates definition 1.1, recall that the nature of sets is given by the membership relation: Sets are objects related by a well-founded, extensional membership relation. Now, it is sufficient to notice that the domains of existence that are coherent with the nature of sets are exactly the transitive ones. One way of seeing this is by means of Mostowski's Collapsing Lemma ([5], page 70): A domain with a well-founded, extensional membership relation is isomorphic to a transitive domain (with true membership). Therefore, the "models" of the nature of sets are the transitive domains, including the empty one.

[1] Since A does not contain the class variable I, it is indeed a simple context.

However, it is also possible to argue directly for the identification between domains of existence coherent with the nature of sets and transitive domains. In fact, in any reasonable domain of existence, if a set exists, then its elements must also exist. The hypothesis that a set may exist, but some of its elements may not, is incoherent with respect to the nature of sets. A set is the collection of its elements, and one such object cannot exist unless all its elements do exist. In other words, any reasonable domain of existence is transitive. On the other hand, any transitive domain is coherent with the nature of sets. Indeed, the statement that a set is the collection of its elements only talks about closure under taking elements. In conclusion, there is nothing in the nature of the membership relation excluding the possibility that the sets that do exist are only those in a given transitive domain. Furthermore, there is nothing in the structure of membership implying that there are sets at all, and it follows that the empty domain is also a reasonable domain, coherent with the nature of sets. Therefore, from the fact that a sentence, assumed to be valid, expresses no more than an aspect of the nature of sets if and only if it holds in every reasonable domain of existence, it follows that definition 1.1 is the correct specification for the intuitive notion of "a valid sentence of the nature of sets".

In this sense, extensionality and regularity are, indeed, the only axioms of the nature of sets. Furthermore, it was shown in [1] that if A is a sentence of the nature of sets, such that A is proved consistent with ZFC by a method of extension of transitive models in ZFC (like forcing without further assumptions), then A is already a theorem of ZFC. Therefore, ZFC is somewhat complete with respect to sentences of the nature of sets.

On the other hand, from the fact that extensionality (or regularity) is an axiom of the nature of sets, it does not follow that it is not an existence axiom. Being an axiom of the nature of sets and being a nonexistence axiom[2] are not the same thing. As was shown in [1] and [2], the axiom of extensionality also expresses a closure property of the universe: Its transitivity. This is a very weak closure property, nevertheless, it is a closure property, and cannot be neglected.

At the end of the paragraph quoted above, Löwe and Steel mention the incompleteness of ZFC with respect to "interesting mathematical assertions". Of course, any consistent axiomatized set theory will be incomplete because of Gödel's first incompleteness theorem, but that is not what Löwe and Steel are talking about. The incompleteness of ZFC is not due to Gödel's theorem alone: This theory allows many transitive models. All those transitive models agree with respect to arithmetic statements (like Gödel's sentence). ZFC suffers from a much more dramatic incompleteness.

As was mentioned above, the axioms of ZFC have the task of specifying (a) the sets

[2]Recall definition 40 in [1]: A is a nonexistence assertion if and only if A is a conditional constructive nonproductive assertion, which means that the relativization of A to whatever domain (including the empty one) is valid.

as objects related by an extensional and well-founded membership relation, and (b) the production of sets, or, in the words of Löwe and Steel, they have the task of limiting the objects under consideration and of expressing closure properties of the universe, respectively. Since the axioms of ZFC do, indeed, restrict the objects under consideration by specifying their nature, it is task (b) that, presumably, is left incomplete by those axioms, and this is the source of the incompleteness that Löwe and Steel are talking about.

According to this view, the search for new axioms does not aim at changing the nature of sets, but at giving a different, presumably more complete, answer to the question "what sets are there?". This question can be further analyzed in two components, according to the aspects (i) and (ii) of the intuitive description given in the very beginning of this section: (i') "how exactly does the production of sets occur at each level?" and (ii') "which levels of the hierarchy do exist?". The search for new axioms can then be understood as an attempt at answering (i') and (ii') in a more satisfactory way.

Therefore, it seems that the systematic analysis of existence and related notions established so far may be useful in order to compare different answers to (i') and (ii'). The aim of the present paper is exactly this: To make use of the apparatus presented in [1] and [2], and to develop new tools for analyzing and comparing collections of new axioms, such as large cardinal axioms, forcing axioms and inner models axioms.

2 Evaluation of New Axioms

The relative degrees of existence requirement and the upper semilattice R were introduced in Part II ([2]).[3] This structure emerged from the analysis of the notion of productivity. Recall that the productivity of a (valid) sentence is, roughly, its power of producing sets in domains of existence. This existential power can be measured by the closure property that a domain should have in order to fulfill the existence requirement of the sentence in question. In order to evaluate the productivity of a sentence, a natural hierarchy of closure properties for domains was used.

[3]Recall that the upper semilattice R, as defined in [2], is obtained as follows. First, define the following pre-ordering on pairs of the form (d, φ), in which $d \in \{0, 1, 2, 3, 4, 5\}$ is an absolute degree and φ is a context:

$$(d, \varphi) \leq_0 (d', \varphi') \text{ if and only if } T^* \vdash \varphi' \wedge C^{d'}(I) \to \varphi \wedge C^d(I).$$

Next, take the quotient by the equivalence relation associated with $\leq_0$: $(d, \varphi) \equiv_0 (d', \varphi')$ if and only if $(d, \varphi) \leq_0 (d', \varphi')$ and $(d', \varphi') \leq_0 (d, \varphi)$. This quotient gives an ordering that is also denoted by $\leq_0$. The semilattice R is constituted by the classes corresponding to simple degrees, that is, classes of the form $[(d, \varphi)]_0$, in which φ is simple, ordered by $\leq_0$. Notice that there is a partial jump operation in R: $[(d, \varphi)]_0 \mapsto [(d + 1, \varphi)]_0$, if $d \prec 5$.

Any sentence in $L(ZF)$ that is consistent with ZFC can, in principle, be considered as a candidate for a new axiom. However, this maximum generality is not desirable here, for at least two reasons: First, the point of a new axiom is not to simply add something true but independent of ZFC, and second, set theorists are focused on very few groups of axioms. With respect to the first point, the search for new axioms is considered here as a program with a very specific target: It aims at providing more satisfactory answers to (i') "how exactly does the production of sets occur at each level?" and (ii') "which levels of the hierarchy do exist?". Regarding the second point, if this analysis is to be useful, it must add some insight and organization to what is actually going on in this program for new axioms, which is something very difficult to achieve with maximum generality.

Therefore, it is desirable to restrict the collection of sentences that can potentially be considered as new axioms, in a way that causes no loss with respect to the first or to the second points raised above. Fortunately, this is possible in the present analysis. Recall that a sentence A admits degree 0 of existence requirement relative to a simple context φ if, under the hypothesis that φ, A holds in every nonempty domain. Since a simple context can only require from a domain that it contains some sets $x_1, ..., x_n$, this basically means that A makes no existential demands on nonempty domains containing the sets $x_1, ..., x_n$, as required by φ, and hence it expresses no closure property and no productivity. Therefore, A is of no help with respect to both (i') "how exactly does the production of sets occur at each level?" and (ii') "which levels of the hierarchy do exist?". Furthermore, nobody has proposed a new axiom of this kind, and hence these sentences can be ignored without prejudice for the present analysis.

Taking these considerations into account, "A is a new axiom" is understood here as meaning that A is a sentence in $L(ZF)$ that is consistent with ZFC, and such that A does *not* admit degree 0 of existence requirement relative to a simple context. In other words, A cannot be classified in the semilattice R with a degree $[(0, \varphi)]_0$, for some simple context φ.

As was remarked shortly after definition 1.1, if A is a sentence of the nature of sets,[4] and is such that A is proved consistent with ZFC by a method of extension of transitive models in ZFC, then A is in fact a theorem of ZFC. Although this indicates that sentences A that admit degree 1 relative to A itself are not very interesting as new axioms, there are sentences A that admit degree 1 relative to a simple context (that is not A) that are interesting. In any case, it is harmless to consider the sentences that already are theorems, and therefore sentences that admit degree 1 relative to a simple context are considered as potential new axioms.

The theory T^* used throughout in [2] is the theory obtained from ZFC by the introduction of *unquantifiable* class variables, as described in [5] (in the first chapter),

[4]For example, any sentence of the form $\forall x_1 ... \forall x_n B$, such that all quantifiers in B are bounded, admits degree 1 relative to itself.

and by adding the appropriate relation and function symbols. T^* is a conservative extension of ZFC. Recall that the main closure properties are denoted by $C^0(I)$, $C^1(I)$, $C^2(I)$, $C^3(I)$, $C^4(I)$ and $C^5(I)$, and that these properties are linearly ordered by strength. For example, $C^4(I)$ means that I is a limit level, and $C^5(I)$ means that I is the universe $\mathbf{V}$. In order to use these concepts in the evaluation of new axioms, it is enough to remark that the new axioms can occur as contexts, and, indeed, as simple contexts, since they say nothing about the domain I. It is assumed, tacitly, that $\mathbf{V}$ has all closure properties, that is, for each $d \in \{0,1,2,3,4,5\}$, $T^* \vdash C^d(\mathbf{V})$. This will be important in what follows.

The main definition in [2] is the following:[5]

Definition 2.1 *Let T^* be the theory described above, and $\varphi(I)$ be a context in T^*. The sentence A in $L(ZF)$ is said to admit the following relative degrees of existence requirement:*

- *degree 0 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in every (nonempty) $\in$-interpretation of $L(ZF)$ in T. That is, if*

$$T^* \vdash \varphi(I) \land \exists x(x \in I) \to A^I;$$

- *degree 1 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in every transitive $\in$-interpretation of $L(ZF)$ in T. That is, if*

$$T^* \vdash \varphi(I) \land \exists x(x \in I) \land \forall x \forall y(x \in I \land y \in x \to y \in I) \to A^I;$$

- *degree 2 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in every supertransitive $\in$-interpretation of $L(ZF)$ in T. That is, if*

$$T^* \vdash \varphi(I) \land \exists x(x \in I) \land \forall x \forall y(x \in I \land (y \in x \lor y \subset x) \to y \in I) \to A^I;$$

- *degree 3 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in every $\in$-interpretation of $L(ZF)$ in T that is a level V_α. That is, if*

$$T^* \vdash \varphi(I) \land \exists x(x \in I) \land Ord(\alpha) \land \forall x(x \in I \leftrightarrow x \in V_\alpha) \to A^I,$$

[5] The clause corresponding to degree 4ω is omitted in view of remark 22 in [2].

in which $Ord(\alpha)$ stands for "α is an ordinal";

- *degree 4 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in every $\in$-interpretation of $L(ZF)$ in T that is a level V_α for α a limit ordinal. That is, if*

$$T^* \vdash \varphi(I) \wedge \exists x(x \in I) \wedge LimOrd(\alpha) \wedge \forall x(x \in I \leftrightarrow x \in V_\alpha) \rightarrow A^I,$$

  in which $LimOrd(\alpha)$ stands for "α is a limit ordinal";

- *degree 5 of existence requirement relative to the context $\varphi(I)$ in T^*, if, under the assumption of $\varphi(I)$, the sentence A holds in the identity interpretation $\boldsymbol{V}$ of $L(ZF)$ in T. That is, if*

$$T^* \vdash \varphi(I) \wedge \forall x(x \in I) \rightarrow A^I.$$

For the sentence A and context φ, the least degree admitted by A relative to φ is said to be the *existence requirement* of A relative to φ, and is denoted by $r(A|\varphi)$. The following results classify some important new axioms according to the existence requirement relative to a simple context.

Proposition 2.2 *If φ is a simple context, then the existence requirement of $\boldsymbol{V} = \boldsymbol{L}$ relative to the simple context $\varphi \wedge \boldsymbol{V} = \boldsymbol{L}$ is 5, that is, $r(\boldsymbol{V} = \boldsymbol{L}|\varphi \wedge \boldsymbol{V} = \boldsymbol{L}) = 5$.*

Proof. It suffices to show that, given an infinite ordinal β, $\mathbf{V} = \mathbf{L}$ cannot hold in all V_α, where α is a limit ordinal such that $\alpha > \beta$. The transitive sets V_α, for α a limit ordinal, with $\alpha > \omega$ and $cf(\alpha) > \omega$, are *adequate* in the sense of [3], page 110.[6] For those sets, the function $\xi \mapsto L_\xi$ is absolute. Therefore, V_α satisfies $V = L$ if and only if $V_\alpha = L_\alpha$. However, $V_\alpha = L_\alpha$ implies that $\alpha = |L_\alpha| = |V_\alpha| = \beth_\alpha$. [7] Since this equality cannot hold for each limit ordinal α above a given β with $cf(\alpha) > \omega$,[8] $\mathbf{V} = \mathbf{L}$ cannot hold in all V_α, where α is a limit ordinal such that $\alpha > \beta$.

■

[6]In order to see this, first notice that Gödel's operations cannot increase rank by an infinite ordinal. Since α is a limit, it follows that V_α is closed. Also, if $x \in V_\alpha$, then the closure of x increases rank at most by ω: Assuming that $rank(x) = \beta$, for $\beta < \alpha$, the closure of x is in $V_{\beta+\omega+1}$. Since $cf(\alpha) > \omega$, it follows that $\beta + \omega + 1 < \alpha$, and the closure of x is in V_α. Finally, if $\gamma < \alpha$ and $\beta < \gamma$, then $L_\beta \in V_\gamma$. Therefore, the sequence $\langle L_\beta : \beta < \gamma \rangle$ is in $\wp(\gamma \times V_\gamma)$. Since α is a limit, it follows that $\wp(\gamma \times V_\gamma) \in V_\alpha$, and, by the transitivity of V_α, $\langle L_\beta : \beta < \gamma \rangle \in V_\alpha$.

[7]Moreover, assuming $\mathbf{V} = \mathbf{L}$ as a context, $V_\alpha = L_\alpha$ if and only if $\alpha = \beth_\alpha$ (see [4], page 180, exercise (3)).

[8]If $\alpha = \beth_\alpha$, then it is easy to see that α is a strong limit cardinal. Therefore, if $\alpha = \omega_{\gamma+1} > \beta$, for an ordinal γ, then α is a limit ordinal and $cf(\alpha) = \alpha > \omega$. Furthermore, since $\omega_{\gamma+1}$ is not a strong limit cardinal, α is different from $\beth_\alpha$.

Proposition 2.3 *Let X be a set. If φ is a simple context, then the existence requirement of $\boldsymbol{V} = \boldsymbol{L}[X]$ relative to the simple context $\varphi \wedge \boldsymbol{V} = \boldsymbol{L}[X]$ is 5, that is, $r(\boldsymbol{V} = \boldsymbol{L}[X] | \varphi \wedge \boldsymbol{V} = \boldsymbol{L}[X]) = 5$.*

Proof. The proof is similar to that of proposition 2.2. It suffices to show that, given an infinite ordinal $\beta > rank(X)$, $\mathbf{V} = \mathbf{L}[X]$ cannot hold in all V_α, where α is a limit ordinal such that $\alpha > \beta$. The transitive sets V_α, for α a limit ordinal with $\alpha > \omega$ and $cf(\alpha) > \omega$, are X-*adequate* in the sense of [3], page 129. For those sets, the function $\xi \mapsto L_\xi[X]$ is absolute. Therefore, V_α satisfies $\mathbf{V} = \mathbf{L}[X]$ if and only if $V_\alpha = L_\alpha[X]$. Taking cardinalities in this equality, $\beth_\alpha = |V_\alpha| = |L_\alpha[X]| = |\alpha|$. Since this equality cannot hold for each limit ordinal α above a given β with $cf(\alpha) > \omega$, $\mathbf{V} = \mathbf{L}[X]$ cannot hold in all V_α, where α is a limit ordinal such that $\alpha > \beta$. ∎

Propositions 2.2 and 2.3 show that $\mathbf{V} = \mathbf{L}$ admits only relative degrees of the form $[(5, \varphi \wedge \mathbf{V} = \mathbf{L})]_0$, and that $\mathbf{V} = \mathbf{L}[X]$ admits only relative degrees of the form $[(5, \varphi \wedge \mathbf{V} = \mathbf{L}[X])]_0$, respectively. These axioms are strongly productive, as shown above: They express a closure property which is truly about $\mathbf{V}$. This kind of axiom basically answers the question (i') "how exactly does the production of sets occur at each level?", and it can also go halfway towards answering (ii') "which levels of the hierarchy do exist?". The addition of the axiom $\mathbf{V} = \mathbf{L}$, for example, causes no change in the nature of the sets, and provides a set theory which is as categorical as a first order set theory can be. The resulting theory, denoted by ZFL, has the condensation property and its transitive models[9] are either L_α, for some ordinal α, or $\mathbf{L}$. Therefore, one cannot add sets to enlarge a transitive model of ZFL, without violating the axioms of this theory: ZFL cannot admit two different answers to question (i'). Furthermore, the resulting theory cannot admit the existence of a measurable cardinal, and hence goes halfway towards answering question (ii'). If one is seeking for completeness, this is the kind of axiom one must consider.

Proposition 2.4 *Let A be the standard sentence expressing "there is a measurable cardinal", and let μ be a constant for the least measurable cardinal. If φ is the simple context $A \wedge (\wp(\wp(\mu)) \in I)$, then $r(A|\varphi) = 1$.*

Proof. Let U be a μ-complete nonprincipal ultrafilter over μ. Assuming φ, it follows that $U \in I$ and, since I is transitive, all elements of U are in I. In particular, if $\langle X_\alpha : \alpha < \mu \rangle$ is a sequence in I such that $X_\alpha \in U$, then $\bigcap_{\alpha<\mu} X_\alpha \in U$, and $\bigcap_{\alpha<\mu} X_\alpha \in I$,

[9] The only reasonable models of a set theory are the transitive ones, since it is absurd to assume that a set exists, but that some of its elements do not. Indeed, the models that are not even isomorphic to a transitive one are non-well-founded. But this means that, from the "outside" perspective, the members of the model are not sets in the usual sense: They are objects related by what is in fact a non-well-founded membership relation.

which shows that U is μ-complete in I. Therefore, A admits degree 1 relative to the simple context $A \wedge (\wp(\wp(\mu)) \in I)$. Since μ is an infinite ordinal, A cannot admit degree 0 relative to a simple context, as shown in [2]. ■

Proposition 2.4 shows that the existence of a measurable cardinal is not a very strong productive assertion, which is a plausible outcome. It does not express a strong closure property, but only the existence of a specific set. Qualitatively, it is *an unconditional nonconstructive productive assertion*, which is the only one of the six, mutually exclusive, qualitative classes of assertions introduced in [1] that is not populated by an axiom of ZFC. This makes this axiom a very interesting potential complement of the axioms of ZFC. If ZFM denotes the theory $ZFC+$ *"there is a measurable cardinal"*, then for all kinds of existence there is an axiom of ZFM expressing it, and there are axioms classified in all different degrees of existence requirement (as in ZFC).

Proposition 2.5 *Let A be the sentence $MA(\aleph_1)$, expressing Martin's axiom. If φ is the simple context $MA(\aleph_1) \wedge \aleph_1 \in I$, then $r(A|\varphi) = 2$.*

Proof. If P is a partially ordered set in I and D is a collection of less than $\aleph_1$ dense subsets of P in I, then there is a D-generic filter G contained in P and, since I is supertransitive, G is in I. Furthermore, from the supertransitivity of I, it follows that G is a D-generic filter in I.[10] This shows that $MA(\aleph_1)$ admits degree 2 relative to the simple context $MA(\aleph_1) \wedge \aleph_1 \in I$. Transitivity alone is clearly insufficient for this. In fact, one can take a transitive set containing $\aleph_1$, P and the family D, such that P is disjoint from $\aleph_1$ and that the dense sets in D are the only subsets of P in I. ■

The evaluation of Martin's Axiom, $MA(\aleph_1)$, provides another example of the relevance of simple relative degrees. Of course, the existence of $\aleph_1$ is not part of the existential import of $MA(\aleph_1)$, but it is a requirement for the statement of $MA(\aleph_1)$ to be meaningful. Therefore, the real existential import of $MA(\aleph_1)$ is what it adds beyond the existence of $\aleph_1$. This can only be evaluated by considering relative degrees, and this shows that, beyond the existence of $\aleph_1$, the axiom expresses a closure property that is a particular case of supertransitivity. Therefore, $MA(\aleph_1)$ adds another kind of construction that is not covered by the usual constructions (such as unions, power sets, etc.) of sets.

3 The Subsemilattice of K-Absolute Degrees

It was already proved that both the axiom of infinity and the axiom of existence of a measurable cardinal are simply reducible to degree 1. Of course, the axiom of ex-

[10] In fact, any subset of G is in I, and, because of the transitivity of I, a nonempty subset of G is nonempty in I.

istence of a measurable cardinal should be considered more complex, in terms of its existence requirement, than the axiom of infinity. Therefore, one may ask if this difference of complexity appears in their classification in R. Let A be the standard sentence expressing *"there is a measurable cardinal"*, and let μ be a constant for the least measurable cardinal, if there is one. Since $T^* \vdash A \to \mu > \omega$, it follows that $(1, \omega \in I) \leq_0 (1, A \wedge \wp(\wp(\mu)) \in I)$, and the converse does not hold. Therefore, one may say that, in fact, this difference of complexity has appeared in the classification.

However, regarding the search for new axioms, it is desirable to have a meaningful comparison not only between classifications of individual sentences, when these classifications are ordered according to $\leq_0$, but also between classifications of arbitrary collections of sentences, identifying differences of complexity between those collections even when they are incomparable in terms of $\leq_0$. For example, it may be informative to compare subcollections of a collection of new axioms that can be added consistently to ZFC with various existence requirements, or to compare the whole classification of this collection of new axioms with the whole classification of ZFC.

In Section 4, the collection of the usual axioms of ZFC is compared to the following collection of two new axioms $\mathbf{V} = \mathbf{L}[U]+$ *"there is a measurable cardinal"*, where $\mathbf{L}[U]$ is the unique inner model of measurability associated with the first measurable cardinal, if there is one. Notice that, in the latter collection, the two axioms interact strongly, which is another reason for considering them collectively in a comparison of complexity. The ordering $\leq_0$ cannot provide any information here. The existence requirements of the axioms of ZFC range from 0 to 5, and the existence requirements of the axioms *"there is a measurable cardinal"* and $\mathbf{V} = \mathbf{L}[U]$ are 1 and 5 relative to the appropriate simple contexts, respectively.

The purpose of this section is to devise some tools for such a comparison, and the forthcoming analysis will show the difference of complexity between the usual axioms of ZFC and the axiom *"there is a measurable cardinal"*. The rough idea is that, given a collection of sentences Γ, one may consider a subsemilattice of R which is sufficient for classifying Γ, in the sense that the classification of Γ in R or in the subsemilattice is, in a sense, equivalent. Now, in order to compare two collections of sentences, one may compare subsemilattices: The bigger the subsemilattice needed for the classification of a collection, the more complex that collection is, in terms of the existence requirements.

Definition 3.1 *Let K be a class term, $K = \{x : \Psi(x)\}$, where $\Psi(x)$ has no class variables and has no free variables other than x. Suppose that $d \in \{0, 1, 2, 3, 4, 5\}$ and that φ is the simple context $\exists x_1 ... \exists x_n(\phi(x_1, ..., x_n) \wedge x_1 \in I \wedge ... \wedge x_n \in I)$. If*

$$T^* \vdash K \subseteq I \to (C^d(I) \wedge \varphi \leftrightarrow C^d(I) \wedge \exists x_1 ... \exists x_n \phi(x_1, ..., x_n)),$$

*then the simple degree $[(d, \varphi)]_0$ is said to be K-*absolute*.*

It must be proved that this notion is well defined, that is, that definition 3.1 depends only on $[(d,\varphi)]_0$, and not on the choice of the representative, and is defined for all classes. In fact, any element of R is of the form $[(d,\varphi)]_0$, in which φ is a simple context. If $(d,\varphi) \equiv_0 (d',\varphi')$, where $d \in \{0,1,2,3,4,5\}$, and φ and φ' are simple contexts, then $d = d'$ and:

$$T^* \vdash C^d(I) \wedge \varphi \leftrightarrow C^d(I) \wedge \varphi'.$$

Replacing all ocurrences of I by $\mathbf{V}$ in this theorem gives:

$$T^* \vdash C^d(\mathbf{V}) \wedge \exists x_1 ... \exists x_n(\phi(x_1,...,x_n) \wedge x_1 \in \mathbf{V} \wedge ... \wedge x_n \in \mathbf{V}) \leftrightarrow \\ C^d(\mathbf{V}) \wedge \exists x_1 ... \exists x_{n'}(\phi'(x_1,...,x_{n'}) \wedge x_1 \in \mathbf{V} \wedge ... \wedge x_{n'} \in \mathbf{V}).$$

Therefore:

$$T^* \vdash \exists x_1 ... \exists x_n \phi(x_1,...,x_n) \leftrightarrow \exists x_1 ... \exists x_{n'} \phi'(x_1,...,x_{n'}).$$

Since

$$T^* \vdash K \subseteq I \rightarrow (C^d(I) \wedge \varphi \leftrightarrow C^d(I) \wedge \exists x_1 ... \exists x_n \phi(x_1,...,x_n)),$$

from the equivalences above, it follows that

$$T^* \vdash K \subseteq I \rightarrow (C^d(I) \wedge \varphi' \leftrightarrow C^d(I) \wedge \exists x_1 ... \exists x_{n'} \phi(x_1,...,x_{n'})).$$

Remark 3.2 *Notice that the convention that* $\boldsymbol{V}$ *has all closure properties, that is, for each* $d \in \{0,1,2,3,4,5\}$, $T^* \vdash C^d(\boldsymbol{V})$, *is used in the above argument. Of course, this is very reasonable convention.*

Remark 3.3 *For each class* K *and each simple context* φ, *the simple degree* $[(5,\varphi)]_0$ *is* K*-absolute.*

Notation 3.4 *Let* Q *be a subsemilattice of* R, φ *be a context (not necessarily simple) and* d *be an absolute degree. If the pair* (d,φ) *is in a class of* Q, *that is, if there is a simple context* ψ *such that* $(d,\varphi) \in [(d,\psi)]_0 \in Q$, *then it is said that* (d,φ) belongs to Q.

Notation 3.5 *Let* Q *be a subsemilattice of* R *and* K *be a class term,* $K = \{x : \Psi(x)\}$, *where* $\Psi(x)$ *has no class variables and has no free variables other than* x. *Denote by* Q_K *the set of degrees in* Q *that are* K*-absolute. If* $c \in \{0,1,2,3,4,5\}$ *is such that for each* $d \succ c$, *and that for each simple context* φ *it holds that* $[(d,\varphi)]_0 \in Q_K$ *if and only if* $[(d,\varphi)]_0 \in Q$, *then* Q_K is said to agree with Q above c.

The simple degrees contained in the subsemilattice Q_K are those whose existential demands on the domain I are uniformly fulfilled by assuming that $K \subseteq I$. The following simple lemma containing basic properties of the above construction is useful:

Lemma 3.6 *Let Q be a subsemilattice of R and let K, K' be class terms, $K = \{x : \Psi(x)\}$, $K' = \{x : \Psi'(x)\}$, where $\Psi(x)$ and $\Psi'(x)$ have no class variables and no free variables other than x. Under these conditions:*

1. *Q_K is a subsemilattice of Q.*
2. *If $T^* \vdash K \subseteq K'$ then $Q_K \subseteq Q_{K'}$.*
3. *$(Q_K)_{K'} = Q_K \cap Q_{K'}$.*
4. *$Q_{\boldsymbol{V}} = Q$.*
5. *Q_K agrees with Q above 4.*
6. *If K is a proper class then Q_K agrees with Q above 2.*
7. *If (d, φ) belongs to Q and φ is a context that contains no occurrence of the class variable I, then $[(d, \varphi)]_0 \in Q_K$.*

Proof. These properties are all immediate consequences of the previous definitions. (1): If $[(d, \varphi)]_0$ and $[(d', \varphi')]_0$ are K-absolute, then their least upper bound $[(\max(d, d'), \varphi \wedge \varphi')]_0$ is K-absolute, because

$$T^* \vdash C^{\max(d,d')}(I) \rightarrow C^d(I) \text{ and } T^* \vdash C^{\max(d,d')}(I) \rightarrow C^{d'}(I).$$

Since the least upper bound of $[(d, \varphi)]_0$ and $[(d', \varphi')]_0$ is in Q_K, it follows that Q_K is a subsemilattice of Q. (2): If $T^* \vdash K \subseteq K'$, then if $[(d, \varphi)]_0$ is K-absolute then $[(d, \varphi)]_0$ is K'-absolute. (3): $(Q_K)_{K'}$ is the set of simple degrees in Q_K which are K'-absolute. Therefore, $(Q_K)_{K'}$ is the set of simple degrees in Q which are K-absolute and K'-absolute, and

$$(Q_K)_{K'} = Q_K \cap Q_{K'}.$$

(4): By definition, all simple degrees are V-absolute (see example 3.7 below). (5): If $d = 5$ then $[(d, \varphi)]_0$ is K-absolute, for any class K. (6): if $d \succ 2$, then $C^d(I)$ implies that I is a level. If K is a proper class, then, since I is a level, $K \subseteq I$ implies that $I = V$. Therefore, if $d \succ 2$, then any simple degree $[(d, \varphi)]_0$ is K-absolute. (7): Notice that if φ is a context that contains no occurrence of the class variable I, then φ is simple. Under these conditions, $[(d, \varphi)]_0$ is trivially K-absolute, and the result follows. ■

Example 3.7 *$R_{\boldsymbol{V}} = R$. In fact, $T^* \vdash \boldsymbol{V} \subseteq I \rightarrow \boldsymbol{V} = I$. Therefore, for each simple context φ, $T^* \vdash \boldsymbol{V} \subseteq I \rightarrow (\varphi \leftrightarrow \exists x_1 ... \exists x_n \phi(x_1, ..., x_n))$. It follows that, for all $d \in \{0, 1, 2, 3, 4, 5\}$, $[(d, \varphi)]_0 \in R_{\boldsymbol{V}}$.*

Example 3.8 *$R_{\boldsymbol{OD}}$ agrees with R above 1. In fact, for each ordinal α, $V_\alpha \in \boldsymbol{OD}$. If $\boldsymbol{OD} \subseteq I$, then, for each ordinal α, $V_\alpha \in I$. Since every set is a subset of some V_α, it follows that, if I is supertransitive, then $I = \boldsymbol{V}$.*

Theorem 3.9 *Let $\theta = (\aleph_1)^{\boldsymbol{L}}$. The semilattice R_{L_θ} contains all classes $[(d,\varphi)]_0$, in which $d \in \{0,1,2,3,4,5\}$ and φ is a simple context of the form $\exists x_1 ... \exists x_n(\phi \wedge x_1 \in I \wedge ... \wedge x_n \in I)$, such that all quantifiers in ϕ are bounded.*

Proof. The theorem follows from the Lévy-Shoenfield Absoluteness Lemma ([3], page 120). In fact,

$$T^* \vdash \exists x_1 ... \exists x_n \phi \leftrightarrow \exists y(\exists x_1 ... \exists x_n(x_1 \in y \wedge ... \wedge x_n \in y \wedge \phi)).$$

Furthermore, the Lévy-Shoenfield Absoluteness Lemma gives:

$$T^* \vdash \exists y(\exists x_1 ... \exists x_n(x_1 \in y \wedge ... \wedge x_n \in y \wedge \phi)) \rightarrow \exists y(y \in L_\theta \wedge \\ \exists x_1 ... \exists x_n(x_1 \in y \wedge ... \wedge x_n \in y \wedge \phi)).$$

Since L_θ is transitive, it follows that:

$$T^* \vdash \exists y(\exists x_1 ... \exists x_n(x_1 \in y \wedge ... \wedge x_n \in y \wedge \phi)) \rightarrow \\ \exists x_1 ... \exists x_n(x_1 \in L_\theta \wedge ... \wedge x_n \in L_\theta \wedge \phi).$$

Therefore, from the equivalence above:

$$T^* \vdash \exists x_1 ... \exists x_n \phi \rightarrow \exists x_1 ... \exists x_n(x_1 \in L_\theta \wedge ... \wedge x_n \in L_\theta \wedge \phi).$$

From this,

$$T^* \vdash L_\theta \subseteq I \rightarrow (\exists x_1 ... \exists x_n \phi \leftrightarrow \exists x_1 ... \exists x_n(x_1 \in I \wedge ... \wedge x_n \in I \wedge \phi)),$$

and hence

$$T^* \vdash L_\theta \subseteq I \rightarrow (C^d(I) \wedge \varphi \leftrightarrow C^d(I) \wedge \exists x_1 ... \exists x_n \phi),$$

for each $d \in \{0,1,2,3,4,5\}$. ■

Recall that the simple degrees contained in the subsemilattice R_{L_θ} are those whose existential demands on the domain I are uniformly fulfilled by assuming that $L_\theta \subseteq I$. Lemma 3.6 shows that this is the case for all simple degrees of the form $[(5,\varphi)]_0$, and for simple degrees $[(d,\varphi)]_0$ in which φ (is simple and) contains no occurrence of the class variable I, and $d \in \{0,1,2,3,4,5\}$. Theorem 3.9 says that this is also the case for all simple degrees $[(d,\varphi)]_0$, in which $d \in \{0,1,2,3,4,5\}$ and φ is a simple context of the form $\exists x_1 ... \exists x_n(\phi \wedge x_1 \in I \wedge ... \wedge x_n \in I)$, such that all quantifiers in ϕ are bounded. These results motivate the following definition:

Definition 3.10 *Let A be a sentence in $L(ZF)$, and Q be a subsemilattice of R. If for each $d \in \{0, 1, 2, 3, 4, 5\}$, for each context φ, such that (d, φ) belongs to R, there is a context ψ, such that (d, ψ) belongs to Q, and such that A admits degree d of existence requirement relative to φ if and only if A admits degree d of existence requirement relative to ψ, then Q is said to be a faithful subsemilattice of R with respect to A.*

The idea is that in order to classify A, it is sufficient to consider the subsemilattice Q. If $T^* \vdash K \subseteq K'$ then, since Q_K is more restrictive than $Q_{K'}$, the collection of new axioms A, such that Q_K is faithful with respect to A, is collectively simpler to classify when compared to the collection of new axioms A, such that $Q_{K'}$ is faithful with respect to A. Of course, if $T^* \vdash K \subseteq K'$ and Q_K is faithful with respect to A, then $Q_{K'}$ is also faithful with respect to A. This makes it possible to compare not only individual new axioms that can be added to ZFC, but collections of new axioms that can be added consistently to ZFC.

Proposition 3.11 *Let A be the standard sentence expressing "there is a measurable cardinal", and let φ be a simple context such that A admits degree 1 relative to φ. The simple degree $[(1, \varphi)]_0$ is not in $R_{\mathbf{L}}$, and hence not in R_{L_θ}.*

Proof. Suppose, on the contrary, that $[(1, \varphi)]_0$ is in $R_{\mathbf{L}}$. The simple context φ is $\exists x_1 ... \exists x_n (\phi \wedge x_1 \in I \wedge ... \wedge x_n \in I)$. It follows that,

$$T^* \vdash \mathbf{L} \subseteq I \rightarrow (C^1(I) \wedge \varphi \leftrightarrow C^1(I) \wedge \exists x_1 ... \exists x_n \phi).$$

By hypothesis,

$$T^* \vdash C^1(I) \wedge \varphi \rightarrow A^I.$$

Therefore,

$$T^* \vdash \mathbf{L} \subseteq I \rightarrow (C^1(I) \wedge \exists x_1 ... \exists x_n \phi \rightarrow A^I).$$

Replacing all occurrences of I by $\mathbf{L}$, it follows that:

$$T^* \vdash C^1(\mathbf{L}) \wedge \exists x_1 ... \exists x_n \phi \rightarrow A^{\mathbf{L}}.$$

Since $\mathbf{L}$ is transitive,

$$T^* \vdash \exists x_1 ... \exists x_n \phi \rightarrow A^{\mathbf{L}}.$$

However, $T^* \vdash \neg A^{\mathbf{L}}$, and hence $T^* \vdash \neg(\exists x_1 ... \exists x_n \phi)$. This contradicts the hypothesis that φ is a simple context. ■

Proposition 3.11 shows that $R_{\mathbf{L}}$ is not a faithful subsemilattice of R with respect to the sentence *"there is a measurable cardinal"*, whereas the results in [1] and [2] show that $R_{\omega+1}$ is a faithful subsemilattice of R with respect to the axioms of ZFC. Proposition 2.2 shows that R_{L_θ} is a faithful subsemilattice of R with respect to $\mathbf{V} = \mathbf{L}$.

A simple result given in [2] is that a possible new axiom A of the form $\exists x_1 ... \exists x_n B$, such that all quantifiers in B are bounded, admit degree 1 of existence requirement relative to the simple context $\exists x_1 ... \exists x_n (B \wedge x_1 \in I \wedge ... \wedge x_n \in I)$. Theorem 3.9 shows that the simple degree $[(1, \exists x_1 ... \exists x_n (B \wedge x_1 \in I \wedge ... \wedge x_n \in I)]_0$ is in R_{L_θ}. Therefore, since A is a new axiom, the subsemilattice R_{L_θ} is a faithful one with respect to A.[11]

Consider A to be a new axiom such that the subsemilattice R_{L_θ} is faithful with respect to A. In this case, A makes no unconditional existential demand that goes beyond L_θ, that is, A doesn't categorically assert the existence of a set that cannot be instantiated in L_θ, as, for example, the existence of a measurable cardinal. Furthermore, since L_θ is a very small subset from the point of view of $\mathbf{V}$, it follows that R_{L_θ} is a very restrictive subsemilattice of $R = R_{\mathbf{V}}$, which is, nevertheless, faithful with respect to all axioms of ZFL and to all new axioms of the form $\exists x_1 ... \exists x_n B$, such that all quantifiers in B are bounded. Therefore, in these conditions, it seems that A is a new axiom that, independently of which degrees are admitted by A in R, is very well behaved with respect to the classification of its existence requirement. For in order to classify A, it suffices to consider the subsemilattice R_{L_θ}.

4 Conclusion

The present paper is not proposing another axiom for set theory: instead, it presents tools for an analysis, in terms of set existence and production of sets, of the search for new axioms itself. Such an analysis is needed, since this problem has been in the hands of set-theorists for a while, and the search for new axioms is now a highly developed, technical branch of set theory. I think that the foundational significance of all this technical material is far from being organized into a systematic analysis. This has certainly not been achieved here, although some tools that can be useful for a systematic reflection on the problem, one that goes beyond intuitive descriptions, are presented and studied.

Consider, for example, the theory obtained from ZFC by adding the axiom *"there is a measurable cardinal"*, and the axiom $\mathbf{V} = \mathbf{L}[U]$, where $\mathbf{L}[U]$ is the unique inner model of measurability associated with the first measurable cardinal, if there is one. Proposition 2.3 shows that $\mathbf{V} = \mathbf{L}[U]$ admits only degree 5 of existence requirement relative to a simple context. Proposition 2.4 shows that *"there is a measurable cardinal"*

[11] This holds for all sentences A of the form above that are consistent with ZFC, unless the sentence A admits degree 0 relative to a simple context that is not in R_{L_θ}, and does not admit degree 0 relative to any simple context in R_{L_θ}. Since, by convention, new axioms cannot admit degree 0 relative to a simple context, the result follows.

admits degree 1 of existence requirement relative to the simple context $A \wedge \wp(\wp(\mu)) \in I$, where A is the usual sentence expressing *"there is a measurable cardinal"*.

It seems very plausible to say that this extension of ZFC is, indeed, an attempt at answering (i') "how exactly does the production of sets occur at each level?", and (ii') "which levels of the hierarchy do exist?". For, with respect to (ii'), an unconditional nonconstructive (weakly) productive assertion[12] is added and is such that it guarantees the existence a measurable level. With respect to (i'), a strongly productive assertion is added which admits only degree 5, and hence expresses a closure property which is truly about $\mathbf{V}$. The resulting theory is categorical, in the appropriate sense: it has the condensation property, and its transitive models that contain $\wp(\wp(\mu))$ are of the kind $L_\alpha[U]$.[13] Therefore, this extension completely answers (i'). Furthermore, in the resulting theory there is only one measurable cardinal, and hence it also gives an answer to (ii').

Recall that "unconditional nonconstructive productive" is the only qualitative class of set existence that is not populated by an axiom of ZFC. Therefore, the resulting theory ($ZFC + \mathbf{V} = \mathbf{L}[U] +$ *"there is a measurable cardinal"*) is naturally exaustive with respect to both the qualitative and quantitative aspects of set existence. Since those two new axioms interact strongly, it follows that an analysis of them as a single collection is also desirable. Proposition 3.11 shows that the semilattice $R_{\mathbf{L}}$ is not faithful for this collection of two new axioms, which indicates that these new axioms are, indeed, on a different level of complexity when compared to the collection of axioms of ZFC, or to any collection of new axioms of the form $\exists x_1 ... \exists x_n B$, such that all quantifiers in B are bounded.

Of course, this analysis is not suggesting that the "true" set theory is the theory obtained from ZFC by adding the axiom *"there is a measurable cardinal"*, and the axiom $\mathbf{V} = \mathbf{L}[U]$, where $\mathbf{L}[U]$ is the unique inner model of measurability associated with the first measurable cardinal, if there is one. However, the present analysis does show that this extension does what may be demanded, in terms of set existence, from a first order set theory. Firstly, it is naturally exaustive with respect to both the qualitative and quantitative aspects of set existence. Secondly, it answers the questions (i') "how exactly does the production of sets occur at each level?", and (ii') "which levels of the hierarchy do exist?", in a satisfactory way.

This brief case study shows that the conceptual apparatus around the notion of set existence, developed in [1], [2] and in the present paper, is meaningful in the context of the search for new axioms. It is, above all, an organizational apparatus that can provide comparative information about collections of new axioms. However, since the analysis

[12]Weakly productive means that it admits degree 1 of existence requirement, and hence the axiom does very little more, in terms of productivity, than guaranteeing the existence of a measurable level.

[13]Notice that the proof that $\mathbf{V} = \mathbf{L}[X]$ admits only degree 5, and the proof of the categoricity of the resulting theory are based on the same lemma.

is developed from the perspective of set existence, it says nothing about the truth of these axioms. For example, in the development of this conceptual apparatus, the "ideal of maximality" of the universe of all sets is never invoked, or even considered. I don't even think that the meaning of "ideal of maximality" is ultimately clarified, but the point is just to illustrate that no criteria for the truth of axioms is under consideration here. I believe that intuition about the truth of a collection of new axioms can only be developed through systematic work with that collection. From this perspective, the fact that there are important set-theorists seriously interested in a specific collection of new axioms, already implies that it is worth analyzing.

References

[1] Freire, R., *On Existence in Set Theory*, Notre Dame Journal of Formal Logic, **53**, nº 4 (2012), pgs. 525–547, 2012

[2] Freire, R., *On Existence in Set Theory, Part II: Relative Productivity*, Notre Dame Journal of Formal Logic, **55**, nº 1 (2014), pgs. 91–105, 2014

[3] Jech, T., *Set Theory*, Academic Press, 1978

[4] Kunen, K., *Set Theory: An Introduction to Independence Proofs*, Elsevier, 1980

[5] Levy, A., *Basic Set Theory*, Dover, 2002

[6] Lowe, B., Steel, J., *An Introduction to Core Model Theory*, in Sets and Proofs, 1999

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