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On Vague Notions and Modalities

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Abstract

We present a precise approach to treat modalities involving vague notions. Vague notions, such as 'generally','many', 'rarely', 'often', etc., occur often in ordinary language and in some branches of science. We introduce some basic ideas about vague notions, quantification and modalities, and address the issue of treating them in a precise manner. Some ideas about multimodal logics are extended to vague modalities. We also give a formal approach to these ideas.

Keywords: vague notions, 'generally', 'rarely', precise reasoning, quantification, modalities.

1 Introduction

We present some ideas about certain vague notions and a way to treat modalities related to them so as to reason in a precise manner about these modalities. The exposition aims to convey the main ideas and results through illustrative examples.

The structure of this paper is as follows. In the remaining of this introductory section, we will try to motivate the issues that will be treated here. In Section 2 we will examine and illustrate some basic ideas about vague notions, quantification and modalities. Section 3 continues examining quantification and modalities, by focusing on the question of how to treat them in a precise way, and suggests a possible approach to this matter. Section 4 introduces, by a concrete case study, some ideas about logics for vague notions. Section 5 examines logics with vague multi modalities. Section 6 gives a more formal treatment for these ideas. Section 7 presents some final comments.

Vague notions, such as 'generally','many', 'rarely', 'often', etc., occur often in ordinary language and in some branches of science. We wish to express such assertions and to reason about them in a precise manner. To this end, it is important to have a clear understanding of the vague notions involved. Here the focus is on modalities with vague notions, but we will also examine quantification dealing with vague notions.

One may ask why bothering with vague notions (such as 'generally', 'rarely', etc) ? One answer has already been anticipated above: they occur often in ordinary language and in some branches of science.

For instance, one may hear from his medical doctor the following assertion:

"A `typical` patient presenting these symptons is `prone` to develop ..."

Other, perhaps somewhat less vague, examples are the following:

- " Swans `generally` are white."
- "Metals `rarely` are liquid under normal temperature conditions"
- " Your order will be delivery `soon`."

The expressions '`generally`', '`rarely`' and '`soon`', occurring in these assertions seem to have the aspect of quantifiers or modalities but they are somehow vague. How to understand them in a precise way?

There also seems to be inferences involving such notions, despite their vagueness, as the next example indicates.

Example 1.1 (Vague inference) *Consider the following two assertions:*

1. *"Boys* `generally` *like sports."*
2. *"***Whoever** *likes sports watches channel SportTV."*

These two assertions seem to lead to the following conclusion:

3. *"Boys* `generally` *watch SportTV".*

How to account for such (apparent) inferences? Are they correct?

These are some of the main questions that motivate the investigation on the vague aspects of quantifiers and modalities. We will focus on the case of vague modalities, using vague quantifiers to introduce some auxiliary ideas.

2 Basic ideas

In this section we introduce some basic ideas about quantification and modalities through examples. Our aim is contrasting vague and non-vague cases and introducing the issue of how to treat the vague cases in a precise manner.

2.1 Quantification with vague notions

We now examine some ideas about quantification: vague and not vague

Consider the following assertion:

(**T**) "**All** prime numbers are odd."

It is a strong (non-vague) assertion: '`all`' does not accept exceptions. That is the reason why the above assertion is false. One can try to circumvent this by trying a "more attenuated" version.

(`Q`) "`Almost all` prime numbers are odd."

This (somewhat vague) version could be understood as:

'`Almost all`' means "**all**, but 2".

In a similar manner, one could have the attenuation:

- of "**No** metal is liquid (under normal temperature conditions)"
- to "`Almost no` metal is liquid (under normal temperature conditions)",

where '`almost no`' might mean "**none**, except mercury".

There are some variants for vague quantification; for instance:

"'**There are nine** planets in the solar system."[1]

Also, there are (more complex) non-vague quantifiers variants; for instance:

"**There are infinitely many** prime numbers."

Now, consider the following (non-vague) assertion:

($\overset{\infty}{\exists}$) "**There are infinitely many** twin primes."[2]

This is an open conjecture.[3]

One can try to "attenuate" ($\overset{\infty}{\exists}$) as follows:

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[1]There has recently been a controversy among astronomers whether Pluto should be considered as a planet or not. This suggests that perhaps the concept of `planet` was not very clear: a vague predicate. But, '**there are (exactly) 9**' objects has a precise meaning, as have '**there are more than 9**' objects, '**there are at least 9**', etc.

[2]Twin primes are numbers such that p and $p+2$ are both primes; e. g. 17 and 19.

[3]This conjecture (of the twin primes) is related to Hilbert's 8^{th} Problem. One knows very large twin primes (such as $571305 \,.\, 2^{7701} \pm 1$ and $1706595 \,.\, 2^{11235} \pm 1$); but it is not known whether there are infinitely many of them.

($\overset{\mathtt{m}}{\exists}$) "There are many twin primes."

Here, in going from the strong version to the attenuated one, the speaker is on a safer side, but the attenuated version is less clear: how one should understand 'there are many'?

2.2 Modalities with vague notions

We now examine some ideas about modalities: vague and not vague.

When one selects the 'shutdown' option in a computer menu, the following message appears on the monitor screen:

> "if you do nothing, the system will shut down **within 120 seconds**."

This is a perfectly clear and precise (non-vague) information: one has exactly 2 minutes to give up turning off the computer.

Now, imagine that the following message appears on the monitor screen:

> 'if you do not take any action, the system will shut down soon."

This (vague) information is not very clear: how much time has one got to give up turning off the computer?

Consider the following assertion:

> "The system **always** works without freezing."

This is a very strong (non vague) assertive: '**always**' ($\Box$) does not admit exceptions; that is why it may be considered somewhat risky. One can try to circumvent this by trying a "softer" version.

> "The system almost always works without freezing."

Here, in going from the strong version to the attenuated one, the manufacturer is on a safer side, but the attenuated version is less clear: how should the user understand 'almost always'? [4]

[4] A single case of system freezing is a counter-example for the strong assertion (with '**always**'). How could one refute the attenuated assertion (with 'almost always')? It is not very clear, but it seems that just one case of freezing would not be enough: one may be need many cases of freezing.

3 Vague quantification and vague modalities

In this section, we are going to examine some ideas about vague quantification and vague modalities, taking back the question of how to treat them in a precise manner, and indicating a possible approach to this matter.

Dealing with a computing system, one can find assertions like the following ones.

(O) "The system `often` works well."

(S) "The message will be answered `soon`."

The expressions '`often`' and '`soon`' in these two assertions appear to present an aspect of modality; but they are somewhat vague. We have perhaps an intuitive understanding of them: for instance, we can think of '`often`' as "`almost` always" or "in `many` situations". But, we still have the question of how to deal with them in a precise manner.

Let us examine non vague analogues of these assertions (O) and (S).

(**A**) "The system **always** works well."

(**E**) "The message will be answered **eventually**."

We know, at least in principle, how to deal with these versions (**A**) and (**E**). We can reduce:

- '**always**' to "at **all** instants";
- '**eventually**' to "at **some** (future) instant".

These explanations reduce modality to quantification as follows:

Modality		*Quantier*
always: $\Box$	$\mapsto$	$\forall$: **all** instants
eventually: $\Diamond$	$\mapsto$	$\exists$: **some** instant (in the future)

If we had vague quantifiers, we could try a analogous reduction:

$$\texttt{often}: \triangledown \quad \mapsto \quad \nabla: \texttt{many}\text{ instants}$$

This is, in broad lines, the idea of the suggested approach to be presented here: to extend the reductions of modalities to quantifiers to vague versions. In order to carry out this plan successfully, we must also explain clearly vague quantification.

Vague notions, like '`generally`', '`rarely`', '`most`', '`several`' etc., appear often in assertions and arguments in ordinary language and in some branches of science [1, 11]. Quantification with vague notions appear often in assertions and arguments in ordinary language and in some branches of science. Besides the examples given in Section 1, we can mention the following ones.

"Brazilians `rarely` love cricket."

"Birds `generally` fly."

"Bodies `generally` expand when heated."

"Metals `rarely` float in water"

We we will present our approach in a stepwise way as follows.

1. In 3.1 (Numerical approach to vague notions), we will introduce a first approach to modalities and quantifiers with vague notions.
2. In 3.2 (Qualitative account for vague quantification), we will try to explain better the notions of vague quantification.
3. In 3.3 (Qualitative approach to vague notions), we will put together these ideas to have a treatment of modalities with vague nations in a general setting.

3.1 Numerical approach to vague notions

We will now examine a first explanation for quantifiers and modalities with vague notions (in a simplified version).

Imagine that we wish to make some assertions about the color of the offsprings of a given animal. Consider a given animal a and the following assertion:

($\forall$) "**All** offsprings of a are black ."

A formulation of this assertion with a modal aspect could be as follows:

($\Box$) "**Necessarily** the offsprings of a are black."

In this case, we have an accessibility relation R between an animal and its offsprings.

The meaning of the formulation ($\Box$) above is as follows:

For each animal a' such that $a\,R\,a'$: a' is black.

Now, consider the following vague assertion:

(q) "The offsprings of a are `almost certainly` black."

We intend to reduce this assertion (q) to the following (still somewhat vague) version:

(g) "The number of black offsprings of a is big."

It still remains to explain what 'big' means. For this purpose, we are going to consider a simplified (but intuitively reasonable) version where 'big' means "at least 75% of the total".

Example 3.1 (Offspring) *Imagine that animal a has 10 offsprings $a_1, \ldots, a_{10}$, so that the black ones are $a_1, \ldots, a_8$, the remaining ones being white:*

$$\underbrace{a_1,\ \ldots,\ a_8}_{black},\quad \underbrace{a_9,\ a_{10}}_{white}$$

In this case, the number of black offsprings of a (which is 8) is considered big. *Hence, the assertion* (g) *is satisfied and so is the assertion with the vague modality* (q)*: "The offsprings are* `almost certainly` *black."*

One could similarly explain assertions like "Viennese generally like music".

3.2 Qualitative account for vague quantification

We are now going to try to clarify some ideas about quantification with vague notions [9, 10].

In 3.1, we have decided to consider 'big' as "at least 75% (of the total)". It seems that this explanation involves some arbitrary decisions: why "at least 75%", instead of "more than 75%", or some other threshold, like 70% or 80%?

Let us try to alleviate such sensation of arbitrariness. For this purpose, we are going to consider a non empty universe V and examine some properties of the sets taken as 'big', i. e. those with at least 75% of the universe. As a matter of fact, it seems more intuitive to start with their duals: the small sets, namely those having less than 25% of the universe.

We have 2 immediate and intuitively reasonable) properties of these small sets.

($\emptyset$) The empty set $\emptyset$ is small (0% < 25%).

(V) The universe V is not small (100% $\not<$ 25%).

Moreover, the small sets have then following (closure) property.

($\subseteq$) A subset X of a small set Y is small.

Here two points should be noticed.[5]

1. The above properties continue to hold if we take 'small' as, say, "less than 20 %" or "at most 30 %" of the total.
2. From these properties of such a family $\mathcal{S}$ of small sets, we can infer properties of the dual family $\mathcal{B}$ of big sets.

These points are summarized in Table 1.

[5]Notice that these properties explain the vague inference in Example 1.1.

	$\mathcal{S}$ (small sets)	$\mathcal{B}$ (big sets)
(+)	$\emptyset \in \mathcal{S}$	$V \in \mathcal{B}$
(-)	$V \notin \mathcal{S}$	$\emptyset \notin \mathcal{B}$
($\subseteq$)	$X \subseteq S \,\&\, S \in \mathcal{S} \Rightarrow X \in \mathcal{S}$	$G \subseteq Y \,\&\, G \in \mathcal{B} \Rightarrow Y \in \mathcal{B}$

Table 1: Properties of small and big sets

On the other hand, it does not seem reasonable to expect the intersection of two big sets to be a big set (neither the union of two small sets to be a small set).

We examine an example which illustrates well the properties mentioned above.

Example 3.2 (Brazilians) *Considering the universe of inhabitants of Brazil, the following two assertions look quite reasonable.*

(M) "`Many` *Brazilians have their moustache shaved" ($M \in \mathcal{B}$)*

(L) "`Many` *Brazilians have their legs shaved" ($L \in \mathcal{B}$)*

Under these conditions it also seems reasonable to state that

($\cup$) *"*`Many` *Brazilians have their moustache shaved <u>or</u> support Flamengo"*

but, it does not seem so reasonable the statement

($\cap$) *"*`Many` *Brazilians have their moustache <u>and</u> their legs shaved".*

Indeed, assertion ($\cup$) *means $M \cup F \in \mathcal{B}$ (which seems to follow from $M \in \mathcal{B}$, as $M \subseteq M \cup F$); whereas the unexpected assertion* ($\cap$) *means $M \cap L \in \mathcal{B}$.*

We have just seen some (reasonable) properties of small and big sets, which serve to provide reasons for accepting some inferences while rejecting others. These explanations, however, are based on having used a threshold to describe small and big sets. Nonetheless, this view is still somewhat restricted. This may be noticed in the next example.

Example 3.3 (Naturals) *For the universe of natural numbers, the following 3 assertions seem reasonable.*[6]

(λ) *"*`Few` *naturals are smaller than 15" ($L \in \mathcal{S}$)*

(δ) *"*`Few` *naturals divide 12" ($D \in \mathcal{S}$)*

[6]Note that these are vague assertions about a sharp domain.

(υ) *"*`Few` *naturals are smaller than 15' or divide 12"* $(A \cup D \in \mathcal{P})$

Indeed we may be tempted to believe that the third assertion (υ) *follows from the first two* (λ) *and* (δ)*. How could we account for this? We have been inclined to expect the family of* small *sets not to be closed under union!*

Here, we can appeal to another "more relaxed" description of small sets. Since the universe of naturals is infinite, it seems reasonable to understand 'small' as being "finite".[7] With such a description, besides the above properties, we also have the closure of the family of small sets under union, i. e, the following property:

if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$, then $X \cup Y \in \mathcal{S}$.

Summarizing, we have seen two ways of describing the vague sets that are to be considered small (and dually the big ones). These two approaches do not establish the same properties. Moreover, there are other reasonable descriptions for these notions [11, 10]. On the other hand, the above examples suggest that certain properties are sometimes desirable, sometimes not. How can we explain this?

We should mention that there are several distinct intuitive ideas of 'big': `several`, `many`, `most`, `almost all`, etc. It does not seem reasonable to expect that all of them sould have the same properties. What we can expect is to obtain a uniform way of describing such vague quantifications.[8]

Towards this goal, we will abstract from the particular descriptions. We will consider two families of subses of a given universe V: families $\mathcal{N}$ (of the negligible sets) and $\mathcal{W}$ (of the important sets), which will correspond to intuitive ideas of '`rarelly`' and '`generally`', respectively.

Of course, these ideas of negligible and important are still more vague. But, this can be a bonus: they are more flexible. What we will do is considering families that may have some properties like the ones mentioned before. In each case, these dual families may have some of the properties summarized in Table 2.

The dual families of negligible and of important sets may have some other properties. We present some of them for the family $\mathcal{W}$ of important sets in Table 3.

3.3 Qualitative approach to vague notions

We will now reconsider the approach to vague notions: modalities and quantifiers.

First, we have used a simplified version for modalities and quantifiers with vague notions (in 3.1). Then, we have encountered a better explanation for notions related

[7] For the universe of real numbers, one may take 'small' as being "denumerable".

[8] In a similar way, there are several meanings for a modality like `always`; nevertheless we have a unified way of characterizing them.

	$\mathcal{N}$ (negligible sets)	$\mathcal{W}$ (important sets)
$(\emptyset)$	$\emptyset \in \mathcal{N}$	$\emptyset \notin \mathcal{W}$
(V)	$V \notin \mathcal{N}$	$V \in \mathcal{W}$
$(\subseteq)$	$X \subseteq N \,\&\, N \in \mathcal{N}$ $\Downarrow$ $X \in \mathcal{N}$	$W \subseteq Y \,\&\, W \in \mathcal{W}$ $\Downarrow$ $Y \in \mathcal{W}$
$(\bullet)$	$N_1 \in \mathcal{N} \,\&\, N_2 \in \mathcal{N}$ $\Downarrow$ $N_1 \cup N_2 \in \mathcal{N}$	$W_1 \in \mathcal{W} \,\&\, W_2 \in \mathcal{W}$ $\Downarrow$ $W_1 \cap W_2 \in \mathcal{W}$

Table 2: Properties of negligible and important sets

Prime:	$X \notin \mathcal{W} \,\&\, Y \notin \mathcal{W} \;\Rightarrow\; X \cup Y \notin \mathcal{W}$
Rejection:	$X \in \mathcal{W} \;\Rightarrow\; \overline{X} \notin \mathcal{W}$
Attraction:	$X \notin \mathcal{W} \;\Rightarrow\; \overline{X} \in \mathcal{W}$

Table 3: Other possible properties of important sets

to vague quantification (in 3.2). Now, we are going to put together these two parts in order to have a general treatment of modalities and quantifiers with vague notions.

Consider a given animal a and the assertion:

"The offsprings of a are `generally` black."

This assertion has been successively interpreted as:

1. "The set of of black offsprings of a is big" (cf. 3.1);

2. "The set of black offsprings of a is important" (cf. 3.2).

In Example 3.1, we have considered an animal and its offspring. We now consider a variant of this example with two animals.

Example 3.4 (Animals) *In Example 3.1, animal a has* 10 *offsprings* $a_1, \ldots, a_{10}$*, of which* $a_1, \ldots, a_8$ *are black (*a_9 *e* a_{10} *being white). Now, consider another animal c having* 20 *offsprings* $c_1, \ldots, c_{20}$*, of which* $c_1, \ldots, c_8$ *are black (the remaining* 12 *ones being white). This situation can be visualized below (where we use* ${}_aR$ *and* ${}_cR$*, respectively, for the set of offspring of animals a and c):*

$$\overbrace{\underbrace{a_1, \ldots, a_8}_{\textit{black}}, \ \underbrace{a_9, a_{10}}_{\textit{white}}}^{{}_aR} \qquad\qquad \overbrace{\underbrace{c_1, \ldots, c_8}_{\textit{black}}, \ \underbrace{c_9, c_{10}, \ldots, c_{20}}_{\textit{white}}}^{{}_cR}$$

In this case we may notice that:

(a) the set $\{a_1, \ldots, a_8\}$ *has* 8 *animals (more than* 75% *of the total* 10*); thus, it is considered as a* big *set of offspring of a;*

(b) the set $\{c_1, \ldots, c_8\}$ *has* 8 *animals (less than* 75% *of the total* 20*); thus, it is not considered as a* big *set of offsprings of c.*

So, a set with 8 *animals is* big*, whereas another set with* 8 *animals is* <u>*not*</u> big*.*

The idea to be emphasized here is that a set being big depends not only on the (size of) the set, but also on the total (the set of offsprings, in the Example 3.4). Recall that 'important' intends to be an abstraction of 'big' (cf. 3.2). Thus, a set is considered important depending not only on the set itself, but also on the universe (the set of reachable animals by the offspring relation, in the above example).

Another example illustrating such a dependency is chess. Here, the reachable set depends on a particular configuration (or situation) of the chessboard.

Example 3.5 (Queen) *First, consider a situation Q_m where the queen is in the center of the chessboard. In this case, the reachable (or menaced) positions by the queen are her horizontal, her vertical and the two diagonals (following the pattern of the United Kingdom's flag): with a total of 27 positions (see Figure 1). Now, consider a situation Q_c where the queen is in corner of the chessboard. In this case, the reachable (or menaced) positions by the queen are her horizontal and her diagonal: with a total of 21 positions (see Figure 2). We see that the reachable set from a given situation depends on the particular situation itself. The queen's accessibility relation has the aspect sketched in Figure 3.*

+			+			+	
	+		+		+		
		+	+	+			
+	+	+	Q	+	+	+	+
		+	+	+			
	+		+		+		
+			+			+	
			+				+

Figure 1: Queen in the center: positions menaced

Q	+	+	+	+	+	+	+
+	+						
+		+					
+			+				
+				+			
+					+		
+						+	
+							+

Figure 2: Queen in the corner: positions menaced

Thus, it is reasonable to consider that the notion important is also related to a situation. Then, for each situation s:

($_sR$) besides the s-reachable set $_sR$, (given by the accessibility relation R),

($\underline{\mathcal{W}}_s$) we associate a family $\underline{\mathcal{W}}_s$ of subsets of $_sR$ (considered important within $_sR$).

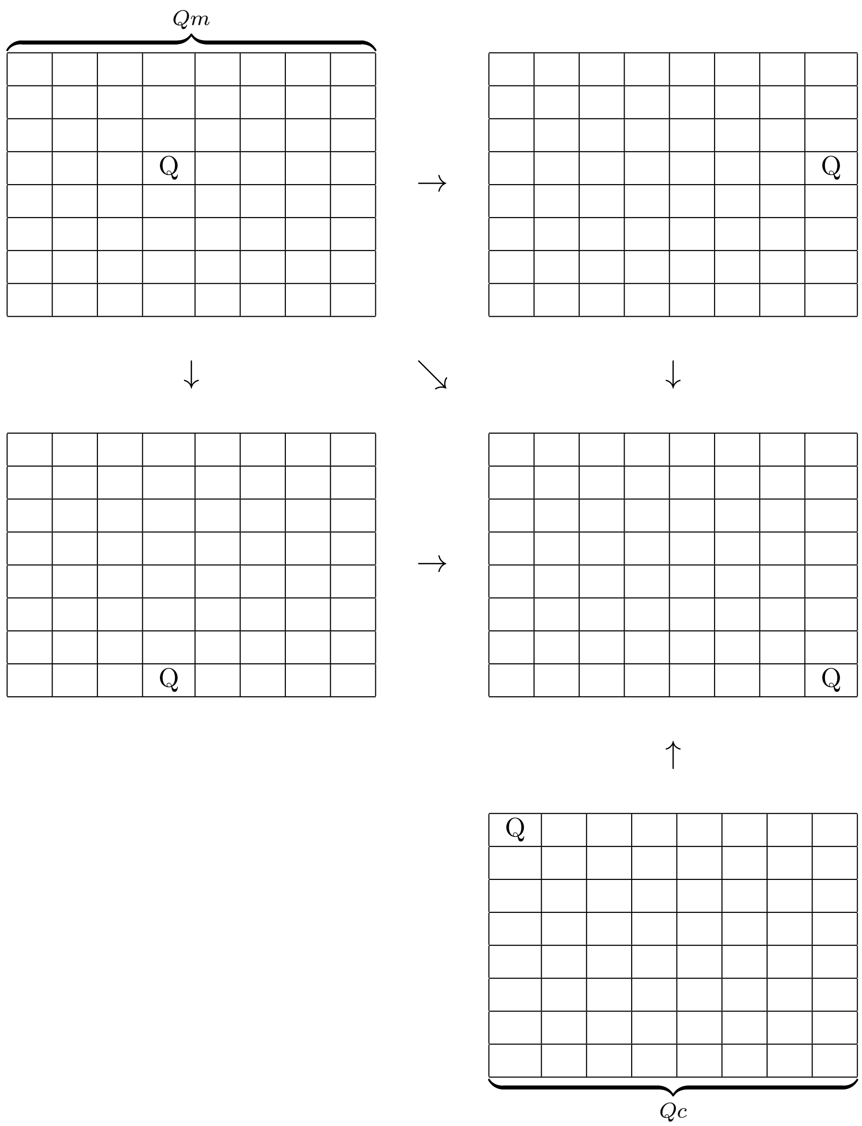


Figure 3: Queen's accessibility relation

4 Logics for vague notions

In this section we examine some ideas on logics for vague notions We will use a concrete example to introduce the main ideas.

4.1 Case study

We are going to consider a case similar to Examples 3.1 and 3.4.

Example 4.1 (Animals and colors) *Consider several animals. Some of them are offspring of others. Some animals may be black or dark.*
We can represent the offspring relation by an arrow between animals as shown below (where an arrow $x \to y$ indicates that y is offspring of x):

$$
\begin{array}{ccccccccccc}
a_2 & & a_3 & & a_4 & \rightarrow & c_2 & & & & c_3 \\
 & \nwarrow & \uparrow & \nearrow & & & & \nwarrow & & \nearrow & \\
a_1 & \leftarrow & a & \rightarrow & a_5 & \rightarrow & c_1 & \leftarrow & c & & \\
 & \swarrow & \downarrow & \searrow & & & & \swarrow & \downarrow & \searrow & \\
a_8 & & a_7 & & a_6 & & c_6 & & c_5 & & c_4 \\
 & \nwarrow & \uparrow & \nearrow & & & & \nwarrow & & \nearrow & \\
 & & e_2 & & & & & & f_2 & & \\
 & & \uparrow & & & & & & \uparrow & & \\
e_1 & \leftarrow & e & \rightarrow & e_3 & \leftarrow & f_1 & \leftarrow & f & & \\
 & & \downarrow & & & & & & & & \\
 & & e_4 & & h & \rightarrow & h' & & & &
\end{array}
$$

Assume that the colors of the animals are as follows..

(b) *The black animals are:*

a and its offsprings $a_1, \ldots, a_6$;	*c and its offsprings $c_1, \ldots, c_4$;*
the offsprings e_1 and e_2 of e;	*f and its offsprings f_1 and f_2.*

(d) *The dark animals are:*

a and its offsprings $a_1, \ldots, a_6$ and a_7;	*c and its offsprings $c_1, \ldots, c_4$ and c_5;*
e and its offsprings e_1, e_2 and e_3;	*f and its offsprings f_1 and f_2.*

If we indicate colors by underlining the dark animals and overlining the black ones, we have the situation shown in Figure 4.

We are going to analyze the case described in Example 4.1 by using:

1. a logic with a vague quantifier (in 4.2),
2. a logic with a vague modality (in 4.3).

$$
\begin{array}{ccccccccccc}
\overline{\underline{a_2}} & & \overline{\underline{a_3}} & & \overline{\underline{a_4}} & \rightarrow & \overline{\underline{c_2}} & & & & \overline{\underline{c_3}} \\
& \nwarrow & \uparrow & \nearrow & & & & \nwarrow & & \nearrow & \\
\overline{\underline{a_1}} & \leftarrow & \overline{\underline{a}} & \rightarrow & \overline{\underline{a_5}} & \rightarrow & \overline{\underline{c_1}} & \leftarrow & \overline{\underline{c}} & & \\
& \swarrow & \downarrow & \searrow & & & & \swarrow & \downarrow & \searrow & \\
a_8 & & \overline{\underline{a_7}} & & \overline{\underline{a_6}} & & c_6 & & \underline{c_5} & & \overline{\underline{c_4}} \\
& \nwarrow & \uparrow & \nearrow & & & & \nwarrow & & \nearrow & \\
& & \overline{\underline{e_2}} & & & & & & \overline{\underline{f_2}} & & \\
& & \uparrow & & & & & & \uparrow & & \\
\overline{\underline{e_1}} & \leftarrow & \overline{\underline{e}} & \rightarrow & \overline{\underline{e_3}} & \leftarrow & \overline{\underline{f_1}} & \leftarrow & \overline{\underline{f}} & & \\
& & \downarrow & & & & & & & & \\
& & e_4 & & h & \rightarrow & h' & & & &
\end{array}
$$

Figure 4: Animals with offspring and colors

4.2 Logic with vague quantifier

We now analyze Example 4.1 in a logic with a vague quantifier. We start with a usual first-order logic [6] and extend it with a vague quantifier [8, 11].

To formalize this example we sill use a first-order language L_F, having (besides variables, etc.) the following (extra-logical) symbols:

bynary predicate R	for	'offspring',
unary predicates B and D	for	the colours 'black' and 'dark' (respectively),
constants a e f	for	the animals a and f (respectively).

Example 4.1 gives a structure for this language L_F: the *structure*

$$\mathfrak{A} := \langle A, R^{\mathfrak{A}}, B^{\mathfrak{A}}, D^{\mathfrak{A}}, a^{\mathfrak{A}}, f^{\mathfrak{A}} \rangle, \text{ where}$$

A	is	the above universe of animals,
$R^{\mathfrak{A}}$	is	the binary 'offspring' relation, as above
$B^{\mathfrak{A}}$	is	the set $\{a, a_1, \ldots, a_6, c, c_1, \ldots, c_4, e_1, e_2, f, f_1, f_2\}$ (of black animals),
$D^{\mathfrak{A}}$	is	the set $\{a, a_1, \ldots, a_6, a_7, c, c_1, \ldots, c_4, c_5, e, e_1, e_2, e_3, f, f_1, f_2\}$,
$a^{\mathfrak{A}}$	is	the animal $a \in A$,
$f^{\mathfrak{A}}$	is	the animal $f \in A$.

With this language L_F, we can symbolize assertions about animals involving offspring relation among them and their colors: black and dark. Examples of such assertions and sentences symbolizing them are as follows:

"The animal a is black"	$\mathrm{B(a)}$
"Animal f is not an offspring of animal a"	$\neg\mathrm{R(a,f)}$
"**Every** offspring of f is dark"	$\forall v\,[\,\mathrm{R}(\mathrm{f},v)\ \rightarrow\ \mathrm{D}(v)\,]$
"**No** offspring of f is dark and not black"	$\neg\exists v\,[\,\mathrm{R}(\mathrm{f},v)\ \wedge \mathrm{D}(v)\ \wedge\ \neg\mathrm{B}(v)\,]$

These four assertions are true and their corresponding sentences hold in structure $\mathfrak{A}$. Calling τ any one of these sentences, we have $\mathfrak{A}\models\tau$.

To interpret a formula with free variables, like $\neg\mathrm{R}(\mathrm{a},v)$, we use an assignment of elements of A to the variables.

Formula $\neg\mathrm{R}(\mathrm{a},v)$ is satisfied by assigning $b_6\in A$ to variable v:

$$\mathfrak{A}\models\neg\mathrm{R}(\mathrm{a},v)\,[\![\,b_6\,]\!],\quad \text{for } (a,b_6)\notin\mathrm{R}^{\mathfrak{A}}\quad \text{(animal } b_6 \text{ is not an offspring of } a\text{).}$$

Now, the assignment of a_8 to variable v does not satisfy this formula:

$$\mathfrak{A}\not\models\neg\mathrm{R}(\mathrm{a},v)\,[\![\,a_8\,]\!],\qquad \text{for } (a,a_8)\in\mathrm{R}^{\mathfrak{A}}\qquad \text{(animal } a_8 \text{ is an offspring of } a\text{).}$$

Similarly, the formula $\mathrm{B}(v)$ is satisfied exactly by the assignment of a black animal to variable v, i. e. the *extension* of formula $\mathrm{B}(v)$ is the set $\mathrm{B}^{\mathfrak{A}}$.

Now, consider the assertion with vague quantification, as:

(g) "Animals `generally` are black".

To symbolize assertions such as this one, we will extend the first order language $\mathrm{L_F}$: we add a new (vague) quantifier ∇, to symbolize 'generally', obtaining $\mathrm{L}_\mathrm{F}^\nabla$.

In this language $\mathrm{L}_\mathrm{F}^\nabla$, the above vague assertion can be symbolized as (γ): $\nabla v\,\mathrm{B}(v)$.

We have seen above that the extension of the formula $\mathrm{B}(v)$ is the set $\mathrm{B}^{\mathfrak{A}}$ of black animals. As seen in Section 3 (cf. 3.3), the vague assertion (g) and the sentence (γ) should be interpreted as:

1. "Black animals form an important set of animals".
2. The extension of the formula $\mathrm{B}(v)$ is an important set of animals.
3. The set $\mathrm{B}^{\mathfrak{A}}$ belongs to the family $\mathcal{W}$ of the important sets of animals.

But, for this, we must provide such a family $\mathcal{W}$ of the important sets of animals: we should enrich the structure

$$\mathfrak{A}\ =\ \langle\,A,\mathrm{R}^{\mathfrak{A}},\mathrm{B}^{\mathfrak{A}},\mathrm{D}^{\mathfrak{A}},\mathrm{a}^{\mathfrak{A}},\mathrm{f}^{\mathfrak{A}}\,\rangle, \text{ for the first-order language } \mathrm{L_F},$$

adding to it a family $\mathcal{W}$ of sets of animals. This would yield a *modulated structure*

$$\mathfrak{A}^{\mathcal{W}}\ :=\ \langle\,A,\mathrm{R}^{\mathfrak{A}},\mathrm{B}^{\mathfrak{A}},\mathrm{D}^{\mathfrak{A}},\mathrm{a}^{\mathfrak{A}},\mathrm{f}^{\mathfrak{A}},\mathcal{W}\,\rangle\quad \text{for the extended language } \mathrm{L}_\mathrm{F}^\nabla.$$

Then, a sentence holds in the modulated structure $\mathfrak{A}^{\mathcal{W}}$ iff its extension belongs to the family $\mathcal{W}$, so $\mathfrak{A}^{\mathcal{W}}\models\nabla v\,\mathrm{B}(v)$ iff $\mathrm{B}^{\mathfrak{A}}\in\mathcal{W}$.

4.3 Logic with vague modality

We are now going to examine Example 4.1 using a logic with a vague modality. We start with a modal logic [5, 4, 2] and extend it by a vague modality.

To symbolize our example, we will use a modal language $\mathrm{L_M}$ with modalities $\Diamond$ and $\Box$, as well as sentential letters (variables), such as b, d, p,...

We can view Example 4.1 as a relational structure, consisting of the following:

set A (of states)	as	the above universe of animals,
(accessibility) relation R	as	the binary relation 'offspring' given above

So, we have a *frame* $\mathfrak{F} \coloneqq \langle A, \mathrm{R} \rangle$, giving the offspring relation between animals.

To describe our example, we choose propositional letters to represent the colors:

letter b for 'black'   letter d for 'dark'

By this, we mean that the animals for which the letter b holds are those having black color; similarly, letter d holds exactly for the animals having dark color. Indicating for which elements of A each propositional letter holds, we have a *valuation* $\mathbf{v}$.[9]

Thus, we have a structure $\langle A, \mathrm{R}, \mathbf{v} \rangle$ for $\mathrm{L_M}$: the *model* $\mathfrak{M} \coloneqq (\mathfrak{F}, \mathbf{v})$.

We can now express some assertions. Some examples are as follows:

"Animal a is black"	b holds at state a
"**Necessarily** the offsprings of f are dark "	$\Box\,\mathrm{d}$ holds at state f
"An offspring of f **cannot** be dark <u>and</u> <u>not</u> black"	$\neg\Diamond(\mathrm{d} \wedge \neg\mathrm{b})$ holds at f

These three assertions are true and the corresponding formulas hold at the corresponding states (animals) in the model $\mathfrak{M}$. In other words, we have[10]

$$\mathfrak{M} \Vdash \mathrm{b}\,[\![\,a\,]\!] \qquad \mathfrak{M} \Vdash \Box\,\mathrm{d}\,[\![\,f\,]\!] \qquad \mathfrak{M} \Vdash \neg\Diamond(\mathrm{d} \wedge \neg\mathrm{b})\,[\![\,f\,]\!].$$

Similarly, we see that the formula b holds exactly for the black animals, i. e. the *extension* $\mathfrak{M}[\![\,\mathrm{b}\,]\!]$ of formula b is the set $V(\mathrm{b})$ of black animals.

Now, consider an assertion with a vague modality, like:

(`f`) "The offsprings of a `frequently` have black color'.

To symbolize such assertions we will extend the modal language $\mathrm{L_M}$: we add a new (vague) modality $\bigtriangledown$, to symbolize `frequently`, yielding the extension $\mathrm{L}_{\mathrm{M}}^{\bigtriangledown}$.

In this language $\mathrm{L}_{\mathrm{M}}^{\bigtriangledown}$, the vague assertion (`f`) can be symbolized as (ϕ): $\bigtriangledown\,\mathrm{b}$.

As seen before, the assertion (`f`) and the formula (ϕ) should be interpreted as:

---

[9]For instance, $\mathbf{v}(\mathrm{b}) = \{a, a_1, \ldots, a_6, c, c_1, \ldots, c_4, e_1, e_2, f, f_1, f_2\}$.

[10]Alternative notations for $\mathfrak{M} \Vdash \varphi\,[\![\,a\,]\!]$ are $\mathfrak{M}, a \Vdash \varphi$ [2] and $(\mathfrak{F}, V, a) \Vdash \varphi$ [7].

"The black offspings of a form an important set of offsprings of a".

We have seen that the extension $\mathfrak{M}[\![\,\mathrm{b}\,]\!]$ of formula b is the set of black animals. So, the set of the black offsprings of a is the restriction of the set $\mathfrak{M}[\![\,\mathrm{b}\,]\!]$ (of black animals) to the set ${}_aR = \{a_1, \ldots, a_8\}$ of offsprings of a, i. e. the set ${}_aR \cap \mathfrak{M}[\![\,p\,]\!]$. It remains the question: is this set an important set of the offspring of a? To answer this question we need a family $\underline{\mathcal{W}}_a$ of important sets of the offsprings of a.

To deal with such assertions with vague modalities, we should associate to each animal a family of *important* sets of its offsprings: we must enrich

the model $\mathfrak{M} = \langle\, A, \mathrm{R}, \mathbf{v}\,\rangle$, for the modal language L_{M},

adding to it a function $\underline{\mathcal{W}}$ that assigns to each animal a family of sets (considered important) of its offsprings. This yields a *modulated model*

$\mathfrak{M}^{\underline{\mathcal{W}}} \coloneqq \langle\, A, \mathrm{R}, \mathbf{v}, \underline{\mathcal{W}}\,\rangle$ for the extended modal language $\mathrm{L}^{\triangledown}_{\mathrm{M}}$.

Alternatively, we can start with the frame $\mathfrak{F} = \langle\, A, \mathrm{R}\,\rangle$ and first enrich it to the *modulated frame* $\mathfrak{F}^{\underline{\mathcal{W}}} \coloneqq \langle\, A, \mathrm{R}, \underline{\mathcal{W}}\,\rangle$, and then add to it a valuation $\mathbf{v}$, to obtain $\mathfrak{M}^{\underline{\mathcal{W}}} = \langle\, A, \mathrm{R}, \mathbf{v}, \underline{\mathcal{W}}\,\rangle$.

Then, a formula holds at a state s of the modulated model $\mathfrak{M}^{\underline{\mathcal{W}}}$ iff its restricted extension belongs to the family $\mathcal{W}_s$, so $\mathfrak{M}^{\underline{\mathcal{W}}} \Vdash \triangledown\, \mathrm{b}\, [\![\, a\,]\!]$ iff ${}_aR \cap \mathfrak{M}[\![\,p\,]\!] \in \mathcal{W}_a$.

5 Logics with vague modalities

We are now going to examine logics with several vague modalities. We will start introducing multimodalities (in 5.1) and then extend the treatment to encompass vague multimodalities (in 5.2). We will also briefly examine extensions to other multimodalities in 5.3.

Multimodal logics have several modalities [3]. This happens to be very natural in certain cases [7]. An example of this appears in the treatment of time by modalities: one modality for the past, another one for the future. The treatment of programs, and of agents in general, also happens to involve several modalities: one for each agent [2].

5.1 Multimodal logics

We are now going to examine multimodal logics: syntax and semantics.

To introduce the ideas we will use our chess example in 3.3. The queen and the king have different moves. We have seen a chessboard configuration Qc, having the queen in a corner of the chessboard, from which 21 squares can be accessed (cf. Figure 2). Now, consider the case of the king in the same corner: a configuration Kc. in this case, the king can only access the 3 contiguous squares.

K	+						
+	+						

Thus, it seems natural to consider an accessibly relation for each piece, say R_Q for the queen, R_K for the king, etc.

First, we are going to examine modal structures corresponding to these ideas: with several accessibility relations. We have a set S (of states or configurations). Each piece has a state-to-sate accessibility relation. Thus, we have a *multimodal frame* $\mathfrak{J} = \langle S, \mathrm{R}_Q, \mathrm{R}_K, \ldots \rangle$, with several accessibility relations. In this manner, each state $s \in S$ gives rise to the reachable set for a given piece; for instance, for the king, ${}_s\mathrm{R}_K$ consists of the states $t \in S$ such that $s\,\mathrm{R}_K\,t$.

To deal with such structures, we will use a multimodal language L. Besides the propositional letters (variables), $p_1, p_2, \ldots$, we have modalities corresponding to the accessibility of each piece. For instance, for the king we have the following modalities, with their intended meanings:

$\Box_K$	at **every** state accessible by the king,
$\Diamond_K$	at **some** state accessible by the king.

In this multimodal language L, we can express some properties of the game. For instance, property of a piece, such as:

$\Diamond_K \top$ expressing "The king has **some** move (in this state)".

We can also express properties of distinct pieces, like:

$\Diamond_K \top \to \Diamond_Q \top$ expressing "<u>If</u> the king has a move, <u>then</u> so does the queen".

More precisely, we will interpret formulas of L in a model. Much as in 4.3, a *model* $\mathfrak{M}$ consists of a frame and a valuation V (indicating for each propositional letter, the set of states where it holds). Thus, the above mentioned intended meanings may be captured by formulas as follows:

$\mathfrak{M} \Vdash \Box_K \varphi\,[\![s]\!]$	iff	for **every** state $t \in {}_s\mathrm{R}_K$, $\mathfrak{M} \Vdash \varphi\,[\![t]\!]$;
$\mathfrak{M} \Vdash \Diamond_K \varphi\,[\![s]\!]$	iff	for **some** state $t \in {}_s\mathrm{R}_K$, $\mathfrak{M} \Vdash \varphi\,[\![t]\!]$.

Imagine that we have a formula ξ of L describing the checkmate configurations:

s is a checkmate configuration iff ξ holds at s (i. e. $\mathfrak{M} \Vdash \xi\,[\![s]\!]$).

Then, we can express some properties of a given configuration s:

all accessible configurations by the king are checkmate by $\mathfrak{M} \Vdash \Box_K\,\xi\,[\![s]\!]$;

some accessible configuration by the king is checkmate by $\mathfrak{M} \Vdash \Diamond_K\,\xi\,[\![s]\!]$.

5.2 Extension to vague multimodalities

We are now going to examine logics with vague multimodalities. We will extend the preceding treatment (in 5.1) to encompass vague multimodalities.

We will use similar ideas to the extension seen to the unimodal case (cf. 4.3). Now, we will extend the multimodal language L, by adding a vague modality for each chess piece. For instance, for the king, we have the following vague modality, with his intended meaning:

∇_K in an important set of states accessible by the king.

Thus, we add to the multimodal language L, vague modalities $\nabla_Q, \nabla_K, \ldots$, to obtain the extended multimodal language L^*.

As before, the idea is that $\nabla_K\,\varphi$ holds at state s iff the set of states $t \in {}_s\mathrm{R}_K$ where φ holds is an important set among the states accessible by the king.

Thus, to interpret formulas with such vague modalities, we need the notion of important sets of accessible states: we should associate to each state a family of important sets for each piece. In other words, we must enrich

the model $\mathfrak{M} = \langle\, S, \mathrm{R}_Q, \mathrm{R}_K, \ldots, V \,\rangle$, for the multimodal language L,

by a function for each piece assigning to each state a family of sets (considered *important* among the accessible ones). This will yield a *multimodulated model*

$\mathfrak{M}^{\underline{\mathcal{W}}^*} \coloneqq \langle\, S, \mathrm{R}_Q, \mathrm{R}_K, \ldots, V, {}_Q\underline{\mathcal{W}}, {}_K\underline{\mathcal{W}}, \ldots \,\rangle$ for the language L^*.

Now, the above intended meaning can be given as follows:

$$\mathfrak{M}^{\underline{\mathcal{W}}^*} \Vdash \nabla_K\,\varphi\,[\![s]\!] \quad \text{iff} \quad \{\, t \in {}_s\mathrm{R}_K \,/\, \mathfrak{M}^{\underline{\mathcal{W}}^*} \Vdash \varphi\,[\![t]\!] \,\} \text{ belongs to family } {}_K\underline{\mathcal{W}}_s.$$

For instance, consider the formula ξ of L describing the checkmate configurations (cf. 5.1). We can express some vague properties of a given configuration s; e. g. "The accessible configurations by the king `generally` are checkmate" by $\mathfrak{M}^{\underline{\mathcal{W}}^*} \Vdash \xi\,[\![s]\!]$.

5.3 Other vague multimodalities

So far, we have treated unary modalities, corresponding to binary accessibility relations. We now briefly indicate how to extend our previous ideas to non-unary modalities, corresponding to other accessibility relations.

Non-unary modalities are quite natural in some cases. An example appears in arrow logic, where non-binary accessibility relations appear naturally [2]. Arrows can be reversed, composed, etc., which leads to some relations comparing arrows. For instance, the arrows $a \to b$ and $a \leftarrow b$ are reverse to each other; an arrow such as $a \to c$ can be decomposed into two others: $a \to b$ and $b \to c$. Note that the relation 'is reversible to' compares 2 arrows, while the relation 'is decomposable to' envolves 3 arrows:

$$\begin{array}{ccc} \text{binary relation} & \qquad & \text{ternary relation} \\ \underbrace{a \to b \;\; a \leftarrow b}_{C} & \qquad & \underbrace{a \to c \;\; a \to b \;\; b \to c}_{D} \end{array}$$

Thus, we are going to consider an *arrow frame* $\mathfrak{S}$, consisting of a set S (of arrows) and the accessibility relations on S: C (binary) e D (ternary).[11] So, for each arrow $s \in S$, ${}_sC$ is a set of arrows and ${}_sD$ is a set of pairs of arrows. In this manner, one has: $(t_1, t_2) \in {}_sD$ iff s can be decomposed into t_1 and t_2. Now, it seems natural to consider the following modalities (with their intended meanings):

$\circ\,\varphi$ holds at s iff for **some** arrow $t \in {}_sC$, φ holds at t;

$\psi \otimes \theta$ holds at s iff for **some** $(t_1, t_2) \in {}_sD$, ψ holds at t_1 and θ holds at t_2.

In this case, is seems natural to consider vague modalities associated to the reverse and to the decomposition.

The vague modality $\triangledown_\circ$ associated to converse is unary and its interpretation is as before (cf. 5.2): it involves, for each arrow $s \in S$, a family ${}_C\underline{\mathcal{W}}_s$ of sets (considered important) of converses, i. e. of subsets of ${}_sC$.

The vague modality $\triangledown_\otimes$ associated to decomposition is more interesting. This vague modality $\triangledown_\otimes$ is binary and its interpretation involves a family of important sets of decompositions of each arrows. But, for that, it suffices to extend the previous ideas of 5.2 to important sets of pairs of arrows. Thus, we associate to each arrow $s \in S$, a family ${}_D\underline{\mathcal{W}}_s$ of subsets (considered important) of ${}_sD$.

In such a model $\mathfrak{M}^\star$, the intended meanings can be given as follows:

$\mathfrak{M}^\star \Vdash \triangledown_\circ\,\varphi\,[\![s]\!]$ iff $\{\, t \in {}_sC \,/\, \mathfrak{M}^\star \Vdash \varphi\,[\![s]\!] \,\}$ belongs to ${}_C\underline{\mathcal{W}}_s$;

$\mathfrak{M}^\star \Vdash \psi \;\triangledown_\otimes\; \theta\,[\![s]\!]$ iff $\{\, (t_1, t_2) \in {}_sD \,/\, \mathfrak{M}^\star \Vdash \psi\,[\![t_1]\!] \,\&\, \mathfrak{M}^\star \Vdash \theta\,[\![t_2]\!] \,\} \in {}_D\underline{\mathcal{W}}_s$.

Thus, one can see how to extend the preceding ideas to other vague modalities.

[11] Arrow logic may also have a unary accessibility relation (for a nullary modality) [2].

6 Logics for vague notions: a formal approach

We will now give a more formal treatment for the ideas presented previously.

6.1 Assigning precise meaning to vague notions

We first recapitulate some of the preceding ideas.

Consider a universe V. In view of the preceding considerations, the interpretation of "the objects of V `generally` have property φ" can be seen to amount to "the set of objects of V having φ belongs to a given family $\underline{\mathcal{W}}_V$ (of important sets of V). Similarly, the interpretation of "the objects of V `rarely` have property φ" can be seen to amount to the set of objects of V having φ belongs to a given family $\underline{\mathcal{N}}_V$ (of negligible sets of V). Thus, '`generally`' and '`rarely`', within a given universe V, can be explained in terms of families $\underline{\mathcal{W}}_V$ of important subsets and $\underline{\mathcal{N}}_V$ of negligible subsets of universe V.

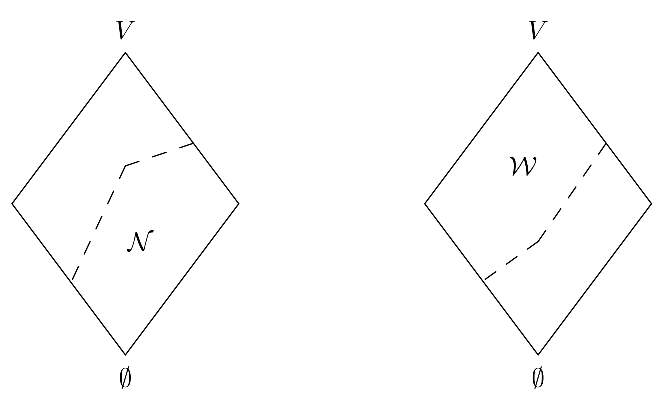
Some general properties can be expected to be shared by several notions corresponding to '`generally`' and '`rarely`'.

On the one hand, the idea of exceptions, involved in understanding "objects `generally` have property φ" as "objects `rarely` fail to have property φ", corresponds to the duality between these families (cf. 3.2): a subset S of the universe V is negligible ($S \in \underline{\mathcal{N}}_V$) iff its complement $V \setminus S$ is important ($\overline{S} \in \underline{\mathcal{W}}_V$).

On the other hand, we may wish non-trivial notions: the existence of negligible and non-negligible sets (as well as important and non-important sets). Now, one would probably regard the empty set as (quite) negligible ($\emptyset \in \underline{\mathcal{N}}_V$) and the universe as non-negligible ($V \notin \underline{\mathcal{N}}_V$).[12] We then have $\underline{\mathcal{N}}_V \neq \emptyset$ and, if $V \neq \emptyset$, also $\underline{\mathcal{N}}_V \neq \wp(V)$. Thus, over a non-empty universe $V \neq \emptyset$, our dual families will both be proper: $\emptyset \neq \underline{\mathcal{N}}_V \neq \wp(V)$ and $\emptyset \neq \underline{\mathcal{W}}_V \neq \wp(V)$.

Other properties, however, would be expected to be shared only by families corresponding to some notions of '`generally`' and '`rarely`'.

[12] One often pictures the negligible and important sets by Hasse diagrams as follows:



In the interpretation of '`generally`' as '`many`', the family of important sets is closed under supersets: a superset $Y \supseteq X$ of an important set $X \in \underline{\mathcal{W}}_V$ is important: $Y \in \underline{\mathcal{W}}_V$ (cf. Example 3.2). Then, the family of important sets is upward closed.

In addition, in the interpretation of '`rarely`' as '`few`', the family of negligible sets, those having few elements, is also closed under union (cf. Example 3.3). Thus the intersection of important sets $X, Y \in \mathcal{W}$ is important: $X \cap Y \in \mathcal{W}$.

In such cases, the family of important sets is a (proper) filter. Also, in other interpretations of '`generally`', the family of important sets is a proper ultrafilter. Thus, these interpretations of '`generally`' give rise to a hierarchy of families. The family of important subsets of the universe is:

($\mathcal{S}$) upward closed in the '`many`' interpretation,

($\mathcal{F}$) a filter in the '`most`' interpretation, and

($\mathcal{U}$) an ultrafilter in other interpretations.

6.2 Basic concepts and results

We will now introduce some logics embodying the preceding ideas. We start from basic modal logic [2] (BML, for short) and extend it as in 4.3.

Our *alphabet* will be that of basic modal logic with a new modality ∇. The *formulas* of our language $\mathrm{L}_{\mathrm{M}}^{\nabla}$ are obtained by closing the set of formulas of BML under the new modality ∇. Our frames and (rooted) models are much as in BML, but we add families of important sets to give semantics for the ∇-operator (cf. 4.3). For a (Kripkean) frame $\mathfrak{F} = \langle S, \mathrm{R} \rangle$ and a state $s \in S$, we use ${}_s\mathrm{R}$ for the set of states R-reachable from s: ${}_s\mathrm{R} \coloneqq \{ t \in S \,/\, (s,t) \in \mathrm{R} \}$. Now, *complex frame* is a frame with a complex $\underline{\mathcal{K}}$: a triple $\mathfrak{F}^{\underline{\mathcal{K}}} = \langle S, \mathrm{R}, \underline{\mathcal{K}} \rangle$, where $\langle S, \mathrm{R} \rangle$ is a frame and complex $\underline{\mathcal{K}}$ is a function mapping each $s \in S$ to a family $\underline{\mathcal{K}}_s \subseteq \wp({}_s\mathrm{R})$. A *complex model* is a complex frame $\mathfrak{F}^{\underline{\mathcal{K}}}$ with a valuation $\mathbf{v}$ (noted $\mathfrak{F}^{\underline{\mathcal{K}}}[\mathbf{v}]$). A *rooted* complex model is a complex model with a distinguished state, as usual. *Satisfaction* of a formula in a rooted complex model (noted $\mathfrak{M}^{\underline{\mathcal{K}}}, s \Vdash \varphi$) is defined as in BML, with the following extra clause:

$$\mathfrak{M}^{\underline{\mathcal{K}}}, s \Vdash \nabla\theta \text{ iff } \{ t \in {}_s\mathrm{R} \,/\, \mathfrak{M}^{\underline{\mathcal{K}}}, t \Vdash \theta \} \in \underline{\mathcal{K}}_s$$

This clause is intended to express the idea that a formula $\nabla\theta$ holds at a state $s \in S$ iff the set of states reachable from s where θ holds is important: is in $\underline{\mathcal{K}}_s$.
We use $\mathfrak{M}^{\underline{\mathcal{K}}}[\varphi]$ for the *extension* of a formula φ (cf. 4.2). With this notation, the above clause for $\nabla\theta$ becomes: $s \in \mathfrak{M}^{\underline{\mathcal{K}}}[\nabla\theta]$ iff ${}_s\mathrm{R} \cap \mathfrak{M}^{\underline{\mathcal{K}}}[\theta] \in \underline{\mathcal{K}}_s$.

We will shortly set up some logics for local vague notions and axiomatize them. For this purpose, we shall first examine some basic ideas.

On the semantical side, we will deal with local consequence relations. It turns out to be convenient to present them in a relativized form.

Consider a class $\boldsymbol{\mathcal{C}}$ of families of sets (e. g. lattices, filters, etc.). By a $\boldsymbol{\mathcal{C}}$-frame we mean a complex frame $\mathfrak{F}^{\underline{\mathcal{K}}} = \langle S, \mathrm{R}, \underline{\mathcal{K}}_s \rangle$, where each family $\underline{\mathcal{K}}_s \subseteq \wp({}_s\mathrm{R})$ is in the given class $\boldsymbol{\mathcal{C}}$ of families. We use $\mathfrak{F}^{\boldsymbol{\mathcal{C}}}$ for the class of $\boldsymbol{\mathcal{C}}$-frames.

We now introduce our notion of *local consequence* relative to a class of frames $\mathfrak{F} \subseteq \mathfrak{F}^{\boldsymbol{\mathcal{C}}}$: $\Gamma \models_{\mathfrak{F}} \varphi$ iff $\mathfrak{F}^{\underline{\mathcal{K}}}[\mathbf{v}], s \Vdash \varphi$, for every complex frame $\mathfrak{F}^{\underline{\mathcal{K}}} \in \mathfrak{F}$ such that $\mathfrak{F}^{\underline{\mathcal{K}}}[\mathbf{v}], s \Vdash \Gamma$. In the special case where $\mathfrak{F}$ consists of all frames, we simplify the notation by using simply $\models_{\boldsymbol{\mathcal{C}}}$.

On the axiomatic side, the notions of theorem and derivability take into account the rules of necessitation, Modus Ponens and uniform substitution.

Consider a normal deductive system for BML [2], having as set of logical axioms the (instances of) tautologies, as well as the following axioms

(K)	$\Box(p \to q) \to (\Box p \to \Box q)$	(distributivity)
(Dl)	$\Diamond p \leftrightarrow \neg\Box\neg p$	(duality)

and the following rules of inference

[MP]	from $(\psi \to \theta)$ and ψ, infer θ	(Modus Ponens)
[Sb]	from ψ, infer an instance θ of ψ	(Uniform Substitution)
[Nc]	from φ, infer $\Box\varphi$	(Necessitation)

We will consider deductive systems parameterized by axioms. Given a set Ξ of formulas of language $\mathrm{L}_{\mathrm{M}}^{\nabla}$, we form a Ξ-*presentation* by adding to our deductive system for BML the formulas in set Ξ as axioms. We say that a formula τ is a Ξ-*theorem* iff τ can be obtained from the axioms in $K \cup \Xi$ by the rules of inference [MP], [Sb] and [Nc]. Now, given a set Γ of formulas, we call formula φ Ξ-*derivable* from set Γ (and write $\Gamma \vdash_{\Xi} \varphi$) iff φ can be obtained from the Ξ-theorems by applications of Modus Ponens [MP] and Uniform Substitution [Sb]. Notice that we have the Deduction Theorem for Ξ-derivability: if $\Gamma \cup \{\psi\} \vdash_{\Xi} \theta$, then also $\Gamma \vdash_{\Xi} \psi \to \theta$.

Also, given a derivability relation $\vdash$, we have a $\vdash$-canonical model $\mathfrak{M}_{\vdash}$: its states are maximal $\vdash$-consistent sets of formulas [2].

6.3 Basic local modal logic for 'generally' and 'rarely'

We now examine the basic local modal logic for ∇: with no restriction on the nature of the local complexes.

By considering the class $\boldsymbol{\mathcal{B}}$ of all families of sets, we obtain the relation $\models_{\boldsymbol{\mathcal{B}}}$ of *basic* consequence. We thus have our *modal local vague logic* (LV, for short).

Now, a formula is satisfied iff its extension is in the corresponding complex (cf. 4.3 and 6.2). This fact can be expressed by the following valid formulas:

$(\leftrightarrow \nabla)$	$\forall v(\psi \leftrightarrow \theta) \to (\nabla v\psi \to \nabla v\psi)$	(Extensionality)
(∇^{α})	$\nabla v\varphi \to \nabla v\varphi')$ for an alphabetic variant φ' of φ	(Variant)

A deductive system for LV is obtained by extending our deductive system for BML by the extensionality and variant schemas. With $B := \{(\leftrightarrow \nabla), (\nabla^{\alpha})\}$, we have B-theorems and the relation $\vdash_B$ of *local vague* derivability. We can now establish soundness and completeness of this deductive system for our logic LV. Soundness $\vdash_B \subseteq \models_{\mathcal{B}}$ is easy to establish, as usual. We shall now prove completeness $\models_{\mathcal{B}} \subseteq \vdash_B$. For this purpose, we consider the $\vdash_B$-canonical model (cf. 6.2), say $\mathfrak{M} = \langle S, \mathrm{R}, \mathbf{v} \rangle$, and extend it by the canonical complex. More precisely, we introduce the spreads of a formula φ: $[\varphi] := \{\Delta \in S \,/\, \varphi \in \Delta\}$ and ${}_{\Gamma}[\varphi] \coloneqq {}_{\Gamma}\mathrm{R} \cap [\varphi]$. We can now define the *canonical complex* $\underline{\mathcal{K}}$ by setting $\underline{\mathcal{K}}_{\Gamma} \coloneqq \{ {}_{\Gamma}[\varphi] \subseteq S \,/\, \nabla\varphi \in \Gamma \}$. We can then establish a satisfaction lemma: $\mathfrak{M}^{\underline{\mathcal{K}}}, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$ (by induction on φ). Thus, completeness follows as usual [2].[13]

Our logic LV places no restriction on the complexes. We may view it as a basic local logic for ∇, where ∇ may be interpreted as '`generally`' or '`rarely`'. In the sequel we will examine logics for some particular classes of complexes.

6.4 Particular local modal logics for 'generally' and 'rarely'

We will now consider logics for some particular versions of 'generally' and 'rarely', corresponding to special classes of families. These logics will be constructed modularly by extending our axiomatization B for basic logic LV with axioms coding properties of the families.

Characterizing a class of families as appropriate to a particular vague notion is a modeling issue. Associated logical problems are axiomatizing the corresponding class of frames and consequence relation. We shall illustrate the latter with some classes of complexes characterized by closure-like properties.

As a simple example, consider the case where the families of important sets are lattices: closed under intersection and union. By considering the class $\mathcal{L}$ of all lattices, we obtain the local lattice frames as well as the relation $\models_{\mathcal{L}}$ of lattice consequence. This gives *modal local lattice logic* (LL, for short). We can axiomatize LL by adding to our axiomatization B for basic logic LV the following two axioms:

$(\nabla\wedge)$	$(\nabla p \wedge \nabla q) \to \nabla(p \wedge q)$	($\cap$-closure)
$(\nabla\vee)$	$(\nabla p \wedge \nabla q) \to \nabla(p \vee q)$	($\cup$-closure)

Axioms $(\nabla\wedge)$ and $(\nabla\vee)$ express closure under $\cap$ and $\cup$, respectively.[14] Thus, they

[13]Indeed, if $\Delta \nvdash \varphi$, then $\Delta \cup \{\neg\varphi\}$ is B-consistent and can be extended to a B-maximal consistent set Γ, then $\mathfrak{M}^{\underline{\mathcal{K}}}, \Gamma \nVdash \varphi$.

[14]In a rooted complex model, formulas $[\nabla\wedge]$ and $[\nabla\vee]$ express if ${}_s\mathrm{R} \cap \mathbf{v}(p) \in \underline{\mathcal{K}}_s$ and ${}_s\mathrm{R} \cap \mathbf{v}(q) \in \underline{\mathcal{K}}_s$, then ${}_s\mathrm{R} \cap (\mathbf{v}(p) \cap \mathbf{v}(q)) \in \underline{\mathcal{K}}_s$ and ${}_s\mathrm{R} \cap (\mathbf{v}(p) \cup \mathbf{v}(q)) \in \underline{\mathcal{K}}_s$.

characterize the lattice frames as the complex frames where they are globally valid: $\mathfrak{F}^{\underline{\mathcal{K}}}$ is a lattice frame iff $\mathfrak{F}^{\underline{\mathcal{K}}} \models \{ (\nabla\wedge), (\nabla\vee) \}$.

6.5 Modal local logics for 'generally'

The logics LV and LL involve families corresponding to '`generally`' and to '`rarely`': no distinction between 'important' and 'negligible' is apparent.

One way of distinguishing 'important' from 'negligible' is: "the universe is important and the empty set is not". We will examine it shortly. First, we consider another way: "a subset of a negligible set is also negligible", i. e. we consider each family of important sets to be closed under supersets of its universe. (See 6.1.)

Call a family $\mathcal{S}$ over universe V *inclusion-closed* iff it is closed under supersets of its universe V. Now, consider the class $\mathcal{S}$ consisting of all inclusion-closed families. We then obtain *local inclusion-closed frames* as well as the relation $\models_{\mathcal{S}}$ of inclusion-closed consequence. This gives *modal local inclusion-closed logic* (LS, for short).

To axiomatize LS, we consider an up-closure axiom, similar to (K) (cf. 6.2) and to $(\leftrightarrow \nabla)$ (cf. 6.3), namely $(\rightarrow \nabla)$: $\Box(p \rightarrow q) \rightarrow (\nabla p \rightarrow \nabla q)$. Let us analyze this axiom. In a complex model, its instances express only a moderate upward-closure (for definable subsets, see below). But, in a frame, it does express superset closure; so it characterizes the inclusion-closed frames as the complex frames where it is globally valid: $\mathfrak{F}^{\underline{\mathcal{K}}}$ is as inclusion-closed frame iff $\mathfrak{F}^{\underline{\mathcal{K}}} \Vdash (\rightarrow \nabla)$.

Axiom $(\rightarrow \nabla)$ is <u>not</u> canonical, but there is a simple constructive remedy. We shall call a presentation *up-closed* iff the up-closure axiom $(\rightarrow \nabla)$ is among its theorems. Up-closed presentations are extensional: axiom $(\leftrightarrow \nabla)$ can be deduced from $(\rightarrow \nabla)$.[15] Consider a complex model $\mathfrak{M}^{\underline{\mathcal{K}}}$ of an up-closed presentation Ξ. We can see that the derivable instances $\Box(\psi \rightarrow \theta) \rightarrow (\nabla\psi \rightarrow \nabla\theta)$ make each family $\underline{\mathcal{K}}_s \subseteq \wp({}_s\mathrm{R})$ closed under <u>definable</u> supersets, but not necessarily under every superset of the universe ${}_s\mathrm{R}$. Thus, we obtain canonical complex $\underline{\mathcal{K}}$ (cf. 6.3) and take its superset closure: we construct $\underline{\mathcal{K}}^{\supseteq}{}_{\Gamma}$ by closing $\underline{\mathcal{K}}_{\Gamma}$ under supersets.[16]

With $S \coloneqq B \cup \{\rightarrow \nabla\}$, we have S-theorems and the relation $\vdash_S$ of *local up-closed* derivability. We clearly have soundness ($\vdash_S \subseteq \models_{\mathcal{S}}$) and completeness ($\models_{\mathcal{S}} \subseteq \vdash_S$) now follows from the preceding remarks.

We have examined some modal local logics for '`generally`': the basic logic LV and its extensions LL and LS. Each one of these logics for '`generally`' (as well as some

[15] In the presence of $(\leftrightarrow \nabla)$, $\Box(\psi \rightarrow \theta) \rightarrow (\nabla\psi \rightarrow \nabla\theta)$ has several equivalent versions, like $\nabla\psi \rightarrow \nabla(\psi \vee \theta)$, $\nabla(\psi \wedge \theta) \rightarrow \nabla\theta$ and $\nabla(\psi \wedge \theta) \rightarrow \nabla(\psi \vee \theta)$. This situation is reminiscent of the inference rule RM [4]. Also, note that a rule such as $\dfrac{\vdash \mathsf{p} \rightarrow \mathsf{q}}{\vdash \nabla\mathsf{p} \rightarrow \nabla\mathsf{q}}$ does not imply $(\rightarrow \nabla)$ thus, failing to characterize the local inclusion-closed frames.

[16] This extension adds no new definable subsets.

similar ones) is a conservative extension of the basic modal logic: for formulas without ∇, each consequence coincides with that of BML.[17]

6.6 Modal local logics for proper 'generally'

We now consider distinguishing 'important' from 'negligible' by proper families.

As seen in 6.1, we can have proper families (with the universe but not the empty set) over non-empty universes. So, let us call a complex $\underline{\mathcal{K}}$ *proper* for a frame $\mathfrak{F} = \langle S, \mathrm{R} \rangle$ iff, for each $s \in S$, ${}_s\mathrm{R} \in \underline{\mathcal{K}}_s$ and, whenever ${}_s\mathrm{R} \neq \emptyset$, $\underline{\mathcal{K}}_s \neq \emptyset$. A *proper frame* is a complex frame $\mathfrak{F}^{\underline{\mathcal{K}}}$ whose complex is proper. The proper frames can be characterized by the formulas $\nabla\top$ and $\Diamond\top \to \neg\nabla\bot$.

By considering the class $\mathfrak{P}$ of all proper frames, we obtain the relation $\models_{\mathfrak{P}}$ of *proper* consequence. We thus have our *modal local proper logic* (LP, for short). We can axiomatize LP by $P \coloneqq B \cup \{\nabla\top, \Diamond\top \to \neg\nabla\bot\}$.

We shall now examine two other examples of logics for proper '`generally`', with complexes giving proper filters and ultrafilters [12, 13].

The filters are the families of subsets closed under supersets and intersection. We can characterize the filter-frames by the previous up-closure axiom ($\to \nabla$) (cf. 6.5) together with $\cap$-closure axiom ($\nabla\wedge$): $(\nabla p \wedge \nabla q) \to \nabla(p \wedge q)$. By considering the proper complex frames $\mathfrak{F}^{\underline{\mathcal{K}}}$ where each $\underline{\mathcal{K}}_s$ is a filter, we have *local proper filter logic* (PF, for short). As before, soundness is easy and completeness not quite so. If we consider the canonical complex $\underline{\mathcal{K}}$ (cf. 6.3), we can see that each $\underline{\mathcal{K}}_\Gamma$ has the finite intersection property: $S_1 \cap \ldots \cap S_n \neq \emptyset$, whenever $S_1, \ldots, S_n \in \underline{\mathcal{K}}_\Gamma$. So, the superset-closure $\underline{\mathcal{K}}^{\supseteq}{}_\Gamma$ is a proper filter.

We now examine the case of proper ultrafilters: maximal proper filters. We can characterize the ultrafilter-frames by the filter axioms together with attracting axiom ($\neg\nabla$): $\neg\nabla p \to \nabla(\neg p)$.[18] By considering the proper complex frames $\mathfrak{F}^{\underline{\mathcal{K}}}$ where each $\underline{\mathcal{K}}_s$ is an ultrafilter, we have *local proper ultrafilter logic* (PU, for short). As before, soundness is easy and completeness is now apparent. If we consider the canonical complex $\underline{\mathcal{K}}$, since each $\underline{\mathcal{K}}_\Gamma$ has the finite intersection property, it can be extended to a proper ultrafilter.

We mention that we can also consider vague logics based on restricted frames, like serial frames (characterized by $\Diamond\top$).

7 Final comments

We now summarize our presentation and make some comments.

We have examined some ideas about vague notions such as 'generally', 'rarely', etc., which often occur in natural language and in some branches of science, and a way

[17] Each model can be expanded with the trivial complex: $\underline{\mathcal{E}}_s := \emptyset$, for every state s.

[18] Alternatively, we can add the prime axiom ($\vee\nabla$): $\nabla(p \vee q) \to (\nabla p \vee \nabla q)$.

to handle modalities corresponding to them in order to have a precise treatment for reasoning about them.

In the first part of the paper (Sections 2 through 5), we have introduced the ideas in an informal manner, motivating them through illustrative examples, with the purpose of showing how intuitive ideas led to the treatment by families of sets. We have presented the basic ideas for vague notions, vague quantification and vague modalities (in Section 2), addressed the question of a precise treatment to vague quantification and vague modality, presented logics with vague modalities (in section 4) and addressed the case of logics with vague multimodalities (in Section 5). Notice that these logics are not fuzzy logics: we address vague assertions about precise domains (cf. Example 3.3).

In the second part of the paper (Section 6) we have treated these ideas in a more formal way. We have examined modal logics, with a new generality operator for expressing and reasoning about some local versions of 'generally'. We have introduced some local generalized modal logics, where the locality aspect corresponds to the intended meaning of the generalized operator as "an important part of the reachable states has the given property".

The various logics presented are obtained as extension of the basic modal logic by addition of axioms which capture properties of a specific family of important sets. The basic modal local logic LV for 'generally' is axiomatized by extending the usual deductive system for basic modal logic [2]. Other modal local logics for 'generally' and 'rarely' can be characterized by some new axioms, which axiomatize them when added to the basic logic LV. These calculi are correct [14].

We hope that the ideas presented in this paper provide a first step towards the development of a modal framework for generalized logics where vague notions can be represented and manipulated in a precise way and the connections among them investigated (e. g. relate 'important' with 'very important', etc.).

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