Reasoning about logic circuits with independent input errors

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Abstract

Having in mind applications to the design and verification of logic circuits, a complete extension of classical propositional logic is presented for reasoning about circuits with possibly erroneous inputs. The pitfalls of extrapolating classical reasoning to such circuits are extensively illustrated. Redundancy is shown to be effective for improving the reliability of such circuits.

Keywords: probabilistic logic, logic circuits with erroneous inputs.

Introduction

In [11] we presented a complete extension of classical propositional logic for reasoning about logic circuits with unreliable gates but error-free inputs, and illustrated the pitfalls of applying classical reasoning to such probabilistic circuits. Herein we assume that the gates are error-free and focus instead on the impact of allowing for errors on the circuit inputs. In due course, we show that the pitfalls found in [11] when applying classical logic to circuits with unreliable gates are already present even if the only source of errors is in the circuit inputs. For instance, *modus ponens* is not sound in general.

Accordingly, our goal here is the development and study of the main properties of an extension of classical propositional logic endowed with randomly-valued propositional variables. Several probabilistic and non-deterministic logics have been proposed in the literature [1, 2, 3, 4, 5, 7, 8, 9, 10] having in mind applications to computing and artificial intelligence, but they do not address our focus: the specific problems that appear when reasoning about circuits with random input errors.

To this end, we adopt the traditional representation of logic circuits by formulas with logical connectives representing gates and propositional variables standing for inputs. For instance, the formula

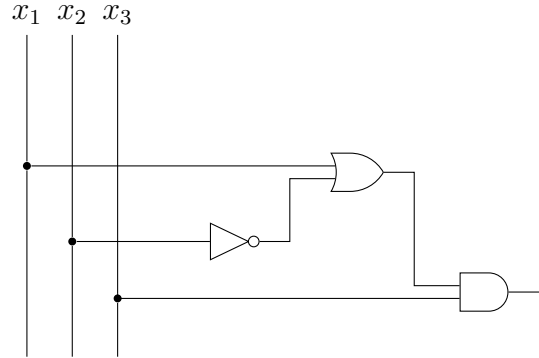$$(x_1 \vee (\neg\, x_2)) \wedge x_3$$

Figure 1: Circuit represented by the formula $(x_1 \vee (\neg\, x_2)) \wedge x_3$.

faithfully represents the circuit depicted in Figure 1.

If the circuit gates and inputs are assumed to be error-free then this representation can be and has been extensively used for applying classical propositional logic to the design and verification of circuits.

When the inputs may be erroneous but with sufficiently small probability of error, we show that redundancy can be used to improve the reliability of the circuit provided that, as assumed in this paper, independent observations of each input are available. The errors in different inputs are also assumed to be independent and every input observation is assumed to have the same probability $\varepsilon < \frac{1}{2}$ of error. Examples of application scenarios where these assumptions hold are given in due course. Under these assumptions it becomes possible to adapt von Neumann's redundancy technique [13] to circuits with possibly erroneous inputs.

Allowing for probabilistic propositional variables representing possibly erroneous inputs of the circuit requires a major semantic shift away from classical propositional logic: instead of valuations we need probabilistic mixtures of valuations. The proposed complete inference calculus includes some rules that may be surprising to the newcomer but that emerged quite naturally when adapting the calculus in [11] to the situation at hand: the key intuition was to look at each possibly erroneous input observation as resulting from applying the unreliable pass-through unary gate to an error-free input.

The language, the semantics and the calculus of the proposed *erroneous-inputs circuit logic* EICL are presented in Section 1 with some examples. Soundness and completeness results are established in Section 2. Although the proposed logic is a conservative extension of the classical propositional logic (as shown in Section 1), it is full of pitfalls for those used to the well known meta-properties of the latter. For instance, as proved at the end of Section 2, the metatheorem of deduction only holds with additional provisos. Application scenarios are briefly discussed in Section 3. Finally, in Section 4 we assess what was achieved and mention some open problems.

1   Erroneous-inputs circuit logic

The envisaged erroneous-inputs circuit logic (EICL) is defined below as an extension of classical propositional logic (PL).

Assume that PL is endowed with the rich *signature* Σ containing the propositional constants ff (falsum) and tt (verum) plus the usual propositional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\equiv$ (equivalence), as well as M_{3+2k} ($3+2k$-ary majority) for each $k \in \mathbb{N}$. Each majority connective returns true if the majority of its inputs is true and, otherwise, it returns false. Most of these signature elements could be introduced as abbreviations from a small set of primitives (like falsum and implication). Nevertheless, we adopt this rich signature in order to simplify the presentation of PL, since no price has to be paid in its axiomatisation and in the subsequent development of EICL.

For each $n \in \mathbb{N}$, we denote by Σ_n the set of n-ary constructors in Σ. Clearly, $\Sigma_0 = \{ff, tt\}$.

Formulas of PL are composed as usual with the elements of Σ and the *propositional variables* in $X = \{x_k : k \in \mathbb{N}\}$. The PL *language* (the set of its *formulas*) and the PL *ground language* (the set of its *ground formulas*, that is, formulas without variables) are denoted by $L(X)$ and L, respectively.

Assuming that the PL semantics is provided by valuations, we write

$$v \Vdash \varphi$$

for stating that valuation $v : X \to \{\bot, \top\}$ satisfies formula $\varphi \in L(X)$, and

$$\Lambda \vDash \varphi$$

for stating that φ is entailed from hypotheses in $\Lambda \subseteq L(X)$. Furthermore, assuming that PL is endowed with a Hilbert calculus including the tautologies (TAUT) as axioms plus modus ponens (MP) as the unique inference rule, we write

$$\Lambda \vdash \varphi$$

for stating that φ can be derived from Λ. Recall that this calculus is strongly sound and complete: $\Lambda \vDash \varphi$ if and only if $\Lambda \vdash \varphi$.

Before proceeding with the presentation of EICL, we also need some notation concerning the first-order theory of ordered real closed fields (denoted by ORCF), having in mind the use of its terms for denoting probabilities and other quantities.

Recall that the first-order signature of ORCF contains the constants 0 and 1, the unary function symbol $-$, the binary function symbols $+$ and $\times$, and the binary predicate symbols $=$ and $<$. As usual, we may write $t_1 \leq t_2$ for $(t_1 < t_2) \vee (t_1 = t_2)$, $t_1 t_2$ for $t_1 \times t_2$ and t^n for the product of t by itself n times. Furthermore, we also use the

following abbreviations for any given $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$: m for the sum of 1 with itself $m - 1$ times; m^{-1} for the unique z such that $m \times z = 1$; and $\dfrac{n}{m}$ for $m^{-1} \times n$. The last two abbreviations might be extended to other terms, but we need them only for numerals.

In order to avoid confusion with the other notions of satisfaction used herein, we adopt $\Vdash^{\mathsf{fo}}$ for denoting satisfaction in first-order logic.

Recall also that the theory ORCF is decidable [12]. This fact will be put to good use in the proposed axiomatisation of EICL. Furthermore, every model of ORCF satisfies the theorems and only the theorems of ORCF (Corollary 3.3.16 in [6]). We shall take advantage of this result in the semantics of EICL for adopting the ordered field $\mathbb{R}$ of the real numbers as the model of ORCF.

With this modicum of PL and of ORCF at hand, we are ready to present the syntax, the semantics and the calculus of EICL.

1.1   Syntax of EICL

The *signature* of EICL is the triple (Σ, ν, μ) where Σ is the PL signature above and both ν and μ are symbols used as follows for denoting probabilities. Each possibly erroneous input observation port is assumed to present the correct input value with probability $\nu > \frac{1}{2}$ and to present the incorrect input value with probability $\varepsilon = 1 - \nu < \frac{1}{2}$. The output produced by a circuit when all its input observation ports present correct inputs is said to be the *correct output* of the circuit. A circuit is to be accepted as good if it produces the correct output value with probability not less than the acceptance threshold μ.

The random *input variables* in $\widetilde{Y} = \{\widetilde{y}_k : k \in \mathbb{N}\}$ are used for representing possibly erroneous input observation ports. In addition, for each $k \in \mathbb{N}$, we use y_k for denoting the correct value of $\widetilde{y}_k$. Thus, $\widetilde{y}_k \equiv y_k$ is intended to be true with probability ν. In the sequel, $Y = \{y_k : k \in \mathbb{N}\}$. It becomes handy to use $\overline{y}_k$ as an abbreviation of $\neg\, y_k$. Clearly, $\widetilde{y}_k \equiv \overline{y}_k$ is intended to be true with probability $\varepsilon = 1 - \nu$.

Given a set W of generators, we denote by $L(W)$ the set of formulas built with propositional constants and connectives in Σ from generators in W. Given $\varphi \in L(X \cup Y)$ and $\psi \in L(X \cup Y \cup \widetilde{Y})$, we write

$$\varphi \sqsubseteq \psi$$

for saying that φ is a possible *outcome* of ψ. This outcome relation is inductively defined as expected:

- $\psi \sqsubseteq \psi$ provided that $\psi \in L(X)$;

- $y_k \sqsubseteq y_k$ for every $k \in \mathbb{N}$;

- $y_k \sqsubseteq \widetilde{y}_k$ for every $k \in \mathbb{N}$;

- $\overline{y}_k \sqsubseteq \widetilde{y}_k$ for every $k \in \mathbb{N}$;

- $c(\varphi_1, \ldots, \varphi_n) \sqsubseteq c(\psi_1, \ldots, \psi_n)$ provided that $n \geq 1$, $c \in \Sigma_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$.

For each such ψ, we denote by Ω_ψ the set $\{\varphi : \varphi \sqsubseteq \psi\}$ of all possible outcomes of ψ.

As already mentioned, terms are needed for denoting probabilities and other quantities. In EICL, by a *term* we mean a univariate polynomial written according to the term syntax of ORCF, using ν as the unique variable. For example, $\nu \times (1 - \nu)^2$ that we may write $\nu(1 - \nu)^2$ is a term of EICL. Symbol μ is also taken as a variable in the context of ORCF but it is not used in EICL terms.

Three kinds of formulas are needed for reasoning about circuits with possibly erroneous inputs:

- *Circuit formulas* or *c-formulas* that are the formulas built with propositional constants and connectives in Σ from generators in $X \cup Y \cup \widetilde{Y}$, that is, the formulas in $L(X \cup Y \cup \widetilde{Y})$. These c-formulas can be used (among other things) for representing circuits with possibly erroneous inputs. For instance, the c-formula

$$(*) \qquad \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2)$$

represents the circuit in Figure 2 with three independent observation ports of each of its two inputs. This circuit achieves a more robust conjunction of its two inputs as shown in due course. Circuit formulas can also be used for asserting relevant properties of circuits. For example, the c-formula

$$(**) \qquad \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

is intended to state that the circuit represented by $(*)$ can be accepted as being good in the sense that it produces the correct output with probability of at least μ. As we shall see, this is the case as long as μ does not exceed

$$3\nu^4 - 2\nu^6.$$

This is much better than what can be achieved, ν^2, without redundancy using the circuit represented by the c-formula

$$\widetilde{y}_1 \wedge \widetilde{y}_2.$$

For instance, if $\nu = \frac{4}{5}$ then μ must not exceed $\frac{11008}{15625} = 0.704512$ and $\frac{16}{25} = 0.64$, respectively, so as to be possible to accept the circuit as good. The detailed comparison between these two situations is made in Section 3.
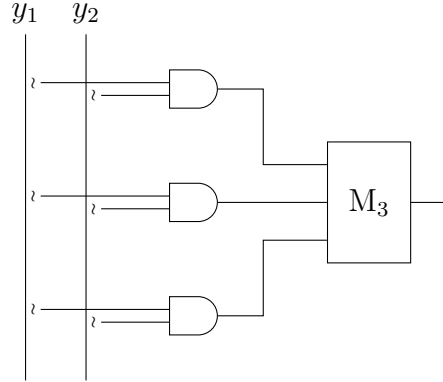
Figure 2: A robust circuit for conjunction.

- *Outcome formulas* or *o-formulas* that are of the general form

$$\Phi \sqsubseteq_P \psi$$

  where ψ is a c-formula, $\Phi \subseteq \Omega_\psi$ and P is an EICL term. Such an o-formula is used with the intent of stating that the probability of the outcome of ψ being in Φ is at least P. For instance,

$$\{y_1 \wedge y_2, y_1 \wedge \overline{y}_2\} \sqsubseteq_\nu \widetilde{y}_1 \wedge \widetilde{y}_2$$

  should be true in any interpretation of EICL because $y_1 \wedge y_2$ and $y_1 \wedge \overline{y}_2$ are both possible outcomes of $\widetilde{y}_1 \wedge \widetilde{y}_2$ (the former when both input observation ports present the correct input values and the latter when the first input port presents the correct input value while the second port is erroneous), the probability of the former is ν^2, the probability of the latter is $\nu(1 - \nu)$, and $\nu^2 + \nu(1 - \nu) = \nu$.

- *Ambition formulas* or *a-formulas* that are of the general form

$$\mu \le P$$

  where P is a term. Such an a-formula can be used for constraining the envisaged probability μ of the correct output. For instance, every EICL interpretation where the a-formula

$$\mu \le 3\nu^4 - 2\nu^6$$

  holds should make the c-formula $(**)$ true, as it is shown in due course.

Given m distinct formulas $\varphi_1, \ldots, \varphi_m$ in Ω_ψ, we may write

$$\varphi_1, \ldots, \varphi_m \sqsubseteq_P \psi$$

for $\{\varphi_1, \ldots, \varphi_m\} \sqsubseteq_P \psi$. We denote by $L^\circ(X \cup Y \cup \widetilde{Y})$ and L^{a} the set of o-formulas and a-formulas, respectively, and by $L^{\mathsf{eic}}(X \cup Y \cup \widetilde{Y})$ the set $L(X \cup Y \cup \widetilde{Y}) \cup L^\circ(X \cup Y \cup \widetilde{Y}) \cup L^{\mathsf{a}}$ of all EICL formulas. Observe that each of these sets is decidable.

Given a c-formula ψ and $\varphi \in \Omega_\psi$, we write

$$\mathfrak{P}[\psi \triangleright \varphi]$$

for the EICL term that provides the probability of φ being the outcome of ψ. This term is inductively defined as follows:

- $\mathfrak{P}[\psi \triangleright \psi]$ is 1 provided that $\psi \in L(X)$;

- $\mathfrak{P}[y_k \triangleright y_k]$ is 1 for each $k \in \mathbb{N}$;

- $\mathfrak{P}[\widetilde{y}_k \triangleright y_k]$ is ν for each $k \in \mathbb{N}$;

- $\mathfrak{P}[\widetilde{y}_k \triangleright \overline{y}_k]$ is $1 - \nu$ for each $k \in \mathbb{N}$;

- $\mathfrak{P}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\displaystyle\prod_{i=1}^n \mathfrak{P}[\psi_i \triangleright \varphi_i]$ for each $n \geq 1$, $c \in \Sigma_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$.

For instance, $\mathfrak{P}[\widetilde{y}_1 \wedge \widetilde{y}_2 \triangleright y_1 \wedge \overline{y}_2]$ is the polynomial $\nu(1 - \nu)$ since, for the given inputs $\widetilde{y}_1$ and $\widetilde{y}_2$, outcome $y_1 \wedge \overline{y}_2$ happens when the observation of $\widetilde{y}_1$ gives the correct value y_1 and the observation of $\widetilde{y}_2$ gives the wrong value $\overline{y}_2$.

1.2   Semantics of EICL

Each interpretation of EICL should provide a valuation to the variables in $X \cup Y$, a model of ORCF and an assignment to the variables ν and μ. The probabilistic semantics of the possibly erroneous inputs in $\widetilde{Y}$ is specified in the definition of EICL satisfaction.

As already mentioned, the choice of the model of ORCF is immaterial since all such models are elementarily equivalent and, so, we adopt once and for all the ordered field $\mathbb{R}$ of the real numbers.

Accordingly, by an EICL interpretation we mean a pair

$$I = (v, \rho)$$

where $v : X \cup Y \to \{\bot, \top\}$ is a valuation and ρ is an assignment over $\mathbb{R}$ such that:[1]

$$\begin{cases} \frac{1}{2} < \rho(\mu) \leq 1 \\ \frac{1}{2} < \rho(\nu) \leq 1. \end{cases}$$

---

[1]Recall that we use ν and μ as variables in the language of ORCF.

In such an interpretation the valuation v provides for each $\widetilde{y}_k$ its correct value, while $\rho(\nu)$ is the probability of this value being observed. Taking into account the independence assumptions on the observations, v and $\rho(\nu)$ together specify a probabilistic mixture of valuations as envisaged.

We now proceed to define *satisfaction*, by the interpretation $I = (v, \rho)$ at hand, of the three kinds of formulas in the language of EICL. This definition closely mimics the one of UCL in [11] since both logics deal with random outcomes of probabilistic formulas. The differences between the two logics are encapsulated in the definition of $\mathfrak{P}[\psi \triangleright \varphi]$.

Starting with c-formulas, we write

$$I \Vdash^{\text{eic}} \psi$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\text{fo}} \mu \leq \sum_{\substack{\varphi \sqsubseteq \psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi].$$

That is, the aggregated probability of the outcomes of ψ that are satisfied by v is at least the value of μ.

Observe that each outcome φ corresponds to a possible valuation (for the relevant input variables) in the probabilistic mixture of valuations specified by $I = (v, \rho)$ and the value of $\mathfrak{P}[\psi \triangleright \varphi]$ given by ρ is the probability of that valuation.

Concerning o-formulas, we write

$$I \Vdash^{\text{eic}} \Phi \sqsubseteq_P \psi$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\text{fo}} P \leq \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi].$$

That is, the collection Φ of possible outcomes of ψ has aggregated probability not smaller than the value of P.

Finally, concerning a-formulas, we write

$$I \Vdash^{\text{eic}} \mu \leq P$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\text{fo}} \mu \leq P.$$

That is, the required probability $\rho(\mu)$ for the correct output being produced by the whole circuit does not exceed the value of P.

The notion of satisfaction is taken to mixed sets of c-formulas, o-formulas and a-formulas with no surprises. Given $\Gamma \subseteq L^{\text{eic}}(X \cup Y \cup \widetilde{Y})$,

$$I \Vdash^{\text{eic}} \Gamma$$

if $I \Vdash^{\text{eic}} \gamma$ for each $\gamma \in \Gamma$. Then, entailment and validity in EICL are also defined as expected. Given $\{\theta\} \cup \Gamma \subseteq L^{\text{eic}}(X \cup Y \cup \widetilde{Y})$, we write

$$\Gamma \models^{\text{eic}} \theta$$

for stating that Γ *entails* θ in the following sense:

$$I \Vdash^{\text{eic}} \theta \text{ whenever } I \Vdash^{\text{eic}} \Gamma, \text{ for every interpretation } I.$$

Finally, we write

$$\models^{\text{eic}} \theta$$

for $\emptyset \models^{\text{eic}} \theta$, saying that formula θ is *valid*, in which case $I \Vdash^{\text{eic}} \theta$ for every interpretation I.

As envisaged, the EICL entailment is a conservative extension of the PL entailment as we now show.

Theorem 1.1 (Conservativeness of EICL) *Let* $\{\varphi\} \cup \Lambda \subseteq L(X)$. *Then,* $\Lambda \models^{\text{eic}} \varphi$ *if and only if* $\Lambda \models \varphi$.

Proof.   We start by proving that

$$(*) \qquad I \Vdash^{\text{eic}} \varphi \text{ if and only if } v \Vdash \varphi$$

where I is an interpretation with valuation v:

$(\rightarrow)$ Assume that $I \Vdash^{\text{eic}} \varphi$. Hence,

$$\mathbb{R}\rho \Vdash^{\text{fo}} \sum_{\substack{\varphi' \in \Omega_\varphi \\ v \Vdash \varphi'}} \mathfrak{P}[\varphi \rhd \varphi'] \geq \mu.$$

Thus, $\{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\} \neq \emptyset$ because $\rho(\mu) > 0$. Moreover, $\Omega_\varphi = \{\varphi\}$. Therefore, $\varphi \in \{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\}$ and, so, $v \Vdash \varphi$.

$(\leftarrow)$ Assume that $v \Vdash \varphi$. Then, $\{\varphi\} = \{\varphi' \in \{\varphi\} : v \Vdash \varphi'\}$. On the other hand, $\{\varphi' \in \{\varphi\} : v \Vdash \varphi'\} = \{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\}$. Therefore,

$$\sum_{\substack{\varphi' \in \Omega_\varphi \\ v \Vdash \varphi'}} \mathfrak{P}[\varphi \rhd \varphi'] \text{ is } \mathfrak{P}[\varphi \rhd \varphi]$$

and, so, it is the polynomial 1. Thus, in order to obtain $I \Vdash^{\text{eic}} \varphi$, we have only to show that $\mathbb{R}\rho \Vdash^{\text{fo}} 1 \geq \mu$ which holds because $\rho(\mu) \leq 1$.

We are ready to prove the envisaged result:

($\rightarrow$) Assume that $\Lambda \vDash^{\text{eic}} \varphi$. Let v be a valuation such that $v \Vdash \Lambda$. Let I be an interpretation with valuation v. Then, $I \Vdash^{\text{eic}} \Lambda$, by $(*)$, and, so, $I \Vdash^{\text{eic}} \varphi$. Thus, $v \Vdash \varphi$, by $(*)$.

($\leftarrow$) Assume that $\Lambda \vDash \varphi$. Let I be such that $I \Vdash^{\text{eic}} \Lambda$. Let v be the valuation in I. Then, $v \Vdash \Lambda$, by $(*)$, and, so, $v \Vdash \varphi$. Thus, once again by $(*)$, $I \Vdash^{\text{eic}} \varphi$.   ∎

The following semantic lemma is quite useful later on. It states that it is possible to transfer entailment of c-formulas to entailment of a-formulas. To this end we need the following notation: given a valuation v and $\psi \in L(X \cup Y \cup \widetilde{Y})$, we write Ω_ψ^v for $\{\varphi \in \Omega_\psi : v \Vdash \varphi\}$.

Proposition 1.2 *Let* $\Gamma \subseteq L^{\mathsf{a}}$, $\psi \in L(X \cup Y \cup \widetilde{Y})$ *and* v *be a valuation. Then,* $\Gamma \vDash^{\text{eic}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$ *whenever* $\Gamma \vDash^{\text{eic}} \psi$.

Proof.   Let v be a valuation. Assume that $\Gamma \vDash^{\text{eic}} \psi$. Let $I = (v, \rho)$ be an interpretation such that $I \Vdash^{\text{eic}} \Gamma$. Observe that

$$\mathfrak{P}[\psi \triangleright \Omega_\psi^v] = \sum_{\varphi \in \Omega_\psi^v} \mathfrak{P}[\psi \triangleright \varphi] = \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi].$$

Moreover, since $I \Vdash^{\text{eic}} \psi$,

$$\mathbb{R}\rho \Vdash^{\text{fo}} \mu \leq \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi].$$

Thus, $I \Vdash^{\text{eic}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$.   ∎

1.3   Hilbert calculus

The calculus of EICL capitalizes on the decidability of the following problems which are used in some provisos:

- membership in the language $L(X \cup Y)$;

- PL validity within $L(X \cup Y)$;

- emptyness of intersection of two finite sets;

- theoremhood in ORCF.

The calculus contains the following axioms and rules:

- the PL tautologies as *axioms*:

TAUT $\dfrac{}{\varphi}$ provided that $\varphi \in L(X \cup Y)$ and $\vDash \varphi$;

- the *modus ponens rule*:

MP $\dfrac{\begin{array}{c}\psi \\ \psi \supset \varphi\end{array}}{\varphi}$ provided that $\varphi \in L(X \cup Y)$;

- the following *o-axioms*:

NO $\dfrac{}{\emptyset \sqsubseteq_0 \psi}$;

VO $\dfrac{}{z \sqsubseteq_1 z}$ provided that $z \in X \cup Y$;

EIO↑ $\dfrac{}{y_k \sqsubseteq_\nu \widetilde{y_k}}$;

EIO↓ $\dfrac{}{\overline{y_k} \sqsubseteq_{(1-\nu)} \widetilde{y_k}}$;

- the following *o-rules*:

CO $\dfrac{\varphi_i \sqsubseteq_{P_i} \psi_i \text{ for } i = 1, \ldots, n}{c(\varphi_1, \ldots, \varphi_n) \sqsubseteq_{\left(\prod_{i=1}^n P_i\right)} c(\psi_1, \ldots, \psi_n)}$;

AO $\dfrac{\Phi_i \sqsubseteq_{P_i} \psi \text{ for } i = 1, 2}{\Phi_1 \cup \Phi_2 \sqsubseteq_{(P_1 + P_2)} \psi}$ provided that $\Phi_1 \cap \Phi_2 = \emptyset$;

WO $\dfrac{\Phi \sqsubseteq_{P_1} \psi}{\Phi \sqsubseteq_{P_2} \psi}$ provided that $\forall \nu \left(\left(\dfrac{1}{2} < \nu \le 1 \right) \supset P_2 \le P_1 \right) \in \mathsf{ORCF}$;

- the following *a-rule* for each $k \in \mathbb{N}$:

WA $\dfrac{\mu \le P_i \text{ for } i = 1, \ldots, k}{\mu \le P}$ provided that

$$\forall \mu \forall \nu \left(\left(\frac{1}{2} < \mu, \nu \le 1 \wedge \bigwedge_{i=1}^k \mu \le P_i \right) \supset \mu \le P \right) \in \mathsf{ORCF};$$

- the following *c-rule* for each $k \in \mathbb{N}^+$:

$$\mathsf{LFT} \quad \dfrac{\begin{cases} \Phi_i \sqsubseteq_{P_i} \psi \\ \mu \leq P_i \end{cases} \text{for } i = 1, \ldots, k}{\psi} \, .$$

with the display above:
$$\bigvee_{i=1}^{k} \left(\bigwedge \Phi_i \right)$$

The EICL calculus above is a close adaptation of the UCL calculus in [11] because both logics deal with probabilistic outcomes of probabilistic formulas.

The reader will wonder why we took the tautologies over $X \cup Y$ as axioms (TAUT) but not their instances with possibly erroneous inputs in $\widetilde{Y}$. In fact, such instances are not valid in general. For example, the instance $\widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1)$ of the tautology $x_1 \supset (x_2 \supset x_1)$ is not valid. Indeed, take an interpretation $I = (v, \rho)$, such that $v(y_1) = \top$, $v(y_2) = \top$, $\rho(\nu) = 0.6$ and $\rho(\mu) = 0.8$. Consider

$$\Phi = \{ y_1 \supset (y_2 \supset y_1), \overline{y}_1 \supset (y_2 \supset y_1), \overline{y}_1 \supset (y_2 \supset \overline{y}_1) \}.$$

Then,

- $v \Vdash \Phi$;

- $v \nVdash y_1 \supset (y_2 \supset \overline{y}_1)$.

Furthermore,

- $\mathfrak{P}[\widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1) \triangleright y_1 \supset (y_2 \supset y_1)] = \nu^2$;

- $\mathfrak{P}[\widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1) \triangleright \overline{y}_1 \supset (y_2 \supset y_1)] = \nu(1 - \nu)$;

- $\mathfrak{P}[\widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1) \triangleright \overline{y}_1 \supset (y_2 \supset \overline{y}_1)] = (1 - \nu)^2$.

Hence,

$$\mathbb{R} \rho \Vdash^{\mathsf{fo}} \left(\sum_{\varphi \in \Phi} \mathfrak{P}[\widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1) \triangleright \varphi] = \frac{19}{25} \right) \wedge \left(\frac{19}{25} < \mu \right)$$

and so $I \nVdash^{\mathsf{eic}} \widetilde{y}_1 \supset (y_2 \supset \widetilde{y}_1)$.

The reader will also wonder if MP is sound when the conclusion is a c-formula. It is not always so since, for example,

$$\widetilde{y}_1, \widetilde{y}_1 \supset \widetilde{y}_2 \nVdash^{\mathsf{eic}} \widetilde{y}_2.$$

Nevertheless, there are situations where MP is sound even in the presence of unreliable connectives in the conclusion. For instance, the dual of MP (conclusion in $L(X \cup Y \cup \widetilde{Y})$ provided that the antecedent of the implication is in $L(X \cup Y)$) is sound.

Derivability and theoremhood in this calculus are defined as usual. Given $\{\theta\} \cup \Gamma \subseteq L^{\text{eic}}(X \cup Y \cup \widetilde{Y})$, we write

$$\Gamma \vdash^{\text{eic}} \theta$$

for stating that θ is *derivable* from Γ, that is, for stating that there is a derivation sequence for obtaining θ from the elements of Γ (as hypotheses) and the axioms, using the rules of the calculus. Furthermore, when $\emptyset \vdash^{\text{eic}} \psi$, written $\vdash^{\text{eic}} \psi$, we say that ψ is a *theorem* of EICL.

For example, the derivation sequence in Figure 3 establishes that

$$\mu \leq 3\nu^4 - 2\nu^6 \vdash^{\text{eic}} \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

where the set Φ mentioned in Figure 3 is composed of the following outcomes:

- $\mathsf{M}_3(\overline{y}_1 \wedge \overline{y}_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(\overline{y}_1 \wedge y_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge \overline{y}_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, \overline{y}_1 \wedge \overline{y}_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, \overline{y}_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge \overline{y}_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge y_2, \overline{y}_1 \wedge \overline{y}_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge y_2, \overline{y}_1 \wedge y_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge y_2, y_1 \wedge \overline{y}_2) \equiv (y_1 \wedge y_2)$;

- $\mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$.

and the sub-derivation $\mathcal{D}_0$ is presented in Figure 4. For the sake of simplification, we omit the other derivations $\mathcal{D}_1$ to $\mathcal{D}_9$ since they are similar.

Useful admissible rules

The calculations of the probabilities in o-formulas in a derivation can be encapsulated using the following admissible rule.

Proposition 1.3 *Let $\psi \in L(X \cup Y \cup \widetilde{Y})$ and $\varphi \in \Omega_\psi$. Then,*

$$\mathsf{SO} \quad \frac{}{\varphi \sqsubseteq_{\mathfrak{P}[\psi \triangleright \varphi]} \psi}$$

is an admissible rule of EICL.

$$1 \qquad \bigwedge \Phi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{TAUT}$$

$$\mathcal{D}_0$$

$$m_0 \qquad \mathsf{M}_3(\overline{y}_1 \wedge \overline{y}_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2) \sqsubseteq_{(1-\nu)^2\nu^4}$$

$$\mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

$$\vdots$$

$$\mathcal{D}_9$$

$$m_9 \qquad \mathsf{M}_3(y_1 \wedge y_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2) \sqsubseteq_{\nu^6}$$

$$\mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

$$m_9{+}8 \qquad \Phi \sqsubseteq_{3\nu^4 - 2\nu^6}$$

$$\mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2) \qquad \text{AO:}_{m_0,\ldots,m_9}$$

$$m_9{+}9 \qquad \mu \leq 3\nu^4 - 2\nu^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{HYP}$$

$$m_9{+}10 \quad \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2) \qquad \text{LFT:}_{1,\, m_9{+}1,\, m_9{+}2}$$

Figure 3: $\mu \leq 3\nu^4 - 2\nu^6 \vdash^{\mathsf{eic}} \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$.

Proof. The proof follows by induction on ψ.

(Basis) There are several cases to consider:

(i) $\psi \in X \cup Y$. Then, φ is ψ. Consider the derivation:

$$1 \quad \varphi \sqsubseteq_1 \psi \quad \text{VO}.$$

Then, the thesis follows since $\mathfrak{P}[\psi \triangleright \varphi] = 1$.

(ii) ψ is $\widetilde{y}_k \in \widetilde{Y}$ and φ is y_k. Consider the derivation:

$$1 \quad \varphi \sqsubseteq_\nu \psi \quad \text{EIO}\uparrow.$$

Then, the thesis follows since $\mathfrak{P}[\psi \triangleright \varphi] = \nu$.

(iii) ψ is $\widetilde{y}_k \in \widetilde{Y}$ and φ is $\overline{y}_k$. We omit the proof of this case since it is similar to the previous case (ii).

(Step) Let ψ be $c(\psi_1, \ldots, \psi_n)$ and φ be $c(\varphi_1, \ldots, \varphi_n)$ with $\varphi_i \in \Omega_{\psi_i}$ for $i = 1, \ldots, n$. Observe that, by induction hypothesis,

$$\frac{}{\varphi_i \sqsubseteq_{\mathfrak{P}[\psi_i \triangleright \varphi_i]} \psi_i}$$

$$1 \quad y_1 \sqsubseteq_\nu \widetilde{y}_1 \qquad\qquad \text{EIO}{\uparrow}$$

$$2 \quad \overline{y}_1 \sqsubseteq_{1-\nu} \widetilde{y}_1 \qquad\qquad \text{EIO}{\downarrow}$$

$$3 \quad y_2 \sqsubseteq_\nu \widetilde{y}_2 \qquad\qquad \text{EIO}{\uparrow}$$

$$4 \quad \overline{y}_2 \sqsubseteq_{1-\nu} \widetilde{y}_2 \qquad\qquad \text{EIO}{\downarrow}$$

$$5 \quad y_1 \wedge y_2 \sqsubseteq_{\nu^2} \widetilde{y}_1 \wedge \widetilde{y}_2 \qquad\qquad \text{CO} : {1,3}$$

$$6 \quad \overline{y}_1 \wedge \overline{y}_2 \sqsubseteq_{(1-\nu)^2} \widetilde{y}_1 \wedge \widetilde{y}_2 \qquad\qquad \text{CO} : {2,4}$$

$$7 \quad \mathsf{M}_3(\overline{y}_1 \wedge \overline{y}_2, y_1 \wedge y_2, y_1 \wedge y_2) \sqsubseteq_{(1-\nu)^2\nu^4}$$
$$\mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \qquad \text{CO} : {6,5,5}$$

$$8 \quad y_1 \sqsubseteq_1 y_1 \qquad\qquad \text{VO}$$

$$9 \quad y_2 \sqsubseteq_1 y_2 \qquad\qquad \text{VO}$$

$$10 \quad y_1 \wedge y_2 \sqsubseteq_1 y_1 \wedge y_2 \qquad\qquad \text{CO} : {8,9}$$

$$11 \quad \mathsf{M}_3(\overline{y}_1 \wedge \overline{y}_2, y_1 \wedge y_2, y_1 \wedge y_2) \equiv (y_1 \wedge y_2) \sqsubseteq_{(1-\nu)^2\nu^4}$$
$$\mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2) \quad \text{CO} : {7,10}$$

Figure 4: Sub-derivation $\mathcal{D}_0$ in Figure 3.

is an admissible rule, for $i = 1, \ldots, n$. Denote by $\mathcal{D}_i$ the derivations in EICL of $\varphi_i \sqsubseteq_{\mathfrak{P}[\psi_i \triangleright \varphi_i]} \psi_i$ for $i = 1, \ldots, n$. Consider the derivation:

$$\begin{array}{ll} & \mathcal{D}_1 \\ m_1 & \varphi_1 \sqsubseteq_{\mathfrak{P}[\psi_1 \triangleright \varphi_1]} \psi_1 \\ & \vdots \\ & \mathcal{D}_n \\ m_n & \varphi_n \sqsubseteq_{\mathfrak{P}[\psi_n \triangleright \varphi_n]} \psi_n \\ m_n{+}1 & c(\varphi_1, \ldots, \varphi_n) \sqsubseteq_{\mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \ldots \times \mathfrak{P}[\psi_n \triangleright \varphi_n]} c(\psi_1, \ldots, \psi_n) \quad \text{CO} : m_1, \ldots, m_n. \end{array}$$

Then, the thesis follows since $\mathfrak{P}[\psi \triangleright \varphi] = \mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \ldots \times \mathfrak{P}[\psi_n \triangleright \varphi_n]$. ∎

Another admissible rule is related with the intuitive fact that the probability of an o-formula should be one when considering the set of all outcomes. In order to show this fact we need the following auxiliary result.

Proposition 1.4 *Let* $\psi \in L(X \cup Y \cup \widetilde{Y})$. *Then,*

$$\left(\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] = 1 \right) \in \mathsf{ORCF}.$$

Proof.   The proof follows straightforwardly by induction on ψ.   ∎

Proposition 1.5 *Let* $\psi \in L(X \cup Y \cup \widetilde{Y})$. *Then,*

$$\Omega\mathsf{O} \qquad \vdash^{\mathsf{eic}} \Omega_\psi \sqsubseteq_1 \psi$$

is an admissible rule of EICL.

Proof.   The proof of this fact follows from Proposition 1.4 taking into account rules
SO, AO and WO.   ∎

The dual of the MP rule mentioned above is also an admissible rule but we omit
the details since it will not be needed in the rest of the paper.

Basic properties of derivability

The following result establishes that any c-formula is derivable as long as the acceptance
threshold μ is appropriately bounded.

Proposition 1.6 *Let* $\psi \in L(X \cup Y \cup \widetilde{Y})$. *Then,*

$$\{\mu \leq \mathfrak{P}[\psi \rhd \Omega_\psi^v] : v \text{ is a valuation}\} \vdash^{\mathsf{eic}} \psi.$$

Proof.   Observe that $\{\Omega_\psi^v : v \text{ is a valuation}\}$ is a finite set. The thesis follows by LFT
taking into account:

(a) $\vdash \bigvee_v \bigwedge \Omega_\psi^v$ since $\bigvee_v \bigwedge \Omega_\psi^v$ is a tautology and by completeness of PL.

(b) $\vdash^{\mathsf{eic}} \Omega_\psi^v \sqsubseteq_{\mathfrak{P}[\psi \rhd \Omega_\psi^v]} \psi$ for every valuation v using NO, SO and AO.

(c) $\{\mu \leq \mathfrak{P}[\psi \rhd \Omega_\psi^v] : v \text{ is a valuation}\} \vdash^{\mathsf{eic}} \mu \leq \mathfrak{P}[\psi \rhd \Omega_\psi^v]$ for every valuation v, using
the extensivity of $\vdash^{\mathsf{eic}}$.   ∎

The next result states that o-formulas involving an implication are theorems of EICL
as long as: (1) we include all the outcomes of the antecedent of the implication and
only one outcome of the consequent; and (2) the probability of the outcome of the
consequent is used in the o-formula.

Proposition 1.7 *Let* $\psi, \psi' \in L(X \cup Y \cup \widetilde{Y})$ *and* $\varphi' \in \Omega_{\psi'}$. *Then,*

$$\vdash^{\mathsf{eic}} \{\varphi \supset \varphi' : \varphi \in \Omega_\psi\} \sqsubseteq_{\mathfrak{P}[\psi' \rhd \varphi']} \psi \supset \psi'.$$

Proof.   Observe that, by rule SO, $\vdash^{\mathsf{eic}} \varphi \supset \varphi' \sqsubseteq_{\mathfrak{P}[\psi \triangleright \varphi] \times \mathfrak{P}[\psi' \triangleright \varphi']} \psi \supset \psi'$ for every $\varphi \in \Omega_\psi$, and, so, $\vdash^{\mathsf{eic}} \{\varphi \supset \varphi' : \varphi \in \Omega_\psi\} \sqsubseteq_{(\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] \times \mathfrak{P}[\psi' \triangleright \varphi'])} \psi \supset \psi'$ by rule AO. Note that, in ORCF,

$$
\begin{aligned}
\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] \times \mathfrak{P}[\psi' \triangleright \varphi'] &= \mathfrak{P}[\psi' \triangleright \varphi'] \times \sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] \\
&= \mathfrak{P}[\psi' \triangleright \varphi']
\end{aligned}
$$

by Proposition 1.4. Hence, by rule WO, the thesis follows.   ∎

Now we identify two interesting situations in which some hypotheses can be disregarded.

Proposition 1.8 *Let $\Gamma_1 \subseteq L^{\mathsf{a}}$, $\Gamma_2 \subseteq L(X \cup Y \cup \widetilde{Y}) \cup L^{\mathsf{o}}(X \cup Y \cup \widetilde{Y})$ and P a term. Then, $\Gamma_1 \vdash^{\mathsf{eic}} \mu \leq P$ whenever $\Gamma_1, \Gamma_2 \vdash^{\mathsf{eic}} \mu \leq P$.*

Proof.   The result follows by straightforward induction on the given derivation of $\mu \leq P$ from $\Gamma_1 \cup \Gamma_2$. In the basis, the conclusion is either an hypothesis or follows by WA over an empty set of premises. So the same derivation is also a derivation of $\mu \leq P$ from Γ_1. Regarding the step, the conclusion follows by rule WA over a non-empty set of premises in the derivation. Hence, the thesis follows by the induction hypothesis and by applying the same rule.   ∎

Proposition 1.9 *Let $\Gamma \subseteq L^{\mathsf{a}} \cup L(X \cup Y \cup \widetilde{Y})$, $\psi \in L(X \cup Y \cup \widetilde{Y})$ and $\Phi \subseteq \Omega_\psi$. Then, $\vdash^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$ whenever $\Gamma \vdash^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$.*

Proof.   The result follows by induction on a derivation of $\Phi \sqsubseteq_P \psi$ from Γ. In the basis, the conclusion follows by NO, VO, EIO↑ and EIO↓. Since they are axioms, the thesis follows immediately. Regarding the step, observe that the rules used to conclude $\Phi \sqsubseteq_P \psi$ all have o-formulas as premises. Hence, by applying the induction hypothesis to each premise, we can conclude that they are theorems. Using the same rule the thesis follows.   ∎

2   Main results

In this section we start by investigating if the proposed calculus does capture the semantics of EICL. First we prove strong soundness, that is:

$$
\text{if} \quad \Gamma \vdash^{\mathsf{eic}} \theta \quad \text{then} \quad \Gamma \models^{\mathsf{eic}} \theta
$$

for every $\Gamma \cup \{\theta\} \subseteq L^{\text{eic}}(X \cup Y \cup \widetilde{Y})$.

Observe that strong completeness for o-formulas is out of question since the entailment is not compact and the EICL calculus is obviously compact. So we only prove that

$$\text{if} \quad \vDash^{\text{eic}} \Phi \sqsubseteq_P \psi \quad \text{then} \quad \vdash^{\text{eic}} \Phi \sqsubseteq_P \psi.$$

Strong completeness for c-formulas and a-formulas is also out of question for the same reason. So, with respect to completeness for such formulas, we prove the following results:

$$\begin{cases} \text{if} & \Gamma \vDash^{\text{eic}} \mu \leq P \quad \text{then} \quad \Gamma \vdash^{\text{eic}} \mu \leq P \\ \text{if} & \Gamma \vDash^{\text{eic}} \psi \quad \text{then} \quad \Gamma \vdash^{\text{eic}} \psi \end{cases}$$

provided that Γ is a finite subset of L^{a}.

Notice that, it is important to allow a finite number of a-formulas as hypotheses, since no connectives are available in the EICL language for combining a-formulas and for combining them with c-formulas.

2.1   Soundness

It is enough to show soundness of tautologies and soundness of each rule in EICL. Then, the result follows by a straightforward induction.

Proposition 2.1 *The rules of* EICL *are sound.*

Proof.   We only show that the rules VO, EIO↑, EIO↓, CO and LFT are sound since the others follow similarly.

(VO) Let $I = (v, \rho)$ be an EICL interpretation and $z \in X \cup Y$. Then $\mathfrak{P}[z \triangleright z] = 1$ by definition of $\mathfrak{P}$. Hence,

$$\mathbb{R}\rho \Vdash^{\text{fo}} 1 \leq \mathfrak{P}[z \triangleright z]$$

and so $I \Vdash^{\text{eic}} z \sqsubseteq_1 z$. Thus VO is sound.

(EIO↑) Let $I = (v, \rho)$ be an EICL interpretation and $\widetilde{y}_k \in \widetilde{Y}$. Then $\mathfrak{P}[\widetilde{y}_k \triangleright y_k] = \nu$ by definition of $\mathfrak{P}$. Hence,

$$\mathbb{R}\rho \Vdash^{\text{fo}} \nu \leq \mathfrak{P}[\widetilde{y}_k \triangleright y_k]$$

and so $I \Vdash^{\text{eic}} y_k \sqsubseteq_\nu \widetilde{y}_k$. Thus EIO↑ is sound.

(EIO↓) The proof that this rule is sound is similar to the one of EIO↑, so we omit it.

(CO) Let $I = (v, \rho)$ be an EICL interpretation, $\psi_1 \ldots, \psi_n \in L(X \cup Y \cup \widetilde{Y})$, $\varphi_1, \ldots, \varphi_n \in L(X \cup Y)$ and $c \in \Sigma$ such that $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$. Assume that

$$I \Vdash^{\text{eic}} \varphi_i \sqsubseteq_{P_i} \psi_i$$

for $i = 1, \ldots, n$. Then,

$$\mathbb{R}\rho \Vdash^{\mathsf{fo}} P_i \leq \mathfrak{P}[\psi_i \triangleright \varphi_i]$$

for $i = 1, \ldots, n$. Taking also into account that $\mathbb{R}\rho \Vdash^{\mathsf{fo}} 0 \leq P_i$ for $i = 1, \ldots, n$ and that $\mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \ldots \times \mathfrak{P}[\psi_n \triangleright \varphi_n] = \mathfrak{P}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$, we have that

$$\mathbb{R}\rho \Vdash^{\mathsf{fo}} P_1 \times \ldots \times P_n \leq \mathfrak{P}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$$

and so $I \Vdash^{\mathsf{eic}} c(\varphi_1, \ldots, \varphi_n) \sqsubseteq_{P_1 \times \ldots \times P_n} c(\psi_1, \ldots, \psi_n)$. Thus the rule CO is sound.

(LFT) Let $I = (v, \rho)$. Assume that

$$\begin{cases} I \Vdash^{\mathsf{eic}} \bigvee_{i=1}^{k} \bigwedge \Phi_i \\ \begin{cases} I \Vdash^{\mathsf{eic}} \Phi_i \sqsubseteq_{P_i} \psi \\ I \Vdash^{\mathsf{eic}} \mu \leq P_i \end{cases} \quad \text{for } i = 1, \ldots, k. \end{cases}$$

Since $\bigvee_{i=1}^{k} \bigwedge \Phi_i \in L(X \cup Y)$, then there is $i \in \{1, \ldots, k\}$ such that $v \Vdash \bigwedge \Phi_i$ (see the proof of Theorem 1.1). Hence,

$$\{\varphi \in \Omega_\psi : v \Vdash \varphi\} \supseteq \Phi_i$$

and, so,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \sum_{\varphi \in \Phi_i} \mathfrak{P}[\psi \triangleright \varphi].$$

By hypothesis,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\varphi \in \Phi_i} \mathfrak{P}[\psi \triangleright \varphi] \geq P_i$$

and

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} P_i \geq \mu.$$

Therefore,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \mu$$

and, so, the thesis follows.   ∎

Theorem 2.2 (Strong soundness of EICL) *Let $\Gamma \cup \{\theta\} \subseteq L^{\mathsf{eic}}(X \cup Y \cup \widetilde{Y})$. Then, $\Gamma \vDash^{\mathsf{eic}} \theta$ whenever $\Gamma \vdash^{\mathsf{eic}} \theta$.*

Proof.   Induction on the length of the given derivation for $\Gamma \vdash^{\mathsf{eic}} \theta$.   ∎

2.2   Completeness

We concentrate now on the completeness of EICL. We start with the weak completeness for o-formulas.

Proposition 2.3 *Let $\psi \in L(X \cup Y \cup \widetilde{Y})$, $\Phi \subseteq \Omega_\psi$ and P a term. Then, $\vdash^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$ if $\models^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$.*

Proof.   Assume that $\models^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$. Let ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \nu \leq 1.$$

Let ρ' be an assignment such that $\rho'(\nu) = \rho(\nu)$ and $\frac{1}{2} < \rho'(\mu) \leq 1$. Then,

$$\mathbb{R}\,\rho' \Vdash^{\mathsf{fo}} \frac{1}{2} < \nu \leq 1.$$

Let v be a valuation and $I = (v, \rho')$. Then, $I \Vdash^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$ and, so,

$$\mathbb{R}\,\rho' \Vdash^{\mathsf{fo}} P \leq \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi].$$

Therefore,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} P \leq \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi].$$

So

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset \left(P \leq \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi] \right) \right) \in \mathsf{ORCF}.$$

On the other hand, by rules NO, SO and AO, $\vdash^{\mathsf{eic}} \Phi \sqsubseteq_{\sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi]} \psi$ and, so, $\vdash^{\mathsf{eic}} \Phi \sqsubseteq_P \psi$ by rule WO. ∎

We now establish the constrained strong completeness for a-formulas.

Proposition 2.4 *Let Γ be a finite subset of L^{a} and P a term. Then, $\Gamma \vdash^{\mathsf{eic}} \mu \leq P$ if $\Gamma \models^{\mathsf{eic}} \mu \leq P$.*

Proof.   Let Γ be $\{\mu \leq P_1, \ldots, \mu \leq P_k\}$. Assume that $\Gamma \models^{\mathsf{eic}} \mu \leq P$. Let ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1 \wedge \bigwedge_{i=1}^{k} \mu \leq P_i$$

and v an arbitrary valuation. Then,

$$(v, \rho) \Vdash^{\mathsf{eic}} \bigwedge_{i=1}^{k} \mu \leq P_i$$

and, so, $(v, \rho) \Vdash^{\mathsf{eic}} \mu \leq P$ since $\Gamma \vDash^{\mathsf{eic}} \mu \leq P$. Hence, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq P$ and, thus, we have proved

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \forall\mu\forall\nu \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \bigwedge_{i=1}^{k} \mu \leq P_i \right) \supset \mu \leq P \right)$$

for every assignment ρ over $\mathbb{R}$. Therefore, $\Gamma \vdash^{\mathsf{eic}} \mu \leq P$, by WA. ∎

Finally, we prove the constrained strong completeness for c-formulas.

Theorem 2.5 (Constrained strong completeness of EICL) *Let Γ be a finite subset of L^{a} and $\psi \in L(X \cup Y \cup \widetilde{Y})$. Then, $\Gamma \vdash^{\mathsf{eic}} \psi$ if $\Gamma \vDash^{\mathsf{eic}} \psi$.*

Proof.   Assume that $\Gamma \vDash^{\mathsf{eic}} \psi$. Let ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1 \wedge \left(\bigwedge \Gamma \right).$$

and $I = (v, \rho)$ an interpretation for some valuation v. Observe that I is in fact an interpretation since $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1$. Hence, $I \Vdash^{\mathsf{eic}} \Gamma$ and so $I \Vdash^{\mathsf{eic}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$ since, by Proposition 1.2, $\Gamma \vDash^{\mathsf{eic}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$ for every valuation v. Therefore, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$. In this way, we showed that

$$\forall\mu\forall\nu \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \left(\bigwedge \Gamma \right) \right) \supset \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v] \right) \in \mathsf{ORCF}$$

for every valuation v. Then, $\Gamma \vdash^{\mathsf{eic}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$ for every valuation v, by WA since Γ is finite. Moreover, $\{\mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v] : v$ is a valuation$\} \vdash^{\mathsf{eic}} \psi$, by Proposition 1.6. Thus, the result follows by idempotence of $\vdash^{\mathsf{eic}}$. ∎

2.3   Metatheorem of deduction

Herein, we discuss the metatheorem of deduction for EICL. It holds only with some provisos. We start by introducing two relevant notions: (1) given a derivation $\theta_1 \ldots \theta_k$ of ψ from Γ, we say that θ_i *depends* on $\gamma \in \Gamma$ in this derivation if either θ_i is γ or θ_i is obtained using a rule with at least one of the premises depending on γ; (2) an application of MP is said to be *classical* if both premises are in $L(X \cup Y)$.

Proposition 2.6 *Let $\Gamma \subseteq L^{\mathsf{a}}$ and ψ and ψ' be distinct formulas in $L(X \cup Y \cup \widetilde{Y})$. Assume that there is a derivation of ψ' from $\Gamma \cup \{\psi\}$ where ψ' depends on ψ and all the applications of MP over dependents of ψ are classical. Then, $\psi \in L(X \cup Y)$.*

Proof.   Let $\theta_1 \dots \theta_k$ be a derivation of $\Gamma, \psi \vdash^{\mathsf{eic}} \psi'$ where ψ' depends on ψ and all the applications of MP over dependents of ψ are classical. The proof follows by induction on k. Since ψ' depends on ψ, ψ' is not obtained by TAUT. Moreover, since ψ' is not ψ, ψ' does not appear as an hypothesis. Hence, we have only to consider two cases: either ψ' is obtained by LFT (we omit the proof of this case since it follows straightforwardly) or ψ' is obtained by MP from θ_i and $\theta_i \supset \psi'$ where either θ_i or $\theta_i \supset \psi'$ depends on ψ in the given derivation. Since ψ' depends on ψ, then, both θ_i and $\theta_i \supset \psi'$ are in $L(X \cup Y)$. If $\theta_i \supset \psi'$ depends on ψ in the given derivation, we need to consider two possibilities: (i) ψ is $\theta_i \supset \psi'$. Then, ψ is in $L(X \cup Y)$; (ii) ψ is not $\theta_i \supset \psi'$. Then, by the induction hypothesis, $\psi \in L(X \cup Y)$. If θ_i depends on ψ, yet again we need to consider two possibilities and apply the same reasoning as when $\theta_i \supset \psi'$ depends on ψ. ∎

Theorem 2.7 (Metatheorem of deduction - MTD)   *Let $\Gamma \subseteq L^{\mathsf{a}}$ and $\psi, \psi' \in L(X \cup Y \cup \widetilde{Y})$. Assume that ψ' fulfills the following proviso: either ψ' is distinct from ψ or $\psi' \in L(X \cup Y)$. Then, $\Gamma \vdash^{\mathsf{eic}} \psi \supset \psi'$ whenever there is a derivation establishing $\Gamma, \psi \vdash^{\mathsf{eic}} \psi'$ where all the applications of MP over dependents of ψ are classical.*

Proof.   Let $\theta_1 \dots \theta_k$ be a derivation of ψ' from $\Gamma \cup \{\psi\}$ where all the applications of MP over dependents of ψ are classical. The proof follows by induction on k.

(Basis) Consider two cases. Either (1) ψ' is obtained by TAUT or (2) ψ' is ψ. Assume that case (1) holds. Then, $\psi' \in L(X \cup Y)$. Take $\Phi = \{\varphi \supset \psi' : \varphi \in \Omega_\psi\}$. Then, by tautological reasoning $\vdash \varphi \supset \psi'$ for every $\varphi \in \Omega_\psi$, and so, $\vdash^{\mathsf{eic}} \bigwedge \Phi$. On the other hand, by Proposition 1.7, since $\mathfrak{P}[\psi' \triangleright \psi'] = 1$, $\vdash^{\mathsf{eic}} \Phi \sqsubseteq_1 \psi \supset \psi'$. Hence, the thesis follows by rule LFT. Assume now that case (2) holds. Then, by hypothesis, $\psi' \in L(X \cup Y)$. The proof is similar to case (1).

(Step) There are two cases to consider. Either ψ' is obtained by LFT from $\bigvee_{i=1}^{n} \bigwedge \Phi_i$, $\Phi_i \sqsubseteq_{P_i} \psi'$ and $\mu \leq P_i$ for $i = 1, \dots, n$ (we omit the proof of this case since it follows straightforwardly), or ψ' is obtained by MP from ψ'' and $\psi'' \supset \psi'$ where $\psi' \in L(X \cup Y)$. We have three sub cases: (a) ψ' depends on ψ in $\theta_1 \dots \theta_k$, and ψ' and ψ are distinct. Then, by Proposition 2.6, $\psi \in L(X \cup Y)$. Moreover, $\psi'' \in L(X \cup Y)$ since all the applications of MP on dependents of ψ are classical. By the induction hypothesis $\Gamma \vdash^{\mathsf{eic}} \psi \supset \psi''$ and $\Gamma \vdash^{\mathsf{eic}} \psi \supset (\psi'' \supset \psi')$. Then, by tautological reasoning, $\Gamma \vdash^{\mathsf{eic}} \psi \supset \psi'$. (b) ψ' does not depend on ψ in $\theta_1 \dots \theta_k$. Then, $\Gamma \vdash^{\mathsf{eic}} \psi'$. By tautological reasoning, $\Gamma \vdash^{\mathsf{eic}} \varphi \supset \psi'$ for every $\varphi \in \Omega_\psi$. The rest of the proof is similar to the one in (1). (c) ψ' is ψ. Then,

$\psi \in L(X \cup Y)$ by hypothesis. The proof is similar to case (1).   ■

Observe that one would have no difficulty in refuting the metatheorem of deduction without the proviso on ψ'. For instance,

$$\widetilde{y}_1 \vdash^{\mathsf{eic}} \widetilde{y}_1$$

by extensivity. On the other hand,

$$\nvdash^{\mathsf{eic}} \widetilde{y}_1 \supset \widetilde{y}_1$$

as can easily be verified, and, so, by soundness of EICL (see Theorem 2.2), $\nvDash^{\mathsf{eic}} \widetilde{y}_1 \supset \widetilde{y}_1$.

3   Application scenarios

One should wonder if the assumptions on which EICL relies (availability of independent observations of each circuit input and independence between observations of different inputs) do have any practical significance.

To this end recall the circuit in Figure 2 represented by c-formula

$$\text{(M3)} \qquad \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2)$$

where we have two inputs y_1 and y_2 that are each of them independently observed thrice, once in each of the three conjunction subcircuits feeding the majority gate.

A typical application scenario would be the control of some industrial process, say a nuclear power station, the inputs reflecting for example the temperature of the fluids in two key pipes using for each of them three independent possibly erroneous threshold sensors. The goal of the circuit would be to set up an alarm if both temperatures exceed a certain critical value.

To this end, the simple conjunction circuit represented by c-formula

$$\text{(M1)} \qquad \widetilde{y}_1 \wedge \widetilde{y}_2$$

(that uses only one sensor on each of the two pipes) would be enough but more error prone than the circuit above where redundancy plays a decisive role.

It is worthwhile to compare in detail the two circuits in order to show how redundancy is used to achieve a higher probability of correct output.

Recall that in Subsection 1.3 we established

$$\mu \leq 3\nu^4 - 2\nu^6 \vdash^{\mathsf{eic}} \mathsf{M}_3(\widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2, \widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

using the derivation presented in Figure 3. Therefore, circuit (M3) produces the correct output of the envisaged conjunction with at least probability[2]

$$3\nu^4 - 2\nu^6$$

while circuit (M1) does so with probability at least

$$\nu^2$$

since

$$\mu \leq \nu^2 \vdash^{\mathsf{eic}} (\widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$$

as derived in Figure 5.

1	$(y_1 \wedge y_2) \equiv (y_1 \wedge y_2)$	TAUT
2	$y_1 \sqsubseteq_\nu \widetilde{y}_1$	EIO↑
3	$y_2 \sqsubseteq_\nu \widetilde{y}_2$	EIO↑
4	$y_1 \wedge y_2 \sqsubseteq_{\nu^2} \widetilde{y}_1 \wedge \widetilde{y}_2$	CO : 2, 3
5	$y_1 \sqsubseteq_1 y_1$	VO
6	$y_2 \sqsubseteq_1 y_2$	VO
7	$y_1 \wedge y_2 \sqsubseteq_1 y_1 \wedge y_2$	CO : 5, 6
8	$(y_1 \wedge y_2) \equiv (y_1 \wedge y_2) \sqsubseteq_{\nu^2} (\widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$	CO : 4, 7
9	$\mu \leq \nu^2$	HYP
10	$(\widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$	LFT : 1, 8, 9

Figure 5: $\mu \leq \nu^2 \vdash^{\mathsf{eic}} (\widetilde{y}_1 \wedge \widetilde{y}_2) \equiv (y_1 \wedge y_2)$.

Observe that for values of ν not exceeding

$$\frac{\sqrt{2}}{2}$$

there is no value of $\mu > \frac{1}{2}$ that entails the acceptance of circuit (M1) as good or the acceptance of (M3). For values of ν above this acceptance threshold, the value of $3\nu^4 - 2\nu^6$ exceeds the value of ν^2 and, so, circuit (M3) is better than circuit (M1) as expected.

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[2]The probability can be higher on favourable values of the inputs y_1 and y_2.

It is also worthwhile to examine how far can we go by increasing the degree of redundancy. In general, when using the circuit represented by the c-formula

$$(M3 + 2k) \qquad \widetilde{\mathsf{M}}_{3+2k}(\widetilde{y}_1 \wedge \widetilde{y}_2, \ldots, \widetilde{y}_1 \wedge \widetilde{y}_2)$$

for obtaining the output of a conjunction with possibly erroneous inputs, we are sure that the output is correct with at least probability

$$(*) \qquad \sum_{i=0}^{k+1} \binom{3+2k}{3+2k-i} (\nu^2)^{3+2k-i}((1-\nu)^2 + 2\nu(1-\nu))^i.$$
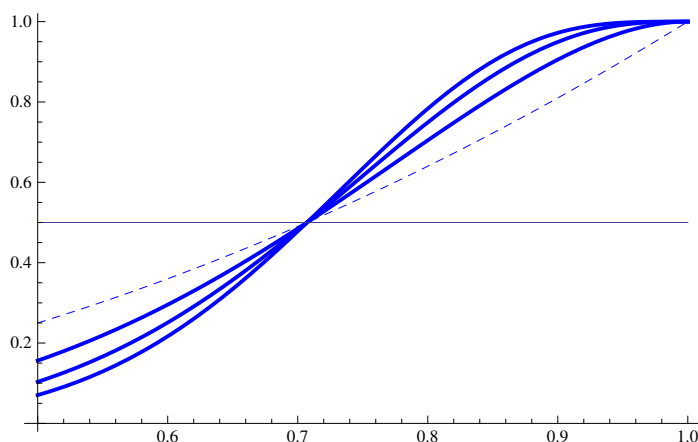
Figure 6: Term (*) for several degrees of redundancy.

Observe that an increase in the redundancy degree does not improve on the acceptance threshold value $\frac{\sqrt{2}}{2}$ for ν. But for ν above this threshold any increase in redundancy does improve the lower bound of the probability of obtaining the correct output, as depicted in Figure 6 where this bound is depicted as a function of ν for circuits (M1) (dashed line), (M3), (M5) and (M7). Hence, it seems that $\frac{\sqrt{2}}{2}$ is a characteristic of the basic circuit at hand.

From the analysis above it is clear that the redundancy technique proposed by von Neumann in [13] for improving the overall reliability of circuits built with unreliable gates but receiving error-free inputs also works when we are dealing with perfect gates but receiving possibly erroneous inputs with sufficiently small probability of error, as long as independent observations of each input are available and the observations of different inputs are also independent.

4   Outlook

Under some reasonable independence assumptions on the possibly erroneous observations of the inputs of logic circuits with perfect gates, we were able to set up a logic (EICL) appropriate for reasoning about such circuits, as a conservative extension of classical propositional logic (PL). For the axiomatic calculus we capitalized on the decidability of the first-order theory of ordered real closed fields. Useful completeness results were established in due course.

The pitfalls of extrapolating classical reasoning to the realm of circuits with possibly erroneous inputs were extensively illustrated. For instance, the metatheorem of deduction was established only with additional provisos that once again show the striking differences between PL and EICL. These differences arise from the referential opacity of the input variables in EICL.

Concerning future work, as a first step, other metaproperties of EICL beyond completeness and the MTD should be investigated, in due course including decidability and other algorithmic issues. In another direction, relaxing the independence assumptions seems to be the most challenging and significant open problem. In particular, allowing for perfect cloning of any observation input, that is, removing the fan-out restriction to 1 implicitly assumed in this paper, should be the first step. Only afterwards will the problem of allowing for both unreliable gates and erroneous inputs be tractable.

Acknowledgments

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