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The Recovering Information Property and the Axiom of Choice

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Abstract

Given an infinite family of non-empty sets and its corresponding cartesian product, we consider the intersection structure of all subsets of the cartesian product which are themselves cartesian products (of subsets of the sets of the initially given family). In this note we show that natural notions involving *cancellative properties* and *uniqueness of representations* in structures of this kind are intrinsically related to the Axiom of Choice. Our motivation for this research was the question of whether it is possible or not to recover the factor information from such subsets in the absence of choice principles, and our main theorem shows that getting such task done in its full generality is impossible in a choiceless context. A restricted, topological version of the main theorem is also presented.

Keywords: axiom of choice, cartesian products, intersection structures, generalized projections.

1 Introduction

Throughout this note, **ZF** is the Zermelo-Fraenkel set theory, **AC** is the Axiom of Choice and **ZFC** = **ZF** + **AC**. We work within **ZF**, meaning that all propositions and theorems are proved to hold in a choiceless context. Our set-theoretical notations are standard. For instance, given any set X we denote by $\mathcal{P}(X)$ the *powerset* of X, which is the set of all subsets of X.

The research of this paper was motivated by the following question. Let $\{X_i : i \in I\}$ be an infinite family of non-empty sets, and *do not assume* the Axiom of Choice. Let $Y \subseteq \prod_{i\in I} X_i$ be a subset of the cartesian product which is also a cartesian product, meaning that Y can be written in the form $Y = \prod_{i\in I} A_i$ for *some* family $\{A_i : i \in I\}$

– where, obviously, we assume $A_i \subseteq X_i$ for every $i \in I$. Under such assumptions, is it possible to ensure that there is a structural feature of the cartesian product $\prod_{i\in I} X_i$ which could recover from any such subset Y the factor information on a fixed, arbitrary coordinate $j \in I$? Of course, this question only makes sense for the sets Y as described with $\emptyset \neq A_i \subseteq X_i$ for every $i \in I$.

Before going further in our discussion, let us introduce a natural terminology for our sets of interest.

Definition 1.1 *Let Y be a subset of a cartesian product $\prod_{i\in I} X_i$, where $\{X_i : i \in I\}$ is an arbitrary family of non-empty sets. Y will be said to be a* **product set** *if for some family $\{A_i : i \in I\}$ of sets satisfying $A_i \subseteq X_i$ for every $i \in I$ we have $Y = \prod_{i\in I} A_i$. If all sets in the family $\{A_i : i \in I\}$ are also non-empty, Y will be said to be a* **product set with non-empty factors***.*

Notice that the family of all product sets in $\mathcal{P}(\prod_{i\in I} X_i)$ may be easily viewed as a *topped intersection structure*.

Definition 1.2 ([1], 2.33) (i) *A family $\mathcal{L}$ of subsets of a given set X is said to be a* **intersection structure** *if it is closed for intersections, meaning that $\bigcap \mathcal{A} \in \mathcal{L}$ for every non-empty subfamily $\emptyset \subsetneq \mathcal{A} \subseteq \mathcal{L}$.*

(ii) *If $\mathcal{L}$ is an intersection structure and one has $X \in \mathcal{L}$, then $\mathcal{L}$ is said to be a* **topped intersection structure***.*

The condition (ii) above may be regarded as the relaxation of (i) obtained by removing the requirement of non-emptiness of the subfamilies to be intersected. Now it should be clear that if X is given by $X = \prod_{i\in I} X_i$ (where $\{X_i : i \in I\}$ is a family of non-empty sets) then $\mathcal{L} = \{Y \subseteq X : Y \text{ is a product set}\}$ is a topped intersection structure. Topped intersection structures (ordered by inclusion) are complete lattices, under suitable definitions of $\bigvee$ and $\bigwedge$ ([1], Corollary 2.32).

It turns out that the focus of our investigating is on the following notion, which we now define as the *Recovering Information Property* of a given family $\{X_i : i \in I\}$:

Definition 1.3 *A family of non-empty sets* $\{X_i : i \in I\}$ *is said to satisfy* **RIP**[1], *the* **Recovering Information Property**, *if for every product set with non-empty factors* $Y \subseteq \prod_{i \in I} X_i$ *and for every fixed* $j \in I$, *there is a well-defined, canonical way of associating the set* Y *to its* $j-th$ *factor.*

As we asked the reader *not to assume* the Axiom of Choice, the expected, quick answer given by "you have only to consider the usual projections" is not necessarily on the table. If **AC** fails, we could be in a case where the family of non-empty sets $\{X_i : i \in I\}$ satisfies $\prod_{i \in I} X_i = \emptyset$. In such a case, there are no answers coming from the projections. [2]

The main result of this paper is showing that **AC** is equivalent to the statement *"Every family of non-empty sets satisfies RIP"* – and therefore the full validity of RIP is impossible in a choiceless context. We will establish such result that by introducing the language of *generalized projections*. Generalized projections will formalize the idea of having a canonical way of recovering factor information – as it appears in the definition of RIP.

Some of the results of this note have an algebraic taste; in some sense, we will show that the Axiom of Choice is equivalent to the presence of algebraic properties such as *cancellative properties* and *uniqueness of representations* in the intersection structure of product sets. Such properties are commonly related to algebraic features of internal direct products.

A restricted, topological version of the main result will also be presented. Such restricted version is related to the well-known *Kuratowski's Theorem* on projections defined in products of two topological spaces – which states that projections which are parallel to a compact factor are closed maps.

We give an end to this introduction by pointing out two "minimal forms" of the Axiom of Choice, meaning that we will be concerned with the precise aspect of families which witness the failure of **AC**.

[1]The author declares he is not entirely sure whether RIP is an unhappy or catchy acronym. Nevertheless, he believes this notion is worthwhile thinking of, as he will devote himself to demonstrate in the rest of this note.

[2]Despite the awkwardness of the situation, if $\prod_{i \in I} X_i = \emptyset$ we fix $j \in I$ and conclude that the sentence

"$\emptyset$ *is a function and for any* $z \in \emptyset$, *there are* x *and* y *such that* $z = \langle x, y \rangle$ *with* $x \in \prod_{i \in I} X_i$ *and* $y \in X_j$"

is vacuously true; therefore we are allowed to consider the empty function as being the projection $\prod_i$ for any $i \in I$. Nevertheless, there are no answers for our questions arising from the projections in this pathological case.

Proposition 1.4 *The following statements are equivalent:*

(i) **AC**.

(ii) *If I is an infinite set and $\{X_i : i \in I\}$ is a family of non-empty sets such that $J = \{j \in I : X_j$ has at least two elements$\}$ is infinite, then $\prod_{i\in I} X_i \neq \emptyset$.*

(iii) *If I is an infinite set and $\{X_i : i \in I\}$ is a family of non-empty sets such that $J = \{j \in I : X_j$ has at least two elements$\}$ is non-empty, then $\prod_{i\in I} X_i \neq \emptyset$.* ■

Indeed, it should be clear that if $\{X_i : i \in I\}$ is a witness of the failure of the Axiom of Choice then I should be infinite and not all (nor all but finitely many) of the X_i's could be singletons.

2 AC and generalized projections

In the following definition we introduce the notion of generalized projections, needed for the presence of RIP. It will be clear from the below definitions that a family of non-empty sets $\{X_i : i \in I\}$ satisfies RIP if, and only if, $\{X_i : i \in I\}$ admits generalized projections.

Definition 2.1 (i) *Let I be a non-empty set, $\mathcal{F} = \{X_i : i \in I\}$ be a family of non-empty sets and let $j \in I$. A partial function from $\mathcal{P}\Big(\prod_{i\in I} X_i\Big)$ into $\mathcal{P}(X_j)$ will be said to be a* **generalized projection in the** $j-th$ **coordinate** *if such partial function assigns, to every product set with non-empty factors, its $j-th$ factor.*

(ii) *Under the same assumptions of (i), we will say that $\mathcal{F} = \{X_i : i \in I\}$* **admits generalized projections** *if for every $j \in I$ there is a generalized projection in the $j-th$ coordinate.*

(iii) *Finally, $\mathcal{F} = \{X_i : i \in I\}$ will be said to have* **an indexed family of generalized projections** *if there is a family $\{\varphi_i : i \in I\}$ of partial functions such that, for every $j \in I$, φ_j is a generalized projection in the $j-th$ coordinate.*

So, a family of non-empty sets $\mathcal{F} = \{X_i : i \in I\}$ has an indexed family of generalized projections if there is a family $\{\varphi_i : i \in I\}$ such that, for every $j \in I$, φ_j is a partial function from $\mathcal{P}\Big(\prod_{i\in I} X_i\Big)$ into $\mathcal{P}(X_j)$ satisfying

$$\varphi_j(\prod_{i\in I} A_i) = A_j$$

whenever $\{A_i : i \in I\}$ is a family of non-empty sets satisfying $A_i \subseteq X_i$ for every $i \in I$. [3]

The following theorem is the main result of this paper; we show that **AC** is equivalent to *"Every family of non-empty sets satisfies RIP"*. The algebraic interpretation, in terms of the corresponding intersection structures, is that **AC** is equivalent to the the full validity of the "uniqueness of representations property" for product sets with non-empty factors.

Theorem 2.2 *The following statements are equivalent:*

(i) **AC**.

(ii) *Every infinite family of non-empty sets has an indexed family of generalized projections.*

(iii) *Every infinite family of non-empty sets admits generalized projections.*

Proof. $(i) \Rightarrow (ii)$. Let $\mathcal{F} = \{X_i : i \in I\}$ be an infinite family of non-empty sets. Assuming the Axiom of Choice, we use the family of usual, canonical projections $\{\prod_i : i \in I\}$ to induce a family of generalized projections: as probably expected, we define

$$\varphi_i(Z) = \textstyle\prod_i[Z]$$

for every $i \in I$ and for every $Z \subseteq \prod_{i\in I} X_i$. Now, let $Y = \prod_{i\in I} A_i$ with all factors non-empty. For a fixed $j \in I$, the inclusion $\varphi_j(Y) \subseteq A_j$ is clear. For the opposite inclusion, let w be any element of A_j and, using **AC**, just take any $f \in \prod_{i\in I\setminus\{j\}} A_i$; the extension of such function to the domain I obtained by adding the pair $\langle j, w\rangle$ give us the desired.

$(ii) \Rightarrow (iii)$. Immediate.

$(iii) \Rightarrow (i)$. Suppose the failure of the Axiom of Choice. Then there is an infinite family of non-empty sets, say $\mathcal{F} = \{X_i : i \in I\}$, satisfying $\prod_{i\in I} X_i = \emptyset$, and, by item (iii) of Proposition 1.4, we may consider $k \in I$ such that X_k has at least two elements. Let p be

[3]Notice that, formally, the condition given by (iii) of Definition 2.1 is stronger than the one given by (ii) since the existence of generalized projections for each coordinate *do not* imply the existence of an *indexed* family of generalized projections – recall that we are *in a choiceless context*. However, as we will see presently in our main theorem, the "every family has" versions of (ii) and (iii) are both equivalent to **AC**.

an element of such X_k[4]. Define a family of non-empty sets $\{A_i : i \in I\}$ in the following way: for $i \neq k$, let $A_i := X_i$, and let A_k be the non-empty set given by $A_k := X_k \setminus \{p\}$. Notice that necessarily one has $\prod_{i\in I} A_i = \emptyset$, since $\prod_{i\in I} A_i \subseteq \prod_{i\in I} X_i = \emptyset$. It follows that there is no possibility of well-defining a generalized projection in the $k-th$ coordinate, since, if we suppose towards a contradiction that f is such a generalized projection, then we would have, by the definition,

$$f(\emptyset) = f\Big(\prod_{i\in I} X_i\Big) = X_k$$

but also

$$f(\emptyset) = f\Big(\prod_{i\in I} A_i\Big) = A_k = X_k \setminus \{p\},$$

which is clearly an absurd. ■

3 AC and the Cancellative Property

Given the kind of uniqueness of representations proved in the previous section, it is very natural to consider the following definition.

Definition 3.1 *A family of non-empty sets $\{X_i : i \in I\}$ is said to satisfy* **CP** *(the* **Cancellative Property***) if whenever $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are families of non-empty sets with $A_i, B_i \subseteq X_i$ for every $i \in I$ such that $\prod_{i\in I} A_i = \prod_{i\in I} B_i$ then necessarily one has $A_i = B_i$ for every $i \in I$.*

The following proposition is essentially a corollary (or even a rephrasing) of the main theorem.

Proposition 3.2 *The following statements are equivalent:*

(i) **AC**.

(ii) *Every non-empty family of non-empty sets satisfies* **CP**.

Proof. $(ii) \Rightarrow (i)$ follows by an argument entirely similar to the one presented for proving the last implication of the main theorem: one has just to let the family $\{B_i : i \in I\}$ be $\{X_i : i \in I\}$ itself. For $(i) \Rightarrow (ii)$, notice that within **ZFC** we are able to proceed with the usual manipulations involving the usual projections and therefore we are done (since $A_j = \prod_j[\prod_{i\in I} A_i] = \prod_j[\prod_{i\in I} B_i] = B_j$ for every $j \in I$). ■

[4]Notice that there are only two arbitrary choices in the argument: that for k and that for p, so this is indeed a **ZF** argument.

4 AC and closed generalized projections for products of compact spaces

It is well-known that if X is a compact space, then for every topological space Y the canonical projection $\prod_Y : X \times Y \to Y$ is a closed map; such result is widely known and it is due to Kuratowski. In fact, even the reciprocal result is true, and, furthermore, none of these implications require any form of choice.

Such interesting fact – the unnecessity of choice in both directions – was pointed out by K. P. Hart (at Math Overflow); in what follows, we give an adaptation of his nice **ZF** argument. In one hand, let X be compact, Y be any topological space, F be a closed subset of $X \times Y$ and $y \in Y \setminus \prod_Y(F)$; then $(X \times Y) \setminus F$ can be written as the union of basic open sets, and so there is a finite family of them, say $\{U_m \times V_m : m \leqslant n\}$, which covers the compact subset $X \times \{y\}$. Of course we may suppose that $y \in V_m$ for every $m \leqslant n$, and therefore $V = \bigcap_{m \leqslant n} V_m$ is an open neighbourhood of y included in $Y \setminus \prod_Y(F)$. On the other hand, suppose X not compact and let $\mathcal{F}$ be a family of non-empty, closed subsets of X with the finite intersection property but with empty intersection; we do not need **AC** for ensuring the existence of such a family. We may also assume that $\mathcal{F}$ is closed for finite intersections. Let $Y = X \cup \{\mathcal{F}\}$, topologized in the following way: points of X are declared isolated and the basic neighbourhoods of $\mathcal{F}$ are all sets of the form $\{\mathcal{F}\} \cup F$ for $F \in \mathcal{F}$. Now let G be the closure of the diagonal of X in $X \times Y$; as $\mathcal{F}$ is non-isolated in Y one has clearly $\mathcal{F} \in \overline{\prod_Y[G]}$, however considering a point $\langle x, \mathcal{F} \rangle \in X \times Y$ it follows from $\bigcap \mathcal{F} = \emptyset$ that there is an open neighbourhood of $\langle x, \mathcal{F} \rangle$ which does not intersect the diagonal – and so $\mathcal{F} \notin \prod_Y[G]$.

In [2], the author and J. P. C. de Jesus have proved that the Axiom of Choice is equivalent to the following topological statement: *"If a product of a non-empty family of sets is closed in a topological (Tychonoff) product, then at least one of the factors is closed"*. Such result was obtained with a slight modification on an argument due to Schechter, who proved in [3] that **AC** is equivalent to the statement *"A product of closures of subsets of topological spaces is equal to the closure of their product (in the Tychonoff topology)"*.

Essentially with the same proof of the main result of [2], the following theorem holds:

Theorem 4.1 *The following statements are equivalent:*

(A) **AC**.

(B) *Let I be an infinite set, $\{X_i : i \in I\}$ be a family of topological spaces and $\{A_i : i \in I\}$ be a family of non-empty sets satisfying $A_i \subseteq X_i$ for all $i \in I$. If $\prod_{i \in I} A_i$ is closed in the*

(Tychonoff) product $\prod_{i \in I} X_i$*, then, for all* $i \in I$*,* A_i *is a closed subset of* X_i*.*

(C) *Let* I *be an infinite set,* $\{X_i : i \in I\}$ *be a family of compact topological spaces and* $\{A_i : i \in I\}$ *be a family of non-empty sets satisfying* $A_i \subseteq X_i$ *for all* $i \in I$*. If* $\prod_{i \in I} A_i$ *is closed in the (Tychonoff) product* $\prod_{i \in I} X_i$*, then, for all* $i \in I$*,* A_i *is a closed subset of* X_i*.*

(D) *Let* I *be an infinite set,* $\{X_i : i \in I\}$ *be a family of compact topological spaces and* $\{A_i : i \in I\}$ *be a family of non-empty sets satisfying* $A_i \subseteq X_i$ *for all* $i \in I$*. If* $\prod_{i \in I} A_i$ *is closed in the (Tychonoff) product* $\prod_{i \in I} X_i$*, then exists* $j \in I$ *such that* A_j *is a closed subset of* X_j*.*

Proof. (Outlined) $(A) \Rightarrow (B)$ follows from Schechter's equivalence and Cancellative Property. $(B) \Rightarrow (C)$ and $(C) \Rightarrow (D)$ are immediate. For $(D) \Rightarrow (A)$, we argue contrapositively: notice that under the failure of **AC** there is an infinite family of non-empty sets $\{A_i : i \in I\}$ with $\prod_{i \in I} A_i = \emptyset$. Let $p \notin \bigcup_{i \in I} A_i$ and for all $i \in I$ let $X_i = A_i \cup \{p\}$ be topologized with the indiscrete (therefore, finite – thus, compact) topology given by $\{\emptyset, X_i\}$. As the empty set is always closed, we are done – since none of the A_i's is closed in the compact space X_i. ■

Defining the notion of *closed generalized projections* in the expected way – i.e., by saying that a partial function f is a *closed generalized projection in the* $j-th$ *coordinate* if f is a generalized projection in the $j-th$ coordinate with the additional property of always assigning a *closed factor* to every *closed* product set with non-empty factors –, then the following proposition (which is a restricted, topological version of our main result) holds:

Proposition 4.2 *The following statements are equivalent:*

(i) **AC**.

(ii) *Every family of compact spaces* $\{X_i : i \in I\}$ *has an indexed family of closed generalized projections.*

(iii) *Every family of compact spaces* $\{X_i : i \in I\}$ *admits closed generalized projections.*

Proof. For $(i) \Rightarrow (ii)$, fix any $j \in I$. By Tychonoff's Theorem (which is equivalent to **AC**), $\prod_{i \in I \setminus \{j\}} X_i$ is compact, and considering that $\prod_{i \in I} X_i \cong (\prod_{i \in I \setminus \{j\}} X_i) \times X_j$ then we have that $\prod_j$ is a closed map by Kuratowski's result, so we may define (exactly in the same way as done before) a function φ_j from $\mathcal{P}\Big(\prod_{i\in I} X_i\Big)$ into $\mathcal{P}(X_j)$ using the usual projection $\prod_j$; it is clear that such a function will be a closed generalized projection. $(ii) \Rightarrow (iii)$ is obvious. The implication $(iii) \Rightarrow (i)$ follows from the equivalence (C) of the previous theorem. ■

The author would like to remark that the usual projections are not in position, in general, of being used when it comes to attempting to seize the equivalence (iii) of Theorem 4.1; as we have commented in the beginning of the paper, the canonical projections could be empty functions in **ZF**. This also justifies the introduction (and investigation) of the notion of generalized projections.

We give an end to the paper presenting the following problem.

Problem 4.3 *Given an arbitrary family of non-empty sets, find topological (or algebraic, or any kind of) hypotheses on the family such that the exhibited hypotheses imply the presence of RIP.*

Of course, the cases of interest of the previous problem are those where the usual, canonical projections, as we expected them to be, are not available.

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