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A System of Hyperintensional Logic

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Abstract

We put forward a new system of hyperintensional propositional logic, which can be summarily described as extending second-order propositional modal logic by the addition of four specifically hyperintensional connections: propositional identity, propositional existence, a notion of propositional constituency, and a notion of grounding. We set up the foundations of the system (syntax and semantics, both of which include some distinctive features); pursue various internal developments (definitions, principles); and investigate some fragments of the full system.

Keywords: hyperintensional logic, hyperintensional propositions, ground, constituency, propositional identity, propositional existence, truthmaking, entailment.

Contents

1 Introduction

The present work sets up and develops a system of 'Hyperintensional Propositional Logic' – i.e. a formal theory concerned with hyperintensional propositions and logical connections between them. The system can be summarily described as extending propositional modal logic (S5) with propositional quantifiers by the addition of some specifically hyperintensional connections: propositional identity, propositional existence, a notion of propositional constituency, and a notion of grounding. – We will present a formal (set-theoretic) semantics for this system; and will pursue various internal (as opposed to meta-theoretical) developments (definitions, principles). We will investigate not only the full system, but also various fragments of special interest.

The full system HPL (as we call it), and even some of its more conspicuous strong fragments (e.g. without modality, or without propositional existence), have very rich conceptual resources, comparable e.g. to the resources of standard set theory. – Indeed one of the main motivations for this work was precisely the idea of constructing an ultra-comprehensive apparatus – a kind of *characteristica universalis* (as far as the subject of hyperintensional propositional logic goes) (although this description should certainly be taken with many pinches of salt). – Accordingly, as with set theory, there is here a practically unlimited space for internal ('intra-theoretical', as opposed to 'meta-theoretical') developments: i.e. one defines notions of increasing complexity, and one proves propositions, also of increasing complexity, concerning such notions; subfields are naturally formed, then sub-subfields and so on. – The internal developments of Hyperintensional Propositional Logic given below are of a relatively rudimentary character and consist only of some initial steps into this unlimited space. I would be happy to see this carried further in future by workers with a taste for such 'mathematical metaphysics'.

– We add here a few general remarks before proceeding to ‘official theorizing’ in the next section.

(1) *On primitive notions.*

(1.1) We use a single classical-propositional-logic primitive connective, viz. the ‘*neconjunction*’ connective N, with $N(\varphi_1, \ldots, \varphi_n)$ supposed to be equivalent to $\neg\wedge(\varphi_1, \ldots, \varphi_n)$. (Hence the neologism ‘neconjunction’, from ‘*ne*gation of *conjunction*’.) This is taken as ‘multigrade’, with the n above ≥ 0. – This procedure permits the greatest possible simplicity for our semantical scheme, where the construction of a set from its elements will serve to ‘model’ the construction of a ‘neconjunctive’ proposition from its ‘neconjuncts’. (The use of N here was suggested of course by the notation in Wittgenstein’s *Tractatus.* Note however that there N stands rather for ‘nedisjunction’ [or equivalently ‘con-negation’].)

(1.2) We will understand the propositional quantifiers in a ‘generalistic’ rather than ‘absolutistic’ sense: i.e. as ranging over some ‘suitably closed’ domain of hyperintensional propositions, rather than over ‘absolutely *all*’ hyperintensional propositions. Thus e.g., as in standard S5 with propositional quantifiers (viz. what Fine 1970 calls S5π+), such a formula as $\exists p(p \wedge \neg\Box p)$ will *not* count as valid here. (Much as $\exists x\exists y\neg(x = y)$ does not count as valid in general predicate logic.)

(1.3) *Propositional existence* (E). We think here of the (hyperintensional) proposition as a complex object the existence of which is tantamount to the existence of its constituents. Thus the existence of an atomic proposition aRb amounts to the existence of a and of R and of b. – This we cannot say in our HPL, since we have no means to speak of the constituents of atomic propositions. But we have also that e.g. the existence of the molecular proposition N(p, q) amounts in effect to the existence of p and of q; and this we *can* say in HPL: $EN(p, q) \leftrightarrow (Ep \wedge Eq)$ will be a valid formula of HPL.

(1.4) *Immediate propositional constituency* (ε). Like any other complexes, propositions have their *constituents*, i.e. objects which enter into the constitution of the proposition. There are the *immediate* constituents, then the immediate constituents of immediate constituents, and so on. E.g. a is an immediate constituent of aRb, which in turn is an immediate constituent of N(aRb); so a is a constituent of N(aRb) though not an immediate constituent. – Again, in HPL we have no means to speak of non-propositions; so our ε stands for *propositional* immediate constituency in a double sense: we must have a proposition ‘on the right’ *and* ‘on the left’. But still there is much we can say ($p\ \varepsilon\ N(p)$, $N(p)\ \varepsilon\ N(N(p))$, $\neg(p\ \varepsilon\ N(N(p)))$, etc.).

(1.5) *Immediate grounding* (θ). Here the restriction to propositions is really no restriction, since immediate grounding in the intended sense already only relates propositions. (Indeed only relates facts [i.e. true propositions]. But still it is of course perfectly appropriate to call it a propositional connection. Necessity too for that matter only holds of facts, but is no less naturally called a propositional connection for that.) A typical example of valid formula here is: $\mathrm{p} \rightarrow \mathrm{p}\ \theta\ \mathrm{N(N(p))}$. Our θ will always take a *single* formula on the left (as well as on the right): it corresponds to what is often called in the grounding literature *partial* (as opposed to *full*) (immediate) grounding. (I have serious doubts as to whether the idea of full grounding is ultimately in good standing. But here is not the place to go into a discussion of this. – Unfortunately full grounding is what tends to have 'center stage' in the recent literature on logic of ground, including the works of Fine [2012a, b] and Correia [2010, 2014, 2017, etc.]. The paper Schnieder 2011 is a laudable exception.)

(On ground-theory see further Batchelor 2010; and on the theory of constituency, or as I call it 'Stoichiology', Batchelor 2013. The present work is in some respects a continuation of these earlier papers. I do not here presuppose familiarity with the earlier papers; but they should certainly also be studied by readers who wish to obtain a fuller appreciation of my ideas on these matters.)

(2) *On the syntax of HPL.* We distinguish sharply between expressions which stand for methods of *formation of compound propositions* and expressions which serve to *make statements about* propositions, where such statements themselves may or may not correspond to propositions in our universe of propositions, and if they do *we* may or may not know what these propositions are. The former may be called *formative* expressions, and the latter *enunciative* expressions. – In our language for HPL we take the neconjunction symbol N as the only primitive formative expression; all other (constant) primitive expressions we take as purely enunciative. (Even N itself is used only enunciatively if applied to formulas involving enunciative expressions.)

Take e.g. propositional identity. Everybody will agree that the formula, say,

$$\mathrm{p} = \mathrm{q} \ \rightarrow\ \mathrm{q} = \mathrm{p}$$

should count as valid. For if the statement that p is identical with q is correct, then so is also of course the statement that q is identical with p. This does not depend on any specificities of different views as to the ultimate nature of the concept of propositional identity, nor on views as to the constitution of putative propositional-identity propositions, nor even on there *being* such propositions at all. – But now take e.g.

$$\neg(\mathrm{p} = \mathrm{q}) \rightarrow \neg((\mathrm{p} = \mathrm{p}) = (\mathrm{q} = \mathrm{q})).$$

Should this count as valid? That now *will* depend on specificities of views as to the nature of identity. If for instance one holds a primitivistic view on identity – that it is a basic, simple notion –, then one will no doubt think that the above formula is always true. (Even two primitivists might however disagree on whether e.g. the formula $(p = q) = (q = p)$ should count as valid. – Note incidentally that the curious restricted symmetry principle $(p = q) \rightarrow ((p = q) = (q = p))$ is much less questionable, having a droll proof by two applications of a principle of substitution of identicals.) But if on the other hand one thinks, as Ramsey did, that identity should be given a Leibnizian definition in terms of possession of the same 'properties' in the sense of 'propositional functions in extension', and that the universal property-quantification is really the conjunction of its instances, then one will *not* think that the above formula is always true. Indeed one will think that the formula

$$(p = p) = (q = q)$$

is always true.

Similar considerations apply to necessity (e.g., again, should $\neg(p = q) \rightarrow \neg(\Box p = \Box q)$ count as valid?), to grounding (including as special case the now much-debated issues of 'iterated ground'), etc.

In all such cases I would say that: (i) disputes are typically fruitless if they try to proceed independently of theories as to the ultimate nature of the notions in question (identity, modality, grounding, or whatever it may be); (ii) under assumption of a specific such theory, the disputed question will usually have a straightforward answer; (iii) at the present stage of inquiry, all such theories must be considered uncertain and tentative; and yet (iv) that it should make sense in *some* way to say that this is identical with that, that it is necessary that so-and-so, etc. – *this* is, at least comparatively, much more certain; and so finally (v) it seems then advisable at present to theorize under the assumption only of *enunciative* significance of such resources (or at any rate to clearly separate theorizing which proceeds only under such weak enunciative assumptions from more speculative theorizing under bolder formative assumptions).

So this will be our procedure in the syntax of HPL. Such expressions as say $(p = p) = (q = q)$, $\Box p = \Box q$, $p\ \theta\ (p\ \theta\ N(N(p)))$, $N(\forall p(p))\ \theta\ N(\forall p(p))$, EEp, $p\ \varepsilon\ (q\ \varepsilon\ r)$, etc. – these will not count as well-formed formulas.

(3) *On the semantics of HPL.* In the familiar possible world semantics for S5, or S5 with propositional quantifiers, we use '*flat*' *sets*, i.e. subsets of a non-empty set W, to 'model' the idea of *intensional propositions* ('sets of worlds'). Thus e.g. the negation, conjunction, disjunction etc. of intensional propositions get modelled by the flat operations of complementation, intersection, union etc. of subsets of W. – Now the

basic idea in our semantics for HPL is to use full '*hierarchical*' *sets*, i.e. sets built up from a basic stock of 'atoms' in the usual 'cumulative' way (with sets of atoms, sets of sets of atoms, sets of some atoms and some sets of atoms, etc. etc.), to model the idea of *hyperintensional propositions* built by neconjunction from some atomic propositions. Thus the *neconjunction* of certain propositions corresponds simply to the *set* of the sets-or-atoms corresponding to the given propositions. – After this the *truth or falsity* of propositions, the *existence* of propositions, and *modality* can be relatively easily dealt with by natural additions to the basic semantic apparatus.

(Someone might complain that this semantics models hyperintensional propositions so closely by hierarchical sets that this HPL becomes little more than set theory in disguise. But just the same point can be made for the possible world semantics for basic modal logic: S5 then, one would say, is some rudimentary, 'flat' set theory in disguise. – In both cases the friend of modality and propositions can give the same answer: that the formal semantics in question exploits a systematic structural *similarity* between modality/propositions and set-theoretic constructions; that this need not be taken to mean that the theory of propositions *is* set theory 'in disguise'; and that however the skeptic about propositions and modality can accept our theory since literally [though not in spirit] it involves nothing but ordinary extensional set theory, and so in this sense one might think that the set-theoretic semantics provides actually a kind of *vindication* of proposition theory. – In any case, I should like to emphasize that our semantics for HPL is in no worse [nor better] standing in these respects than the usual possible world semantics for basic modal logic.)

Underlying the whole construction is, of course, a predilection for an atomistic metaphysics where all propositions are truth-functions of atomic propositions. This does not mean however that readers who do not share this predilection ought then to immediately throw away the paper in disgust. For, *first*, it is no doubt interesting to see where this predilection might lead, at least in the spirit of exploratory, hypothetico-deductive investigations. In particular, I think it is remarkable how much *enunciation* is possible here with at the same time so little *formation*. *Secondly*, the theory developed here, and in particular our semantical scheme, may well be useful as a starting point for extensions corresponding to richer apparatuses of supposed methods of propositional formation. And *thirdly*, dependence on this atomistic scheme is far from ubiquitous in the present work. Often we need only the existence of truth-functional compounds (not also the *non*-existence of any *other* compounds); and often not even that.

(4) *Higher-order resources*. A neconjunction 'does' two different things. First, it collects certain objects (propositions) into a compound object (the neconjunctive proposition) having those objects as immediate constituents (much like sets are sup-

posed to do). And secondly, it 'says' something – it has a kind of 'import' or 'significance' (and thus will be true, or false). (The set in contrast just collects some objects into a compound and that's it. There is nothing corresponding to the import of a proposition, and accordingly nothing corresponding to the true/false dichotomy. – The proposition, we might say, is like a *talking set*.) – Now by considering neconjunctions in their 'purely collecting capacity' (where the import, although there, is unimportant, or the significance insignificant, or the talking unheard or unheeded), we obtain a straightforward simulation of set-theoretic resources within the language of HPL – namely: instead of something like 'There is set S s.t. ... $p \in S$ ...', we can use 'There is proposition q s.t. ... $p \, \varepsilon \, q$ ...'. (There are also somewhat similar *ground*-theoretic simulations in terms of θ.)

This corresponds of course to relatively 'small' sets – e.g. we cannot collect *all* the propositions of our universe into a new proposition still inside our universe (the formula $\neg\exists p \forall q(q \, \varepsilon \, p)$ is valid in HPL). To obtain something corresponding to *arbitrary* sets of propositions we would need to add further resources to our formal language. Perhaps the most obvious addition would be a stock of set variables (for arbitrary sets of propositions) and the membership-sign $\in$ (so now $\exists S \forall p(p \in S)$ would be a valid formula). But this seems unpleasantly 'hybrid': we may just as well ('technically'), and more purely ('conceptually'), add instead variables for arbitrary neconjunctions of propositions in our basic universe. We can then say that there is e.g. a neconjunction of all 'propositions' only it is not itself a 'proposition' but (say) a 'proper statement'. We would have a 'proposition-statement theory' along similar lines to the more familiar case of 'set-class theory'. We can even consider such a formula as $\exists p(p \vee \neg p)$ as now representing a 'statement' in our (extended) universe, viz. the disjunction of all propositions of the form $p \vee \neg p$, i.e. the neconjunction of all propositions of the form $\neg(p \vee \neg p)$. However even $\forall p(p \vee \neg p)$ is already 'problematic', since the neconjunction of all propositions of the form $p \vee \neg p$, being a 'proper statement', cannot be 'neconjoined' (negated). – This leads us to a third alternative here, viz.: again we add 'statement-variables' to the language, but now interpret them in terms of not just *one* extra level of arbitrary neconjunctions 'on top of' the basic universe of propositions, but a *denumerable sequence* of extra levels (an n-th level for each $n \geq 1$). Then each formula of classical propositional logic with propositional quantifiers will correspond either to a proposition or to a proper statement (w.r.t. a given assignment of propositional values to its free propositional variables).

Since all these alternatives (and other obvious variations on them) strike me as somewhat artificial, I will stick here with the basic version of HPL as just proposition theory. It should be clear however that my semantical scheme for HPL is immediately adaptable to these richer languages.

2 Basic notions: Syntax

We give now the basic syntactic notions for our system of Hyperintensional Propositional Logic (HPL). – The basic symbols of the system are (besides parentheses and the comma):

Propositional variables: p, q, r, s, p′, q′, . . .

Neconjunction: N

Metaphysical necessity: □

Propositional quantifier: ∀

Propositional identity: =

Propositional existence: E

Immediate propositional constituency: ε

Immediate ground: θ

– What we will call *transparent formulas* are defined by the clauses: (i) propositional variables are transparent formulas; and (ii) if $X_1 \ldots X_n$ ($n \geq 0$) are transparent formulas then so is $N(X_1, \ldots, X_n)$.

We will use A, B, C, . . . as metavariables for transparent formulas.

The general notion of *formula* is then defined by the clauses:

(1) Transparent formulas are formulas.

(2) If A and B are *transparent* formulas, then A = B, EA, A ε B and A θ B are formulas.

(3) If $X_1 \ldots X_n$, Y ($n \geq 0$) are formulas, then so are $N(X_1, \ldots, X_n)$, □Y and ∀p(Y) (for any propositional variable p).

– We will use φ, ψ, χ, . . . as metavariables for arbitrary formulas.

We say that a formula is an *opaque formula* if it is not a transparent formula.

The intuitive idea here is roughly that transparent formulas are formulas where we know what is the corresponding hyperintensional proposition (given assignment of hyperintensional propositions to the variables), whereas opaque formulas are formulas where we do *not* assume that we know what (if any) is the corresponding hyperintensional proposition, although we do assume that a meaningful statement is being made.

Remark. Fine (2012a, b) has influentially distinguished between the '*pure* logic of ground', which includes only general laws of ground (transitivity, irreflexity, etc. etc.) independent from systematic grounding connections involving specific methods of formation of logically complex propositions, and the '*impure* logic of ground', which *does* treat also of such connections (so we can here say that a true proposition grounds its double negation, or a disjunction of which it is disjunct, etc. etc.). – Our HPL is of course an '*impure*' logic of ground, constituency etc. The formulas of the '*pure*' *fragment* of HPL can be defined as in the above definition of formula only replacing 'transparent formulas' in clauses (1) and (2) by 'propositional variables'. (In Fine's own paper [2012b] on pure logic of ground, the outer conceptual apparatus [beyond the inner formation of 'atomic' grounding statements] is very rudimentary. But of course this is not required by purity itself.)

The 'pure fragment' of HPL can of course be further 'fragmented' by omission of primitives. Thus e.g. omitting E and ε (and if desired = which is definable from the other resources) gives a 'pure (modal) logic of ground'; and omitting further □ a 'pure (classical) logic of ground'. Or again, omitting ε and θ gives a 'pure logic of propositional identity and existence' – very close to the formal theory of Fine 1980.

(In Fine 1980 propositional variables occupy 'name position' and one has a 'first-order modal language' of familiar kind; there is also the addition of a propositional-truth predicate so that we can 'get off the ground' [not just with identity or existence] and write say $Tp \rightarrow Tp$, the simple $p \rightarrow p$ not being of course a well-formed formula there. Still, there are straightforward translations back and forth between the two languages, as indeed Fine himself in effect observes [pp. 159–160]. Even our full 'impure' use of transparent formulas [rather than just variables] can be reproduced in the 'first-order theory' framework by addition of 'function-symbols' for neconjunction: either a single 'multigrade' function-symbol forming a term from any n (≥ 0) terms, or alternatively an n-ary neconjunction function-symbol for each n (≥ 0). – Indeed this correspondence may perhaps make more easily understandable to some readers the idea of our distinction between transparent formulas [corresponding to terms in the first-order language] and opaque formulas [corresponding roughly to formulas in the first-order language].) ⊣

3 Basic notions: Semantics

We define four notions of 'frame': *basic frames*, suitable to HPL without modality and without propositional existence; *modal frames*, suitable to HPL with modality but without propositional existence; *existential frames*, suitable to HPL with propositional existence but without modality; and *full frames*, suitable to full HPL.

We begin by defining a notion of 'Zermelo universe', which will be involved in all four kinds of frame.

A *Zermelo universe* is a pair $\langle \mathrm{At}, \mathrm{Mol} \rangle$ where:

(1) At ('atoms') is an arbitrary (not necessarily non-empty) set of non-sets.

(2) Mol ('molecules') is a collection of sets, with $\mathrm{At} \in \mathrm{Mol}$, satisfying the following closure conditions:

(i) (Vertical analytic closure.) $\forall x \in \mathrm{Mol}$: $\forall y \in x$: If $y \notin \mathrm{At}$ then $y \in \mathrm{Mol}$.

(ii) (Horizontal analytic closure.) $\forall x \in \mathrm{Mol}$: $\forall y \subseteq x$: $y \in \mathrm{Mol}$.

(iii) (Vertical synthetic closure.)

(iii-A) (Element-Element Replacement.) If some molecules and/or atoms are 'indexed' by the elements of a molecule, then their collection is a molecule. I.e.:

$\forall x \in \mathrm{Mol}$: $\forall$ function f from x to $\mathrm{At} \cup \mathrm{Mol}$: $\{f(y) : y \in x\} \in \mathrm{Mol}$.

(iii-B) (Part-Element Replacement.) If some molecules and/or atoms are 'indexed' by the *subsets* of a molecule, then their collection is a molecule. I.e.:

$\forall x \in \mathrm{Mol}$: $\forall$ function f from $\mathcal{P}(x)$ to $\mathrm{At} \cup \mathrm{Mol}$: $\{f(y) : y \subseteq x\} \in \mathrm{Mol}$.

(iv) (Horizontal synthetic closure.)

(iv-A) (Element-Part Replacement.) If some molecules are indexed by the elements of a molecule, then their *union* is a molecule. I.e.:

$\forall x \in \mathrm{Mol}$: $\forall$ function f from x to Mol: $\bigcup\{f(y) : y \in x\} \in \mathrm{Mol}$.

(iv-B) (Part-Part Replacement.) If some molecules are indexed by the *subsets* of a molecule, then their *union* is a molecule. I.e.:

$\forall$x $\in$ Mol: $\forall$ function f from $\mathcal{P}$(x) to Mol: $\bigcup\{f(y) : y \subseteq x\} \in$ Mol.

Remarks. (1) This definition amounts to roughly the same as Zermelo's (1930) notion of 'normal domain', or the notion of 'Grothendieck universe'. There one has the more familiar conditions corresponding to the Power-Set and Union axioms instead of my more general conditions (iii-B) and (iv-A). (And nothing corresponding to my condition (iv-B), which of course follows from (iii-B) and (iv-A), but was nevertheless included above for the sake of conceptual symmetry.) My reason for this departure is that the conditions as formulated here, as four forms of 'Replacement', are plausibly regarded as having a *purely synthetic* character (if we consider only the objects to be collected or united [not the 'indices'] as the 'premisses' of the 'ontological inference'); whereas Power-Set and Union are more naturally regarded as hybrid, analytico-synthetic principles (and hence, plausibly, less fundamental principles).

(2) Note also that, given the requirement that At $\in$ Mol, of course Mol (unlike At) must necessarily be non-empty; and also Mol is forced to contain 'large' sets if At is itself a 'large' set. E.g. if At is infinite then Mol must contain e.g. $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}$. (Whereas if At is finite, Mol may or may not contain ω.)

(3) Here *possible worlds* will correspond to possibly true attributions of truth-values to atoms, and *intensions* (intensional propositions) as usual to sets/disjunctions of possible worlds. Now if the requirement that At $\in$ Mol was dropped from the definition above (and nothing else changed), intensions – even possible worlds themselves – might not have any corresponding hyperintensional proposition (in a 'frame'); so even say the formula of S5 with propositional quantifiers 'saying' that 'there are world-propositions' (valid w.r.t. the standard possible world semantics for S5 with propositional quantifiers) would not be valid here. ⊣

A *basic frame* is a triple $\langle$At, Mol, $w_0\rangle$ where:

(1) $\langle$At, Mol$\rangle$ is a Zermelo universe.

(2) w_0 ('actual world') is a function from At to the set of truth-values {T, F}.

A *modal frame* is a quadruple $\langle$At, Mol, w_0, W$\rangle$ where:

(1) $\langle$At, Mol, $w_0\rangle$ is a basic frame.

(2) W ('possible worlds') is a set of functions from At to {T, F}, with $w_0 \in$ W.

An *existential frame* is a quadruple $\langle \mathrm{At}, \mathrm{Mol}, \mathrm{w}_0, \mathcal{E} \rangle$ where:

(1) $\langle \mathrm{At}, \mathrm{Mol}, \mathrm{w}_0 \rangle$ is a basic frame.

(2) $\mathcal{E}$ ('existence-set' function) is a function from At to $\{\{\mathrm{w}_0\}, \{\,\}\} = \mathcal{P}(\{\mathrm{w}_0\})$.

(Intuitively, $\mathcal{E}$ counts an atom as existent or non-existent, according as it sends it to $\{\mathrm{w}_0\}$ or $\{\,\}$.)

And finally a *full frame* – or briefly, *frame* – is a quintuple $\langle \mathrm{At}, \mathrm{Mol}, \mathrm{w}_0, \mathrm{W}, \mathcal{E} \rangle$ where:

(1) $\langle \mathrm{At}, \mathrm{Mol}, \mathrm{w}_0, \mathrm{W} \rangle$ is a modal frame.

(2) $\mathcal{E}$ ('existence-set' function) is a function from At to $\mathcal{P}(\mathrm{W})$.

– To avoid awkwardness, we often state definitions etc. in terms of full frames only, but it should always be clear whether or not they are applicable in the more limited contexts of basic frames or modal frames or existential frames.

Relative to a frame $\mathcal{F} = \langle \mathrm{At}, \mathrm{Mol}, \mathrm{w}_0, \mathrm{W}, \mathcal{E} \rangle$, we define:

$\mathrm{HypProp} =_{\mathrm{df}} \mathrm{At} \cup \mathrm{Mol}$.

$\mathrm{IntProp} =_{\mathrm{df}} \mathcal{P}(\mathrm{W})$.

(If more than one frame is being considered in a given context, we may use obvious differentiating notation such as $\mathcal{F}$-HypProp etc. The same point applies to many other definitions below.) The elements of HypProp we call *hyperintensional propositions*, and the elements of IntProp we call *intensional propositions* (of course, relative to $\mathcal{F}$).

With basic frames and existential frames, the role of W can be systematically played by $\{\mathrm{w}_0\}$; and so the many important notions defined in the context of full frames involving the idea of *intension* can still be applied, although now, in this 'limiting case' of in effect one-world possible-world domains, it is more natural to speak of *extension*. (Thus instead of the IntProp above we can write ExtProp $[= \mathcal{P}(\{\mathrm{w}_0\})]$; instead of the Int, σ-Int etc. below, we can write Ext, σ-Ext, etc.)

The *intension* (or 'truth-set') *function* is defined for atoms in the obvious way: for

$\alpha \in \mathrm{At}$:

$\mathrm{Int}(\alpha) = \{w \in W : w(\alpha) = T\}$.

We can then extend the functions Int and $\mathcal{E}$ to all hyperintensional propositions in the natural way: for $X \in \mathrm{Mol}$:

$\mathrm{Int}(X) = -\bigcap\{\mathrm{Int}(x) : x \in X\}$.

$\mathcal{E}(X) = \bigcap\{\mathcal{E}(x) : x \in X\}$.

– A *model* is a pair $\langle \mathcal{F}, \sigma \rangle$ where $\mathcal{F}$ is a frame and σ (‘interpretation-function’) is a function from the set of propositional variables to HypProp.

(Thus this is the notion of a ‘full model’. With $\mathcal{F}$ taken instead as a basic frame, or modal frame, or existential frame, we have the notion of ‘basic model’, or ‘modal model’, or ‘existential model’.)

The interpretation-function σ can be extended in the obvious way to non-atomic transparent formulas:

$\sigma(N(A_1, \ldots, A_n)) = \{\sigma(A_1), \ldots, \sigma(A_n)\}$.

Thus all *transparent* formulas get ‘*hyperintensional* values’. Next we define ‘*intensional* values’ for *all* formulas:

$\sigma\text{-Int}(A) = \mathrm{Int}(\sigma(A))$.

$\sigma\text{-Int}(A = B) = W$ or $\emptyset$ according as $\sigma(A) = \sigma(B)$ or not.

$\sigma\text{-Int}(EA) = \mathcal{E}(\sigma(A))$.

$\sigma\text{-Int}(A\, \varepsilon\, B) = W$ or $\emptyset$ according as $\sigma(A) \in \sigma(B)$ or not.

$\sigma\text{-Int}(A\, \theta\, B) = \sigma\text{-Int}(B)$ or $\sigma\text{-Int}(A)$ or $\emptyset$, according as: $\sigma(B) \in \mathrm{Mol}$ and $\mathrm{card}(\sigma(B)) = 1$ and $\sigma(A) \in^2 \sigma(B)$; or $\sigma(B) \in \mathrm{Mol}$ and $\mathrm{card}(\sigma(B)) > 1$ and $\sigma(A) = \{X\}$ for some $X \in \sigma(B)$; or neither condition is satisfied.

(The first case is that where $\sigma(B)$ is a conjunction of the form $\wedge(\sigma(A), \ldots)$, i.e. $N(N(\sigma(A), \ldots))$: then the immediate grounding claim is equivalent to the obtaining of $\sigma(B)$ (which here implies also the obtaining of $\sigma(A)$). And the second case is the 'broadly disjunctive' case where σ(B) is a neconjunction with two or more neconjuncts, and σ(A) is the negation of one such neconjunct: here the immediate grounding claim is equivalent to the obtaining of σ(A) (which here implies the obtaining of σ(B)). – So this clearly correctly accounts for the cases where σ(B) is either a neconjunction with single neconjunct itself a non-empty neconjunction, or a neconjunction with more than one neconjunct. And in all *other* cases it is *impossible* that anything be immediate ground of σ(B). For these other cases can be classified as: (1) σ(B) is neconjunction with single neconjunct *not itself a non-empty neconjunction*, i.e. either atomic – i.e. σ(B) is negation-of-atom – or the empty neconjunction – i.e. σ(B) is the empty *con*junction N(N( )) –; or (2) σ(B) is the empty neconjunction; or (3) σ(B) is not a neconjunction at all but an atom. And it is clear that in all these cases nothing *can* immediately ground σ(B).)

(One can of course say, with much plausibility, that the complexity of the present clause is a clear indication that θ is more reasonably taken as a defined rather than primitive expression. And indeed as we will see it is definable from the other resources of HPL. Here we have taken it as primitive for the benefit of convenient study of ground-theoretic fragments of HPL.)

$\sigma\text{-Int}(N(\varphi_1, \ldots, \varphi_n)) = -\bigcap\{\sigma\text{-Int}(\varphi_1), \ldots, \sigma\text{-Int}(\varphi_n)\}$.

$\sigma\text{-Int}(\Box\varphi) = W$ or $\emptyset$ according as $\sigma\text{-Int}(\varphi) = W$ or not.

$\sigma\text{-Int}(\forall p\varphi) = \bigcap\{\sigma'\text{-Int}(\varphi) : \sigma' \text{ is p-variant of } \sigma\}$.

– We say that model $\langle \mathcal{F}, \sigma \rangle$ *verifies* formula φ if $w_0 \in \sigma\text{-Int}(\varphi)$. And we say that φ is *valid*, or ⊨ φ, if every model verifies φ; that φ is *satisfiable* if some model verifies φ; that Γ *implies* φ, or Γ ⊨ φ, if every model which verifies every formula in Γ also verifies φ; and so on. – If φ is a *sentence* (closed formula), we may also say that a frame $\mathcal{F}$ verifies φ if every (or equivalently, some) model based on $\mathcal{F}$ verifies φ.

Remark (on 'supervalidity'). Some valid formulas will be 'artefacts' of our choice of neconjunction as single primitive truth-functional connection. E.g. (with the natural definitions of the relevant expressions, soon to be given explicitly):

$\models \wedge(p) = \vee(p)$

$\models \wedge(p) = \neg\neg p$

$\models \forall p\, \forall q\, (\mathrm{Mol}(p) \wedge \mathrm{Mol}(q) \wedge \forall r(r\, \varepsilon\, p \leftrightarrow r\, \varepsilon\, q) \rightarrow p = q)$

$\vDash \forall p\ \exists! q\ \forall r\ (r\ \varepsilon\ q \leftrightarrow r = p).$

Many other valid formulas by contrast have a more ‘robust’ character and would remain valid with any reasonable variation in the choice of primitive truth-functional connections (and the corresponding adjustments in the semantics).

A precise notion of ‘robust validity’ or ‘supervalidity’ can be defined in terms of a fixed list of alternative sets of primitive connections. (So a supervalid formula is defined as a formula which is valid under all such alternative schemes. But note that a precise definition here requires a distinction between a formula ‘as written’ and its definitional expansion. It is the formula-as-written that is supervalid, but this means that all its definitional expansions are valid in their respective schemes.) (There are also of course corresponding variations of other semantical notions such as satisfiability, consequence etc.) But doubts as to which set of primitive connections should be ‘elected’ as the set of ‘absolute’ primitives are now replaced by doubts as to which sets should be elected as ‘main candidates’! – I will not pursue this matter further here, but just state dogmatically that the following seems to me to be perhaps the most plausible list of such ‘main candidates’: (i) neconjunction only, (ii) nedisjunction only, (iii) negation and conjunction, (iv) negation and disjunction, (v) negation, conjunction and disjunction, and (vi) neconjunction and nedisjunction.

It is easily seen that none of the above-displayed formulas (as written) is supervalid w.r.t. this list. – Still, e.g. (to take a variant of the last formula above) the formula ‘saying’ that either for every proposition there is exactly *one* proposition having it as single immediate constituent, or for every proposition there are exactly *two* propositions having it as single immediate constituent, or for every proposition there are exactly *three* propositions having it as single immediate constituent – *this* formula *is* supervalid. (*One* holds in schemes (i) and (ii); *two* in (iii), (iv) and (vi); and *three* in (v).) – So I guess one might feel with some justification that we have replaced a ‘parochial’ (w.r.t. neconjunction as single primitive) notion of validity not by a ‘global’ notion but by a ‘hexa-parochial’ notion. – However this may be, there remain of course (for what they are worth) legitimate *relative* notions of supervalidity – one for each non-empty collection of (truth-functionally ‘complete’) sets of truth-functional connections. ⊣

Remark (on nomenclature for subsystems of HPL). The most rudimentary (natural) fragment of HPL is Classical Propositional Logic (in terms of N): this we call CPL. Then there is S5, i.e. CPL plus modality. Next there is CPL^2, i.e. CPL with the addition of propositional quantifiers, and $S5^2$, i.e. S5 with the addition of propositional quantifiers. And if to any of these four systems we add some of our four hyperintensional connections (=, E, ε, θ), we accordingly adjoin the names of the connections to the name of the system: thus CPL=, $CPL^2\varepsilon$, $S5^2\theta$, S5=E, etc.

If we want to refer to the *language* of a system, we prefix $\mathcal{L}$ to our name for the system: thus e.g. $\mathcal{L}S5^2$ is the language of $S5^2$.

Sometimes we want to refer to a subsystem which is close to the whole of HPL, and then it may be more convenient to speak in 'subtractive' terms – e.g. 'E-free HPL' instead of $S5^2$=εθ, or 'non-modal HPL' instead of CPL^2=Eεθ. ⊣

Remark (on 'purely ontological' fragments of HPL). As we have already said, we may distinguish two aspects of a proposition: (i) its '*purely ontological*' aspect (as a certain complex object made up from certain constituents in a certain way); and (ii) its *import*, what it 'says'. – It is perhaps not too far off to say that 'extensional' and 'intensional' systems of propositional logic – such as CPL, CPL^2, S5, $S5^2$ –, understood (as one *can* understand them) as theories of hyperintensional propositions, focus entirely on the aspect of the import of propositions, in complete abstraction from the aspect of their inner ontic constitution. In HPL by contrast, the inner ontic constitution is of course much in evidence; but the import is also considered. It is then natural to raise the question of isolating a natural fragment or fragments of HPL where *only* the ontic constitution of propositions is considered, in complete abstraction from their import. – It seems to me that the following should be considered as the main strong fragments of HPL of this kind.

(A) *Syntax*: – Rechristen transparent formulas as *terms*. Define 'atomic formula' in the obvious way in terms of such 'terms' and ε and = as the basic (binary) 'predicates'. (θ is left out of the language, as import-involving; E is left out here but will be added in the system (B) below.) Then from such atomic formulas, *formulas* in general are constructed by N and ∀. (□ is left out. It could have been included but would be effectless since the atomic formulas are here all 'rigid' and also there are no 'varying domains' for the quantifier.) *Semantics*: – A frame is now reduced to just a Zermelo universe ⟨At, Mol⟩; the interpretation function σ gives an element of At ∪ Mol to each propositional variable, which induces values in the obvious way for all terms. Then the notion of a model (i.e. Zermelo universe + interpretation function) verifying a formula is defined in the obvious way. – This a natural 'purely ontological' hyperintensional propositional logic; but, as the alert reader will no doubt already have noticed, it is a very slight notational variant of general set theory (in Zermelian sense)! (Or alternatively, a keen propositionalist might say: general set theory is a notational variant of this purely ontological hyperintensional propositional logic.) ε corresponds of course to ∈, and N inside terms to the braces {...} expressing set formation. – This exact correspondence should not be surprising: for we are supposing that the construction of neconjunctive propositions from atomic propositions is exactly structurally similar to the construction of sets from urelements in a Zermelo universe; so if we ignore the import of propositions and consider only their ontic structure, we have nothing but a doppelganger of sets. (Talking sets are just like sets, if we ignore the talking!)

(Note that despite sameness of primitive symbols, this system is *not* $\mathrm{CPL}^2\varepsilon{=}$, but a fragment thereof. Here e.g. $\forall p(p)$ is not a formula, nor is $N(p = q, N(p))$.)

(B) Here we add the existence symbol E as basic (unary) predicate to the syntax, otherwise as in (A). And in the semantics we have frame $\langle \mathrm{At}, \mathrm{Mol}, \mathcal{E}\rangle$ where $\langle \mathrm{At}, \mathrm{Mol}\rangle$ is Zermelo universe and $\mathcal{E}$ is function from At to the set of 'existence-values' {Existence, Non-Existence} (which one may define as $\{1, 0\}$, or whatever). Then $\mathcal{E}$ can be extended to molecules in the obvious way: a molecule is assigned Existence if so are all its elements; otherwise it is assigned Non-Existence. – The rest of the definitions of basic semantical notions is as before in (A). – The correspondence is now to general set theory with an existence predicate – with the natural view that a set exists iff all its elements exist, and no constraints as to existence or non-existence of urelements. – Here one might with more reason consider a (straightforward) modal extension, since the existence of atoms (and hence of sets/propositions built therefrom) is plausibly taken as not in general rigid. The result is in effect the theory of Fine 1980 minus its truth-predicate and plus our ε. ⊣

4 Definitions: (ε, θ)-free notions

(1) *Truth-functional connectives.* The familiar truth-functional connectives can be defined from N in obvious way:

$$\neg\varphi \;=_{df}\; N(\varphi).$$

$$\wedge(\varphi, \psi, \ldots) \;=_{df}\; \neg N(\varphi, \psi, \ldots).$$

$$\vee(\varphi, \psi, \ldots) \;=_{df}\; N(\neg\varphi, \neg\psi, \ldots).$$

$$\varphi \to \psi \;=_{df}\; N(\varphi, \neg\psi).$$

$$\varphi \leftrightarrow \psi \;=_{df}\; (\varphi \to \psi) \wedge (\psi \to \varphi).$$

We write also $\varphi \wedge \psi$ and $\varphi \vee \psi$ as short for respectively $\wedge(\varphi, \psi)$ and $\vee(\varphi, \psi)$.

(2) *Modal connectives.* As usual we define:

$\Diamond\varphi \ =_{df} \ \neg\Box\neg\varphi.$

$\varphi \strictif \psi \ =_{df} \ \Box(\varphi \to \psi).$

$\varphi \strictiff \psi \ =_{df} \ \Box(\varphi \leftrightarrow \psi).$

$\Delta\varphi \ =_{df} \ \Box\varphi \vee \neg\Diamond\varphi.$

$\nabla\varphi \ =_{df} \ \neg\Delta\varphi.$

(3) *Existential quantifier; facts; world-propositions.*

$\exists p\, \varphi \ =_{df} \ \neg\forall p\, \neg\varphi.$

$\forall f\, \varphi(f)$ [‘For every fact f, $\varphi(f)$’] $=_{df} \ \forall p\, (p \to \varphi(p)).$

(Where p is the alphabetically first propositional variable not occurring in $\varphi(f)$. – In future definitions this kind of specification will be left tacit.)

$\exists f\, \varphi(f) \ =_{df} \ \exists p\, (p \wedge \varphi(p))$ (or: $\neg\forall f\, \neg\varphi(f)$).

$WP(\varphi)$ [‘φ is a world-proposition’] $=_{df} \ \Diamond(\varphi \wedge \forall p\, (p \to (\varphi \strictif p))).$

$\forall w\, \varphi(w) \ =_{df} \ \forall p\, (WP(p) \to \varphi(p)).$

$\exists w\, \varphi(w) \ =_{df} \ \exists p\, (WP(p) \wedge \varphi(p))$ (or: $\neg\forall w\, \neg\varphi(w)$).

(4) *Difference; numerical quantifiers.*

$A \neq B \ =_{df} \ \neg(A = B).$

Then weak and strict numerical propositional quantifiers $\exists_n p\, \varphi$, $\exists!_n p\, \varphi$ can be defined in familiar way. We can also define similar notions $\exists_n^{(\strictiff)} p\, \varphi$, $\exists!_n^{(\strictiff)} p\, \varphi$ using $\strictiff$ in the definitions instead of =. (In all cases we omit the n if n = 1.) Also with quantification over facts, or world-propositions, etc., there are of course in all cases the similar definitions. Thus e.g. we have $\vDash \neg\exists! w(w)$; indeed for any n, $\vDash \exists_n w(w)$; but $\vDash \exists!^{(\strictiff)} w(w)$.

(5) *Priorian modalities; content inclusion; E-analytic and E-synthetic implication, and E-equivalence.*

$\Box^{s}$A [‘It is strongly necessary that A’] $=_{df}$ $\Box$(A ∧ E(A)).

$\Box^{k}$A [‘It is weakly necessary that A’] $=_{df}$ $\Box$(E(A) → A).

$\Diamond^{s}$A [‘It is strongly possible that A’] $=_{df}$ $\Diamond$(A ∧ E(A)).

$\Diamond^{k}$A [‘It is weakly possible that A’] $=_{df}$ $\Diamond$(E(A) → A).

(These notions correspond approximately to Prior’s strong and weak modalities. See Prior 1957 and Fine 1977.)

A ⊑ B [‘The basic content (simple non--necessarily-existing constituents) of A is included in the basic content of B’] $=_{df}$ E(B) ⥽ E(A).

A (⥽, ⊒) B [‘A E-analytically implies B’] $=_{df}$ (A ⥽ B) ∧ (B ⊑ A).

A (⥽, ⊑) B [‘A E-synthetically implies B’] $=_{df}$ (A ⥽ B) ∧ (A ⊑ B).

(E.g. ⊨ (p ∧ q) (⥽, ⊒) p but not so for E-synthetic implication, and ⊨ p (⥽, ⊑) (p ∨ q) but not so for E-analytic implication.)

A (⥼⥽, ⊒⊑) B [‘A is E-equivalent to B’] $=_{df}$ (A ⥼⥽ B) ∧ (E(A) ⥼⥽ E(B)).

(These notions of E-analytic implication and E-equivalence correspond approximately to the ideas of ‘analytic implication’ and ‘analytic equivalence’ commonly associated with the name of Parry [see e.g. Fine 1986].)

There are also of course the ‘mixed’ notions (⥽, ⊒⊑) and (⥼⥽, ⊑), where the modal quasi-order relation goes with the existential equivalence-relation, or the modal equivalence-relation with the existential quasi-order relation.

5 Definitions: Stoichiological notions

(1) *Identity and immediate ground.* Although included among our primitives of HPL for the sake of convenient study of fragments, identity and immediate ground are readily definable in $\mathcal{L}$CPL^{2}ε:

A = B $=_{df}$ ∀p (A ε p ↔ B ε p).

(An alternative definiens is A ε ¬B. This has the advantage of greater brevity and avoidance of quantifiers, but the disadvantage of not being in the ‘*pure* logic of constituency' [in the sense similar to Fine's 'pure logic of ground'].)

$$A\ \theta\ B \ =_{df}\ (A \wedge B) \ .\!\wedge.\ (\exists!p(p\ \varepsilon\ B) \wedge A\ \varepsilon^2\ B) \vee (\exists_2 p(p\ \varepsilon\ B) \wedge \exists p\ \varepsilon\ B\ (A = \neg p)).$$

(Here ε^2 means of course immediate constituent of some immediate constituent – see below for the formal definition.) – Indeed, as already mentioned, it may be very plausibly said that the much greater complexity of the clause for σ-Int(A θ B) (compared to the other clauses) in our semantics for HPL is the mark of a notion which ought to be defined and not taken as primitive.

Remark. Note that here the immediate grounds of a true disjunction (with two or more disjuncts) are not the true disjuncts but rather their double negations. For e.g. p ∨ q is here N(¬p, ¬q), and so the immediate grounds (supposing p ∨ q true) will be ¬¬p and/or ¬¬q. More generally, for N(¬p, q, . . .), supposing p true, we will have ¬¬p as immediate ground rather than just p. (In the case of single disjunct though, i.e. ∨(p) = N(¬p) [a.k.a. ¬¬p], p itself is the immediate ground.) – There is here some room for debate as to whether we shouldn't just 'go straight to p' (as indeed we do, as just mentioned, in the case of single disjunct).

We will not here try to decide which of the two notions is 'best'. The alternative notion, which we may call θ′, can be defined here (in $\mathcal{L}CPL^2\varepsilon$) by a formula like the one above for A θ B except that ∃p ε B (A = ¬p) at the end is replaced by (using Neg(A) ['A is negative'] for ∃p (A = ¬p)):

$$\exists p\ \varepsilon\ B\ (\neg Neg(p) \wedge A = \neg p) \vee \exists p\ \varepsilon\ B\ (Neg(p) \wedge p = \neg A).$$

We then have e.g. $\vDash p \rightarrow (p\ \theta'\ (p \vee q))$.

Incidentally, it is also possible to define θ′ in $\mathcal{L}CPL^2\theta$ (where = is readily definable; see further below):

$$A\ \theta'\ B \ =_{df}\ (A \wedge B) \wedge \exists p(p\ \theta\ B) \wedge [(\exists p(B = N(p)) \wedge A\ \theta\ B) \vee (\neg\exists p(B = N(p)) \ .\!\wedge.\ (A\ \theta\ B \wedge \neg\exists p(A = \neg\neg p)) \vee \exists p\ \theta\ B(p = \neg\neg A))].$$

(The second disjunct in the outer disjunction inside the square brackets corresponds to the case of a neconjunction with two or more neconjuncts. Then there a non--doubly-negative θ-ground is still a θ′-ground, and for a doubly negative θ-ground we take its 'double negatum' as the θ′-ground.) – I do not see a reverse definition of θ in $\mathcal{L}CPL^2\theta'$, and suspect that there may be none. Interchanging θ and θ′ in the above-displayed definition until the first disjunct in the outer disjunction inside the square brackets gives a suitable beginning, but it is unclear whether this might be 'completed' for the key case of neconjunction with two or more neconjuncts. We can say that (i) for a *non-negative* θ'-ground, we take its double negation as θ-ground, and (ii) for a *doubly negative* θ'-ground, we again take *its* double negation as θ-ground. However with a *singly negative* θ'-ground, i.e. of form ¬p where p is not negative, we can't know whether this is a θ'-ground of B because p is a (false) neconjunct of B, or because ¬¬p is a (false) neconjunct of B; and the θ-ground to be taken in these two cases is different, viz. ¬p in the first case and ¬¬¬p in the second.

– The 'specifics' of these considerations depend of course on our adoption of neconjunction as single primitive connection. But systematically similar considerations would apply in other cases – e.g. if we took ¬ and ∧ as primitives, or ¬ and ∨, or nedisjunction (in the latter two cases the debatable clauses concerning now *con*junctive cases), or even say all of ¬, ∧, ∨. (But note that with e.g. ¬ and ∧, we can at least 'always take the negation' uniformly including the unit case – i.e. we can say that [true] ¬∧(¬p) has ¬¬p as immediate ground.) ⊣

(2) *Levels of constituency; part, constituency; atomicity and molecularity; purity; etc*. Levels of constituency can be defined inductively thus:

$A\ \varepsilon^1\ B\ =\ A\ \varepsilon\ B$

$A\ \varepsilon^{n+1}\ B\ =\ \exists p\ (A\ \varepsilon\ p \wedge p\ \varepsilon^n\ B)$.

ε-trans(A) ['A is ε-transitive'] $=_{df}$ $\forall p\ (p\ \varepsilon^2\ A \rightarrow p\ \varepsilon\ A)$.

$A \subseteq B$ ['A is a part of B'] $=_{df}$ $\forall p\ (\varepsilon\text{-trans}(p) \wedge B\ \varepsilon\ p \rightarrow A\ \varepsilon\ p)$.

$A \subset B$ ['A is a constituent of B'] $=_{df}$ $A \subseteq B \wedge A \neq B$.

Remarks. (1) General constituency (C) is of course the ancestral (alias transitive closure) of immediate constituency (ε). So the obvious definitions would be in terms of a general pattern for defining the ancestral of a relation, in 'second-order' terms. (As is well known there is no *general* way of defining the ancestral of an arbitrary relation in 'first-order' terms. [Proof is easy exercise using Compactness.]) Here we have managed to give a 'first-order' definition by using the 'purely collecting' capacity of neconjunctions. – Note also that this is very similar to the definition of 'ancestral member' in first-order set theory.

(2) We thus manage to define C in terms of ε – more specifically, in $\mathcal{L}\mathrm{CPL}^2\varepsilon$. It seems clear that the reverse procedure is not possible, i.e. that ε is *not* definable in

(what we may naturally call) $\mathcal{L}$CPL^{2}C – indeed not even in $\mathcal{L}$S5^{2}CE. For: for every Zermelo universe ⟨At, Mol⟩, there exists a permutation π of At ∪ Mol such that π is an automorphism of the relational structure ⟨At ∪ Mol, C⟩ but not an automorphism of the relational structure ⟨At ∪ Mol, ε⟩. E.g. π might be the function which, 'inside' any object of At ∪ Mol, changes NNNN( ) to N(NNN( ), NN( ), N( )) and (simultaneously) vice versa. Obviously this permutation does not change atoms (π(α) = α for all α ∈ At); and so the statement above remains true if we extend the relevant structures by addition of the other frame-components w_0, W, $\mathcal{E}$. And from this (together with various 'bureaucratic' facts) the indefinability of ε in $\mathcal{L}$S5^{2}CE should follow. (Fine 1992 p. 48 fn. 16 contains a similar argument for the set-theoretic case of ∈ and its ancestral.) Note that the slightly simpler permutation transposing ('inside' any object) NNN( ) and N(NN( ), N( )), although also an automorphism of C and not of ε, is less convenient here since it 'affects truth-values' (as NNN( ) is necessarily false and N(NN( ), N( )) necessarily true).

(3) Similar remarks apply also to our definition (to be given later) of general ground (G) from immediate ground (θ). In particular, to show the indefinability of θ in $\mathcal{L}$CPL^{2}G, indeed in $\mathcal{L}$S5^{2}GE, we may use the 'uniform transposition' of NNNNNN( ) and N(NNN( ), N( )), which is automorphism of G but not of θ. ⊣

A is a mediate constituent of B $=_{df}$ ∃p (A C p ∧ p C B).

A overlaps B $=_{df}$ ∃p (p $\underline{C}$ A ∧ p $\underline{C}$ B).

A is disjoint from B $=_{df}$ ¬(A overlaps B).

Mol(A) ['A is molecular'] $=_{df}$ ∃p(p C A) ∨ A = N( ).

At(A) ['A is atomic'] $=_{df}$ ¬Mol(A).

Pure(A) $=_{df}$ ¬∃p $\underline{C}$ A: At(p).

Elem(A) ['A is elementary'] $=_{df}$ At(A) ∨ ∃p (At(p) ∧ A = ¬p).

AtFact(A) ['A is an atomic fact'] $=_{df}$ At(A) ∧ A.

Similarly for MolFact(A) and ElemFact(A).

(3) *Essential part; C-analytic and C-synthetic implication; hyper-rigidity; possible worlds*. All the definitions above are in $\mathcal{L}$CPL^{2}ε. The following definitions by contrast use also modality (except for the definition of MNCS).

A $!\underline{\mathrm{C}}$ B [‘A is an essential part of B’] $=_{df}$ $\forall p \strictiff B$: A $\underline{\mathrm{C}}$ p.

(Thus we have e.g.: $\vDash \neg(q\ \underline{\mathrm{C}}\ p) \rightarrow \neg(q\ !\underline{\mathrm{C}}\ (p \wedge (q \vee \neg q)))$.)

A ($\strictif$, Ɔ) B [‘A C-analytically implies B’] $=_{df}$ $(A \strictif B) \wedge (B\ \mathrm{C}\ A)$.

A ($\strictif$, C) B [‘A C-synthetically implies B’] $=_{df}$ $(A \strictif B) \wedge (A\ \mathrm{C}\ B)$.

– And similarly with respectively $\underline{\text{Ɔ}}$ and $\underline{\mathrm{C}}$ instead of Ɔ and C.

$A \leq^{At\underline{C}} B$ $=_{df}$ $\forall p\ (At(p) \wedge p\ \underline{\mathrm{C}}\ A \rightarrow p\ \underline{\mathrm{C}}\ B)$.

$A \geq^{At\underline{C}} B$ $=_{df}$ $B \leq^{At\underline{C}} A$.

$A =^{At\underline{C}} B$ $=_{df}$ $\forall p\ (At(p) \rightarrow (p\ \underline{\mathrm{C}}\ A \leftrightarrow p\ \underline{\mathrm{C}}\ B))$.

A ($\strictif$, $\geq^{At\underline{C}}$) B [‘A At$\underline{\mathrm{C}}$-analytically implies B’] $=_{df}$ $(A \strictif B) \wedge (A \geq^{At\underline{C}} B)$.

A ($\strictif$, $\leq^{At\underline{C}}$) B [‘A At$\underline{\mathrm{C}}$-synthetically implies B’] $=_{df}$ $(A \strictif B) \wedge (A \leq^{At\underline{C}} B)$.

A ($\strictiff$, $=^{At\underline{C}}$) B [‘A is At$\underline{\mathrm{C}}$-equivalent to B’] $=_{df}$ $(A \strictiff B) \wedge (A =^{At\underline{C}} B)$.

Again, to be thorough we may mention that there are also of course the ‘mixed’ notions ($\strictif$, $=^{At\underline{C}}$) and ($\strictiff$, $\leq^{At\underline{C}}$) where the modal quasi-order relation goes with the stoichiological equivalence-relation, or the modal equivalence-relation with the stoichiological quasi-order relation.

$!\Delta A$ [‘A is hyper-rigid’] $=_{df}$ $\Delta A \wedge \forall p\ \mathrm{C}\ A\ (\Delta p)$. (I.e. $\forall p\ \underline{\mathrm{C}}\ A\ (\Delta p)$.) (Or equivalently: $\forall p\ (At(p) \wedge p\ \underline{\mathrm{C}}\ A \rightarrow \Delta p)$.)

MNCS(A) [‘A is a maximal non-contradictory state’] $=_{df}$ $\exists p$: $\forall q\ \varepsilon\ p\ Elem(q) \wedge \forall q\ (At(q) \rightarrow (q\ \varepsilon\ p\ \underline{\vee}\ \neg q\ \varepsilon\ p)) \wedge A = N(p)$.

(Curiously, here in the definiens we ‘start’ by considering p in its ‘purely collecting capacity’, selecting one out of each pair of an atomic proposition and its negation, but then, ‘at the end’, we ‘remember’ that p is a neconjunction, so that to say that A conjoins these propositions we can just put A = N(p).)

PW(A) [‘A is a possible world’] $=_{df}$ $\Diamond A \wedge MNCS(A)$.

Thus with modal-stoichiological resources we manage to select a sort of 'canonical representative' for each strict-equivalence class of world-propositions. We now have of course $\vDash \exists! p\ (PW(p) \wedge p)$. – Likewise, disjunctions (i.e. neconjunctions of negations) of possible worlds provide 'canonical representatives' for each strict-equivalence class of hyperintensional propositions:

Can(A) ['A is a canonical proposition'] $=_{df}$ $Mol(A) \wedge \forall p\ \varepsilon\ A\ \exists q\ (PW(q) \wedge p = \neg q)$.

We then have of course

$\vDash \forall p\ \exists! q\ (Can(q) \wedge (q \mathrel{\varepsilon\!\!-\!\!\ni} p))$.

$\vDash \forall p\ \forall q\ [Can(p) \wedge Can(q) \rightarrow ((p = q) \leftrightarrow (p \mathrel{\varepsilon\!\!-\!\!\ni} q))]$.

– MNCS, PW and Can must, by virtue of the character of their definitions, apply only to transparent formulas. We may however also define:

$MNCS^{\varepsilon\!-\!\ni}(\varphi)$ $=_{df}$ $\exists p\ (MNCS(p) \wedge (p \mathrel{\varepsilon\!\!-\!\!\ni} \varphi))$.

$PW^{\varepsilon\!-\!\ni}(\varphi)$ $=_{df}$ $\exists p\ (PW(p) \wedge (p \mathrel{\varepsilon\!\!-\!\!\ni} \varphi))$.

It is also possible of course to define $Can^{\varepsilon\!-\!\ni}$ in similar style; but this is not very useful because for *any* formula φ we have $\vDash Can^{\varepsilon\!-\!\ni}(\varphi)$. (See below the 'Intensional Comprehension Principle'.) Indeed for any expression $\Phi(A, B, \dots)$ meaningful for transparent formulas A, B, . . . , we may define the notion

$\Phi^{\varepsilon\!-\!\ni}(\varphi, \psi, \dots)$ $=_{df}$ $\exists p, q, \dots\ (\Phi(p, q, \dots) \wedge (p \mathrel{\varepsilon\!\!-\!\!\ni} \varphi) \wedge (q \mathrel{\varepsilon\!\!-\!\!\ni} \psi) \wedge \dots)$

– for what it may be worth. (There is also a universal notion: $\forall p, q, \dots\ ((p \mathrel{\varepsilon\!\!-\!\!\ni} \varphi) \wedge (q \mathrel{\varepsilon\!\!-\!\!\ni} \psi) \wedge \dots \rightarrow \Phi(p, q, \dots)))$.)

Remarks and Questions. (1) Of course $\vDash WP(p) \leftrightarrow PW^{\varepsilon\!-\!\ni}(p)$.

(2) It seems clear that $\forall \varphi \in \mathcal{L}HPL$: $\nvDash WP(\varphi)$ and (equivalently) $\nvDash PW^{\varepsilon\!-\!\ni}(\varphi)$.

(3) Clearly $\forall$ transparent formula A of $\mathcal{L}HPL$: $\nvDash MNCS(A)$ and (a fortiori) $\nvDash PW(A)$.

(4) $\vDash MNCS^{\varepsilon\!-\!\ni}(\forall p(At(p) \rightarrow p))$ and $\vDash MNCS^{\varepsilon\!-\!\ni}(\forall p(At(p) \rightarrow \neg p))$.

(5) It seems that $\forall\varphi \in \mathcal{L}\mathrm{CPL}^2\varepsilon$: If $\vDash \mathrm{MNCS}^{\varepsilon{-}\exists}(\varphi)$ then φ is equivalent to one of the two formulas indicated in (4).

(6) In $\mathcal{L}\mathrm{S5}^2\varepsilon$ we have also:

$\vDash \mathrm{MNCS}^{\varepsilon{-}\exists}(\forall p(\mathrm{At}(p) \wedge \nabla p \rightarrow p))$.

$\vDash \mathrm{MNCS}^{\varepsilon{-}\exists}(\forall p(\mathrm{At}(p) \wedge \nabla p \rightarrow \neg p))$.

Are there any *other* $\mathcal{L}\mathrm{S5}^2\varepsilon$ formulas (not equivalent to either of these two or the two in (4)) with this property?

(7) Any other such formulas when we move to full $\mathcal{L}$HPL (i.e., in effect, add E)? (Note that such naive candidates as $\forall p(\mathrm{At}(p) \rightarrow (\mathrm{E}p \leftrightarrow p))$ don't 'work', since E is 'non-rigid'.) ⊣

6 Definitions: Truthmaking, entailment, etc.

The following definitions are nearly all still in $\mathcal{L}\mathrm{CPL}^2\varepsilon$, but seemed worthy of consignation to a separate section.

Note on notation: In the present section and in the corresponding 'Principles' section below (sect. 10), we revert to the 'old' symbols ⊃ and ≡ for material implication and material equivalence, reserving → and ↔ for relevantistic notions of entailment and mutual entailment.

(1) *States*.

$\mathrm{State}(A) \;=_{df}\; \exists\Gamma\, (\forall p \in \Gamma\ \mathrm{Elem}(p) \wedge A = \wedge(\Gamma))$.

I.e. a state is a conjunction of (zero or more) elementary propositions. Here and in some other places below we use the set notation for the sake of increased readability; the definiens above stands for $\exists q\, (\mathrm{Mol}(q) \wedge \forall p\ \varepsilon\ q\ \mathrm{Elem}(p) \wedge A = \mathrm{N}(q))$.

$\forall\Sigma\ \varphi(\Sigma) \;=_{df}\; \forall p\, (\mathrm{State}(p) \supset \varphi(p))$.

Similarly for $\exists\Sigma\ \varphi(\Sigma)$.

(2) *Truthmaking and falsity-making*. We define truthmaking and falsity-making recursively as follows. (Recursive definitions such as the present one are allowable in HPL since we have in effect the higher-order resources needed to justify the procedure

in the usual way. Thus here for A and B any transparent formulas of $\mathcal{L}$HPL, both ‘A is tm of B’ and ‘A is fm of B’ stand for specific formulas of $\mathcal{L}$HPL.) (We often use ‘tm’ to abbreviate ‘truthmaker’ or ‘is truthmaker of’, and similarly for ‘fm’.)

For atomic proposition α:

Σ is tm of α iff $\Sigma = \wedge(\alpha)$.

Σ is fm of α iff $\Sigma = \wedge(\neg\alpha)$.

For molecular proposition $N(\Gamma)$:

Σ is tm of $N(\Gamma)$ iff Σ is fm of some element of Γ.

Σ is fm of $N(\Gamma)$ iff Σ is fusion of tms of the respective elements of Γ.

(Here *fusion* of certain states means the state whose conjuncts are the conjuncts of the given states.)

A is semiregular truthmaker (stm) of B $=_{df}$ $\exists\Gamma$ ($\Gamma \neq \emptyset \wedge \forall p \in \Gamma$(p is tm of B) $\wedge$ A = Fusion(Γ)).

I.e. A is a fusion of one or more tms of B. (Note that of course a semiregular truthmaker need not be a truthmaker simpliciter.) – Definition of ‘A is sfm of B’ is exactly similar: just replace tm in the definiens above by fm.

A is regular truthmaker (rtm) of B $=_{df}$ $\exists\Gamma$ ($\Gamma \neq \emptyset \wedge \forall p \in \Gamma$(p is tm of B) $\wedge$ A = Fusion(Γ)) $\vee$ $\exists\Sigma_1, \Sigma_2$ (Σ_1 tm B $\wedge$ Σ_2 tm B $\wedge$ A is superconjunction of Σ_1 $\wedge$ A is subconjunction of Σ_2).

I.e. A is either a fusion of one or more tms of B or an intermediate between two tms of B. – Again, definition of ‘A is rfm of B’ is exactly similar: just replace tm in the definiens above by fm. (The other ‘semiregular’ notion with just the condition on intermediates is also of some interest; but I guess we already have enough on our plate here.)

Both stm/sfm and rtm/rfm can alternatively be defined recursively like tm/fm above, only with the suitable modifications to the clauses for neconjunctions.

(3) *Entailment, etc.*

A $=^{tm}$ B $=_{df}$ $\forall\Sigma$ (Σ tm A $\equiv$ Σ tm B).

Similarly for $=^{fm}$, $=^{stm}$, $=^{sfm}$, $=^{rtm}$, $=^{rfm}$.

A $\leq^{tm}$ B $=_{df}$ $\forall\Sigma$ (Σ tm A $\supset$ Σ tm B).

Similarly for $\leq^{fm}$, $\leq^{stm}$, $\leq^{sfm}$, $\leq^{rtm}$, $\leq^{rfm}$.

A $\cong^{tm}$ B $=_{df}$ $\forall\Sigma$ (Σ includes [i.e. is superconjunction of] tm of A $\equiv$ Σ includes tm of B).

Similarly for $\cong^{fm}$, $\cong^{stm}$, $\cong^{sfm}$, $\cong^{rtm}$, $\cong^{rfm}$.

A $\preccurlyeq^{tm}$ B $=_{df}$ $\forall\Sigma$ (Σ includes tm of A $\supset$ Σ includes tm of B).

Similarly for $\preccurlyeq^{fm}$, $\preccurlyeq^{stm}$, $\preccurlyeq^{sfm}$, $\preccurlyeq^{rtm}$, $\preccurlyeq^{rfm}$.

Of these various concepts the following are particularly important and thus will receive here additional 'names':

A $\rightarrow$ B ['A entails B'] $=_{df}$ A $\preccurlyeq^{tm}$ B.

This is, as is easily seen, equivalent to $\forall\Sigma$ (Σ *is* tm of A $\supset$ Σ includes tm of B).

A $\leftrightarrow$ B ['A mutually entails B'] $=_{df}$ A $\cong^{tm}$ B.

This is of course equivalent to A $\rightarrow$ B **.**$\wedge$**.** B $\rightarrow$ A. It is also equivalent to A $\cong^{fm}$ B, as well as (therefore) equivalent to A $\cong^{tm}$ B $\wedge$ A $\cong^{fm}$ B.

A $\Leftrightarrow^{+}$ B ['A is positively regularly equivalent to B'] $=_{df}$ A $=^{rtm}$ B.

A $\Leftrightarrow^{-}$ B ['A is negatively regularly equivalent to B'] $=_{df}$ A $=^{rfm}$ B.

A $\Leftrightarrow$ B ['A is (fully) regularly equivalent to B'] $=_{df}$ A $\Leftrightarrow^{+}$ B $\wedge$ A $\Leftrightarrow^{-}$ B.

A $\sim^{+}$ B ['A is positively semiregularly equivalent to B'] $=_{df}$ A $=^{stm}$ B.

A $\sim^{-}$ B ['A is negatively semiregularly equivalent to B'] $=_{df}$ A $=^{sfm}$ B.

A $\sim$ B ['A is (fully) semiregularly equivalent to B'] $=_{df}$ A $\sim^{+}$ B $\wedge$ A $\sim^{-}$ B.

$A \approx^{+} B$ ['A is positively exactly equivalent to B'] $=_{\text{df}}$ $A =^{\text{tm}} B$.

$A \approx^{-} B$ ['A is negatively exactly equivalent to B'] $=_{\text{df}}$ $A =^{\text{fm}} B$.

$A \approx B$ ['A is (fully) exactly equivalent to B'] $=_{\text{df}}$ $A \approx^{+} B \wedge A \approx^{-} B$.

Remarks. (1) The equivalences $\leftrightarrow$, $\Leftrightarrow$, $\sim$, $\approx$ correspond to four notions of 'moderately hyperintensional' proposition, successively finer but all intermediate between the coarse-grained intensional propositions and the ultra-fine--grained structured propositions (which is what we think of the variables of $\mathcal{L}$HPL as ranging over) in their full hyperintensional glory. Propositions of these four intermediate kinds can in each case be defined (à la Fine) as the ordered pair consisting of the set of truthmakers and the set of falsity-makers, in the appropriate sense of truthmaking and falsity-making. (In the case corresponding to $\leftrightarrow$, this is: superstates of tms, superstates of fms.) Or the same 'information' can also easily be encoded if we like by ('radically') hyperintensional propositions of appropriate kind.

(2) The origins of the above definitions of (i) truthmaking, falsity-making, entailment, mutual entailment and (ii) regular tm and fm and positive regular equivalence etc. and (iii) positive semiregular equivalence etc. are to be found in the ingenious semantic constructions of respectively (i) van Fraassen 1969 and (ii) Fine 2016 and (iii) Correia 2016 (the general idea of something like regular equivalence remounting to Angell 1977, 1989, and of course the general idea of entailment to Anderson, Belnap etc.). (See also Batchelor 2022 Chs. 4 & 5 for my own work in this area.) Note however the difference in context: here truthmaking is a relation between certain worldly items (states) and other worldly items (propositions, *an sich*), as are entailment, mutual entailment, etc. (now between proposition and proposition). Nothing linguistic or semantic is involved. (In Fine 2017 ['A theory of truthmaker content'] too the context is worldly like here, despite various differences in other respects.) – The case is comparable to the more familiar case of possible worlds and related notions: there is the context of possible world *semantics* with notions like a formula being true in a world w.r.t. a model etc. etc.; and the analogous but purely worldly notions of a *proposition* (*an sich*) being true in a world, of a proposition strictly implying another, and so on.

(3) Just as the resources of possible world semantics, originally devised for ordinary modal logic, can be more fully exploited in the richer systems of so-called 'hybrid logic' (where we have also world-variables and symbol for truth in a world), so also similarly the resources of truthmaker semantics, originally devised (in van Fraassen's paper) for a (rudimentary) form of relevance logic, can be more fully exploited in a richer system of what we might perhaps call *Truthmaker Logic*. Here in the primitive vocabulary, in addition to the propositional variables and the truth-

functional connective or connectives (neconjunction, or more 'traditionally' negation, conjunction and disjunction), we would have state variables and function symbols for fusion and common-part (for formation of 'state terms'), and symbols for truthmaking and falsity-making. We may then distinguish: the quantifier-free system; the system with quantification over propositions and states; and the system with quantification over propositions and states and also quantification over sets of propositions and sets of states. In each case a modal extension of the system might also be considered. – Logics for entailment and various other notions (defined as above) would then be fragments of appropriate versions of this Truthmaker Logic. ⊣

(4) *Existential import*. I use this term here not of course for anything to do with syllogistics (God forbid), but for the idea of the (full) 'existential implications' of a proposition. Thus it is natural to think that the existential import of an atomic proposition Pa consists in the existence of a and the existence of P – or equivalently, the existence of the proposition Pa itself. (For if a has the quality P then surely it must exist; and if the quality P is instantiated then surely *it* must exist too.) On the other hand the existential import of a negative elementary proposition such as ¬Pa appears to be 'null'. (Surely existence of a is not implied by its *lacking* quality P, nor existence of P by its failing to be instantiated by a.) The existential import of a conjunction of elementary propositions is surely the 'sum' of the existential imports of the conjuncts, i.e. (by the preceding considerations) the conjunction of attributions of existence to the atomic conjuncts. – But it should not be thought that the existential import of a molecular proposition is always so easily determined (or even determined at all) by the existential import of the components. E.g. the existential import of ¬Pa is null, as is the existential import of ¬Qb; but the existential import of ¬¬Pa should surely be the same as that of Pa, and of ¬¬Qb same as that of Qb. Nor should it be thought that the existential import is always 'categorical', i.e. given by a conjunction of attributions of existence. For take e.g. Pa ∨ Qb: this does not imply the existence of a, nor of P, nor of b, nor of Q; but it could not obtain if *none* of these things existed. Here the proper existential import should be given by the disjunction: Either a and P exist (or equivalently E(Pa)) or b and Q exist (equivalently, E(Qb)).

These remarks certainly give some 'guidelines' on the idea of existential import. But if we try to give a *general* definition of the existential import of a proposition, the task can be very puzzling. Or at least it can be so until we hit upon the idea of truthmaking, which points the way to a clear and solid definition. For the overall import of a proposition can be clearly organized through the disjunction of its truthmakers; each of these, being a mere conjunction of elementary propositions, as it were wears its ('categorical') existential import on its sleeve (viz.: the conjunction of attributions of existence to the atomic conjuncts); and the existential import of the whole proposition is then taken as the disjunction of the existential imports of its truthmakers. – So we are led to the following definitions.

ExistForce(Σ) ['the existential force of state Σ (obtains)'] $=_{df}$ ∀p (At(p) ∧ p ε^2 Σ ⊃ Ep).

ExistImp(A) ['the existential import of proposition A (obtains)'] $=_{df}$ ∃Σ (Σ tm A ∧ ExistForce(Σ)).

Clearly ⊨ ∀Σ (ExistForce(Σ) ⥼⥽ ExistImp(Σ)).

Note the hyperintensional character of this notion of existential import: it can easily happen that strictly equivalent propositions have non--strictly-equivalent existential imports. E.g. we have:

⊨ At(p) ∧ At(q) ∧ ¬(Ep ⥼⥽ Eq) ⊃ ¬(ExistImp(p ∧ ¬p) ⥼⥽ ExistImp(q ∧ ¬q)).

Even strictly equivalent propositions with the same atomic propositions as constituents might have non-equivalent existential imports:

⊨ At(p) ∧ At(q) ∧ ¬(Ep ⥽ Eq) ⊃ ¬(ExistImp(p ∧ ¬p ∧ ¬q) ⥼⥽ ExistImp(p ∧ ¬p ∧ q)).

These examples show also that the role played in our definitions above by the 'subtle DNF' consisting of the disjunction of *truthmakers* of a proposition could not have been played instead by the more familiar-looking but less subtle concept of the disjunction of maximal non-contradictory states corresponding to the given proposition, or even the subtler but not subtle-enough concept of the corresponding disjunction of non-contradictory states 'in' the atomic propositions which actually occur in the given proposition. For in the examples above the corresponding disjunction of such kinds is always the empty disjunction, and so the corresponding existential import obtained by striking out negated atoms and replacing asserted ones by the corresponding attributions of existence – this would also be the empty disjunction (thus necessarily false); whereas we want the existential import of e.g. Pa ∧ ¬Pa to be E(Pa) (which may very well be possibly true, of course), and the existential import of Qb ∧ ¬Qb to be E(Qb), and of Pa ∧ ¬Pa ∧ ¬Qb to be E(Pa), and of Pa ∧ ¬Pa ∧ Qb to be E(Pa) ∧ E(Qb), etc.

7 Definitions: Ground-theoretic notions

(1) *Identity*. Propositional identity can be defined not only in $\mathcal{L}CPL^2\varepsilon$ (as we saw), but also in $\mathcal{L}CPL^2\theta$:

$A = B \ =_{df}\ (A \wedge B) \wedge \forall p\,(A\,\theta\,p \leftrightarrow B\,\theta\,p)\ .\!\vee\!.\ (\neg A \wedge \neg B) \wedge \forall p\,(\neg A\,\theta\,p \leftrightarrow \neg B\,\theta\,p)$.

(As before with $A\ \varepsilon\ \neg B$, here too it is possible to give a quantifier-free alternative definiens – e.g.

$A\,\theta\,\neg\neg B \vee \neg A\,\theta\,\neg\neg\neg B$, or alternatively

$N(A, \neg A)\,\theta\,\neg\neg N(B, \neg B)$.

Unlike before, however, here there is no difference as to 'purity' or 'impurity', all these definitions being now in the 'impure logic of ground'. Nor do I see any 'pure' alternative.)

Remark. It seems clear that ε itself however is *not* definable in $\mathcal{L}CPL^2\theta$, nor even in $\mathcal{L}S5^2\theta E$ (i.e. in effect $\mathcal{L}(HPL - \varepsilon)$). For take the 'uniform transposition' of NNNN( ) and N(NN( ), N( )): this is automorphism of θ but not of ε. Indeed this is not even an automorphism of C; so even C is not definable in $\mathcal{L}(HPL - \varepsilon)$. ⊣

(2) *Levels of grounding; ground, weak ground; total grounds; fundamentality; etc.*

$A\,\theta^1\,B \ = \ A\,\theta\,B$.

$A\,\theta^{n+1}\,B \ = \ \exists p\,(A\,\theta\,p \wedge p\,\theta^n\,B)$.

$\theta\text{-trans}(A) \ =_{df}\ \forall p\,(p\,\theta^2\,A \rightarrow p\,\theta\,A)$.

$A\ \underline{G}\ B$ ['A weakly grounds B', or 'A is a weak ground of B'] $=_{df}\ (A \wedge B) \wedge \forall p\,(\theta\text{-trans}(p) \wedge B\,\theta\,p \rightarrow A\,\theta\,p)$.

$A\ G\ B$ ['A grounds B', or 'A is a ground of B'] $=_{df}\ A\ \underline{G}\ B \wedge A \neq B$.

A is a mediate ground of B $=_{df}\ \exists p\,(A\ G\ p \wedge p\ G\ B)$.

$A_1, \ldots, A_n$ are the complete grounds of B $=_{df}\ A_1\ G\ B \wedge \ldots \wedge A_n\ G\ B \wedge \forall p\,(p\ G\ B \rightarrow p = A_1 \vee \ldots \vee p = A_n)$.

A fact can of course have infinitely many grounds. To obtain in the present context a notion closer to general idea of the complete grounds of a fact we may appeal to the 'purely collecting capacity' of propositions (neconjunctions) and define (using also ε):

A collects (precisely) the complete grounds of B $=_{df}$ ∀p (p G B ↔ p ε A).

Grounded(A) $=_{df}$ ∃p (p G A).

Ungrounded(A) $=_{df}$ ¬∃p (p G A).

FundFact(A) [‘A is a fundamental fact’] $=_{df}$ A ∧ Ungrounded(A).

We then have: ⊨ FundFact(p) ↔ (ElemFact(p) ∨ p = NN( )). Ground-theoretic equivalents or near-equivalents of the stoichiological notions of elementary proposition, and atomic proposition/fact, and molecular proposition/fact, can also be straightforwardly defined.

A UG B [‘A is an ultimate ground of B’] $=_{df}$ A G B ∧ FundFact(A).

– Similarly for U$\underline{G}$ (ultimate weak ground).

(3) *Sufficient ground; essential ground; G-analytic and G-synthetic implication; Finean transcendentality.* We pass on now to some *modal* ground-theoretic notions.

$A_1, \ldots, A_n$ are sufficient grounds for B $=_{df}$ A_1 G B ∧ . . . ∧ A_n G B ∧ (A_1 ∧ . . . ∧ A_n ⥽ B).

A collects sufficient grounds for B $=_{df}$ ∀p ε A (p G B) ∧ (∀p ε A (p) ⥽ B).

A !G B [‘A is essential ground for B’] $=_{df}$ ∀p ⥼⥽ B (A G p).

A (⥽, G^{-1}) B [‘A G-analytically implies B’] $=_{df}$ (A ⥽ B) ∧ (B G A).

A (⥽, G) B [‘A G-synthetically implies B’] $=_{df}$ (A ⥽ B) ∧ (A G B).

– And similarly with respectively $\underline{G}^{-1}$ and $\underline{G}$ instead of G^{-1} and G.

A $\leq^{U\underline{G}}$ B $=_{df}$ ∀p (p U$\underline{G}$ A → p U$\underline{G}$ B).

A $\geq^{U\underline{G}}$ B $=_{df}$ B $\leq^{U\underline{G}}$ A.

A $=^{U\underline{G}}$ B $=_{df}$ ∀p (p U$\underline{G}$ A ↔ p U$\underline{G}$ B).

A (⥽, $\geq^{U\underline{G}}$) B [‘A U$\underline{G}$-analytically implies B’] $=_{df}$ (A ⥽ B) ∧ (A $\geq^{U\underline{G}}$ B).

A (⥽, $\leq^{U\underline{G}}$) B [‘A U$\underline{G}$-synthetically implies B’] $=_{df}$ (A ⥽ B) ∧ (A $\leq^{U\underline{G}}$ B).

A (⥼⥽, $=^{U\underline{G}}$) B [‘A is U$\underline{G}$-equivalent to B’] $=_{df}$ (A ⥼⥽ B) ∧ (A $=^{U\underline{G}}$ B).

– Again there are also of course the ‘mixed’ notions (⥽, $=^{U\underline{G}}$) and (⥼⥽, $\leq^{U\underline{G}}$).

The following definition (already given in Batchelor 2010) tries to explicate within modal ground-theory the idea of transcendental (or ‘unworldly’) fact proposed in Fine 2005.

TrFact(A) [‘A is a transcendental fact’] $=_{df}$ $\Box A \wedge \Box\forall p\,(p \; G \; A \rightarrow \Box p)$.

(An equivalent formulation is: $A \wedge \Box\forall p\,(p \; U\underline{G} \; A \rightarrow \Box p)$.)

8 Principles: (ε, θ)-free principles

(1) *Modal protothetic*. In the $\mathcal{L}S5^2$ fragment of $\mathcal{L}$HPL, the valid formulas are precisely the valid formulas of $S5^2$, also known (after Fine 1970) as S5π+, i.e. the standard system of ‘modal protothetic’ or ‘second-order propositional modal logic’, corresponding to the straightforward extension of the possible world semantics to deal with propositional quantifiers.

(2) *Identity*. In $\mathcal{L}$CPL= we have valid formulas corresponding to familiar general laws of identity, such as

$\vDash p = p$, $\quad \vDash p = q \rightarrow q = p$, $\quad \vDash p = q \wedge q = r \rightarrow p = r$,

and all instances of

$\vDash p = q \wedge \varphi(p) \rightarrow \varphi(q)$.

In CPL^2= we have also, for each $n \geq 0$,

$\vDash \forall p_1 \ldots p_n \, \exists q \, (q = N(p_1, \ldots, p_n))$.

Or more generally, for any transparent formula A we have an instance of ‘comprehension principle’:

$\vDash \exists q \, (q = A)$.

In $S5^2$= we have:

$\vDash \Delta(p = q)$.

(3) *Existence*. The fundamental principle here is:

$\vDash EA(p_1 \ldots p_n) \leftrightarrow Ep_1 \wedge \ldots \wedge Ep_n$.

More generally, but resorting also to ε, we have:

$\vDash Ep \leftrightarrow \forall q\, \underline{C}\, p\, (Eq)$.

$\vDash Ep \leftrightarrow \forall q\, (At(q) \wedge q\, \underline{C}\, p \rightarrow Eq)$.

$\vDash Mol(p) \rightarrow (Ep \Leftrightarrow \forall q\, C\, p\, (Eq))$,

and same with C replaced by ε, or again by atomic constituent.

(4) *Intensional Comprehension Principle.* This is in fact a general principle of HPL. Where φ is any formula of $\mathcal{L}$HPL,

$\vDash \exists p\, (p \Leftrightarrow \varphi)$.

Since as mentioned before $\vDash \forall p\, \exists! q\, (Can(q) \wedge (q \Leftrightarrow p))$, we have also for any formula φ of $\mathcal{L}$HPL,

$\vDash \exists! p\, (Can(p) \wedge (p \Leftrightarrow \varphi))$.

– A fully formal proof of the validity of all instances of the Intensional Comprehension schema would be somewhat laborious; but the basic ideas are simple enough (recall that = and θ are definable from the rest of $\mathcal{L}$HPL; so it is enough to consider $\mathcal{L}S5^2\varepsilon E$-formulas):

(1) (Universal) quantifications may be regarded as the conjunctions of instances.

(2) Necessitations of specific propositions (or of specific conjunctions of propositions etc., resulting from the expansions in (1)), being rigid, can be replaced by say N( ) or N(N( )) as appropriate.

(3) Same for ε connecting specific propositions.

(4) E applied to a given specific proposition can be replaced by the canonical proposition corresponding to the existence-set of the given proposition.

The result of this construction is a neconjunctional compound built from atomic propositions; it need not of course be an actual proposition in our universe, since the 'conjunctions' in step (1) are 'too big'; but like *any* truth-functional compound built from certain atoms, 'big' or 'small', it will be strictly equivalent to a disjunction of possible state-descriptions in those atoms – which disjunction *is* 'small', i.e. is a proposition inside our universe.

9 Principles: Stoichiological principles

(1) *Basic Propositional Stoichiology ($CPL^2\varepsilon$).* Here we have as valid formulas the obvious translations of all the standard axioms of 'general set theory' (with urelements) in more or less the sense of Zermelo 1930 – i.e. the usual axioms of ZFC set theory with urelements except for the Axiom of Infinity. E.g. corresponding to the Axiom of Pairs we have:

$$\vDash \forall p\ \forall q\ \exists r\ (p\ \varepsilon\ r \wedge q\ \varepsilon\ r),$$

or if we prefer

$$\vDash \forall p\ \forall q\ \exists r\ \forall s\ (s\ \varepsilon\ r \leftrightarrow s = p \vee s = q).$$

– The first version of course immediately follows from the second; and the second follows from the first together with the appropriate instance of the Separation Schema

$$\vDash \forall p\ \exists q\ \forall r\ (r\ \varepsilon\ q \leftrightarrow r\ \varepsilon\ p \wedge \varphi(r)).$$

When appropriate we must make restrictions to molecular propositions (as opposed to atomic ones), in a way exactly corresponding to restrictions to sets (as opposed to urelements). So e.g. corresponding to the Axiom of Extensionality we have:

$$\vDash \forall p\ \forall q\ (\mathrm{Mol}(p) \wedge \mathrm{Mol}(q) \wedge \forall r(r\ \varepsilon\ p \leftrightarrow r\ \varepsilon\ q) \rightarrow p = q).$$

– We have also, corresponding to the requirement on frames that At $\in$ Mol, the principle

$$\vDash \exists p\ \forall q\ (\mathrm{At}(q) \rightarrow q\ \varepsilon\ p).$$

– In addition to these principles concerning (for the most part) *what propositions there are*, we have also the following fundamental principle concerning the *truth* of molecular propositions (to which nothing corresponds in the context of sets):

$\vDash \forall p\, (\mathrm{Mol}(p) \rightarrow (p \leftrightarrow \neg\forall q\, \varepsilon\, p\, (q)))$.

(2) *Existence*. As already mentioned in the preceding section, here the conspicuous fundamental principle is something like:

$\vDash \forall p\, (\mathrm{Mol}(p) \rightarrow (Ep \leftrightarrow \forall q\, \varepsilon\, p\, (Eq)))$.

(3) *Modality*. Rigidity principles:

$\vDash \Delta(p\, \varepsilon\, q)$, $\vDash \Delta(p\, C\, q)$, $\vDash \Delta(p\, \underline{C}\, q)$.

Hence also $\vDash \Delta At(p)$, $\vDash \Delta Mol(p)$, etc. – From this plus the 'Rule of Necessitation' (which is validity-preserving in HPL) and other basic modal principles, we easily extend the principles given above for the basic connection between the truth or existence of a molecular proposition and the truth or existence of its immediate constituents to their strengthening with $\leftrightarrows$ instead of the mere $\leftrightarrow$.

Remark and Question. It would be interesting to have a comprehensive *axiomatization* of full HPL (and/or conspicuous fragments). I believe that the stoichiological principles indicated in the present section, supplemented by standard postulates for $S5^2$ and equivalential axioms corresponding to the stoichiological definitions of propositional identity and immediate grounding, should go a long way towards such a comprehensive axiomatization. (Or we can stick to just $\mathcal{L}S5^2\varepsilon E$ and treat = and θ as ordinary defined symbols.) Note however that a *complete* axiomatization (i.e. where the theorems are precisely the valid formulas) can surely not be hoped for, as no doubt the incompletability phenomenon affects HPL as much as e.g. first-order set theory. ⊣

10 Principles: Truthmaking, entailment, etc.

(1) *Truthmaking and falsity-making*. We focus on tm and fm; but all principles below hold also with rtm and rfm or stm and sfm in place of resp. tm and fm.

$\vDash \Delta(p \text{ tm } q)$, $\vDash \Delta(p \text{ fm } q)$.

$\vDash (p \text{ tm } q) \supset (p \dashv q)$.

$\vDash (p \text{ fm } q) \supset (p \strictif \neg q)$.

$\vDash p \strictiff \exists\Sigma\ (\Sigma \wedge (\Sigma \text{ tm } p))$.

$\vDash \neg p \strictiff \exists\Sigma\ (\Sigma \wedge (\Sigma \text{ fm } p))$.

It may exceptionally occur that a state is both tm and fm of a proposition (which state must then be necessarily false by the preceding principles):

$\vDash At(p) \supset [(p \wedge \neg p) \text{ tm } (p \wedge \neg p) \wedge \neg(p \wedge \neg p)] \wedge [(p \wedge \neg p) \text{ fm } (p \wedge \neg p) \wedge \neg(p \wedge \neg p)]$.

(More thoroughly: $p \wedge \neg p$ is the only tm, and the fms are $p \wedge \neg p$, $\wedge(\neg p)$, and $\wedge(p)$.) Indeed this might happen even with a true proposition:

$\vDash At(p) \supset [(p \wedge \neg p) \text{ tm } (p \vee \neg p) \vee (p \wedge \neg p)] \wedge [(p \wedge \neg p) \text{ fm } (p \vee \neg p) \vee (p \wedge \neg p)]$.

(More thoroughly: $p \wedge \neg p$ is the only fm, and the tms are $p \wedge \neg p$, $\wedge(p)$, and $\wedge(\neg p)$.)

$\vDash Pure(p) \supset \neg\exists\Sigma\ (\Sigma \text{ tm } p) \vee \neg\exists\Sigma\ (\Sigma \text{ fm } p)$.

$\vDash Pure(p) \wedge p \supset \forall\Sigma\ (\Sigma \text{ tm } p \equiv \Sigma = \wedge(\ ))$.

$\vDash Pure(p) \wedge \neg p \supset \forall\Sigma\ (\Sigma \text{ fm } p \equiv \Sigma = \wedge(\ ))$.

(2) *Entailment, etc.*

$\vDash \Delta(p \rightarrow q)$, $\vDash \Delta(p \leftrightarrow q)$, $\vDash \Delta(p \Leftrightarrow^{+} q)$, $\vDash \Delta(p \Leftrightarrow^{-} q)$, $\vDash \Delta(p \Leftrightarrow q)$, $\vDash \Delta(p \sim^{+} q)$, $\vDash \Delta(p \sim^{-} q)$, $\vDash \Delta(p \sim q)$, $\vDash \Delta(p \approx^{+} q)$, $\vDash \Delta(p \approx^{-} q)$, $\vDash \Delta(p \approx q)$.

For $A, B \in \mathcal{L}$CPL: $\vDash A \rightarrow B$ iff $A \rightarrow B$ is a ‘tautological entailment’ in the sense of Belnap 1959 and Anderson and Belnap 1962, or equivalently valid in the sense of the truthmaker semantics of van Fraassen 1969. (Here we have multigrade N as single $\mathcal{L}$CPL primitive connective, whereas Anderson and Belnap and van Fraassen have the more common primitives unary $\neg$ and binary $\wedge$ and $\vee$; but the adaptation of their definitions to the present scheme is straightforward.) In particular we have of course: $\nvDash (p \wedge \neg p) \rightarrow q$, $\nvDash p \rightarrow (q \vee \neg q)$. – Similarly, $\vDash A \sim^{+} B$, resp. $\vDash A \sim B$, iff $A \sim^{+} B$, resp. $A \sim^{+} B \wedge \neg A \sim^{+} \neg B$, is valid in the sense of Correia 2016. The relation between our concepts of positive regular equivalence etc. and equivalence in the sense of Angell or of Fine 2016 is less straightforward. E.g. the formulas $p \vee (p \wedge q \wedge r)$ and $p \vee (p \wedge q) \vee (p \wedge q \wedge r)$ are Angell–Fine equivalent, but $\nvDash p \vee (p \wedge q \wedge r) \ .\!\Leftrightarrow^{+}\!. \ p \vee$

$(p \wedge q) \vee (p \wedge q \wedge r)$ (take p and q as distinct atomic propositions and r as N( )). But for ‘purity-free’ (sic!) propositions $\Leftrightarrow^{+}$ and $\Leftrightarrow^{-}$ (and hence $\Leftrightarrow$) seem to coincide in extension and to agree with the equivalences of Angell–Fine logic. That is to say: For A, B formulas of $\mathcal{L}$CPL in the same variables $p_1 \dots p_n$ and without 0-ary applications of N (or of $\wedge$, $\vee$ if these are our primitives), we seem to have both

(i) $\vDash \neg(N(\ )\ \underline{C}\ p_1) \wedge \dots \wedge \neg(N(\ )\ \underline{C}\ p_n)\ {:}{\supset}{:}\ A \Leftrightarrow^{+} B\ .{\equiv}.\ A \Leftrightarrow^{-} B$,

and (ii) A is equivalent to B in the sense of Angell–Fine logic iff

$\vDash \neg(N(\ )\ \underline{C}\ p_1) \wedge \dots \wedge \neg(N(\ )\ \underline{C}\ p_n)\ .{\supset}.\ A \Leftrightarrow^{+} B$.

Remarks. (1) From this it follows that, if we modified the syntax of HPL by requiring that N should always apply to a *positive* number of formulas, and the semantics for HPL by disallowing sets involving the empty set from the domain of admissible values for the propositional variables (which is what would accord with the [non-absurd] view that a neconjunction requires *one or more* neconjuncts), then $\Leftrightarrow^{+}$, $\Leftrightarrow^{-}$ and $\Leftrightarrow$ (defined as above) would perfectly coincide in extension and be in perfect agreement with the equivalences of Angell–Fine logic.

(2) With HPL as is, however, such perfect agreement does not seem to be attainable w.r.t. *any* reasonable notion of ‘same truthmakers’ which we can define in the context of HPL. For we would have to then have that $p \vee (p \wedge q \wedge r)$ and $p \vee (p \wedge q) \vee (p \wedge q \wedge r)$ always have the same truthmakers in the sense in question, whence so do in particular $p \vee (p \wedge q \wedge N(\ ))$ and $p \vee (p \wedge q) \vee (p \wedge q \wedge N(\ ))$, whence so do p and $p \vee (p \wedge q)$ (since $p \wedge q \wedge N(\ )$ surely has no truthmakers), which however are not Angell–Fine equivalent. ⊣

– We have the following series of ‘subsumptions’ between different kinds of propositional equivalence:

$\vDash p \leftrightarrow q\ .{\supset}.\ p \leftrightarrows q$. (Also $\vDash p \rightarrow q\ .{\supset}.\ p \dashv q$.)

$\vDash p \Leftrightarrow q\ .{\supset}.\ p \leftrightarrow q$.

$\vDash p \sim q\ .{\supset}.\ p \Leftrightarrow q$.

$\vDash p \approx q\ .{\supset}.\ p \sim q$.

(Similarly for the notions of positive and negative regular, semiregular and exact equivalence, which we omit here from explicit discussion to avoid [or diminish] boredom.) But the reverse implications are not valid. We have e.g.

$\vDash p \vee (p \wedge q) \mathrel{.\!\leftrightarrow\!.} p, \qquad \nvDash p \vee (p \wedge q) \mathrel{.\!\Leftrightarrow\!.} p.$

We have even similar examples with the same variables on the two sides:

$\vDash p \vee (p \wedge \neg p) \mathrel{.\!\leftrightarrow\!.} p, \qquad \nvDash p \vee (p \wedge \neg p) \mathrel{.\!\Leftrightarrow\!.} p.$

And we have e.g.

$\vDash p \vee (p \wedge q \wedge (r \vee \neg r)) \mathrel{.\!\Leftrightarrow\!.} p \vee (p \wedge q) \vee (p \wedge q \wedge (r \vee \neg r)),$

$\nvDash p \vee (p \wedge q \wedge (r \vee \neg r)) \mathrel{.\!\sim\!.} p \vee (p \wedge q) \vee (p \wedge q \wedge (r \vee \neg r)).$

$\vDash p \vee q \mathrel{.\!\sim\!.} p \vee q \vee (p \wedge q), \qquad \nvDash p \vee q \mathrel{.\!\approx\!.} p \vee q \vee (p \wedge q).$

In the special case of a *pure* frame, i.e. with At = $\emptyset$, there is a general collapse of equivalences:

$\vDash \forall p\, \mathrm{Pure}(p) \supset [(p \equiv q \mathrel{.\!\equiv\!.} p \leftrightsquigarrow q) \wedge (p \leftrightsquigarrow q \mathrel{.\!\equiv\!.} p \leftrightarrow q) \wedge (p \leftrightarrow q \mathrel{.\!\equiv\!.} p \Leftrightarrow q) \wedge (p \Leftrightarrow q \mathrel{.\!\equiv\!.} p \sim q) \wedge (p \sim q \mathrel{.\!\equiv\!.} p \approx q)].$

But otherwise, i.e. with at least one atomic proposition α, there will be cases of strict equivalence without mutual entailment (e.g. $\alpha \vee \neg\alpha$ and NN( )), and of mutual entailment without regular equivalence (e.g. α and $\alpha \vee (\alpha \wedge \neg\alpha)$), and of regular equivalence without semiregular equivalence (e.g. NN( ) $\vee (\alpha \wedge \neg\alpha)$ and NN( ) $\vee\, \alpha \vee (\alpha \wedge \neg\alpha)$), and of semiregular equivalence without exact equivalence (e.g. $\alpha \vee \neg\alpha$ and $\alpha \vee \neg\alpha \vee (\alpha \wedge \neg\alpha)$). (Material equivalence might here still coincide with strict equivalence: this happens iff all atomic propositions are rigid.)

$\vDash p \wedge p \mathrel{.\!\approx^{+}\!.} p.$

(For $\vDash p \wedge p = \wedge(p) = \mathrm{NN}(p)$.) This is an interesting difference from the linguistic context of truthmaker semantics where, in the basic notion of truthmaking (similar to our tm), a truthmaker of ‘$p \wedge p$’ (fusion of a tm of ‘p’ and a tm of ‘p’) need not be a truthmaker of ‘p’. (Although actually this can be avoided by changing the relevant semantic clause by referring now to the *set* of conjuncts.)

Another ‘anomaly’ of basic and semiregular truthmaking in truthmaker semantics however does appear equally here in the ‘worldly’ context – viz. the fact that $\approx^{+}$ and $\sim^{+}$ – and here even $\Leftrightarrow^{+}$ – may be destroyed by negation (or dualization):

$\vDash$ p ∧ (q ∨ r) $\approx^+/\sim^+/\Leftrightarrow^+$ p ∧ q .∨. p ∧ r.

$\nvDash$ ¬(p ∧ (q ∨ r)) $\approx^+/\sim^+/\Leftrightarrow^+$ ¬(p ∧ q .∨. p ∧ r).

$\nvDash$ p ∨ (q ∧ r) $\approx^+/\sim^+/\Leftrightarrow^+$ p ∨ q .∧. p ∨ r.

(E.g., for the cases with $\approx^+$ and $\sim^+$: with the variables interpreted as distinct atomic propositions, the states corresponding to p ∧ q and p ∧ r will be tms/stms of the r.h.s. but not of the l.h.s. And for the case with $\Leftrightarrow^+$: with p and q interpreted as distinct atomic propositions and r as N( ), the state corresponding to p ∧ q will be rtm of the r.h.s. but not of the l.h.s. Note here again a difference between our $\Leftrightarrow^+$ and Angell–Fine equivalence, for which this distribution principle holds.) (The 'full' notions A ≈ B and A ∼ B and A ⇔ B are not of course subject to this 'anomaly'.)

Note that $\preccurlyeq^{\mathrm{tm}}$, i.e. →, and $\preccurlyeq^{\mathrm{rtm}}$ do not differ in extension:

$\vDash$ p → q .≡. p $\preccurlyeq^{\mathrm{rtm}}$ q.

– 'Paradoxes' of entailment:

$\vDash$ ∨( ) → p.

$\vDash$ p → ∧( ).

Thus with our formulation of $\mathcal{L}$CPL the 'Variable Sharing Property' for entailments between $\mathcal{L}$CPL formulas does not hold in unrestricted form. (Other violations are of course $\vDash$ ∨( ) → ∨( ), $\vDash$ ∧( ) → ∧( ), $\vDash$ ∨( ) → ∧( ), etc. But these are hardly 'paradoxical', as there is no 'content' [in the form of variables] at all involved. In the 'paradoxes' above on the other hand there is 'content' involved, but no 'common content' in antecedent and consequent. There are also of course 'paradoxes' with variables on *both* sides [but no shared variables], such as $\vDash$ p ∧ ∨( ) → q, $\vDash$ p → ∧( ) ∨ q, etc.) The 'paradoxes' of material implication motivated the introduction of the concept of strict implication; but then there were the 'paradoxes' of strict implication, which motivated the introduction of the concept of entailment; but now there are the 'paradoxes' of entailment. One is tempted to say: 'Well, I suppose everything will have *its* paradoxes!' – But note that, *first*, in all these cases the 'paradoxes' are not *really* paradoxes (hence the inverted commas) but simple laws governing the notions in question – given what the notions *are*, it would be paradoxical if these laws *didn't* hold! One may agree that these are bona fide logical notions as far as they go, even if one might want *also* to have other notions with somewhat different behaviour. And *secondly*: the present violations of the unrestricted Variable Sharing Property are very limited: in particular we have (as is easily seen): ∀ formulas A, B of $\mathcal{L}$CPL where (in primitive notation) N( ) does not occur: If ⊨ A → B then A and B share at least one propositional variable. Nor are these violations 'unprincipled': ∨( ) has *no* tms, so of course every tm of ∨( ) includes a tm of p; and so on. Also it is a sort of basic tenet of relevance logic that entailment (like other, 'more classical' notions) obeys the principles that a proposition entails a conjunction iff it entails every conjunct, and a proposition is entailed by a disjunction iff it is entailed by every disjunct; and certainly any proposition is entailed by every disjunct of the empty disjunction, and entails every conjunct of the empty conjunction! – So these 'paradoxes' of entailment are so *principled* and *limited* that I would not think it sensible to seek for other notions (just) to 'overcome' these 'paradoxes'.

(3) *Existential import*.

⊨ At(p) .⊃. ExistImp(p) ⥶ Ep.

⊨ ∃q (At(q) ∧ p = ¬q) ⊃ □ExistImp(p).

⊨ □∀p (At(p) ∧ p ⊃ Ep) ⊃ ∀p (p ⥷ ExistImp(p)).

For suppose the antecedent and that p holds (in a world). Then ∨{Σ : Σ tm p} also holds; whence so does the disjunction of *positive* parts of truthmakers of p (i.e. omitting the negated atoms). Therefore, by the antecedent, the corresponding disjunction of conjunctions of existence-attributions also holds; but this disjunction is in effect ExistImp(p).

It follows that (in contrast with the case of necessarily *false* propositions where as we saw the existential import can easily differ):

⊨ □∀p (At(p) ∧ p ⊃ Ep) .⊃. □p ∧ □q ⊃ □ExistImp(p) ∧ □ExistImp(q) ∧ [therefore] (ExistImp(p) ⥶ ExistImp(q)).

11 Principles: Classical ground-theoretic principles

We saw that the usual axioms of ZFC set theory with urelements, except for the Infinity Axiom, translate into correct principles for ε in $CPL^2ε$. We will now see that often they also translate into correct principles for θ in $CPL^2θ$.

– The Axiom of Extensionality is an exception: indeed we have the *negation* of its θ-translation as a valid formula:

(*Anti-Extensionality*) $\vDash \exists f\, \exists g\, (\text{Grounded}(f) \wedge \text{Grounded}(g) \wedge \forall h\, (h\, \theta\, f \leftrightarrow h\, \theta\, g) \wedge f \neq g)$.

E.g. take $\wedge(\wedge(\ ))$ (i.e. NNNN( )) and $N(N(\ ), \wedge(\ ))$: the immediate grounds are the same, viz. $\wedge(\ ) = NN(\ )$ only, but the two facts are of course different.

(*Foundation*, a.k.a. *Regularity*) $\vDash \forall f\, (\text{Grounded}(f) \rightarrow \exists g\, (g\, \theta\, f \wedge \neg\exists h\, (h\, \theta\, g \wedge h\, \theta\, f)))$.

(*Pairing*) $\vDash \forall f\, \forall g\, \exists h\, \forall i\, (i\, \theta\, h \leftrightarrow i = f \vee i = g)$.

(E.g. take $\wedge(f, g)$.)

(*Separation Schema*) $\vDash \forall f\, \exists g\, \forall h\, (h\, \theta\, g \leftrightarrow h\, \theta\, f \wedge \varphi(h))$.

(E.g. take the *conjunction* of the immediate grounds of f which satisfy condition φ.)

(*Union*) $\vDash \forall f\, \exists g\, \forall h\, (h\, \theta\, g \leftrightarrow h\, \theta^2\, f)$.

(E.g. take the conjunction of the immediate grounds of immediate grounds of f.)

– A more general principle says that if the facts satisfying φ are 'G-bounded' then there is a fact of which they are the immediate grounds:

$\vDash \exists f\, \forall g\, (\varphi(g) \rightarrow g\, G\, f) \rightarrow \exists f\, \forall g\, (g\, \theta\, f \leftrightarrow \varphi(g))$.

(E.g. take the conjunction of the facts satisfying φ.)

(*Choice*) $\vDash \forall f$: $\text{Grounded}(f) \wedge \forall g\, \theta\, f\, (\text{Grounded}(g)) \wedge \forall g\, \forall h\, (g\, \theta\, f \wedge h\, \theta\, f \wedge g \neq h \rightarrow \neg\exists i\, (i\, \theta\, g \wedge i\, \theta\, h)) \rightarrow \exists g\, \forall h\, \theta\, f\, \exists! i\, (i\, \theta\, g \wedge i\, \theta\, h)$.

(Choose an immediate ground of each immediate ground of f, and conjoin these facts.)

(*Replacement Schema*) $\vDash \forall f$: $\forall g\, \theta\, f\, \exists! h\, \varphi(g, h) \rightarrow \exists g\, \forall h\, (h\, \theta\, g \leftrightarrow \exists i\, \theta\, f\, (\varphi(i, h)))$.

(Take the conjunction of the facts h such that $\varphi(g, h)$ for some $g\, \theta\, f$.)

– The direct correlate of the Power-Set Axiom fails in an interesting way. Put

$A \subseteq^{\theta} B$ [‘A is a θ-subproposition of B’] $=_{df}$ $\forall f\,(f\ \theta\ A \rightarrow f\ \theta\ B)$.

(If this holds and both A and B are facts, we may say also that ‘A is a θ-subfact of B’.) Then the direct correlate of the Power-Set Axiom is

$\forall f\ \exists g\ \forall h\ (h\ \theta\ g \leftrightarrow h \subseteq^{\theta} f)$.

But this is not valid. Take e.g. a fact f = ¬¬¬p, whose only immediate ground is ¬p; so a fact is a θ-subfact of f iff it has no grounds or has only ¬p as immediate ground; but this includes e.g. all facts of the form N(p, q) where q is an arbitrary fact; and so the θ-subfacts of f are not ‘bounded’ and so there can be no fact having them all as immediate grounds.

One way to ‘repair’ the principle is to require not *all* θ-subfacts as immediate grounds of the new fact, but only one ‘representative’ for each subcollection of immediate grounds of the original fact. I.e. we have:

$\vDash \forall f\ \exists g\ \forall h \subseteq^{\theta} f\ \exists i\ \theta\ g\ \forall j\ (j\ \theta\ i \leftrightarrow j\ \theta\ h)$.

(E.g. take the conjunction of: for each subset S of the set of immediate grounds of f, the conjunction of the facts in S.)

12 Principles: Modal ground-theoretic principles

(1) *Necessitation from Immediate Grounds*. The idea here is that: a grounded fact is always necessitated by the joint obtaining of its immediate grounds (i.e. strictly implied by their conjunction). – But some care is needed for a proper formal statement of this idea: there must be here a kind of ‘back-reference to the actual world from within the scope of a modal operator’, for what does the necessitating is the conjunction of the *de facto* immediate grounds, not the conjunction of what *would be* the immediate grounds in alternative circumstances. Thus the naive formulation

$\text{Grounded}(p) \rightarrow (\forall q((q\ \theta\ p) \rightarrow q) \mathrel{-\!\!3} p)$

is *not* a valid formula: if p is say ¬¬r where r is a contingent truth, then the strict implication above does not hold, since if p had been false (as it *can* be) it would be vacuously true that all its immediate grounds are true.

Proper formulations can be given in at least two different ways here. One is the following $\mathcal{L}S5^2\varepsilon\theta$ formula using the ‘purely collecting capacity’ of neconjunctions:

$\vDash$ Grounded(p) $\rightarrow \exists$s [$\forall$q (q ε s $\leftrightarrow$ q θ p) $\wedge$ ($\forall$q ε s (q) $\prec$ p)].

And the other is the following $\mathcal{L}S5^2\theta$ formula (so preferable to the above at least in so far as it does not use ε) where we make straightforward 'back-reference to the actual world' at the appropriate place:

$\vDash$ Grounded(p) $\rightarrow \exists$w [w $\wedge$ ($\forall$q (w $\prec$ (q θ p) $.\!\rightarrow\!.$ q) $\prec$ p)].

– There is also a similar principle of '*Necessitation from ALL Grounds*'. Proper formulations are given by simply replacing θ by G in the two valid formulas displayed above. Of course Necessitation from Immediate Grounds implies Necessitation from All Grounds.

(2) *Implicational Connectedness of (Immediate) Ground and Grounded.* A grounded fact can be of 'broadly conjunctive' character or of 'broadly disjunctive' character (or both, as with double negation). In the former case the grounded fact will imply the immediate ground, and in the latter case the immediate ground will imply the grounded fact. So:

$\vDash$ p θ q $\rightarrow$ (p $\prec$ q $.\!\vee\!.$ q $\prec$ p).

(3) *Semi-Rigidity of Immediate Ground.* 'If a fact is an immediate ground of another, then it is so still in any world where both obtain':

$\vDash$ p θ q $\rightarrow$ ((p $\wedge$ q) $\prec$ (p θ q)).

The corresponding statement for general ground (G) is of course not valid. Take e.g. p $\vee$ (q $\wedge$ r), with p, q, r say independent atomic facts: then as it happens q is a ground of p $\vee$ (q $\wedge$ r), but these are both true without the grounding holding in a world where p is true, q true, but r false. – In such cases as we may say the actual θ-chain is 'broken' (in another world). What we do have is the restricted schema ($n \geq 2$):

$\vDash p_1\ \theta\ p_2\ \theta\ p_3\ \theta \ldots \theta\ p_n \rightarrow (p_1 \wedge \ldots \wedge p_n \prec p_1\ G\ p_n)$.

This follows of course from the earlier principle of Semi-Rigidity of Immediate Ground.

(4) *Strengthened forms of Semi-Rigidity of Immediate Ground.* Actually we can say more: for according as the grounded fact is of 'broadly conjunctive' or 'broadly disjunctive' character, the obtaining of the grounded fact alone, or of the grounding fact alone, will necessitate the grounding connection. So we have:

⊨ p θ q → [p ⥽ (p θ q) .∨. q ⥽ (p θ q)].

– Since grounding is of course ‘factive’ (only facts can ground or be grounded), this can be strengthened further to

⊨ p θ q → [p ⥼⥽ (p θ q) .∨. q ⥼⥽ (p θ q)].

(Also in the initial form of the principle in (3) above ⥽ can be replaced by ⥼⥽. The same is *not* true however for the subsequent schema with G: consider e.g. (p ∧ q) ∨ (p ∧ r) with p, q, r independent atomic facts – then p θ (p ∧ q) θ ¬¬(p ∧ q) θ ((p ∧ q) ∨ (p ∧ r)) [= N(¬(p ∧ q), ¬(p ∧ r))], but p G ((p ∧ q) ∨ (p ∧ r)) does *not* strictly imply p ∧ (p ∧ q) ∧ . . . ; for the grounding could go only ‘via’ p ∧ r.)

13 Conditions on frames

It is interesting to consider various kinds of special conditions on frames, and the question of to what extent such conditions can be ‘represented’ by formulas of ℒHPL. The present section is illustrative of this, though far from exhaustive – obviously much more can be done along these lines.

A (basic, modal, existential, or full) frame $\mathcal{F}$ = ⟨At, Mol, w_0, . . . ⟩ is *Cantorian* if Mol contains as an element some infinite set.

Proposition. $\mathcal{F}$ is Cantorian iff $\mathcal{F}$ verifies the sentence

∃p: ∃q (q ε p) ∧ ∀q ε p ∃r ε p (q ε r). □

A (modal, or full) frame $\mathcal{F}$ = ⟨At, Mol, w_0, W, . . . ⟩ is *Tractarian* if W is the set of *all* functions from At to {T, F}. (I.e. all truth-value combinations for atomic propositions are considered as possible, as in Wittgenstein’s *Tractatus*.)

Proposition. $\mathcal{F}$ is Tractarian iff $\mathcal{F}$ verifies the sentence

∃p: ∀q (q ε p ↔ At(q)) ∧
∀q [Mol(q) ∧ ∀r ε q (r ε p) → ◇(∀r ε q (r) ∧ ∀r (r ε p ∧ ¬(r ε q) → ¬r))]. □

Remarks and Questions. The ℒS5^2 formulas which are HPL-valid are precisely the ones which are valid w.r.t. the usual possible world semantics for S5^2. But there are ℒS5^2 formulas which are valid w.r.t. Tractarian frames and yet not HPL-valid or equivalently possible-world-semantics valid. An example is (where w, v, u, z are world-proposition variables):

$\neg\exists w\ \exists v\ \exists u$: $\neg(w ⧟ v) \wedge \neg(w ⧟ u) \wedge \neg(v ⧟ u) \wedge \forall z\ ((z ⧟ w) \vee (z ⧟ v) \vee (z ⧟ u))$.

I.e., roughly speaking: The number of worlds is not 3. – For if the atomic propositions are independent then the number of worlds must be here 2 to the power of the number of atomic propositions, and 3 is not 2 to the power of anything.

The similar situation would occur in a more elementary 'atomic proposition' (rather than 'possible world') semantics for $S5^2$, where instead of having an arbitrary non-empty set W ('worlds') and attributing elements of $\mathcal{P}(W)$ to the propositional variables, we have rather an arbitrary (possibly empty) set A ('atomic propositions') and attribute elements of $\mathcal{P}(\mathcal{P}(A))$ to the propositional variables.

I conjecture that:

(1) An $\mathcal{L}S5^2$ formula φ is valid w.r.t. Tractarian frames iff φ is valid w.r.t. this 'atomic proposition semantics'.

(2) A complete (and of course sound) axiomatization for such valid formulas can be obtained by enriching the usual axiomatization of $S5^2$ (as e.g. in Fine 1970) by the instances of the schema generalizing the above formula in the obvious way to state that 'The number of worlds is not n' for n not a power of 2.

(3) For any $\mathcal{L}S5^2$ formula φ, the following conditions are equivalent:

(i) φ is valid w.r.t. Tractarian frames with infinitely many atoms;

(ii) φ is valid w.r.t. atomic proposition semantics models with infinite set of atoms;

(iii) φ is valid w.r.t. possible world semantics models with infinite set of worlds. ⊣

An alternative conception of atomic propositions holds that they come in 'clusters' consisting of attributions (to given individual or individuals) of the different 'determinate' properties (e.g. specific colours) falling under a certain 'determinable' (e.g. colour), so that *within* a cluster it is necessary that there is exactly one truth, but '*across*' clusters there is complete modal independence. (See e.g. Prior 1949 and Fine 2011.) This suggests the following definition.

A (modal, or full) frame $\mathcal{F} = \langle At, Mol, w_0, W, \ldots \rangle$ *has determinable/determinate* *structure*, or more briefly *is determinable/determinate*, if there exists a partition $\mathcal{P}$ of At (i.e. collection of disjoint non-empty subsets of At whose union is At) such that $W = \{w \in {}^{At}\{T, F\} : \forall\Gamma \in \mathcal{P}\ \exists!\alpha \in \Gamma\ (w(\alpha) = T)\}$.

This condition too corresponds to a single sentence of $\mathcal{L}$HPL, indeed of $\mathcal{L}S5^2\varepsilon$. Since the sentence is rather long, we will use for heuristic convenience variables Γ, Δ, etc. for 'sets' (i.e. officially neconjunctions) of propositions, and variables X, Y, etc. for 'sets of sets' (i.e. neconjunctions of neconjunctions) of propositions, and accordingly ∈ in place of the official ε. We use also some other obvious abbreviations.

Proposition. $\mathcal{F}$ is determinable/determinate iff $\mathcal{F}$ verifies the sentence:

$\exists X$: X is partition of At $\wedge\ \forall\Gamma\ [(\exists\Delta \subseteq \text{At}\colon \Gamma = \Delta \cup \{\neg p : p \in \text{At} - \Delta\})\ .\!\!\rightarrow\!\!.\ \Diamond\forall p \in \Gamma\ (p) \leftrightarrow \forall\Delta \in X\ \exists!p \in \Delta\ (p \in \Gamma)]$,

where 'X is partition of At' stands for

$\exists\Gamma\ (\Gamma \in X) \wedge \forall\Gamma \in X\ \forall p \in \Gamma\ \text{At}(p) \wedge \forall\Gamma \in X\ \exists p \in \Gamma \wedge \forall p\ (\text{At}(p) \rightarrow \exists!\Gamma \in X\ (p \in \Gamma))$. □

Remark. Note that no frame at all is both Tractarian and determinable/determinate. When At = ∅, the frame is trivially Tractarian and trivially *not* determinable/determinate (since there is then *no* partition of At i.e. collection of disjoint *non-empty* subsets of At ...). If At is a singleton {α}, the frame is: Tractarian and not determinable/determinate if W contains w_0 and the *other* truth-value assignment to α; determinable/determinate and not Tractarian if $W = \{w_0\} = \{\{\langle\alpha, T\rangle\}\}$; and neither Tractarian nor determinable/determinate if $W = \{w_0\} = \{\{\langle\alpha, F\rangle\}\}$. Finally, suppose At has two or more elements and is Tractarian: so in particular distinct atoms α and β are always independent; so a candidate partition for determinable/determinate structure would have to consist of unit-blocks; but then every atom would be necessary, contradicting Tractarianess. ⊣

An (existential, or full) frame $\mathcal{F} = \langle$At, Mol, w_0, maybe W, $\mathcal{E}\rangle$ is *attribute-actualist* if ∀α ∈ At: ∀w ∈ W (or ∈ $\{w_0\}$ in case of mere existential frame): If w(α) = T then w ∈ $\mathcal{E}$(α). (I.e. atomic *facts* must exist.) (For suppose [property-actualism, or better attribute-actualism] that 'individuals' cannot have basic properties or enter into basic relations unless they exist; then if p is an atomic *fact* then its individuals will exist, and so p itself will exist if we equate existence of the proposition with existence of its constituents. [The property or relation in p is presumably either a necessary existent, or at least a contingent existent given that it is instantiated.])

Proposition. (1) An existential frame $\mathcal{F}$ is attribute-actualist iff $\mathcal{F}$ verifies the sentence

$$\forall p\,(At(p) \wedge p \rightarrow Ep).$$

(2) A full frame $\mathcal{F}$ is attribute-actualist iff $\mathcal{F}$ verifies the sentence

$$\Box\forall p\,(At(p) \wedge p \rightarrow Ep). \quad \Box$$

14 Weak fragments: Propositional identity

We consider here the weak fragments of HPL where only propositional identity is added to the basic resources of CPL, or alternatively of S5. We are then in a comparatively very 'controlled environment', i.e. the conceptual apparatus is very weak compared to the full resources of our HPL, and so this will embolden us to try the 'experiment' of extending the scope of transparency by allowing 'free rein' to the identity sign. (So strictly speaking these will not be exactly fragments of HPL but extensions of fragments.) Thus the formulas of what we will call CPL=(+) are defined by:

(1) Propositional variables are formulas;

(2) If $X_1, \ldots, X_n$ ($n \geq 0$) are formulas then so is $N(X_1, \ldots, X_n)$;

(3) If X and Y are formulas then so is $X = Y$.

And the formulas of S5=(+) are defined by the same clauses plus: (4) If X is formula then so is $\Box X$.

Questions concerning the 'identity of identities' (i.e. the internal constitution of identity-propositions) and the 'identity of necessitations' (constitution of modal propositions) then becomes statable, and the issue arises of how our formal semantics will deal with such questions.

One option is to construct a semantics which remains neutral as to any questions which might receive different answers under different (reasonable) theories of the constitution of identity-propositions and modal propositions. Thus e.g. neither the formula $(p = q) = (q = p)$ nor the formula $(p \neq q) \rightarrow ((p = q) \neq (q = p))$ would count as valid under such a semantics; although $(p = q) \rightarrow ((p = q) = (q = p))$ *would* count as valid since it follows from a principle of substitution of identicals which no reasonable theory could deny. (The semantics in Bloom & Suszko 1972 is of this neutral kind.)

Another option is to assume some specific views as to the constitution of identity-propositions and modal propositions and build the semantics based on that. The assumptions may be based on conviction, or simply on curiosity as to where they might lead.

For instance, one might assume a primitivistic view on both identity and necessity: that identity and necessity are simple, unanalyzable notions that enter as basic 'building blocks' in the corresponding propositions. Within such a view, there is still room for disagreement, notably on whether or not there is some kind of 'order' or something similar among the 'relata' of identity in the identity-proposition, and correspondingly whether $(p \neq q) \rightarrow ((p = q) \neq (q = p))$ or else $(p = q) = (q = p)$ should count as valid.

Or again, one might assume some non-primitivistic views on identity or modality or both. Then the supposed analyses may or may not correspond to something we can 'write' in the limited resources of CPL=(+) or S5=(+). E.g. the Leibnizian analysis of identity in terms of possession of the same properties (in some sense of 'property'), or the analysis of necessity in terms of truth in all possible worlds – these do not correspond to anything we can write here. On the other hand a certain neat 'truth-functionalist' analysis would correspond very directly to the validity of the following $\mathcal{L}$S5=(+) formulas:

$p = q \mathbin{.\!\rightarrow\!.} (p = q) = N(p, Np).$

$p \neq q \mathbin{.\!\rightarrow\!.} (p = q) = \wedge(p, q, Np, Nq).$

$\Box p \mathbin{.\!\rightarrow\!.} \Box p = N(p, Np).$

$\neg\Box p \mathbin{.\!\rightarrow\!.} \Box p = \wedge(p, Np).$

Or one might prefer to think that, unlike truth-functions, identity and necessity have some kind of 'transcendental' character but still want to use a variant of the above 'reduction' of necessity:

$\Box p \mathbin{.\!\rightarrow\!.} \Box p = (p = p).$

$\neg\Box p \mathbin{.\!\rightarrow\!.} \Box p = (p \neq p).$

(Or alternatively one might want to 'reduce' identity to modality: one can then just insert $\Box$ before $N(p, Np)$ and before $\wedge(p, q, Np, Nq)$ in the above 'truth-functionalist' reductions of identity.)

– I consider several such different approaches worth pursuing; also comparisons of the resulting systems would be interesting. Here however I will develop only one approach, namely the approach based on primitivism about both identity and necessity, and where there is not supposed to be order or anything like it among the terms of the identity-proposition, so that (p = q) = (q = p) will count as valid (much indeed as N(p, q) = N(q, p) has counted and will continue to be counted as valid for us). This is not at all because I am inclined to a primitivist view on either identity or necessity (I am not); but rather because I think that, if we are going to take seriously identity and necessity as proposition-forming connections, then the primitivist view seems a natural choice for starting-point for speculative investigations. On the other hand the thesis of absence of order or the like in identity-propositions, modulo primitivism about identity, *is* something I am inclined to hold. (See Batchelor 2018 for an extensive discussion of such issues.)

– We proceed now to the presentation of our semantics for CPL=(+). The modal extension to S5=(+) will be given immediately afterwards.

Intuitively, the basic ideas are as follows. *Propositions* are supposed to have essentially the same structure as *formulas* of CPL=(+) (as far as their composition by finitary truth-functions and identity is concerned), except that (i) difference in the order of terms in an identity-formula is not supposed to correspond to any difference in the corresponding proposition, and (ii) same for differences in order and/or (positive) number of occurrences of neconjuncts in a neconjunction. – So we will set up a language which is 'isomorphic' to $\mathcal{L}$CPL=(+) in order to interpret the formulas of $\mathcal{L}$CPL=(+). This new language we will call *the onto-language*, since it is meant to approximately model the *propositions*, in *the world*, which will serve to interpret our 'linguistic language' $\mathcal{L}$CPL=(+). Truth of identities in the onto-language will then be understood as literal identity of the expressions flanking the identity-sign modulo only the differences indicated in (i) and (ii) above. (It would have been possible to consider $\mathcal{L}$CPL=(+) itself as the onto-language; but the procedure adopted here seems to me more perspicuous.)

The formulas, or better *sentences*, of this onto-language are then defined as follows. There are the *atomic sentences* α, β, γ, δ, α', β', ...; and other sentences (*molecular sentences*) are built by applications of = (to two sentences) and N (to n sentences, $n \geq 0$). Thus the sentences of the onto-language are exactly like the formulas of $\mathcal{L}$CPL=(+), except that now we have these atomic sentences α, β, ... in lieu of the propositional variables p, q, ...

For φ and ψ sentences of the onto-language, we say that φ is *orthographic variant* of ψ (or that φ and ψ are orthographic variants, since the notion is clearly symmetric) if φ can be obtained from ψ by zero or more applications of the following types of transformation:

(i) interchange of sentences flanking an occurrence of =;

(ii) changing $N(\psi_1, \ldots, \psi_n)$ to $N(\varphi_1, \ldots, \varphi_k)$ provided $\{\psi_1, \ldots, \psi_n\} = \{\varphi_1, \ldots, \varphi_k\}$.

– Now by a *valuation* (for the onto-language) we mean an assignment of truth-values to the atomic sentences.

A valuation induces then an 'extended valuation' for all sentences of the onto-language:

(i) the truth-values of *atomic* sentences are given;

(ii) truth-values of N compounds are determined in obvious way;

(iii) $\varphi = \psi$ is true iff φ is orthographic variant of ψ. (Note that this is independent of truth-values of components. Thus $\varphi = \psi$ is always either true under all valuations or false under all valuations.)

So much for the onto-language itself. Coming now to the semantics for CPL=(+) proper, we define an *interpretation* as a function from the propositional variables to sentences of the onto-language. Such an interpretation σ automatically induces an extended function from all formulas of $\mathcal{L}$CPL=(+) to sentences of the onto-language:

$\sigma(\varphi = \psi) \;=\; \sigma(\varphi) = \sigma(\psi)$.

$\sigma(N(\varphi_1, \ldots, \varphi_n)) \;=\; N(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$.

For interpretation σ, valuation v, formula φ of $\mathcal{L}$CPL=(+):

$\langle\sigma, v\rangle$ verifies $\varphi \quad =_{df} \quad \sigma(\varphi)$ is true under v.

$\models \varphi \quad =_{df} \quad \forall\sigma, v$: $\langle\sigma, v\rangle$ verifies φ.

Γ ($\subseteq \mathcal{L}$CPL=(+)) is satisfiable $\quad =_{df} \quad \exists\sigma, v$: $\forall\varphi \in \Gamma$: $\langle\sigma, v\rangle$ verifies φ.

Γ implies $\varphi \;=_{df}\; \forall\sigma, v$: If $\langle\sigma, v\rangle$ verifies all the formulas in Γ then $\langle\sigma, v\rangle$ verifies φ.

– We have then e.g.:

$\models (p = q) = (q = p)$.

$\models p \neq q \mathbin{.\!\!\rightarrow\!\!.} (p = p) \neq (q = q)$.

Remarks and Questions. (1) I conjecture that, for formulas of the restricted language $\mathcal{L}$CPL=, the notion of consequence (Γ implies φ) (hence also validity, satisfiability) as defined here in this 'syntactical semantics' coincides extensionally with the notion of consequence defined in terms of our earlier 'set semantics' ('basic frames').

(2) The set semantics can be extended to full $\mathcal{L}$CPL=(+) by use of the ideas in the 'truth-functionalist reductions' mentioned in the previous discussion above. The definition of hyperintensional values would be extended to identity-formulas by the condition:

$\sigma(\varphi = \psi) = \{\sigma(\varphi), \{\sigma(\varphi)\}\}$ or $\{\{\sigma(\varphi), \{\sigma(\varphi)\}, \sigma(\psi), \{\sigma(\psi)\}\}\}$, according as $\sigma(\varphi) = \sigma(\psi)$ or not.

This would of course yield a different system: e.g. the formula

$(p = p) = N(p, Np)$

would be valid in this set semantics, but not in the syntactical semantics – indeed in the syntactical semantics the *negation* of this formula is valid. – Nor is there surely any *other* way, essentially different from this reductionistic way, of extending our set semantics to $\mathcal{L}$CPL=(+) without the addition of further apparatus. – Note also that in this set-semantics CPL=(+) there is a straightforward '*reduction*' to CPL=, in the sense that $\forall\varphi \in \mathcal{L}$CPL=(+) $\exists\psi \in \mathcal{L}$CPL= such that (in this set-semantics) $\models \varphi \leftrightarrow \psi$. The following example should suffice to illustrate the reduction method: an identity $A = (B = C)$ is here equivalent to:

$B = C \wedge A = N(B, \neg B) \mathbin{.\!\!\vee\!\!.} \neg(B = C) \wedge A = \wedge(B, C, \neg B, \neg C)$.

– In *syntactical*-semantics CPL=(+), by contrast, it seems clear that e.g. $p = (q = r)$ is *not* equivalent to any $\mathcal{L}$CPL= formula, so that there is no such 'reduction'.

(3) $\mathcal{L}$CPL= and $\mathcal{L}$CPL=(+) are very simple languages and so *decidability* for validity in systems in such languages can very realistically be hoped for. – So *are* the systems described here decidable? (CPL= with syntactical semantics, CPL= with set semantics [if this gives different valid formulas], CPL=(+) with syntactical semantics,

and CPL=(+) with set semantics as in remark (2) [this is of course equi-decidable with CPL= with set semantics in view of the existence of (mechanical) reduction method].)

(4) It would be interesting to have an *axiomatization* of (syntactical-semantics) CPL=(+). Some obvious candidate axiom-schemas here (to be adjoined to postulates for CPL) are:

Orthographic Variants Schema: $\varphi = \psi$, for φ orthographic variant of ψ.

Identity Criterion for Neconjunctions:

$N(\varphi_1, \ldots, \varphi_n) = N(\psi_1, \ldots, \psi_k) .\!\leftrightarrow\!. (\varphi_1 = \psi_1 \vee \ldots \vee \varphi_1 = \psi_k) \wedge \ldots \wedge (\varphi_n = \psi_1 \vee \ldots \vee \varphi_n = \psi_k) \wedge (\psi_1 = \varphi_1 \vee \ldots \vee \psi_1 = \varphi_n) \wedge \ldots \wedge (\psi_k = \varphi_1 \vee \ldots \vee \psi_k = \varphi_n)$.

Identity Criterion for Identities:

$(\varphi_1 = \varphi_2) = (\psi_1 = \psi_2) .\!\leftrightarrow\!. (\varphi_1 = \psi_1 \wedge \varphi_2 = \psi_2) \vee (\varphi_1 = \psi_2 \wedge \varphi_2 = \psi_1)$.

Principle of Substitution of Identicals:

$(\varphi = \psi) \to (\chi = \chi')$, where χ' results from χ by replacement of occurrences of φ by ψ.

(5) Comparisons with some other systems of Classical Propositional Logic with propositional identity in the literature (e.g. Bloom & Suszko 1972 [and later literature following up on this paper], Church 1984) might be interesting. ⊣

We sketch now the extension of the above syntactical semantics to the modal system S5=(+).

The (new) *onto-language* is exactly as before only now the language also contains the unary connective $\Box$. A *valuation* is now a non-empty *set* of truth-value assignments to the atomic sentences, with designated element. This then has a natural extension to a set of truth-value assignments with designated element for *all* sentences. (The value for $\Box\varphi$ in an assignment in the set is T iff the value of φ is T in all assignments in the set. The value of $\varphi = \psi$ in an assignment is T iff φ and ψ are orthographic variants in the sense exactly as before.) A valuation v *verifies* a sentence φ of the onto-language if φ receives T in the designated element of the extended valuation induced by v.

An *interpretation* is a function from the propositional variables to sentences of the onto-language. Again this automatically extends to a function from all *formulas* of $\mathcal{L}$S5=(+) to sentences of the onto-language. Then the definitions of '⟨σ, v⟩ verifies φ', validity etc. are exactly as before.

We have then e.g.:

$\models \Delta(p = q)$.

$\models (\Box p = \Box q) \rightarrow (p = q)$.

Remarks and Questions. The previous Remarks and Questions in classical case all have close analogues here: (1) I conjecture the equivalence of this syntactical semantics and the set semantics for $\mathcal{L}$S5=; (2) The set semantics can be extended to full $\mathcal{L}$S5=(+) using the 'truth-functionalist reductions' (and also a semantics for $\mathcal{L}$CPL=(+) might be extended to a semantics for $\mathcal{L}$S5=(+) using the 'reduction' of necessity *to identity* mentioned above); (3) There are the questions of *decidability*; (4) It would be interesting to have an axiomatization of S5=(+); (5) Comparisons with other literature might be interesting. – Concerning (4), note that:

(i) A natural addition to the earlier 'classical' axiom-schemas is: *Identity Criterion for Necessity Propositions*: $(\Box p = \Box q) \leftrightarrow (p = q)$.

(ii) With reasonable postulates, the rigidity of identity ($\Delta(p = q)$) can be derived as a theorem: By the Orthographic Variants Schema, we have $\vdash p = p$; so by Necessitation, $\vdash \Box(p = p)$; then from this by Substitution of Identicals (using the instance $p = q \rightarrow (\Box(p = p) \leftrightarrow \Box(p = q))$) we have $\vdash p = q \rightarrow \Box(p = q)$; so by Necessitation $\vdash \Box(p = q \rightarrow \Box(p = q))$; and so by S5 (recalling that $\Box(p \rightarrow \Box p)$ is equivalent to Δp) $\vdash \Delta(p = q)$. (Note incidentally that this argument illustrates the interesting general fact that there cannot be an extension of S5, with the Necessitation Rule in force, by a 'half-rigid' connection, i.e. one where true attributions are always necessary but not false attributions always impossible or vice versa. $\Box(\neg p \rightarrow \Box\neg p)$ is also equivalent to Δp.) ⊣

15 Weak fragments: Propositional existence

We consider now the weak fragments of HPL where only propositional existence is added to CPL, or alternatively to S5. Again (as in the fragments with propositional identity), in this 'controlled environment' we will extend the scope of transparency by allowing 'free reign' to the existence symbol; and we will adopt 'methodological primitivism' about both existence and necessity as guide for the construction of the semantics.

Thus now the *formulas* of what we will call CPLE(+) are defined by:

(1) Propositional variables are formulas;

(2) If $X_1, \ldots, X_n$ are formulas ($n \geq 0$) then so is $N(X_1, \ldots, X_n)$;

(3) If X is formula then so is EX.

And for the formulas of S5E(+) we add: (4) If X is formula then so is $\Box X$.

Semantics for CPLE(+). In addition to the familiar truth-values T and F, we use here the *existence-values* E (existence) and Ȩ̸ (non-existence), and the combined *truth-existence values* ET (existent truth), $\not{E}$T(non-existent truth), EF (existent falsity), $\not{E}$F (non-existent falsity).

An *interpretation* is then an assignment of truth-existence values to the propositional variables. This is of course tantamount to a pair consisting of (i) an assignment of truth-values and (ii) an assignment of existence-values; and sometimes it is more convenient to think in terms of such an assignment-pair.

An interpretation σ then induces an assignment of truth-existence values (or corresponding assignment-pair) to all formulas in the obvious way:

(i) $N(\varphi_1, \ldots, \varphi_n)$ exists iff all of $\varphi_1, \ldots, \varphi_n$ exist;

(ii) $N(\varphi_1, \ldots, \varphi_n)$ is true iff not all of $\varphi_1, \ldots, \varphi_n$ are true;

(iii) $E\varphi$ exists iff φ exists;

(iv) $E\varphi$ is true iff φ exists.

We say that interpretation σ *verifies* formula φ if $\sigma(\varphi) \in \{ET, \not{E}T\}$. φ is *valid*, or $\vDash \varphi$, if every interpretation verifies φ. Γ *implies* φ if every interpretation which verifies every formula in Γ verifies φ. Γ is *satisfiable* if there exists interpretation which verifies every formula in Γ.

As in other, more familiar cases (e.g. the standard truth-value semantics for CPL), here what an interpretation σ assigns to a formula φ 'depends only' on what σ assigns to the propositional variables which actually occur in φ (i.e. $\forall\sigma, \sigma'$: if σ and σ' agree on such variables then $\sigma(\varphi) = \sigma'(\varphi)$). So whether or not a given formula is valid can be mechanically decided by means of a '*truth-existence table*'. E.g.:

p	N(p)	Ep	EN(p)	Ep ↔ EN(p)
ET	EF	ET	ET	ET
ɆT	ɆF	ɆF	ɆF	ɆT
EF	ET	ET	ET	ET
ɆF	ɆT	ɆF	ɆF	ɆT

So Ep ↔ EN(p) is valid, or as we might also say a *truth-existence tautology*. (Of course ↔ is here defined in terms of N, and in constructing the table above we use the truth-existence value rule for ↔ derived from the definition and the basic rule for N.)

Remarks. (1) It seems clear that, for formulas of the restricted language $\mathcal{L}$CPLE, consequence in the sense of the present 'truth-existence value semantics' coincides in extension with consequence in the sense of our earlier 'set semantics' ('existential frames').

(2) *Reduction of CPLE(+) to CPLE*. Clearly for every formula φ of $\mathcal{L}$CPLE(+) there exists a formula ψ of $\mathcal{L}$CPLE s.t. $\vDash \varphi \leftrightarrow \psi$. For we can just say replace each occurrence of formula $E\chi(p_1, \ldots, p_n)$ in φ not in the scope of a further occurrence of E – replace this, I say, by $Ep_1 \wedge \ldots \wedge Ep_n$ (including as limiting case $\wedge(\ )$ if $n = 0$, i.e. if χ contains no variables). (So indeed even applications of E to non-atomic formulas are not needed.)

(3) *Axiomatization*. The obvious axiomatization of CPLE(+) is by (in addition to postulates suitable to CPL) the schemas

$$EN(\varphi_1, \ldots, \varphi_n) \leftrightarrow \wedge(E\varphi_1, \ldots, E\varphi_n)$$

(including as limiting case $EN(\ ) \leftrightarrow \wedge(\ )$, i.e. in effect $EN(\ )$) and

$$EE\varphi \leftrightarrow E\varphi.$$

It is straightforward to prove by Kalmár's method that every valid formula of $\mathcal{L}$CPLE(+) is a theorem of this axiom-system. (I.e. the axiom-system is 'weakly complete'. Since no doubt Compactness holds here, 'strong completeness' follows.)

(4) *Logical function theory*. For $n \geq 0$, an n-ary *truth-existence function* is a function f from $\{ET, ɆT, EF, ɆF\}^n$ to $\{ET, ɆT, EF, ɆF\}$ satisfying the condition that $f(v_1, \ldots, v_n) \in \{ET, EF\}$ iff $\forall i = 1, \ldots, n$: $v_i \in \{ET, EF\}$ (i.e. the value of the function 'exists' iff all the arguments do). We may say that a truth-existence function f is

a *truth-function* if the truth-value of a value of f is always determined by the truth-values of the arguments (in the obvious sense of these words); and that f is a *purely existential* function if the *truth*-value of a value of f is always determined by the *existence*-values of the arguments; and that f is *hybrid* if f is neither truth-function nor purely existential function. Thus e.g. N^n (i.e. n-ary neconjunction) is always a truth-function; E (i.e. the unary truth-existence function corresponding to the connective E) is a purely existential function; and $\neg p \wedge Ep$ (i.e. the function expressed by this formula) is hybrid. – The set $\{N^n : n \geq 0\} \cup \{E\}$ is 'functionally complete', i.e. every truth-existence function is expressed by some formula of $\mathcal{L}$CPLE(+). For given an arbitrary truth-existence function f, we can take the disjunction of the 'characteristic formulas' of rows of the truth-existence table corresponding to f which (the rows) have T (i.e. ET or $\not{E}$T) in the value-column. (If e.g. a row gives $\not{E}$T to p and EF to q and these are all the variables present, then its 'characteristic formula' is $(\neg Ep \wedge p) \wedge (Eq \wedge \neg q)$.) Also e.g. $\{N^0, N^2, E\}$ is complete, as all N^n for $n > 0$ are definable from N^2 alone. The mere $\{N^2, E\}$ however is *not* complete, since no zero-ary function is definable (in what seems to me to be the most natural sense of the term in this context) from non--zero-ary functions alone. (But if 'truth-existence function' were redefined so as to exclude zero-ary functions altogether then $\{N^2, E\}$ would be complete.) ⊣

Semantics for S5E(+). An *interpretation* is now a non-empty *set* of truth-existence value assignments to the propositional variables, with designated element. This then induces a set-of-assignments-with-designated-element for all formulas in a natural way: $\Box\varphi$ exists w.r.t. an assignment in the set iff φ exists; $\Box\varphi$ is true w.r.t. an assignment in the set iff φ is true w.r.t. all assignments in the set; and the conditions of existence and truth for N and E are as before. Interpretation σ *verifies* formula φ if the designated element of (extended) σ gives ET or $\not{E}$T to φ. φ is *valid* if every interpretation verifies φ; and similarly for consequence and satisfiability.

Again, whether an interpretation σ verifies a formula φ depends only on the restriction of σ to the variables actually occurring in φ. This can be represented in what we may call the *truth-existence-modal table* for the formula φ, constructed as follows. We write the subformulas of φ, right to left, in decreasing order of complexity until we reach variables (as with truth-tables); then under the variables we write first a 'full subtable' consisting of *all* the attributions of truth-existence values to the variables (as in the classical truth-existence table), then all the 'subtables' where some (but not all) of the rows of the full subtable are omitted (a subtable then represents a non-empty set of assignments of truth-existence values to the variables, and a row within such subtable the interpretation, i.e. set of assignments with designated element); we then 'fill in' the truth-existence values for the compound formulas, left to right, column by column, in accordance with the conditions indicated above for extending an interpretation. So the formula φ is valid, or a '*truth-existence-modal tautology*', iff all entries in its value-column are ET or E̸T.

As example we give here the truth-existence-modal tables for the statements of ‘plain’ necessity $\Box$p, strong necessity $\Box^{s}$p (= $\Box$(Ep ∧ p)), and weak necessity $\Box^{k}$p (= $\Box$(Ep → p)):

p	$\Box$p	Ep	Ep ∧ p	$\Box$(Ep ∧ p)	Ep → p	$\Box$(Ep → p)
ET	EF	ET	ET	EF	ET	EF
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
EF	EF	ET	EF	EF	EF	EF
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
ET	EF	ET	ET	EF	ET	EF
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
EF	EF	ET	EF	EF	EF	EF
ET	EF	ET	ET	EF	ET	ET
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T
ET	EF	ET	ET	EF	ET	EF
EF	EF	ET	EF	EF	EF	EF
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
EF	EF	ET	EF	EF	EF	EF
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
ET	ET	ET	ET	EF	ET	ET
E̸T	E̸T	E̸F	E̸F	E̸F	E̸T	E̸T
ET	EF	ET	ET	EF	ET	EF
EF	EF	ET	EF	EF	EF	EF
ET	EF	ET	ET	EF	ET	ET
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
EF	EF	ET	EF	EF	EF	EF
E̸T	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T
EF	EF	ET	EF	EF	EF	EF
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸F
ET	ET	ET	ET	ET	ET	ET
E̸T	E̸T	E̸F	E̸F	E̸F	E̸T	E̸T
EF	EF	ET	EF	EF	EF	EF
E̸F	E̸F	E̸F	E̸F	E̸F	E̸T	E̸T

Remarks and Questions. (1) I conjecture that, for $\mathcal{L}$S5E formulas, consequence in the sense of this truth-existence-modal value semantics coincides in extension with consequence in the sense of our earlier set semantics.

(2) *Reduction of S5E(+) to S5E.* Again, clearly $\forall\varphi \in \mathcal{L}$S5E(+) $\exists\psi \in \mathcal{L}$S5E: $\vDash \varphi \leftrightarrow \psi$. As before in classical case, we can just replace outermost existence-attributions by the corresponding conjunctions of existence-attributions to variables.

(3) *Axiomatization.* To the previous axiomatization of CPLE(+) we add postulates suitable to S5 and the axiom-schema

$$E\Box\varphi \leftrightarrow E\varphi.$$

The earlier completeness argument also extends straightforwardly enough.

(4) *Logical function theory.* For $n \geq 0$, an n-ary *truth-existence-modal function* is a function f assigning a truth-existence value to each non-empty *set* of n-tuples of truth-existence values with designated element, and satisfying the condition that a value of f is in {ET, EF} iff all terms of the designated element of the argument are in {ET, EF}. This corresponds now to a truth-existence-modal table. Similar remarks as before in classical case apply again here. The 'characteristic formula' of a row is now the conjunction of:

(i) the conjunction of attributions of truth-existence values to the variables corresponding to the given row;

(ii) the possibilizations of (i.e. results of prefixing $\Diamond$ to) such conjunctions corresponding to the rows present in the subtable; and

(iii) the impossibilizations of such conjunctions corresponding to the rows absent from the subtable.

Then again disjunctions of such formulas will serve to express arbitrary truth-existence-modal functions. So e.g. $\{N^0, N^2, E, \Box\}$ is 'functionally complete'. ⊣

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