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The Dialogical Take on Martin-Löf's Proof of the Axiom of Choice[1]

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Dedicated to Prof. Roshdi Rashed

Abstract

The aim of the paper is to develop a new constructivist approach to the game theoretical interpretation of AC based on the CTT-proof of Per Martin-Löf (1980). More precisely, Clerbout and Rahman showed that the CTT- understanding of AC, that stresses the type dependence involved by the function that constitutes the proof-object of the antecedent, can be seen as the result of both an "outside-inside" approach to meaning. It is this approach to meaning, so we claim, that provides a natural dialogical interpretation to AC, where the (intensional) function involved — understood as rules of correspondence produced by the players' interaction — constitutes a play object for the (first-order) universal quantifier that occurs in the antecedent of the formal expression of this axiom.

Keywords: axiom of choice, constructive type theory, games, philosophy of logic, philosophy of mathematics.

Introduction

It has been said, and rightly so, that the principle of set theory known as the Axiom of Choice (AC) "is probably the most interesting and in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's Axiom of Parallels which was introduced more than two thousand years ago" (Fraenkel/Bar-Hillel and Levy [1973]).

According to Ernst Zermelo's formulation of 1904 AC amounts to the claim that, given any family $\mathcal{A}$ of non-empty sets, it is possible to select a single element from each

[1] The present paper combines the main results of Clerbout/Rahman (2014) and Jovanovic (2014, 2015).

member of $\mathcal{A}$. The selection process is carried out by a function $\mathbf{f}$ with domain in $\mathcal{M}$, such that for any nonempty set $\mathcal{M}$ in $\mathcal{A}$, then $\mathbf{f}(\mathcal{M})$ is an element of $\mathcal{M}$. The axiom has been resisted from its very beginnings and triggered heated foundational discussions concerning among others, mathematical existence and the notion of mathematical object in general and of function in particular. However, with the time, the foundational and philosophical reticence faded away and was replaced by a kind of praxis-driven view by the means of which AC is accepted as a kind of postulate (rather than as an axiom the truth of which is manifest) necessary for the practice and development of mathematics.

Recently the foundational discussions around AC experienced an unexpected revival when Per Martin-Löf showed (around 1980) that in constructive logic (that does not presuppose the excluded middle) the axiom of choice is logically valid (however in its intensional version) and that this logical truth naturally (almost trivially) follows from the constructive meaning of the quantifiers involved — it is this "evidence" that makes it an axiom rather than a postulate. The extensional version can also be proved but then, either third excluded or unicity of the function must be assumed. Martin-Löf's proof, for which he was awarded with the prestigious Kolmogorov price, showed that at the root of the old discussions an old conceptual problem was at stake, namely the tension between intension and extension.

An even more recent development studies the game theoretical interpretation of AC brought forward by Jaakko Hintikka by 1996[2], though he did not consider Martin-Löf's proof — presumably so because Hintikka is not favorable to constructivist approaches. The aim of the paper is to develop a new constructivist approach to the game theoretical interpretation of AC based on the CTT-proof. More precisely, Clerbout and Rahman showed that the CTT-understanding of AC, that stresses the type dependence involved by the function that constitutes the proof-object of the antecedent, can be seen as the result of an "outside-inside" approach to meaning.[3] It is this approach to meaning, so we claim, that provides a natural dialogical interpretation to AC, where the (intensional) function involved — understood as rules of correspondence produced by the players' interaction — constitutes a play object for the (first-order) universal quantifier that occurs in the antecedent of the formal expression of this axiom. Different to Hintikka's own game-theoretical approach the dialogical take on AC does not require a non-axiomatisable language such as the one underlying Independent Friendly Logic (IF-logic). As pointed out by Jovanovic (2014) the dialogical approach to CTT supports Hintikka's claims that a game theoretical justifies Zermelo's axiom of choice in a first-order way perfectly acceptable for the constructivists, however, no underlying IF-semantics is required. Moreover, Hintikka's own formulation of AC, when spelled out, yields the CTT-formulation of Martin-Löf, that is constructivist after all. Sum-

[2]See for example Hintikka (1996, 2001).

[3]For a thorough discussion on the issue see Jovanovich (2014, 2015).

ming up, though Hintikka is right in stressing the perspicuity of the game theoretical interpretation of AC he is wrong in relation to the theory of meaning required for this interpretation. One of the main reasons behind Hintikka's criticism of the constructivist approach is that he assumes that the rejection of the classical understanding of the AC by the constructivists has its roots in the rejection of a function that is not a recursive one. However, as thoroughly discussed by Thierry Coquand (2014), already Arend Heyting (1960) pointed out that recursive functions cannot (without circularity) be used to define constructivity and finally Erret Bishop (1967) showed that recursive functions are not at all needed to develop constructive mathematics. The very point of the rejection by the constructivists of the classical take on AC is their (the classical) assumption that the function at stake is an extensional one. For short, the CTT-proof of AC is based on the intensional take on functions and this is what the dialogical interpretation displays. We will conclude with some reflections on the conceptual link between the constructivist notion of function as rule of correspondence and dialogical interaction and that might restate some of Hintikka's remarks albeit in a different frame.

1 Martin-Löf on the axiom of choice

It is well known that this axiom was first introduced by Zermelo in 1904 in order to prove Cantor's theorem that every set can be rendered to be well ordered. Zermelo gave two formulations of this axiom, one in 1904 and a second one in 1908. It is the second formulation that is relevant for our discussion, since it is related to both, Martin-Löfs and the game theoretical formalization:

> *A set S that can be decomposed into a set of disjoint parts $A, B, C, \ldots$ each of the containing at least one element, possesses at least one subset S_1 having exactly one element with each of the parts $A, B, C, \ldots$ considered.* (Zermelo, 1908)

The Axiom attracted immediately much attention and both of its formulations were criticized by constructivists such as René-Louis Baire, Émile Borel, Henri-Léon Lebesgue and Luitzen Egbert Jan Brower. The first objections were related to the non-predicative character of the axiom, where a certain choice function was supposed to exist without showing constructively that it does. However, the axiom found its way into the ZFC set theory and was finally accepted by the majority of mathematicians because of its usefulness in different branches of mathematics.

Martin-Löf produced a proof of the axiom in a constructivist setting bringing together two seemingly incompatible perspectives on this axiom, namely

> Bishop's surprising observation from 1967: *A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence.*
> The proof by Diaconescu (in 1975) and by Goodman and Myhle (in 1978) that the Axiom of Choice implies Excluded Middle.

In his paper of 2006 Martin-Löf shows that there are indeed some versions of the axiom of choice that are perfectly acceptable for a constructivist, namely the ones where the choice function is defined *intensionally*. In order to see this the axiom must be formulated within the frame of a CTT-setting. Indeed such a setting allows comparing the extensional and the intensional formulation of the axiom. It is in fact the extensional version that implies Excluded Middle, whereas the intensional version is compatible with Bishop's remark:

> [...] *this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional.* (Martin-Löf, 2006, p.349)

In CTT the truth of the axiom actually follows rather naturally from the meaning of the quantifiers:
Take the proposition $(\forall x : A)B(x)$ where $B(x)$ is of the type proposition provided x is an element of the set A. If the proposition is true, then there is a proof for it. Such a proof is in fact a function that for every element x of A renders a proof of $B(x)$. This is how Bishop's remark should be understood: the truth of a universal amounts to the existence of a proof, and this proof is a function. Thus, the truth of a universal, amount in the constructivist account, to the existence of a function. From this the proof of the axiom of choice can be developed quite straightforwardly. If we recall that in the CTT-setting

> the existence of a function from A to B amounts to the existence of proof-object for the universal *every A is B*, and that
>
> the proof of the proposition $B(x)$, existentially quantified over the set A amounts to a pair such that the first element of the pair is an element of A and the second element of the pair is a proof of $B(x)$;

a full-fledged formulation of the axiom of choice — where we make explicit the set over which the existential quantifiers are defined — follows:

$$(\forall x : A)(\exists y : B(x))C(x, y) \rightarrow (\exists f : (\forall x : A)B(x))(\forall x : A)C(x, f(x))$$

The proof of Martin-Löf (1980, p. 50-51) is the following

The usual argument in intuitionistic mathematics, based on the intuitionistic interpretation of the logical constants, is roughly as follows: to prove $(\forall x)(\exists y)C(x,y) \rightarrow (\exists f)(\forall x)C(x,f(x))$*, assume that we have a proof of the antecedent. This means we have a method which, applied to an arbitrary* x*, yields a proof of* $(\exists y)C(x,y)$*. Let* f *be the method which, to an arbitrarily given* x*, assigns the first component of this pair. Then* $C(x,f(x))$ *holds for an arbitrary* x*, and hence, so does the consequent. The same idea can be put into symbols getting a formal proof in intuitionistic type theory. Let* A *: set,* $B(x)$ *: set*$(x : A)$*,* $C(x,y)$ *: set*$(x : A, y : B(x))$*, and assume* $z : (\Pi x : A)(\Sigma y : B(x))C(x,y)$*. If* x *is an arbitrary element of* A*, i.e.* $x : A$*, then by* Π*-elimination we obtain*

$$Ap(z,x) : (\Sigma y : B(x))C(x,y)$$

We now apply left projection to obtain

$$p(Ap(z,x)) : B(x)$$

and right projection to obtain

$$q(Ap(z,x)) : C(x, p(Ap(z,x))).$$

By λ*-abstraction on* x *(or* Π*-introduction), discharging* $x : A$*, we have*

$$(\lambda x)p(Ap(z,x)) : (\Pi x : A)B(x)$$

and by Π*-equality*

$$Ap((\lambda x)p(Ap(z,x),x) = p(Ap(z,x)) : B(x).$$

By substitution [making use of $C(x,y)$ *: set*$(x : A, y : B(x))$*,] we get*

$$C(x, Ap((\lambda x)p(Ap(z,x),x) = C(x, p(Ap(z,x)))$$

[that is, $C(x, Ap((\lambda x)p(Ap(z,x),x) = C(x,p(Ap(z,x)))$ *: set]*
and hence by equality of sets

$$q(Ap(z,x)) : C(x, Ap((\lambda x)p(Ap(z,x),x)$$

where $((\lambda x)p(Ap(z,x))$ *is independent of* x*. By abstraction on* x

$$((\lambda x)p(Ap(z,x)) : (\Pi x : A)C(x, Ap((\lambda x)p(Ap(z,x),x)$$

We now use the rule of pairing (that is Σ*-introduction) to get*

$$(\lambda x)p(Ap(z,x)),(\lambda x)q(Ap(z,x)) : (\Sigma f : (\Pi x : A)B(x))(\Pi x : A)C(x, Ap(f,x))$$

(note that in the last step, the new variable f is introduced and substituted for $((\lambda x)p(Ap(z,x))$ *in the right member). Finally by abstraction on z, we obtain*

$$(\lambda z)((\lambda x)p(Ap(z,x)),((\lambda x)q(Ap(z,x)):(\Pi x:A)(\Sigma y:B(x))C(x,y)\rightarrow (\Sigma f:(\Pi x:A)B(x))(\Pi x:A)C(x,Ap(f,x)).$$

(For the formal demonstration spelled out as a natural-deduction-tree see Appendix III.)

Curiously, as pointed out by Jovanovic (2014) if we spell out Hintikka's own formulation of AC by making explicit at the object language level the domain and codomain of the function involved Martin-Löf's formulation of AC comes out. Moreover, Hintikka's remark that the validity of AC results from the fact that a winning strategy for the antecedent amounts to the existence of a suitable function seems to sum up the idea behind the proof displayed above. It is curious since Martin-Löf's proof is developed within a constructivist setting that Hintikka rejects. Moreover, Martin-Löf (2006) shows that what is wrong with the axiom — from the constructivist point of view — is its extensional formulation — that Hintikka seems to assume. That is:

$$(\forall x:A)(\exists y:B(x))C(x,y)\rightarrow(\exists f:(\forall x:A)B(x))(Ext(f)\&(\forall x:A)C(x,f(x))$$

where $Ext(f)=(\forall i,j:A)(i=_A j\rightarrow f(i)=f(j))$

Thus, from the constructivist point of view, what is really wrong with the classical formulation of the axiom of choice is the assumption that from the truth that *all of the A are B* we can obtain a function that satisfies extensionality. In fact, as shown by Martin-Löf (2006), the classical version holds, even constructively, if we assume that there is only one such choice function in the set at stake!:

$$(\forall x:A)(\exists! y:B(x))C(x,y)\rightarrow(\exists f:(\forall x:A)B(x))(Ext(f)\&(\forall x:A)C(x,f(x))$$

Let us retain that

- If we take $(\forall x:A)(\exists y:B(x))C(x,y)\rightarrow(\exists f:(\forall x:A)B(x))(\forall x:A)C(x,f(x))$ to be the formalization of the axiom of choice, then that axiom is not only unproblematic for constructivists but it is also a theorem. But this formalization is a full-fledged formulation of the version Hintikka's adopts.[4] Certainly, the point is that the CTT-formulation stresses explicitly that the choice function at stake has been defined by means of intensional equality but Hintikka seems to assume extensionality. In fact it is the CTT-explicit language that allows a fine-grained

[4]Indeed, Martin-Löf's formalization follows from making explicit in Hintikka's formulation $\forall x\exists yC(x,y)\rightarrow\exists f\forall xC(x,f(x))$ the range of its quantifiers, that is: $\forall x$ quantifies over, say the set A, $\exists y$ quantifies over, say the set $B(x)$, and $\exists f$, over the set $(\forall x:A)B(x)$.

distinction between the, on the surface, equivalent formulations. This is due to the expressive power of CTT that allows to express at the object language level properties that in other settings are left implicit in the metalanguage. This leads us to the second point.

- According to the constructivist approach functions are identified as proof-objects for propositions and are given in *object-language*, as the objects of a certain type. Understood in that way, functions belong to the lowest-level of entities and there is no jumping to higher order. Once more, the truth of a first order-universal sentence, amounts to the existence of a function that is defined by means of the elements of the set over which the universal quantifies and the first-order expression $B(x)$. The existence of such a function is the CTT-way to express at the object language level, that a given universal sentence is true.

Thus, Hintikka is right in defending that we need only first-order language, but this does not really support his attachment to the classical understanding of it. But what about his claim of the importance of a game theoretical interpretation? This takes us to the next section.

2 The dialogical proof of AC[5]

The dialogical approach to logic is not a specific logical system but rather a rule-based semantic framework in which different logics can be developed, combined and compared. An important point is that the rules that fix meaning are of more than one kind. This feature of its underlying semantics quite often motivated the dialogical framework to be understood as a *pragmatist* semantics.[6] More precisely, in a dialogue two parties argue about a thesis respecting certain fixed rules. The player that states the thesis is called Proponent (**P**), his rival, who contests the thesis is called Opponent (**O**). In its original

[5]The proof stems from Clerbout/Rahman (2014).

[6]The main original papers are collected in Lorenzen/Lorenz (1978). For an historical overview see Lorenz (2001). Other papers have been collected more recently in Lorenz (2008, 2010a,b). A detailed account of recent developments since Rahman (1993), can be found in Rahman/Keiff (2005), Keiff (2009) and Rahman (2012). For the underlying metalogic see Clerbout (2013a,b). For textbook presentations: Kamlah/Lorenzen (1972, 1984), Lorenzen/Schwemmer (1975), Redmond/Fontaine (2011) and Rückert (2011a). For the key role of dialogic in regaining the link between dialectics and logic, see Rahman/Keff (2010). Keiff (2004a,b, 2007) and Rahman (2009) study Modal Dialogical Logic. Fiutek et al. (2010) study the dialogical approach to belief revision. Clerbout/Gorisse/Rahman (2011) studied Jain Logic in the dialogical framework. Popek (2012) develops a dialogical reconstruction of medieval *obligationes*. Rahman/Tulenheimo (2009) study the links between GTS and Dialogical Logic. For other books see Redmond (2010) — on fiction and dialogic — Fontaine (2013) — on intentionality, fiction and dialogues — and Magnier (2013) — dynamic epistemic logic and legal reasoning in a dialogical framework.

form, dialogues were designed in such a way that each of the plays ends after a finite number of moves with one player winning, while the other loses. Actions or moves in a dialogue are often understood as speech-acts involving *declarative utterances* or *posits* and *interrogative utterances* or *requests*. The point is that the rules of the dialogue do not operate on expressions or sentences isolated from the act of uttering them. The rules are divided into particle rules or rules for logical constants (*Partikelregeln*) and structural rules (*Rahmenregeln*). The structural rules determine the general course of a dialogue game, whereas the particle rules regulate those moves (or utterances) that are requests and those moves that are answers (to the requests).[7]

Crucial for the dialogical approach are the following points:[8]

> The distinction between *local* (rules for logical constants) and *global* meaning (included in the structural rules that determine how to play).
> The player independence of local meaning.
> The distinction between the play level (local winning or winning of a play) and the strategic level (existence of a winning strategy).
> A notion of validity that amounts to winning strategy *independently of any model* instead of winning strategy for *every* model. The distinction between non formal and formal plays — the latter notion concerns plays that are played independently of knowing the meaning of the elementary sentences involved in the main thesis.

Recent developments in dialogical logic show that the CTT approach to meaning is very natural to game theoretical approaches where (standard) metalogical features are explicitly displayed at the object language-level.[9] Thus, in some way, this vindicates, albeit in quite of a different manner, Hintikka's plea for the fruitfulness of game-theoretical semantics in the context of epistemic approaches to logic, semantics and the foundations of mathematics. In fact, from the dialogical point of view, those actions that constitute the meaning of logical constants, such as choices, are a crucial element of its full-fledged (local) semantics. Indeed, if meaning is conceived as being constituted during interaction, then all of the actions involved in the constitution of the meaning of an expression should be rendered explicit. They should all be part of the object language. The roots of this perspective are based on Wittgenstein's *Un-Hintergehbarkeit der Sprache* — one of the tenets of Wittgenstein that Hintikka explicitly rejects, a rejection that he shares with the supporters of model-theorical approaches to meaning. According to this perspective of Wittgenstein, language-games are purported to accomplish the task of displaying this "internalist feature of meaning". Furthermore, one of the main insights of Kuno Lorenz' (1970, pp. 74-79) interpretation of the relation between the

[7]For a brief presentation of standard dialogical logic see Appendix I.
[8]Cf. Rahman (2012).
[9]Cf. Rahman/Clerbout (2013, 2014), Clerbout/Rahman (2014).

so-called *first* and *second* Wittgenstein is based on a thorough criticism of the meta-logical approach to meaning. Similar criticism has been raised by G. Sundholm (1997, 2001) who points out that the standard model-theoretic approaches to meaning turn semantics into a meta-mathematical formal object where syntax is linked to semantics by the assignation of truth values to uninterpreted strings of signs (formulae). Language does not any more *express content* but it is rather conceived as a system of signs that speaks *about* the world — provided a suitable metalogical link between signs and world has been fixed. Moreover, Sundholm (2013) shows that the cases of quantifiers dependences that motivate Hintikka's IF-logic can be rendered in the frame of CTT. What we add to Sundholm's remark is that even the game theoretical interpretation of these dependences can be given a CTT formulation, provided this is developed within a dialogical framework.

In fact, in his 1988 paper, Ranta linked for the first time game-theoretical approaches with CTT. Ranta took Hintikka's Game Theoretical Semantics as a case study. Ranta's idea was that in the context of game-based approaches, a proposition is a set of winning strategies for the player positing the proposition.[10] Now in game-based approaches, the notion of truth is to be found at the level of such winning strategies. This idea of Ranta's should therefore enable us to apply safely and directly methods taken from constructive type theory to cases of game-based approaches.

But from the perspective of game theoretical approaches, reducing a game to a set of winning strategies is quite unsatisfactory, all the more when it comes to a theory of meaning. This is particularly clear in the dialogical approach in which different levels of meaning are carefully distinguished. There is thus the level of strategies which is a level of meaning analysis, but there is also a level prior to it which is usually called the level of plays. The role of the latter level for developing an analysis is, according to the dialogical approach, crucial, as pointed out by Kuno Lorenz in his (2001) paper:

> [...] *for an entity [A] to be a proposition there must exist a dialogue game associated with this entity* [...] *such that an individual play where A occupies the initial position* [...] *reaches a final position with either win or loss after a finite number of moves* [...]

For this reason we would rather have propositions interpreted as sets of what we shall call *play-objects*, reading an expression

$$p : \varphi$$

as "p is a play-object for φ".

Thus, Ranta's work on proof objects and strategies constitutes the end not the start of the dialogical project.

[10]That player can be called Player 1, Myself or Proponent.

We can present here thoroughly neither standard dialogical logic nor the CTT-version of it. However, the essential features for the understanding of the paper can be found in two appendices that we attach to our article. Now, before developing exhaustively the winning strategy for the intensional axiom of choice, let us formulate the idea behind the dialogical approach by emulating Martin-Löf's (1984, p. 50)[11] own presentation of the informal constructive demonstration of it.

From the dialogical point of view the point is that **P** can copy-cat **O**'s choice for y in the antecedent for his defence of $f(x)$ in the consequent since both are equal objects of type $B(x)$, for any $x : A$. Thus, a winning strategy for the implication follows simply from the meaning of the antecedent. This meaning is defined by the dependences generated by the interaction of choices involving the embedding of an existential quantifier in a universal one:

- Let us assume that the Opponent launches an attack on the implication and accordingly posits its antecedent — the play object for the antecedent being $L^{\rightarrow}(p)$. Let us further assume that with her challenge **O** resolves the instruction $L^{\rightarrow}(p)$, by choosing v.

- Then for any $x : A$ chosen by **P**, there must be a play object for the right component of v ($R^{\forall}(v)$), occurring in the antecedent.

- However, the play object $R^{\forall}(v)$ (the right component of v) is a play object for an existential and is thus composed by two play objects such that the first one ($L^{\exists}(R^{\forall}(v))$), for any $x : A$ is of type $B(x)$ and its right component, is, for any $x : A$, of type $C(x, L^{\exists}(R^{\forall}(v)))$.

- Now, let **P** choose precisely the same play object v for his defence of the existential in the consequent — the play object for the consequent being $R^{\rightarrow}(p)$. Accordingly, the left play object for the existential in the consequent, is, for any $x : A$, of type $B(x)$. Thus, the left component of the play object for the existential in the consequent is of the same type as the left component of the existential in the antecedent. Moreover, since **P** copies (while defending the existential) the choice of **O** (while resolving $R^{\rightarrow}(p)$) — namely v — we are entitled to say that, the left component of the play object for the existential in the consequent is exactly the same in $B(x)$ as the left component of the existential occurring in the antecedent — i.e. $y = v(x) : B(x)$.

[11]See too Bell (2009, p. 203-204) who makes use of the notation of Tait (1994) that is very close to that of the instructions of the dialogical frame, provided they occur in the core of strategy — that is, when they occur in those expressions that constitute a winning strategy. Indeed, Tait's functions π and π' corresponds to our left and right instuctions — though we differentiate instructions for each logical constant adding an exponential to identify them. However, we do not have explicitly the function σ of Tait, though the result of the substitution of an instruction with a pair of embedded instructions — what we call it's resolution — will yield the pair of its components.

- Now, since in the antecedent y in $C(x, y)$ is of type $B(x)$, for any $x : A$, and since, as already mentioned, y is equal to $v(x)$ in $B(x)$, then it follows that $C(x, y)$ in the antecedent is, for any $x : A$, intensionally equal to $C(x, v(x))$ in the type *set*. More generally, and independently of **O**'s particular choice for the play object for the antecedent, and independently of **O**'s particular choice of x, $C(x, y)$ and $C(x, f(x))$ are two equal sets (for any $x : A$ and for $y : B(x)$).

From the two last steps it follows that **P** can copy cat the play object for the antecedent into the play object for the consequent, so that provides a winning strategies and the play objects in question are those relevant for the demonstration: one can then say that they are proof objects.

We will only deploy the plays that have been extracted of the extensive tree of all the plays. These plays constitute the so-called core of the strategy (that is, of the dialogical proof)[12], and they are triggered by the Opponent options at move 9 when challenging the existential posited by the Proponent at move 8. Since **O**'s repetition rank is 1, she cannot perform both challenges within one and the same play, hence the distinction between the following two plays. The first play corresponds in the demonstration to the introduction of the universal in the consequent, under the assumption of the antecedent. The second play develops all the points of the informal demonstration described above:

First play: Opponent's 9^{th} move asks for the left play object for the existential quantification on f

Description:
Move 3: After setting the thesis and establishing the repetition ranks **O** launches an attack on material implication.
Move 4: **P** launches a counterattack and asks for the play object that corresponds to $L^{\rightarrow}(p)$.
Moves 5, 6: **O** responds to the challenge of 4. **P** posits the right component of the material implication.
Moves 7, 8: **O** asks for the play object that corresponds to $R^{\rightarrow}(p)$. **P** responds to the challenge by choosing the pair (v, r) where v is the play object chosen to substitute the variable f and r the play object for the right component of the existential.
Move 9: **O** has here the choice to ask for the left or the right component of the existential. The present play describes the development of the play triggered by the left choice.
Moves 10-26: Follow from a straightforward application of the dialogical

[12]For the process of their extraction and for the proof that these plays render the corresponding CTT demonstration see Clerbout/Rahman (2014).

rules. Move 26 is an answer to move 13, since **P** decided to have enough information to apply the characteristic copy-cat method imposed by the formal rule.

Move 27-28: **O** asks for the play object that corresponds to the instruction posited by **P** at move 26 and **P** answers and wins by applying copy-cat to **O**'s move 25. Notice that 28 is not a case of function substitution: it is simply the resolution of an instruction.

O			**P**		
	H1:$C(x,y) : set(x : A, y : B(x))$ H2:$B(x) : set(x : A)$			$p : (\forall x : A)(\exists y : B(x))C(x,y) \rightarrow$ $(\exists f : (\forall x : A)B(x))(\forall x : A)C(x,f(x))$	0
1	m:=1			n:=2	2
3	$L^{\rightarrow}(p) : (\forall x : A)(\exists y : B(x))C(x,y)$	0		$R^{\rightarrow}(p) : (\exists f : (\forall x : A)B(x))$ $(\forall x : A)C(x,f(x))$	6
5	$v : (\forall x : A)(\exists y : B(x))C(x,y)$		3	$L^{\rightarrow}(p)/?$	4
7	$R^{\rightarrow}(p)/?$	6		$(v,r) : (\exists f : (\forall x : A)B(x))$ $(\forall x : A)C(x,f(x))$	8
9	$L_?$	8		$L^{\exists}(v,r) : (\forall x : A)B(x)$	10
11	$L^{\exists}(v,r)/?$	10		$v : (\forall x : A)B(x)$	12
13	$L^{\forall}(v) : A$	12		$R^{\forall}(v) : B(w)$	26
15	$w : A$		13	$L^{\forall}(v)/?$	14
19	$R^{\forall}(v) : (\exists y : B(w))C(w,y)$		5	$L^{\forall}(v) : A$	16
17	$L^{\forall}(v)/?$	16		$w : A$	18
21	$(t_1,t_2) : (\exists y : B(w))C(w,y)$		19	$R^{\forall}(v)/?$	20
23	$L^{\exists}(t_1,t_2) : B(w)$		21	$L_?$	22
25	$t_1 : B(w)$		23	$L^{\exists}(t_1,t_2)/?$	24
27	$R^{\forall}(v)/?$	26		$t_1 : B(w)$	28

Second play: Opponent's 9th move asks for the right play object for the existential quantification on f

Description:

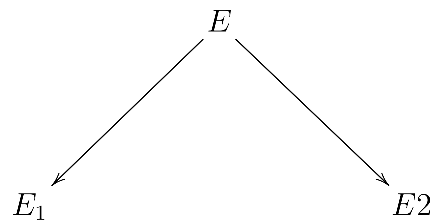
Move 9: Until move 9 this play is the same as the previous. In the present play, in move 9 the Opponent chooses to ask for the right-hand side of the existential posited by **P** at 8.

Moves 10-34: The Proponent substitutes the variable f by the instruction correspondent to to the left-hand component of the existential, i.e., $L^{\exists}(v,r)$. By this **P** accounts for the dependence of the right-hand part on the left-hand component. The point is that the local meaning of the existential requires this dependence of the right component to the left component even if in this play the Opponent, due to the restriction on rank 1, she can ask only for the right-hand part.

	O			**P**	
	H1:$C(x,y)$:$set(x{:}A, y{:}B(x))$ H2:$B(x)$:$set(x{:}A)$			p:$(\forall x{:}A)(\exists y{:}B(x))C(x,y) \rightarrow (\exists f{:}(\forall x{:}A)B(x))(\forall x{:}A)C(x,f(x))$	0
1	m:=1			n:=2	2
3	$L^{\rightarrow}(p)$:$(\forall x{:}A)(\exists y{:}B(x))C(x,y)$	0		$R^{\rightarrow}(p)$:$(\exists f{:}(\forall x{:}A)B(x))(\forall x{:}A)C(x,f(x))$	6
5	v:$(\forall x{:}A)(\exists y{:}B(x))C(x,y)$		3	$L^{\rightarrow}(p)/?$	4
7	$R^{\rightarrow}(p)/?$	6		(v,r):$(\exists f{:}(\forall x{:}A)B(x))(\forall x{:}A)C(x,f(x))$	8
9	$R_?$	8		$R^{\exists}(v,r)$:$(\forall x{:}A)C(x, L^{\exists}(v,r)(x))$	10
11	$L^{\exists}(v,r)/?$	10		$R^{\exists}(v,r)$:$(\forall x{:}A)C(x,v(x))$	12
13	$R^{\exists}(v,r)/?$	12		r:$(\forall x{:}A)C(x,v(x))$	14
15	$L^{\forall}(r)$:A	14		$R^{\forall}(r)$:$C(x,v(w))$	32
17	w:A		15	$L^{\forall}(r)/?$	16
21	$R^{\forall}(v)$:$(\exists y{:}B(x))C(x,y)$		5	$L^{\forall}(v)$:A	18
19	$L^{\forall}(v)/?$	18		w:A	20
23	(t_1,t_2):$(\exists y{:}B(x))C(x,y)$		21	$R^{\forall}(v)/?$	22
25	$L^{\exists}(t_1,t_2)$:$B(w)$		23	$L_?$	24
27	t_1:$B(w)$		25	$L^{\exists}(t_1,t_2)/?$	26
29	$R^{\exists}(t_1,t_2)$:$C(w,t_1)$		23	$R_?$	28
31	t_2:$C(w,t_1)$		29	$R^{\exists}(t_1,t_2)/?$	30
33	$R^{\forall}(r)/?$	32		t_2:$C(w,v(w))$	34
35	$v(w)/?$	34		t_2:$C(w,t_1)$ $\langle C(w,t_1) = C(w,t_1/v(w))$:$set\rangle$	42
41	$C(w,t_1) = C(w,t_1/v(w))$:set		H1$?_{subs}$	$v(w) = t_1$:$B(w)$	36
37	$v(w) = t_1$:$B(w)?$	36		$sic(39)$	40
39	$v(w) = t_1$:$B(w)$	5, 18, 21,25		$?\forall$-eq	38

The conceptually interesting moves start with 35, where the Opponent asks **P** to substitute the function. As already pointed out, in order to respond to 35 the Opponent's move 31 is not enough. Indeed the Proponent needs also to posit $C(w,t_1) = C(w,t_1/v(w))$: *set*. **P** forces **O** to concede this equality (41), on the basis of the substitutions w/x and t_1/y on H1 (we implemented the substitution directly in the answer of **O**) given the ∀-equality $v(w) = t_1$ in $B(w)$ (36), and given that this ∀-equality yields the required set equality. Moreover, **P**'s posit of the ∀-equality (36) is established and defended by moves 38-40.

We can now compose both plays and build up a **P**-winning strategy for AC:



where E is the sequence

H1:		$C(x,y)$:$set(x$:A, y:$B(x))$
H2:		$B(x)$:$set(x$:$A)$
0	**P**	p:$(\forall x$:$A)(\exists y$:$B(x))C(x,y) \to (\exists f$:$(\forall x$:$A)B(x))(\forall x$:$A)C(x,f(x))$
1	**O**	n:=1
2	**P**	m:=1
3	**O**	$L^{\to}(p)$:$(\forall x$:$A)(\exists y$:$B(x))C(x,y)$ [? 0]
4	**P**	$L^{\to}(p)/?$ [? 3]
5	**O**	v:$(\forall x$:$A)(\exists y$:$B(x))C(x,y)$ [4]
6	**P**	$R^{\to}(p)$:$(\exists f$:$(\forall x$:$A)B(x))(\forall x$:$A)C(x,f(x))$ [3]
7	**O**	$R^{\to}(p)/?$ [? 6]
8	**P**	(v,r):$(\exists f$:$(\forall x$:$A)B(x))(\forall x$:$A)C(x,f(x))$ [7]

E_1 is the sequence

9	**O**	$L_?$ [? 8]
10	**P**	$L^{\exists}(v,r)$:$(\forall x$:$A)B(x)$ [9]
11	**O**	$L^{\exists}(v,r)/?$ [? 10]
12	**P**	v:$(\forall x$:$A)B(x)$ [11]
13	**O**	$L^{\forall}(v)$:A [? 12]
14	**P**	$L^{\forall}(v)/?$ [? 13]
15	**O**	w:A [14]
16	**P**	$L^{\forall}(v)$:A [? 5]
17	**O**	$L^{\forall}(v)/?$ [? 16]
18	**P**	w:A [17]
19	**O**	$R^{\forall}(v)$:$(\exists y$:$B(x))C(w,y)$ [16]
20	**P**	$R^{\forall}(v)/?$ [? 19]
21	**O**	(t_1,t_2):$(\exists y$:$B(w))C(w,y)$ [20]
22	**P**	$L_?$ [? 21]
23	**O**	$L^{\exists}(t_1,t_2)$:$B(w)$ [22]
24	**P**	$L^{\exists}(t_1,t_2)/?$ [? 23]
25	**O**	t_1:$B(w)$ [22]
26	**P**	$R^{\forall}(v)$:$B(w)$ [13]
27	**O**	$R^{\forall}(v)/?$ [? 26]
28	**P**	t_1:$B(w)$ [27]

and E_2 is the sequence

9	**O**	$R_?$ [? 8]
10	**P**	$R^{\exists}(v,r){:}(\forall x{:}A)C(x, L^{\exists}(v,r)(x))$ [9]
11	**O**	$L^{\exists}(v,r)/?$ [? 10]
12	**P**	$R^{\exists}(v,r){:}(\forall x{:}A)C(x, v(x))$ [11]
13	**O**	$R^{\exists}(v,r)/?$ [? 12]
14	**P**	$r{:}(\forall x{:}A)C(x, v(x))$ [13]
15	**O**	$L^{\forall}(v){:}A$ [? 14]
16	**P**	$L^{\forall}(v)/?$ [? 15]
17	**O**	$w{:}A$ [16]
18	**P**	$L^{\forall}(v){:}A$ [? 5]
19	**O**	$L^{\forall}(v)/?$ [? 5]
20	**P**	$w{:}A$ [19]
21	**O**	$R^{\forall}(v){:}(\exists y{:}B(x))C(w, y)$ [18]
22	**P**	$R^{\forall}(v)/?$ [? 21]
23	**O**	$(t_1, t_2){:}(\exists y{:}B(w))C(w, y)$ [22]
24	**P**	$L_?$ [? 23]
25	**O**	$L^{\exists}(t_1, t_2){:}B(w)$ [24]
26	**P**	$L^{\exists}(t_1, t_2)/?$ [? 25]
27	**O**	$t_1{:}B(w)$ [26]
28	**P**	$R_?$ [? 23]
29	**O**	$R^{\exists}(t_1, t_2){:}C(w, t_1)$ [28]
30	**P**	$R^{\exists}(t_1, t_2)/?$ [? 29]
31	**O**	$t_2{:}C(w, t_1)$ [30]
32	**P**	$R^{\forall}(r){:}C(w, v(w))$ [15]
33	**O**	$R^{\forall}(r)/?$ [? 32]
34	**P**	$t_2{:}C(w, v(w))$ [33]
35	**O**	$v(w)/?$ [? 34]
36	**P**	$v(w) = t_1{:}B(w)$ [? H1]
37	**O**	$v(w) = t_1?$ [? 36]
38	**P**	? $\forall$-*eq* [? 5, 18, 21,25]
39	**O**	$v(w) = t_1{:}B(w)$ [38]
40	**P**	*sic* [39]
41	**O**	$C(w, t_1) = C(w, t_1/v(w)){:}set$ [36]
42	**P**	$t_2{:}C(w, t_1)\langle C(w, t_1) = C(w, t_1/v(w)){:}set\rangle$ [35]

Notation: In order to identify the dialogical source of each move we make use of [? n] to indicate the attacked line and [m] to indicate the challenge of player X that triggered the posited defence of Y.

3 Conclusions: Functions as rules of correspondence, as interactions

As mentioned in the introduction Coquand (2014) pointed out that recursive functions not only cannot provide a non-circular definition of constructivity, they also are not needed for doing constructive mathematics. What we need is to understand functions as rules of correspondence. Rules of correspondence only make sense if we know how the correspondences are to be carried out and so this supports that the rule-conception of function is the one that leads to the epistemic perspective characteristics of constructivism. Now, let us come once more to what we take to be the positive insights of Hintikka's proposal. Hintikka proposes the idea that the game theoretical interpretation of AC makes its truth evident because of the interplay between the outside-inside reading that is carried out by the interaction of the players. Moreover, Hintikka (2001) seems to think that the rejection of the classical formulation comes from the fact that the constructivists require a notion of *knowing which* or *who, a knowledge of objects*:

> *The crucial notion, in other words, is not knowing that but knowing what (which, who, where, ...) in brief knowing + an indirect question, that is, knowledge of objects (...).* Hintikka (2001), p. 10.

> *Now we can also begin to see the relationship between intuitionism and constructivism. The basic difference in fact allows a simple formulation. An intuitionist of the classical variety wants to restrict his or her attention to known mathematical objects. A constructivist wants to restrict his or her attention to effective or otherwise constructible objects. But is this a distinction with difference? More obviously has to be said here. The two coincide if and only if it is a necessary and sufficient condition for a mathematical object like a function to be knowable that it be constructible. Is this perhaps the case?* Hintikka (2001), p. 15.

Hintikka seems here to realize that to define what a constructive function is might be a difficult issue. Further on in that paper he proposes to replace the notion of constructivity by the notion of an effective winning strategy. And with it he means *human playable*. Certainly he seems to fall in the circularity mentioned by Coquand — what does the existential mean in *there exists an effective strategy*? — but we do think that some interesting ideas can be extracted. It looks indeed very tempting to understand rules of correspondence as those that are carried out by players during an interaction. From the dialogical (and more generally from the game theoretical) point of view a function is the result of a player choosing an object of the domain and the defender choosing the suitable match. This can be seen as carrying out a rule of correspondence. However, the problem is that more has to be said to make an intensional function out of these interactions — also extensionality is brought forward by interaction after all. The next task to study in the dialogical framework is to show how to prove extensionality assuming third excluded but no unicity. This study should help to pinpoint the notion of function that results from players' interaction.

Let us finish with two general philosophical issues on the notion of *human-playable games* and the *outside-inside* approach to meaning. The precise links between them are still to be worked out.

(1) From the constructivist point of view, in order to understand a play, and a winning strategy (i.e., a game-theoretical proof) constituted by the relevant plays, it is not enough to know the rules of the game, not even enough to believe that there is a winning strategy behind the moves: what we need is to be able to describe the moves in such a way that it makes their contribution to the winning strategy understandable. A proof beyond our capacities to describe it does not produce knowledge at all. This is what Hintikka's use of Wittgenstein's notion of *human-playable games* amounts to. Moreover, this comes very close to some remarks of Poincaré when comparing chess and mathematics:

> *Si vous assistez à un partie d'échecs, il ne vous suffira pas, pour comprendre la partie, de savoir les règles de la marche des pièces. Cela vous permettrait seulement de reconnaître que chaque coup a été joué conformément à ces règles et cet avantage aurait vraiment bien peu de prix. C'est pourtant ce que ferait le lecteur d'un livre de Mathématiques, s'il n'était que logicien. Comprendre la partie, c'est toute autre chose; c'est savoir pourquoi le joueur avance telle pièce plutôt que telle autre qu'il aurait pu faire mouvoir sans violer les rgles du jeu. C'est apercevoir la raison intime qui fait de fait de cette série de coups successifs une sorte de tout organisé. A plus forte raison, cette faculté est-elle nécessaire au joueur lui-même, c'est-à-dire à l'inventeur.* Poincaré (1905, pp-30-31).[13]

Based in this and other texts Gerhard Heinzmann (1985, 1986, 1995, 2013) and Michel Detlefsen (1992) develop the idea that Poincaré, while criticizing the purely formal approach to proof in mathematics of those he called the *logicisists*, is aiming

[13] *If you are present at a game of chess, it will not suffice, for the understanding of the game, to know the rules for moving the pieces. That will only enable you to recognize that each move has been made in conformity with these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book on mathematics would do if he were a logician only. To understand the game is wholly another matter; it is to know why the player moves this piece rather than that other which he could have moved without breaking the rules of the game. It is to perceive the inward reason which makes of this series of successive moves a sort of organized whole. This faculty is still more necessary for the player himself, that is, for the inventor.* Poincaré (2014, pp. 23-24).

at an epistemological approach to proof in mathematics that deserves to call him a pre-intuitionist. From this perspective the criticism of Poincaré to the logical (purely formal) understanding of proof is that their notion of proof is lacking its epistemological role of producing conceptual insight. This takes to the next general philosophical point:

(2) One further deep remark by Heinzmann and Detlefsen is that Poincaré's epistemological understanding of the role of proof in mathematics is very close to Kant's conception of the role of inference as building the conceptual architectonic of a science. Mathematics, as every science, constitutes a whole structure of concepts, an Architectonic, in Poincaré's words *"une sorte de tout organisé"*; and the role of inference is to extend this structure or build new links within it. It is important to notice that this kind of holism might lead us to another important insight of Kant's: it is the judgement that provides the fundamental unit of knowledge, and so the meaning of each substantial expression is derivative from its role in a judgement and not the other way round. As pointed out by Robert Brandom (2000, p. 13)[14] this top-down approach to meaning, that contests the compositional standard model-theoretic semantics, led Gottlob Frege to the formulation of his notorious *contextuality principle* and led Wittgenstein to privilege for his theory of *meaning as use*, those linguistic expressions, namely sentences, that make a move in the language-game. This is, so we claim, the very point and origin of the *outside-inside* approach to meaning provided by the game-theoretical approach. Moreover and more precisely, the dialogical frame shares with the Kantian conception the understanding of the outside-inside approach to meaning as the result of deploying an interplay of entitlements and commitments. To make use again of Brandom's formulation:

> *Kant takes the judgement to be the minimal unit of experience (and so of awareness in his discursive sense) because it is the first element in the traditional logical hierarchy that one can take* **responsibility** *for.* Brandom (2000, p. 13)

Though Hintikka blurs the game-theoretical contribution of the outside-inside approach to meaning by adopting a model theoretical semantics for atomic propositions, his overall ideas on the foundations of mathematics do provide us with an insight that, so we claim, can be deployed by the dialogical perspective. The philosophical task ahead is to study further the links between the outside-inside strategy with the concept of human-playable in the context of the notion of constructivity. We are looking forward to accomplish the task.

[14]In fact this is the main idea that animates the whole project of Brandom's *pragmatist rationalism*.

Appendix I: Standard Dialogical Logic[15]

Let L be a first-order language built as usual upon the propositional connectives, the quantifiers, a denumerable set of individual variables, a denumerable set of individual constants and a denumerable set of predicate symbols (each with a fixed arity).

We extend the language L with two labels $\mathbf{O}$ and $\mathbf{P}$, standing for the players of the game, and the question mark '?'. When the identity of the player does not matter, we use variables $\mathbf{X}$ or $\mathbf{Y}$ (with $\mathbf{X} \neq \mathbf{Y}$). A *move* is an expression of the form '$\mathbf{X}$-e', where e is either a formula φ of L or the form '$?[\varphi_1, \ldots, \varphi_n]$'.

We now present the rules of dialogical games. There are two distinct kinds of rules named particle (or local) rules and structural rules. We start with the particle rules.

Previous move	$\mathbf{X}$-$\varphi \wedge \psi$	$\mathbf{X}$-$\varphi \vee \psi$	$\mathbf{X}$-$\varphi \to \psi$	$\mathbf{X}$-$\neg\varphi$
Challenge	$\mathbf{Y}$-$?[\varphi]$ or $\mathbf{Y}$-$?[\psi]$	$\mathbf{Y}$-$?[\varphi, \psi]$	-φ	$\mathbf{Y}$-φ
Defence	$\mathbf{X}$-φ resp. $\mathbf{X}$-ψ	$\mathbf{X}$-φ or $\mathbf{X}$-ψ	$\mathbf{X}$-ψ	$--$

Previous move	$\mathbf{X}$-$\forall\varphi$	$\mathbf{X}$-$\exists\varphi$
Challenge	$\mathbf{Y}$-$?[\varphi(a/x)]$	$\mathbf{Y}$-$?[\varphi(a_1/x), \ldots, \varphi(a_n/x)]$
Defence	$\mathbf{X}$-$\varphi(a/x)$	$\mathbf{X}$-$\varphi(a_i/x)$ with $1 \leq i \leq n$

In this table, the a_i's are individual constants and $\varphi(a_i/x)$ denotes the formula obtained by replacing every free occurrence of x in φ by a_i. When a move consists in a question of the form '$?[\varphi_1, \ldots, \varphi_n]$', the other player chooses one formula among $\varphi_1, \ldots, \varphi_n$ and plays it. We can thus distinguish between conjunction and disjunction on the one hand, and universal and existential quantification on the other hand, in terms of which player has a choice. In the cases of conjunction and universal quantification, the challenger chooses which formula he asks for. Conversely, in the cases of disjunction and existential quantification, the defender is the one who can choose between various formulas. Notice that there is no defence in the particle rule for negation.

Particle rules provide an abstract description of how the game can proceed locally: they specify the way a formula can be challenged and defended according to its main logical constant. In this way we say that these rules govern the local level of meaning. Strictly speaking, the expressions occurring in the table above are not actual moves because they feature formulas schemata and the players are not specified. Moreover, these rules are indifferent to any particular situations that might occur during the game. For these reasons we say that the description provided by the particle rules is abstract. The words "challenge" and "defence" are convenient to name certain moves according

[15]The following brief presentation of standard dialogical logic has been extracted from Nicolas Clerbout (2013b).

to their relationship with other moves. Such relationships can be precisely defined in the following way. Let Σ be a sequence of moves. The function p_Σ assigns a position to each move in Σ, starting with 0. The function F_Σ assigns a pair $[m, Z]$ to certain moves N in Σ, where m denotes a position smaller than $p_\Sigma(N)$ and Z is either C or D, standing respectively for "challenge" and "defence". That is, the function F_Σ keeps track of the relations of challenge and defence as they are given by the particle rules. A *play* (or dialogue) is a legal sequence of moves, i.e., a sequence of moves which observes the game rules. The rules of the second kind that we mentioned, the structural rules, give the precise conditions under which a given sentence is a play. The *dialogical game* for φ, written $D(\varphi)$, is the set of all plays with φ as the thesis (see the Starting rule below). The structural rules are the following:

SR0 (Starting rule) Let φ be a complex formula of L. For every $\pi \in D(\varphi)$ we have:

- $p_\pi(\mathbf{P}\text{-}\varphi) = 0$,
- $p_\pi(\mathbf{O}\text{-}n\!:= i) = 1$,
- $p_\pi(\mathbf{P}\text{-}m\!:= j) = 2$.

In other words, any play π in $D(\varphi)$ starts with $\mathbf{P}$-φ. We call φ the *thesis* of the play and of the dialogical game. After that, the Opponent and the Proponent successively choose a positive integer called *repetition rank*. The role of these integers is to ensure that every play ends after finitely many moves, in a way specified by the next structural rule.

SR1 (Classical game-playing rule)

- Let $\pi \in D(\varphi)$. For every M in π with $p_\pi(M) > 2$ we have $F_\pi(M) = [m', Z]$ with $m' < p_\pi(M)$ and $Z \in \{C, D\}$.
- Let r be the repetition rank of player $\mathbf{X}$ and $\pi \in D(\varphi)$ such that
  - the last member of π is a $\mathbf{Y}$ move,
  - M_0 is a $\mathbf{Y}$ move of position m_0 in π,
  - $M_1, \ldots, M_n$ are $\mathbf{X}$ moves in π such that $F_\pi(M_1) = \ldots = F_\pi(M_n) = [m_0, Z]$.

  Consider the sequence[16] $\pi' = \pi * N$ where N is an $\mathbf{X}$ move such that $F_{\pi'}(N) = [m_0, Z]$. We have $\pi' \in D(\varphi)$ only if $n < r$.

The first part of the rule states that every move after the choice of repetition ranks is either a challenge or a defence. The second part ensures finiteness of plays by setting

[16] We use $\pi * N$ to denote the sequence obtained by adding move N to the play π.

the player's repetition rank as the maximum number of times he can challenge or defend against a given move of the other player.

SR2 (Formal rule) Let ψ be an elementary sentence, N be the move **P**-ψ and M be the move **O**-ψ. A sequence π of moves is a play only if we have: if $N \in \pi$ then $M \in \pi$ and $p_\pi(M) < p_\pi(N)$.

A play is called *terminal* when it cannot be extended by further moves in compliance with the rules. We say it is **X** terminal when the last move in the play is an **X** move.

SR3 (Winning rule) Player **X** wins the play π only if it is **X** terminal.

Consider for example the following sequences of moves: **P**$-Qa \rightarrow Qa$, **O**-n: $= 1$, **P**-m: $= 12$, **O**-Qa, **P**-Qa. We often use a convenient table notation for plays. For example, we can write this play as follows:

O			**P**		
				$Qa \rightarrow Qa$	0
1	n:=1			m:=12	2
3	Qa	(0)		Qa	4

The numbers in the external columns are the positions of the moves in the play. When a move is a challenge, the position of the challenged move is indicated in the internal columns, as with move 3 in this example. Notice that such tables carry the information given by the functions p and F in addition to represent the play itself.

However, when we want to consider several plays together — for example when building a strategy — such tables are not that perspicuous. So we do not use them to deal with dialogical games for which we prefer another perspective. The *extensive form* of the dialogical game $D(\varphi)$ is simply the tree representation of it, also often called the game-tree. More precisely, the extensive form E_φ of $D(\varphi)$ is the tree (T, l, S) such that:

i) Every node t in T is labelled with a move occurring in $D(\varphi)$.

ii) $l : T \rightarrow N$.

iii) $S \subseteq T^2$ with:

- There is a unique t_0 (the root) in T such that $l(t_0) = 0$, and t_0 is labelled with the thesis of the game.
- For every $t \neq t_0$ there is a unique t' such that $t'St$.
- For every t and t' in T, if tSt' then $l(t') = l(t) + 1$.

- Given a play π in $D(\varphi)$ such that $p_\pi(M') = p_\pi(M) + 1$ and t, t' respectively labelled with M and M', then tSt'.

A *strategy* for Player **X** in $D(\varphi)$ is a function which assigns an **X** move M to every non terminal play π with a **Y** move as last member such that extending π with M results in a play. An **X** strategy is *winning* if playing according to it leads to **X**'s victory no matter how **Y** plays.

A strategy can be considered from the viewpoint of extensive forms: the extensive form of an **X** strategy σ in $D(\varphi)$ is the tree-fragment $E_{\varphi,\sigma} = (T_\sigma, l_\sigma, S_\sigma)$ of E_φ such that:

i) The root of $E_{\varphi,\sigma}$ is the root of E_φ.

ii) Given a node t in E_φ labelled with an **X** move, we have that $tS_\sigma t'$ whenever tSt'.

iii) Given a node t in E_φ labelled with a **Y** move and with at least one t' such that tSt', then there is a unique $\sigma(t)$ in T_σ where $tS_\sigma\sigma(t)$ and $\sigma(t)$ is labelled with the **X** move prescribed by σ.

Here are some examples of results which pertain to the level of strategies.[17]

- Winning **P** strategies and leaves. *Let w be a winning* **P** *strategy in $D(\varphi)$. Then every leaf in $E_{\varphi,w}$ is labelled with a* **P** *signed atomic sentence.*

- Determinacy. *There is a winning* **X** *strategy in $D(\varphi)$ if and only if there is no winning* **Y** *strategy in $D(\varphi)$.*

- Soundness and Completeness of Tableaux. *Consider first-order tableaux and first-order dialogical games. There is a tableau proof for φ if and only if there is a winning* **P** *strategy in $D(\varphi)$.*

By soundness and completeness of the tableau method with respect to model-theoretical semantics, it follows that existence of a winning **P** strategy coincides with validity: *There is a winning* **P** *strategy in $D(\varphi)$ if and only if φ is valid.*

[17] These results are proven, together with others, in Clerbout (2013b).

Appendix II: The Dialogical Approach to CTT[18]

The Formation of Propositions

In standard dialogical systems there is a presupposition that the players use well-formed formulas (wff's). One can check well-formedness at will, but only via the usual metareasoning by which one checks that the formula indeed observes the definition of wff. The first addendum we want to make is to allow players to question the status of expressions, in particular to question the status of something as actually standing for a proposition. Thus we start with rules giving a dialogical explanation of the *formation* of propositions. These are local rules added to the particle rules which give the local meaning of logical constants.

Let us make a remark before displaying the formation rules. Because the dialogical theory of meaning is based on argumentative interaction, dialogues feature expressions which are not posits or sentences. They also feature requests used as challenges, as illustrated by the formation rules below and the particle rules in the next section. Now, by the *No entity without type* principle, the type of these actions, which type we shall write "formation-request", should be specified during a dialogue.

Posit	**Challenge** [when different challenges are possible, the challenger chooses]	**Defence**
$\mathbf{X}!\Gamma{:}set$	$\mathbf{Y}\ ?_c\Gamma$	$\mathbf{X}!a_1{:}\Gamma, \mathbf{X}!a_2{:}\Gamma, \ldots$ $\mathbf{X}$ gives the canonical elements of Γ; provides a generation method $a_i{:}\Gamma \Rightarrow a_j{:}\Gamma$; provides the equality rules
$\mathbf{X}!\varphi \vee \psi{:}prop$ (similar applies for the rest of the propositional connectives)	$\mathbf{Y}\ ?_{F_{\vee 1}}$ or $\mathbf{Y}\ ?_{F_{\vee 2}}$	$\mathbf{X}!\varphi{:}prop$ $\mathbf{X}!\psi{:}prop$
$\mathbf{X}!(\forall x{:}A)\varphi(x){:}prop$	$\mathbf{Y}\ ?_{F_{\forall 1}}$ or $\mathbf{Y}\ ?_{F_{\forall 2}}$	$\mathbf{X}!A{:}set$ $\mathbf{X}!\varphi(x){:}prop\ (x{:}A)$

By definition the *falsum* symbol $\bot$ is of type *prop*. Therefore the formation of a posit of the form $\bot$ cannot be challenged.

The next rule is not a formation rule *per se* but rather a substitution rule.[19]

Posit-substitution

There are two cases in which $\mathbf{Y}$ can ask $\mathbf{X}$ to make a substitution in the context $x_i{:}A_i$. The first one is when in a standard play a variable (or a list of variables) occurs

[18]The present overview on the dialogical approach to CTT is based on Rahman/Clerbout (2013, 2014).

[19]It is an application of the original rule from CTT given in Ranta (1994, p. 30).

in a posit with a proviso. Then the challenger posits an instantiation of the proviso:

Posit	**Challenge**	**Defence**
$\mathbf{X}\ !\ \pi(x_1, \ldots, x_n)\ (x_i{:}A_i)$	$\mathbf{Y}\ !\ \tau_1{:}A_1, \ldots, \tau_n{:}A_n$	$\mathbf{X}\ !\ \pi(\tau_1, \ldots, \tau_n)$

The second case is in a formation-play. In such a play the challenger simply posits the whole assumption as in Move 7 of the example below:

Posit	**Challenge**	**Defence**
$\mathbf{X}\ !\ \pi(\tau_1, \ldots, \tau_n)\ (\tau_i{:}A_i)$	$\mathbf{Y}\ !\ \tau_1{:}A_1, \ldots, \tau_n{:}A_n$	$\mathbf{X}\ !\ \pi(\tau_1, \ldots, \tau_n)$

Play objects

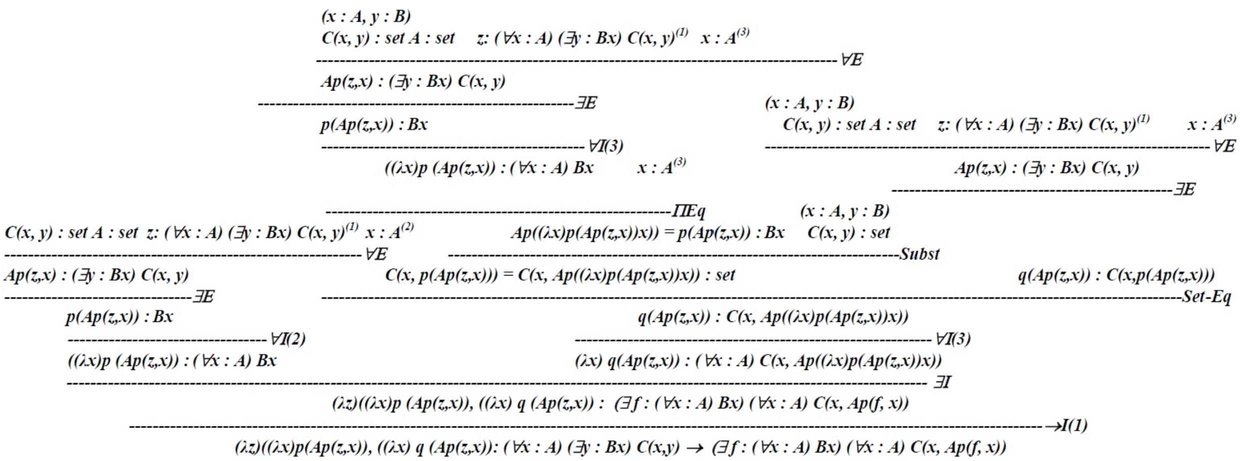
The idea is now to design dialogical games in which the players' posits are of the form "$p{:}\varphi$" and acquire their meaning in the way they are used in the game — i.e., how they are challenged and defended. This requires, among others, to analyse the form of a given play-object p, which depends on φ, and how a play-object can be obtained from other, simpler, play-objects. The standard dialogical semantics for logical constants gives us the needed information for this purpose. The main logical constant of the expression at stake provides the basic information as to what a play-object for that expression consists of:

> A play for $\mathbf{X}\ \varphi \vee \psi$ is obtained from two plays p_1 and p_2, where p_1 is a play for $\mathbf{X}\ \varphi$ and p_2 is a play for $\mathbf{X}\ \psi$. According to the particle rule for disjunction, it is the player $\mathbf{X}$ who can switch from p_1 to p_2 and vice-versa.
>
> A play for $\mathbf{X}\ \varphi \wedge \psi$ is obtained similarly, except that it is the player $\mathbf{Y}$ who can switch from p_1 to p_2.
>
> A play for $\mathbf{X}\ \varphi \to \psi$ is obtained from two plays p_1 and p_2, where p_1 is a play for $\mathbf{Y}\ \varphi$ and p_2 is a play for $\mathbf{X}\ \psi$. It is the player $\mathbf{X}$ who can switch from p_1 to p_2.
>
> The standard dialogical particle rule for negation rests on the interpretation of $\neg\varphi$ as an abbreviation for $\varphi \to \bot$, although it is usually left implicit. It follows that a play for $\mathbf{X}\ \neg\varphi$ is also of the form of a material implication, where p_1 is a play for $\mathbf{Y}\ \varphi$ and p_2 is a play for $\mathbf{X}\ \bot$, and where $\mathbf{X}$ can switch from p_1 to p_2.

As for quantifiers, we are dealing with quantifiers for which the type of the bound variable is always specified. We thus consider expressions of the form $(Qx{:}A)\varphi$, where Q is a quantifier symbol.

Posit	**Challenge**	**Defence**
X φ (where no play-object has been specified for φ)	**Y** ? play-object	**X** $p{:}\varphi$
X $p{:}\varphi \vee \psi$	**Y** $?_{prop}$	**X** $\varphi \vee \psi{:}prop$
	Y $?[\varphi, \psi]$	**X** $L^{\vee}(p){:}\varphi$ Or **X** $R^{\vee}(p){:}\psi$ **[the defender has the choice]**
X $p{:}\varphi \wedge \psi$	**Y** $?_{prop}$	**X** $\varphi \wedge \psi{:}prop$
	Y $?[\varphi]$ Or **Y** $?[\psi]$ **[the challenger has the choice]**	**X** $L^{\wedge}(p){:}\varphi$ Respectively **X** $L^{\wedge}(p){:}\psi$
X $p{:}\varphi \rightarrow \psi$	**Y** $?_{prop}$	**X** $\varphi \rightarrow \psi{:}prop$
	Y $L^{\rightarrow}(p){:}\varphi$	**X** $R^{\rightarrow}(p){:}\psi$
X $p{:}\neg\varphi$	**Y** $?_{prop}$	**X** $\neg\varphi{:}prop$
	Y $L^{\perp}(p){:}\varphi$	**X** $R^{\perp}(p){:}\perp$
X $p{:}(\exists x{:}A)\varphi$	**Y** $?_{prop}$	**X** $(\exists x{:}A)\varphi{:}prop$
	Y $?_{L}$ Or **Y** $?_{R}$ **[the challenger has the choice]**	**X** $L^{\{\cdots\}}(p){:}A$ Respectively **X** $R^{\{\cdots\}}(p){:}\varphi(L(p))$
X $p{:}(\forall x{:}A)\varphi$	**Y** $?_{prop}$	**X** $(\forall x{:}A)\varphi{:}prop$
	Y $L^{\forall}(p){:}A$	**X** $R^{\forall}(p){:}\varphi(L(p))$
X $p{:}B(k)$ (for atomic B)	**Y** $?_{prop}$	**X** $B(k){:}prop$
	Y ?	**X** $sic(n)$ (**X** indicates that **Y** posited it at move n)

Appendix III: Martin-Löf's demonstration of AC as a natural-deduction-tree



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